Invariance of Poisson point processes by moment identities with statistical applications

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Abstract

This paper reviews nonlinear extensions of the Slivnyak-Mecke formula as moment identities for functionals of Poisson point processes, and some of their applications. This includes studying the invariance of Poisson point processes under random transformations, as well as applications to distribution estimation for random sets in stochastic geometry, random graph connectivity, and density estimation for neuron membrane potentials in Poisson shot noise models.

Key words: Poisson point process; moments; stochastic geometry; random-connection model; filtered shot noise processes; Gram-Charlier expansions; neuron membrane potentials.

Mathematics Subject Classification (2010): 60G57; 60G55; 60D05; 60G40; 60G48; 60H07.

1 Introduction

Computing the moments and cumulants of random variables has important applications in probability and statistics, e.g. for the estimation of distributions. In this paper

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we review the computation of moments of random functionals of Poisson point processes via combinatorial identities that extend the Slivnyak-Mecke formula to higher order moments, see [Pri12b, Pri16], and present some applications.

For this, we derive moment identities for stochastic integrals using sums over partitions. Those identities are used to derive criteria for invariance of Poisson random measures under random transformations, and for distribution estimation of the cardinality of random sets based on a Poisson point process.

The moments of random functionals can be used to estimate random graph connectivity in the random connection model using probability generating functions. This approach is based on the computation of the moments of k-hop path counts as sums over non-flat partitions, using extensions of moment identities to multiparameter processes, see [BRSW17].

The calculation of moments can also be applied to estimate the skewness and kurtosis of probability distributions, and to approximate probability densities via Edgeworth and Gram-Charlier expansions. Examples are provided using stochastic differential equations in Poisson shot noise models, with an application to the estimation of probability densities of neuron membrane potentials.

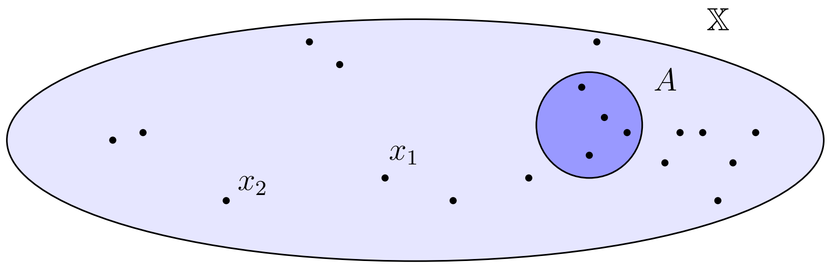
This paper is organized as follows. In Section 2 we review moment identities for Poisson stochastic integrals with random integrands. In Section 3, such identities are specialized to indicator functions of random sets, for application in stochastic geometry. Section 4 deals with applications to the statistics of k-hop counts in the random-connection model, using multiparameter stochastic integrals for the analysis of random graph connectivity. Section 5 considers the moments of Poisson shot noise processes, with an application to the modeling of membrane potential distributions.

2 Moments of Poisson point processes

We consider a Poisson point process with intensity measure $\sigma(dx)$ on the space

$$\Omega^{\mathbb{X}} := \big\{ \xi = \{x_i\}_{i\in I} \subset \mathbb{X} \ : \ \#(A \cap \xi) < \infty \text{ for all compact } A \in \mathcal{B}(\mathbb{X}) \big\}$$

of locally finite configurations on a subset $\mathbb{X} \subset \mathbb{R}^d$, where $\xi(A) = \#\{k \ : \ x_k \in A\}$ denotes the count of configuration points that belong to a measurable subset $A \subset \mathbb{X}$.



For all compact disjoint subsets $A_1, \ldots, A_n$ of $\mathbb{X}$, $n \geq 1$, the mapping

$$\xi \mapsto (\xi(A_1), \ldots, \xi(A_n))$$

is a vector of independent Poisson distributed random variables on $\mathbb{N}$ with respective intensities $\sigma(A_1), \ldots, \sigma(A_n)$. As a consequence, the Poisson stochastic integral with respect to the Poisson random measure with intensity $\sigma(dx)$ on $\mathbb{X}$ has the moment generating function

$$\mathbb{E}\left[\exp\left(\sum_{x\in\xi} h(x)\right)\right] = \exp\left(\int_{\mathbb{X}} (\mathrm{e}^{h(x)} - 1)\sigma(dx)\right). \tag{2.1}$$

Slivnyak-Mecke identity

The Slivnyak-Mecke ([Sli62], [Mec67]) identity allows one to compute the moments of first order stochastic integrals of random integrands as

$$\mathbb{E}\left[\sum_{x\in\xi} u(x,\xi)\right] = \mathbb{E}\left[\int_{\mathbb{X}} \varepsilon_x^+ u(x,\xi)\sigma(dx)\right], \tag{2.2}$$

where ε_x^+ is the addition operator defined on random variables F on $\Omega^{\mathbb{X}}$ as

$$\varepsilon_x^+ F(\xi) = F(\xi \cup \{x\}), \qquad x \in \mathbb{X}.$$

Nonlinear Slivnyak-Mecke identities

Next, we show how the Slivnyak-Mecke identity can be used to derive a covariance formula with random integrands. We have

$$\mathbb{E}\left[\sum_{x_1\in\xi} u_1(x_1,\xi) \sum_{x_2\in\xi} u_2(x_2,\xi)\right] \ = \ \mathbb{E}\left[\sum_{x_1\in\xi}\left(\sum_{x_2\in\xi} u_2(x_2,\xi)\right) u_i(x_1,\xi)\right]$$

$$= \mathbb{E}\left[\int_{\mathbb{X}} \epsilon^+_{x_1}\left(\sum_{x_2\in\xi} u_2(x_2,\xi)u_1(x_1,\xi)\right)\sigma(\mathrm{d}x_1)\right],$$

with

$$\epsilon^+_{x_1}\sum_{x_2\in\xi} u_2(x_2,\xi) = \sum_{x_2\in\xi}\epsilon^+_{x_1}u_2(x_2,\xi) + \epsilon^+_{x_1}u_2(x_1,\xi).$$

Hence, another application of (2.2) yields

$$\begin{aligned}
&\mathbb{E}\left[\sum_{x_1\in\xi} u_1(x_1,\xi)\sum_{x_2\in\xi} u_2(x_2,\xi)\right]\\
&= \mathbb{E}\left[\int_{\mathbb{X}}\sum_{x_2\in\xi}\epsilon^+_{x_1}(u_1(x_1,\xi)u_2(x_2,\xi))\sigma(\mathrm{d}x_1)\right] + \mathbb{E}\left[\int_{\mathbb{X}}\epsilon^+_{x_1}(u_1(x_1,\xi)u_2(x_1,\xi))\sigma(\mathrm{d}x_1)\right]\\
&= \mathbb{E}\left[\int_{\mathbb{X}^2}\epsilon^+_{x_1}\epsilon^+_{x_2}(u_1(x_1,\xi)u_2(x_2,\xi))\sigma(\mathrm{d}x_1)\sigma(\mathrm{d}x_2)\right] + \mathbb{E}\left[\int_{\mathbb{X}}\epsilon^+_{x_1}(u_1(x_1,\xi)u_2(x_1,\xi))\sigma(\mathrm{d}x_1)\right].
\end{aligned}$$

Proposition 2.1 below, see Theorem 1 in [Pri16], can be regarded as a nonlinear extension of the Slivnyak-Mecke formula (2.2) with random integrands $u : \mathbb{X}\times\Omega^{\mathbb{X}} \longrightarrow \mathbb{R}$. The sum (2.3) runs over the partitions $\pi_1,\ldots,\pi_k$ of $\{1,\ldots,n\}$, where $|\pi_i|$ denotes the cardinality of the block π_i, $i=1,\ldots,k$. Given $\mathfrak{z}_n = (z_1,\ldots,z_n)\in\mathbb{X}^n$ we will use the shorthand notation $\varepsilon^+_{\mathfrak{z}_n}$ for the operator

$$(\varepsilon^+_{\mathfrak{z}_n}F)(\xi) = F(\xi\cup\{z_1,\ldots,z_n\}), \qquad \xi\in\Omega^{\mathbb{X}},$$

for F a random variable on $\Omega^{\mathbb{X}}$.

Proposition 2.1 *Let $u : \mathbb{X}\times\Omega^{\mathbb{X}} \longrightarrow \mathbb{R}$ be a (measurable) process. For all $n\geq 1$ we have*

$$\mathbb{E}\left[\left(\sum_{x\in\xi} u(x,\xi)\right)^n\right] = \sum_{\rho\in\Pi[n]}\mathbb{E}\left[\int_{\mathbb{X}^{|\rho|}}\epsilon^+_{\mathfrak{z}_{|\rho|}}\prod_{l=1}^{|\rho|} u^{|\rho_l|}(z_l)\sigma^{\otimes|\rho|}(d\mathfrak{z}_{|\rho|})\right], \qquad (2.3)$$

where the sum runs over all partitions ρ of $\{1,\ldots,n\}$ with cardinality $|\rho|$.

See [DF14] for an extension of (2.3) to point processes admitting Papangelou intensities, and [BRSW17] for an extension to multiparameter processes. This result can be more generally stated as the next joint moment identity for Poisson stochastic integrals with random integrands, cf. Proposition 7 in [Pri16].

Proposition 2.2 *Let $u_1, \ldots, u_p : \mathbb{X} \times \Omega^{\mathbb{X}} \longrightarrow \mathbb{R}$ be random processes, $p \geq 1$. For all $n_1, \ldots, n_p \geq 0$ and $n := n_1 + \cdots + n_p$, We have*

$$\begin{aligned} & \mathbb{E}\left[\left(\sum_{x_1 \in \xi} u_1(x_1, \xi)\right)^{n_1} \cdots \left(\sum_{x_p \in \xi} u_p(x_p, \xi)\right)^{n_p}\right] \\ & = \sum_{k=1}^{n} \sum_{\pi_1 \cup \cdots \cup \pi_k = \{1, \ldots, n\}} \mathbb{E}\left[\int_{\mathbb{X}^k} \epsilon_{x_1}^+ \cdots \epsilon_{x_k}^+ \left(\prod_{j=1}^{k} \prod_{i=1}^{p} u_i^{l_{i,j}^n}(x_j, \xi)\right) \sigma(\mathrm{d}x_1) \cdots \sigma(\mathrm{d}x_k)\right], \end{aligned}$$

where the sum runs over all partitions $\pi_1, \ldots, \pi_k$ of $\{1, \ldots, n\}$ and the power $l_{i,j}^n$ is the cardinality

$$l_{i,j}^n := |\pi_j \cap (n_1 + \cdots + n_{i-1}, n_1 + \cdots + n_i]|, \qquad i = 1, \ldots, k, \quad j = 1, \ldots, p.$$

Proposition 2.2 implies in particular the next joint moment identity. Let $f_1, \ldots, f_p : \mathbb{X} \longrightarrow \mathbb{R}$ be deterministic functions, $p \geq 1$. Then, for any bounded random variable F and $n_1, \ldots, n_p \geq 0$ and $n := n_1 + \cdots + n_p$, we have

$$\begin{aligned} & \mathbb{E}\left[F \left(\sum_{x_1 \in \xi} f_1(x_1)\right)^{n_1} \cdots \left(\sum_{x_p \in \xi} f_p(x_p)\right)^{n_p}\right] \\ & = \sum_{k=1}^{n} \sum_{\pi_1 \cup \cdots \cup \pi_k = \{1, \ldots, n\}} \int_{\mathbb{X}^k} \mathbb{E}\left[\epsilon_{x_1}^+ \cdots \epsilon_{x_k}^+ F\right] \prod_{j=1}^{k} \prod_{i=1}^{p} f_i^{l_{i,j}^n}(x_j) \sigma(\mathrm{d}x_1) \cdots \sigma(\mathrm{d}x_k). \end{aligned}$$

3 Random sets in stochastic geometry

We consider possibly random sets $\mathrm{A}(\xi)$ such that

$$\{\xi \in \Omega^{\mathbb{X}} \ : \ \mathrm{A}(\xi) \subset K\} \in \mathcal{F} \quad \text{for all } K \in \mathcal{K}(X),$$

and let $N(\mathrm{A}(\xi))$ denote the cardinality of $\xi \cap \mathrm{A}(\xi)$. The next proposition is a factorial moment identity for $N(\mathrm{A})$, see Proposition 2.1 in [BP14].

Proposition 3.1 *Let $\mathrm{A}(\xi)$ be a random measurable subset of $\mathbb{X}$. For all $n \geq 1$ and sufficiently integrable random variable F, we have*

$$\mathbb{E}\left[F \ N(\mathrm{A})_{(n)}\right] = \mathbb{E}\left[\int_{X^n} \varepsilon_{\mathfrak{x}_n}^+ (F \mathbf{1}_{\mathrm{A}^n}(x_1, \ldots, x_n)) \, \sigma^{\otimes n}(dx_1, \ldots, dx_n)\right],$$

where $N(\mathrm{A})_{(n)} = N(\mathrm{A})(N(\mathrm{A}) - 1)(N(\mathrm{A}) - n + 1)$ denotes the descending factorial of $N(\mathrm{A})$, $n \geq 1$.

Given K in the collection $\mathcal{K}(X)$ of compact subsets of $\mathbb{X}$, let

$$\mathcal{F}_K := \sigma(\xi(U) \ : \ U \subset K, \ \sigma(U) < \infty)$$

denote the sigma-algebra generated by $\xi \mapsto \xi(U)$, with $U \subset K$ and $\sigma(U) < \infty$. We recall that a random compact set S is called a *stopping set* if

$$\{\xi \in \Omega^{\mathbb{X}} \ : \ \mathrm{S}(\xi) \subset K\} \in \mathcal{F}_K \quad \text{for all } K \in \mathcal{K}(X).$$

In other words, modifying the configuration ξ outside of $\mathrm{S}(\xi)$ does not affect $\mathrm{S}(\xi)$ itself, see [Zuy99] and Definition 2.27 page 335 of [Mol05].

In the sequel, we consider stopping sets S satisfying the following monotonicity and stability conditions.

i) The stopping set S is non-increasing in the sense that

$$\mathrm{S}(\xi \cup \{x\}) \subset \mathrm{S}(\xi), \qquad \xi \in \Omega^{\mathbb{X}}, \quad x \in X.$$

ii) The stopping set S is stable in the sense that

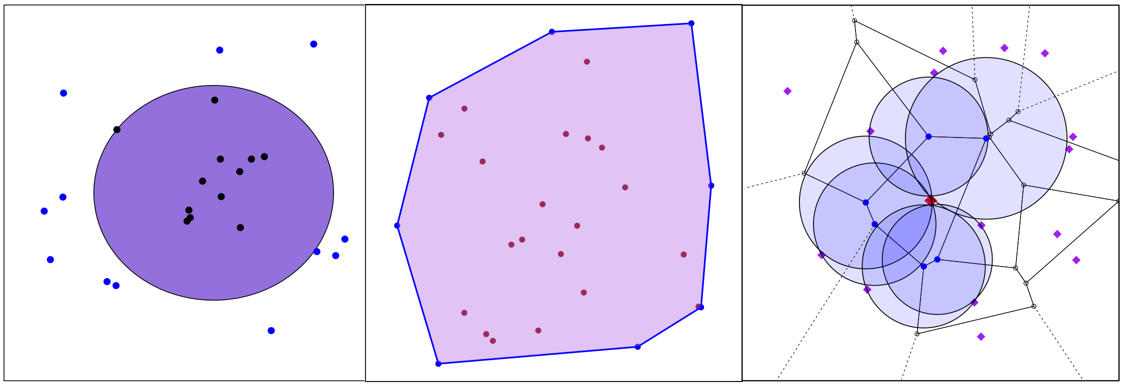
$$x \in \mathrm{S}(\xi) \Longrightarrow x \in \mathrm{S}(\xi \cup \{x\}), \qquad \xi \in \Omega^{\mathbb{X}}, \quad x \in X.$$

Examples of stopping sets satisfying the above conditions can be given as follows:

- The minimal closed ball $\mathrm{S} = B_m$ centered at 0 and containing exactly $m \geq 1$ points, see Figure 1-(a).

- The complement S of the open convex hull $\overline{\mathrm{S}}$ of a Poisson point process inside a convex subset of finite σ-measure in $\mathbb{R}^d$, see Figure 1-(b).

- The Voronoi flower S, which is the union of closed balls centered at the vertices of the Voronoi polygon, containing the point 0 and exactly two other process points, see Figure 1-(c).



(a) Disc B_m with $m = 5$. (b) Convex hull. (c) Voronoi flower.

Figure 1: Examples of stopping sets.

- The complement S of the union of open cones generated by a Boolean-Poisson model on a set of finite σ-measure in $\mathbb{R}^d$, see Figure 2.

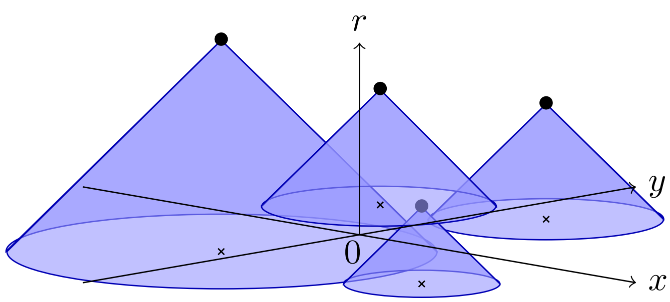


Figure 2: Cones generated by a Boolean-Poisson model.

- Other examples of stopping sets include the Voronoi sausage or the Delaunay lunes, see e.g. [CQZ03] and [Cow06].

From (3.2) and Proposition 3.1 we obtain the next factorial moment identity.

Proposition 3.2 *Let $\overline{\mathrm{S}}$ be the complement of a stable, non-increasing stopping set* S*. For all $n \geq 1$, we have*

$$\mathbb{E}\left[F\, N(\overline{\mathrm{S}})_{(n)}\right] = \mathbb{E}\left[\int_{\overline{\mathrm{S}}^n} \varepsilon^+_{x_1} \cdots \varepsilon^+_{x_n} F\ \sigma(dx_1) \cdots \sigma(dx_n)\right],$$

for F a bounded random variable.

Given S a stopping set, we consider the stopped sigma-algebra generated by S defined as

$$\mathcal{F}_{\mathrm{S}} := \sigma(B \in \mathcal{F}\ :\ B \cap \{\xi \in \Omega^{\mathbb{X}}\ :\ \mathrm{S}(\xi) \subset K\} \in \mathcal{F}_K,\ K \in \mathcal{K}(X)),$$

see Definition 1 in [Zuy99]. As a consequence of Proposition 3.2, we obtain the following invariance result, see Propositions 4.1-4.2 and Corollary 5.2 in [Pri15].

Corollary 3.3 *Consider* $\mathrm{S}(\xi)$ *a* stable *and* non-increasing *stopping set and* $F(\xi)$ *a non-negative* $\mathcal{F}_{\mathrm{S}}$*-measurable random variable with* $\mathbb{E}\left[\mathrm{e}^{z\sigma(\overline{\mathrm{S}})}(1+z)^{\xi(\overline{\mathrm{S}})}F(\xi)\right] < \infty$ *for some* $z > 0$*. We have the Girsanov identity*

$$\mathbb{E}[F(\xi)] = \mathbb{E}\left[\mathrm{e}^{-z\sigma(\overline{\mathrm{S}})}(1+z)^{\xi(\overline{\mathrm{S}})}F(\xi)\right]. \tag{3.1}$$

Relation (3.1) yields the following conditional Laplace transform for $\mathrm{S}(\xi)$ a stable and non-increasing stopping set:

$$\mathbb{E}[\mathrm{e}^{-z\sigma(\mathrm{S})} \mid \xi(\mathrm{S}) = n] = \frac{1}{(1+z)^n}\frac{\mathbb{P}_z(\xi(\mathrm{S}) = n)}{\mathbb{P}(\xi(\mathrm{S}) = n)}, \qquad z > 0, \quad n \in \mathbb{N},$$

where $\mathbb{P}_z$ denotes the Poisson point process distribution with intensity $z\sigma(dx)$, which is consistent with the gamma-type results of Theorem 2 of [MZ96] and Theorem 2 of [Zuy99], and this recovers the gamma distribution of $\sigma(\mathrm{S})$ conditionally to $\xi(\mathrm{S}) = n$, when $\mathbb{P}_z(\xi(\mathrm{S}) = n)$ does not depend on $z > 0$.

Corollary 3.4 *Let* S *be a non-increasing and stable stopping set. Then the complement* $\overline{\mathrm{S}}$ *of* S *satisfies*

$$\mathbb{P}(N(\overline{\mathrm{S}}) = n \mid \mathcal{F}_{\mathrm{S}}) = \frac{\mathrm{e}^{-(\sigma(\overline{\mathrm{S}}))}}{n!}(\sigma(\overline{\mathrm{S}}))^n, \qquad n \geq 0.$$

Proof. We note that the complement $\overline{\mathrm{S}}$ of a stable and non-increasing stopping set S fulfills the condition

$$\varepsilon^+_{\mathfrak{x}_n}(\mathbf{1}_{\overline{\mathrm{S}}}(x_1)\cdots\mathbf{1}_{\overline{\mathrm{S}}}(x_n)) = \mathbf{1}_{\overline{\mathrm{S}}}(x_1)\cdots\mathbf{1}_{\overline{\mathrm{S}}}(x_n), \quad x_1,\ldots,x_n \in X, \quad n \geq 1, \tag{3.2}$$

and apply the factorial moment identity of proposition 3.2. □

Corollary 3.4 shows in particular that, given the stopping set S, the count $N(\overline{\mathrm{S}})$ is a Poisson random variable with intensity $\sigma(\overline{\mathrm{S}})$, see Theorem 3.1 of [BR16], and [Pri12a], when S is the closed complement of the Poisson convex hull $\overline{\mathrm{S}}$. From Corollary 3.4 we can construct an alternative estimator

$$\mathbb{P}(N(\overline{\mathrm{S}}) = n \mid \mathcal{F}_{\mathrm{S}}) = \frac{(\sigma(\overline{\mathrm{S}}))^n}{n!}\mathrm{e}^{-\sigma(\overline{\mathrm{S}})}. \tag{3.3}$$

of the distribution $\mathbb{P}(N(\overline{\mathrm{S}}) = n)$ of the number of Poisson vertices inside the complement $\overline{\mathrm{S}}$ of a stopping set S, in addition to the standard sampling estimator $\mathbf{1}_{\{N(\overline{\mathrm{S}})=n\}}$, see [Pri21] for numerical experiments where the performances of the estimators $\mathbf{1}_{\{N(\overline{\mathrm{S}})=n\}}$ and (3.3) are compared via their respective variances given by $\mathbb{P}(N(\overline{\mathrm{S}}) = n)(1 - \mathbb{P}(N(\overline{\mathrm{S}}) = n))$, and $\mathbb{E}\left[(\sigma(\overline{\mathrm{S}}))^{2n}\mathrm{e}^{-2\sigma(\overline{\mathrm{S}})}\right]/n!^2 - (\mathbb{P}(N(\overline{\mathrm{S}}) = n))^2$.

4 Multiparameter integrals in random graphs

In this section we consider joint moment identities for multiparameter processes $(u_{z_1,\ldots z_r})_{(z_1,\ldots z_r)\in\mathbb{X}^r}$.

- Let $\Pi[n\times r]$ denote the set of partitions of

$$\Delta_{n\times r} := \{1,\ldots,n\}\times\{1,\ldots,r\} = \big\{(k,l)\ :\ k=1,\ldots,n,\ l=1,\ldots,r\big\}.$$

- Given $\rho = \{\rho_1,\ldots,\rho_m\}$ a partition of $\Delta_{n\times r}$, let $\zeta^\rho\ :\ \Delta_{n\times r}\longrightarrow\{1,\ldots,m\}$ let

$$\zeta^\rho(k,l) = p \ \text{ if and only if } (k,l)\in\rho_p,$$

  denote the index of the block ρ_p containing (k,l).

In the next proposition, see Theorem 3.1 in [BRSW17], we use the notation

$$\epsilon^+_{\mathfrak{z}_k} u(z_1,\ldots,z_k,\xi) := u(z_1,\ldots,z_k,\xi\cup\{z_1,\ldots,z_k\}),\quad \mathfrak{z}_n = (z_1,\ldots,z_n)\in\mathbb{X}^n, \tag{4.1}$$

for $(u(z_1,\ldots,z_k,\xi))_{z_1,\ldots,z_k\in\mathbb{X}}$ a multiparameter process.

Proposition 4.1 *We have*

$$\mathbb{E}\left[\left(\sum_{z_1,\ldots,z_r\in\xi} u(z_1,\ldots,z_r,\xi)\right)^n\right] = \sum_{\rho\in\Pi[n\times r]}\mathbb{E}\left[\int_{\mathbb{X}^{|\rho|}}\epsilon^+_{\mathfrak{z}_{|\rho|}}\prod_{k=1}^n u(z^\rho_{\pi_k})\sigma^{\otimes|\rho|}(d\mathfrak{z}_{|\rho|})\right],$$

where $z^\rho_{\pi_k} := (z_{\zeta^\rho(k,1)},\ldots,z_{\zeta^\rho(k,r)})$ *and* $\pi_k := \{(k,1),\ldots,(k,r)\}$, $k=1,\ldots,n$.

When $n=1$, this yields the multivariate version of the Georgii, [NZ79] identity

$$\mathbb{E}\left[\sum_{z_1,\ldots,z_r\in\xi} u(z_1,\ldots,z_r,\xi)\right] = \sum_{\rho\in\Pi[1\times r]}\mathbb{E}\left[\int_{\mathbb{X}^r}\varepsilon^+_{\mathfrak{z}_{|\rho|}} u(z_{\zeta^\rho(1,1)},\ldots,z_{\zeta^\rho(1,r)})\sigma^{\otimes|\rho|}(d\mathfrak{z}_{|\rho|})\right].$$

We write $\pi \preceq \sigma$ when a partition $\pi \in \Pi[n \times r]$ is finer than another partition $\sigma \in \Pi[n \times r]$, i.e. when every block of π is contained in a block of σ. We also write $\rho \wedge \pi = \hat{0}$ when $\mu = \hat{0} := \{\{1,1\}, \ldots, \{n,r\}\}$ is the only partition $\mu \in \Pi[n \times r]$ such that $\mu \preceq \pi$ and $\mu \preceq \rho$, i.e. $|\pi_k \cap \rho_l| \leq 1$ for $k = 1, \ldots, n$, $l = 1, \ldots, |\rho|$. The moment identity in the next proposition is written as a sum over partitions $\rho \in \Pi[n \times r]$ such that the partition diagram $\Gamma(\pi, \rho)$ is *non-flat*, see Chapter 4 of [PT11], where $\pi := (\pi_1, \ldots, \pi_n) \in \Pi[n \times r]$ is given by $\pi_k := \{(k,1), \ldots, (k,r)\}$, $k = 1, \ldots, n$.

Proposition 4.2 *Assume that $u(z_1, \ldots, z_r, \xi) = 0$ whenever $z_i = z_j$, $1 \leq i \neq j \leq r$, $\xi \in \Omega^{\mathbb{X}}$. We have*

$$\mathbb{E}\left[\left(\sum_{z_1,\ldots,z_r \in \xi} u(z_1, \ldots, z_r, \xi)\right)^n\right] = \sum_{\substack{\rho \in \Pi[n \times r] \\ \rho \wedge \pi = \hat{0}}} \mathbb{E}\left[\int_{\mathbb{X}^{|\rho|}} \epsilon^+_{\mathfrak{z}_{|\rho|}} \prod_{k=1}^n u(z^\rho_{\pi_k}) \sigma^{\otimes|\rho|}(d\mathfrak{z}_{|\rho|})\right],$$

where the sum is over non-flat *partition diagrams $\Gamma(\pi, \rho)$, with $z^\rho_{\pi_k} := (z_{\zeta^\rho(k,1)}, \ldots, z_{\zeta^\rho(k,r)})$ and $\pi_k := \{(k,1), \ldots, (k,r)\}$, $k = 1, \ldots, n$.*

Figure 3 shows an example of a non-flat partition of $\Pi[n \times r]$ with $n = 3$ and $r = 2$, which is tagged using the four symbols $\triangle$, $\square$, $\pentagon$, $\bigcirc$, with $\pi_3 = \{(3,1),(3,2)\}$, $\pi_2 = \{(2,1),(2,2)\}$, $\pi_1 = \{(1,1),(1,2)\}$, and $\triangle = \{(1,2),(2,1),(3,2)\}$, $\bigcirc = \{(1,1),(3,1)\}$, $\pentagon = \{(2,2)\}$.

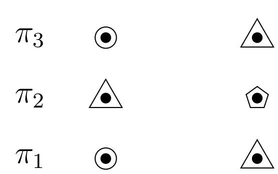


Figure 3: Example of a non-flat partition of $\Pi[3 \times 2]$.

Figure 4 illustrates the non-flat partition technique in the case $n = 3$ and $r = 2$, by displaying 6 out of the 87 multigraphs occurring in the computation of the case of the third moment of the 3-hop count based on possible combinations of common nodes in the product (4.2), together with each corresponding non-flat partition of $[3 \times 2] = \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2)\}$, and every path in the multigraph is followed from the blue node x to the red node y.

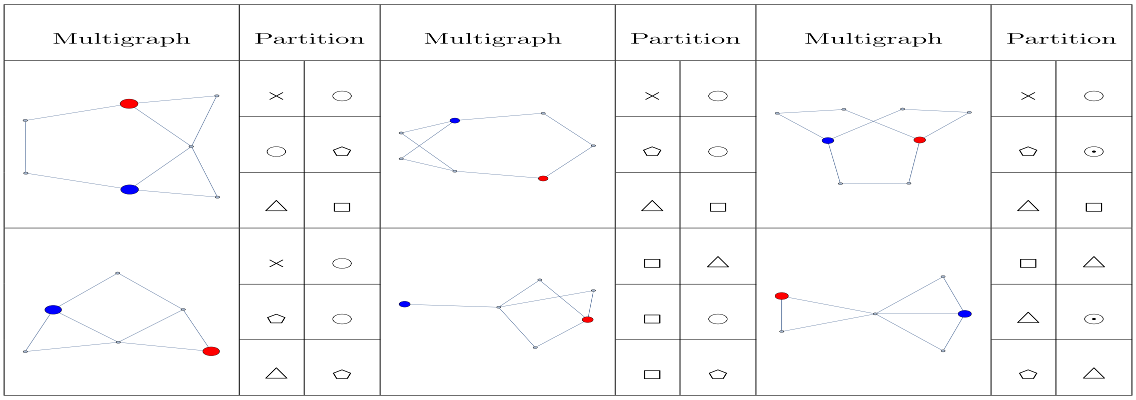


Figure 4: Matching of non-flat partitions of $[3 \times 2]$ to multigraphs with identification of common nodes.

Random-connection model

In the random-connection model, two vertices $x \neq y$ of the Poisson point process ξ of nodes on $\mathbb{X} \subset \mathbb{R}^d$ are independently connected with the probability $H(x, y)$ given ξ in the probability space $\Omega^{\mathbb{X}}$, where $H : X \times X \longrightarrow [0, 1]$ is a connection function, see Figure 5. In particular, the 1-hop count $\mathbb{1}_{\{x \leftrightarrow y\}}$, where $x \leftrightarrow y$ means that $x \in X$ is connected to $y \in X$, is a Bernoulli random variable with parameter $H(x, y)$ and we have the relation

$$\mathbb{E}\left[\epsilon^+_{\mathfrak{z}_r} \prod_{i=0}^{r} \mathbb{1}_{\{z_i \leftrightarrow z_{i+1}\}}(\xi) \,\middle|\, \xi \right] = \prod_{i=0}^{r} H(z_i, z_{i+1})$$

for any subset $\{z_0, \ldots, z_{r+1}\}$ of distinct elements of $\mathbb{X}$, where $\epsilon^+_{\mathfrak{z}_r}$ is the addition operator of point process nodes at the locations $\mathfrak{z}_r = \{z_1, \ldots, z_r\}$, see (4.1).

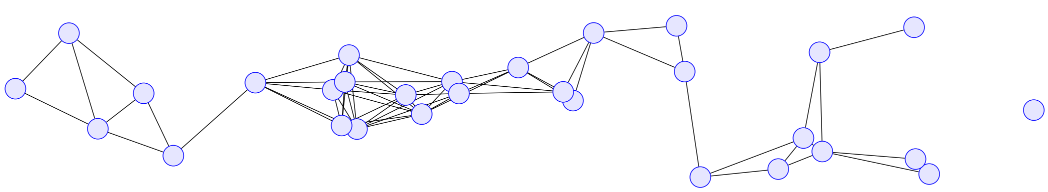


Figure 5: Random-connection graph.

Given $x, y \in \mathbb{X}$ two vertices in $\mathbb{X}$, the count $N_r^{x,y}$ of $(r+1)$-hop paths from x to y as particular cases, i.e. the number of $(r+1)$-hop sequences $z_1, \ldots, z_r \in \xi$ of vertices connecting x to y in the random graph is the multiparameter stochastic integral

$$N_{r+1}^{x,y} = \sum_{z_1,\ldots,z_r \in \xi} u(z_1, \ldots, z_r)$$

over the vertices of the point process ξ, of the multiparameter r-process

$$u(z_1, \ldots, z_r, \xi) := \mathbb{1}_{\{z_i \neq z_j,\ 1 \leq i < j \leq r\}} \mathbb{1}_{\{z_1,\ldots,z_r \in \xi\}} \prod_{i=0}^{r} \mathbb{1}_{\{z_i \leftrightarrow z_{i+1}\}}(\xi)$$

which vanishes on the diagonals in $\mathbb{X}^r$, with $z_0 := x$ and $z_{r+1} := y$. Computing the moments of N_r requires to raise N_r to a given power, creating product terms of the form

$$\prod_{l=1}^{n} u(z_1^{(l)}, \ldots, z_r^{(l)}; \xi) \tag{4.2}$$

where $(z_1^{(l)}, \ldots, z_r^{(l)})$ denotes the sequence of points appearing in the l-th product term. For example, computing the second moment of a 3-hop count requires to identify and count the 7 possible multigraphs that can connect x to y via two 3-hop paths with possible common nodes as in Figure 6, see also Figure 2 in [KGK18], in which every path in each multigraph is followed from the blue node x to the red node y. The difficulty in dealing with common nodes is that they break the independence property in the product (4.2), and as such they have to be dealt with separately.

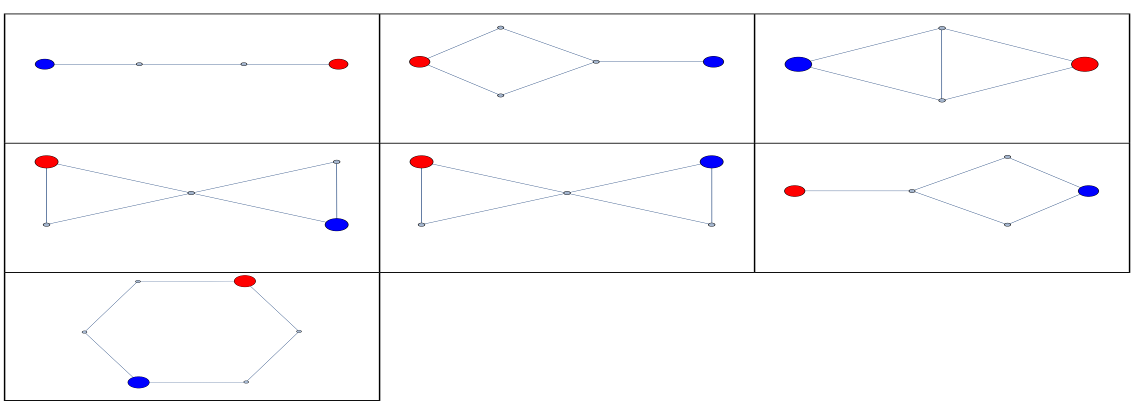


Figure 6: The seven possible ways to join two nodes via two 3-hop paths and their common nodes.

The next proposition, which is a direct consequence of Proposition 4.1, provides a general expression for the moments of the count $N_{r+1}^{x,y}$ of $(r+1)$-hop paths, see [Pri19].

Proposition 4.3 *The moment of order n of the $(r+1)$-hop count between $x, y \in \mathbb{X}$ is given by*

$$\mathbb{E}\left[\left(N_{r+1}^{x,y}\right)^n\right] = \sum_{\substack{\rho\in\Pi[n\times r]\\ \rho\wedge\pi=\hat{0}}} \mathbb{E}\left[\int_{\mathbb{X}^{|\rho|}} \prod_{l=1}^{n}\prod_{i=0}^{r} H^{1/n_{l,i}^{\rho}}(z_{\zeta^\rho(l,i)}, z_{\zeta^\rho(l,i+1)})\sigma^{\otimes|\rho|}(d\mathfrak{z}_{|\rho|})\right],$$

where $z_0 = x$, $z_{r+1} = y$, $\zeta^\rho(l,0) = 0$, $\zeta^\rho(l, r+1) = r+1$, and

$$n_{l,i}^{\rho} := \#\big\{(p,j) \in \{1,\ldots,n\}\times\{0,\ldots,r\} \ : \ \{\zeta^\rho(l,i), \zeta^\rho(l,i+1)\} = \{\zeta^\rho(p,j), \zeta^\rho(p,j+1)\}\big\}.$$

In particular, the first order moment of the $(r+1)$-hop count between $x \in \mathbb{X}$ and $y \in Y$ is given as

$$\begin{aligned} H^{(r+1)}(z_0, z_{r+1}) &:= \mathbb{E}\left[\sum_{z_1,\ldots,z_r\in\xi} u(z_1,\ldots,z_r,\xi)\right] \\ &= \int_{\mathrm{R}^d}\cdots\int_{\mathrm{R}^d}\prod_{i=0}^{r} H(z_i, z_{i+1})\sigma(dz_1)\cdots\sigma(dz_r), \quad z_0, z_{r+1}\in\mathbb{R}^d. \end{aligned}$$

The 2-hop count between $x \in \mathbb{X}$ and $y \in Y$ is given by the first order integral

$$\sum_{z\in\xi} u(z,\xi) = \sum_{z\in\xi} \mathbb{1}_{\{x\leftrightarrow z\}}\mathbb{1}_{\{z\leftrightarrow y\}}(\xi) = \sum_{z\in\xi} \mathbb{1}_{\{x\leftrightarrow z\}}\mathbb{1}_{\{z\leftrightarrow y\}},$$

and its moment of order n is

$$\begin{aligned} \mathbb{E}\left[\left(\sum_{z\in\xi} u(z,\xi)\right)^n\right] &= \sum_{\rho\in\Pi[n\times 1]} \int_{\mathbb{X}^{|\rho|}} \prod_{l=1}^{|\rho|} (H(x,z_l)H(z_l,y))\sigma^{\otimes|\rho|}(dz_1,\ldots,dz_{|\rho|}) \\ &= \sum_{k=1}^{n} S(n,k)\left(\int_{\mathrm{R}^d} H(x,z)H(z,y)\sigma(dz)\right)^k \\ &= \sum_{k=1}^{n} S(n,k)\left(H^{(2)}(x,y)\right)^k, \end{aligned}$$

which shows that the 2-hop count between $x \in \mathbb{X}$ and $y \in Y$ is a Poisson random variable with mean $H^{(2)}(x,y)$.

Variance of 3-hop counts

When $n = 2$ and $r = 3$ Proposition 4.3 allows us to compute the variance of the 3-hop count between $x \in \mathbb{X}$ and $y \in Y$, as follows:

$$\mathrm{Var}\left[N_3^{x,y}\right] = H^{(3)}(x,y) + 2\int_{\mathbb{X}} H(x,z_1)H^{(2)}(z_1,y)H^{(2)}(z_1,y)\sigma(dz_1) \tag{4.3}$$

$$+ 2 \int_{\mathbb{X}} H(x,z_1) H^{(2)}(x,z_1) H^{(2)}(z_1,y) H(z_1,y) \sigma(dz_1)$$
$$+ \int_{\mathbb{X}^2} H(x,z_1) H(z_1,z_2) H(z_2,y) H(x,z_2) H(z_1,y) \sigma^{\otimes 2}(dz_1,dz_2),$$

In the case of a Poisson point process with flat intensity $\sigma(dx) = \lambda dx$ on $\mathbb{X}$, $\lambda > 0$ with a Rayleigh fading function $H(x,y)$ of the form

$$H_\beta(x,y) := \mathrm{e}^{-\beta \|x-y\|^2}, \qquad x,y \in \mathbb{R}^d, \quad \beta > 0,$$

we have

$$H_\beta^{(2)}(x,y) = \lambda \int_{\mathbb{R}^d} H_\beta(x,z) H_\beta(z,y) dz = \lambda \left(\frac{\pi}{2\beta} \right)^{d/2} \mathrm{e}^{-\|x-y\|^2/2},$$

and (4.3) recovers the variance

$$\begin{aligned} \mathrm{Var}\left[N_3^{x,y}\right] &= 2\lambda^3 \left(\frac{\pi^3}{8\beta^3} \right)^{d/2} \mathrm{e}^{-\beta\|x-y\|^2/2} + \lambda^2 \left(\frac{\pi^2}{3\beta^2} \right)^{d/2} \mathrm{e}^{-\beta\|x-y\|^2/3} \\ &\quad + 2\lambda^3 \left(\frac{\pi^3}{12\beta^3} \right)^{d/2} \mathrm{e}^{-3\beta\|x-y\|^2/4} + \lambda^2 \left(\frac{\pi^2}{8\beta^2} \right)^{d/2} \mathrm{e}^{-\beta\|x-y\|^2} \end{aligned}$$

of 3-hop counts between $x \in \mathbb{X}$ and $y \in Y$, see Theorem II.2 of [KGK18]. The knowledge of moments can provide accurate numerical estimates of the probability $P(N_k^{x,y} > 0)$ of at least one k-hop path by expressing it as a series of factorial moments, see [KGK18].

5 Moments of Poisson shot noise processes

We consider a Poisson point process $\xi(dx)$ with intensity measure $\sigma(dt,d\theta)$ on $\mathbb{X} = \mathbb{R} \times S$, where $S = [0,N]$, and the N shot noise processes given by

$$Q_k(t,\xi) = \sum_{(s_j,\theta_j)\in\xi} g_k(t - s_j, \theta_j), \qquad k = 1,\ldots,N,$$

where the shot noise kernels $g_k(u,\theta)$ are such that $g_k(u,\theta) = 0$ for all $u < 0$ and $\theta \in S$. In this framework, the moment generating function of $Q_k(t,\xi)$ is given from (2.1) as

$$\mathbb{E}\left[\exp\left(Q_k(t,\xi)\right)\right] = \exp\left(\int_{(-\infty,t]\times S} (\mathrm{e}^{g_k(t-u,\theta)} - 1) \sigma(du,d\theta) \right).$$

Consider the Poisson shot noise stochastic differential equation

$$\tau \frac{dY_N}{dt}(t,\xi) = -Y_N(t,\xi) + \sum_{k=1}^{N} (w_k - Y_N(t,\xi)) Q_k(t,\xi), \tag{5.1}$$

where $\tau > 0$ and $w_1, \ldots, w_n \in \mathbb{R}$, whose solution is the filtered shot noise process

$$\begin{aligned} Y_N(t,\xi) &= \frac{1}{\tau} \sum_{k=1}^{N} w_k \int_{-\infty}^{t} Q_k(z,\xi) \mathrm{e}^{-\int_z^t Q_0(u,\xi)du} dz \\ &= \frac{1}{\tau} \int_{-\infty}^{t} \mathrm{e}^{-\int_z^t Q_0(s,\xi)ds} \sum_{(s_j,\theta_j)\in\xi} f^{(w)}(z - s_j, \theta_j) dz, \quad t \in \mathbb{R}, \end{aligned} \tag{5.2}$$

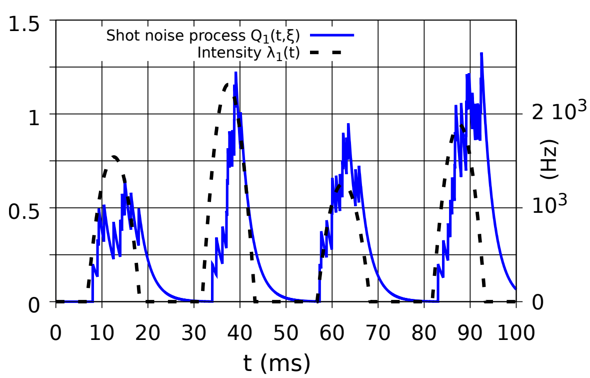
where

$$Q_0(u,\xi) := \frac{1}{\tau} + \frac{1}{\tau} \sum_{k=1}^{N} Q_k(u,\xi) = \frac{1}{\tau} + \frac{1}{\tau} \sum_{(s_j,\theta_j)\in\xi} f(u - s_j, \theta_j),$$

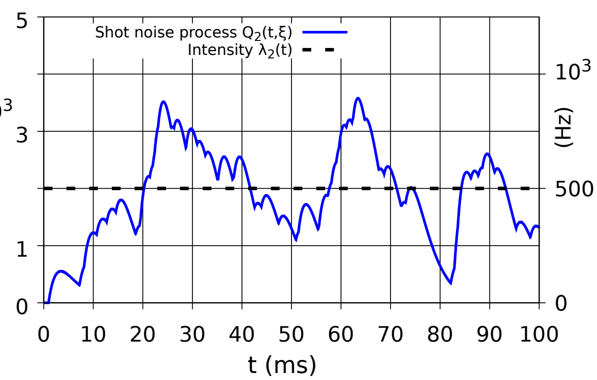
with

$$f(z,\theta) := \sum_{k=1}^{N} g_k(z,\theta) \quad \text{and} \quad f^{(w)}(z,\theta) := \sum_{k=1}^{N} w_k g_k(z,\theta), \quad z \in \mathbb{R}, \ \theta \in S,$$

see e.g. § 2.1 of [BD15a] and [BD15b]. The following numerical examples use the parameters of the double source model of [BD15a] for the modeling of neuron membrane potentials, where $N = 2$, $\lambda_2(t) = 500$Hz and $\lambda_1(t)$ is a periodic function of time, $t \in [0, 100]$, see Figure 7.



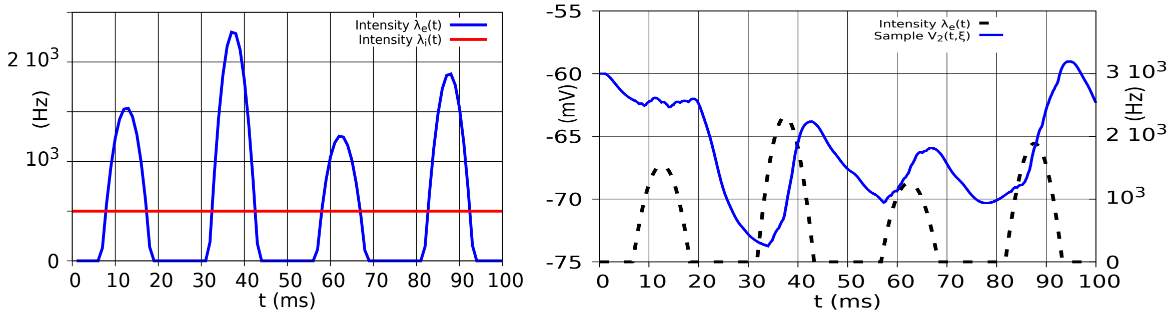
(a) Shot noise process with intensity $\lambda_1(t)$.



(b) Shot noise process with intensity $\lambda_2(t)$.

Figure 7: Shot noise processes $Q_1(t,\xi)$ and $Q_2(t,\xi)$.

Figure 8 presents the graphs of the intensities $\lambda_1(t)$, $\lambda_2(t)$ and a numerical simulation of $V_2(t,\xi)$ in the double source model.



(a) Intensities $\lambda_1(t)$ and $\lambda_2(t)$. (b) Mean and standard deviation of $V_2(t,\xi)$.

Figure 8: Sample of $V_2(t,\xi)$ with mean, standard deviation and intensities $\lambda_1(t)$, $\lambda_2(t)$.

Computation of joint moments

The next proposition gives a general formula for the computation of the joint moments of $Y_N(t_1,\xi),\ldots,Y_N(t_n,\xi)$ in the multiple source model as a direct consequence of (5.2).

Proposition 5.1 *We have the joint moment identity*

$$\mathbb{E}[Y_N(t_1,\xi)\cdots Y_N(t_n,\xi)] = \frac{1}{\tau^n}\int_{-\infty}^{t_1}\cdots\int_{-\infty}^{t_n} m_{n,N}(z_1,\ldots,z_n;t_1,\ldots,t_n)dz_1\cdots dz_n,$$

where

$$m_{n,N}(z_1,\ldots,z_n;t_1,\ldots,t_n) := \mathbb{E}\left[\prod_{k=1}^{n}\left(\mathrm{e}^{-\int_{z_l}^{t_l}Q_0(u,\xi)du}\sum_{(u_j,\theta_j)\in\xi} f^{(w)}(z_k-u_j,\theta_j)\right)\right].$$

The functions $m_{n,N}(z_1,\ldots,z_n;t_1,\ldots,t_n)$ can be evaluated from Proposition 2.2 as a sum over the set $\Pi[n]$ of partitions $\pi=\{\pi_1,\ldots,\pi_k\}$ of $\{1,\ldots,n\}$ with cardinality $k=|\pi|=1,\ldots,n$, as

$$m_{n,N}(z_1,\ldots,z_n;t_1,\ldots,t_n)$$
$$=\mathbb{E}\left[\mathrm{e}^{-\sum_{l=1}^{n}\int_{z_l}^{t_l}Q_0(u,\xi)du}\right]\sum_{\pi\in\Pi[n]}\prod_{j=1}^{|\pi|}\int_{(-\infty,\hat{z}_{\pi_j}]\times S}\prod_{l=1}^{n}\mathrm{e}^{-\frac{1}{\tau}\int_{z_l}^{t_l}f(u-y,\eta)du}\prod_{i\in\pi_j}f^{(w)}(z_i-y,\eta)\sigma(\mathrm{d}y,\mathrm{d}\eta),$$

$(z_1,\ldots,z_n)\in(-\infty,t_1]\times\cdots\times(-\infty,t_n]$, with $\hat{z}_{\pi_j}=\min_{i\in\pi_j}z_i$, where, by (2.1), we have

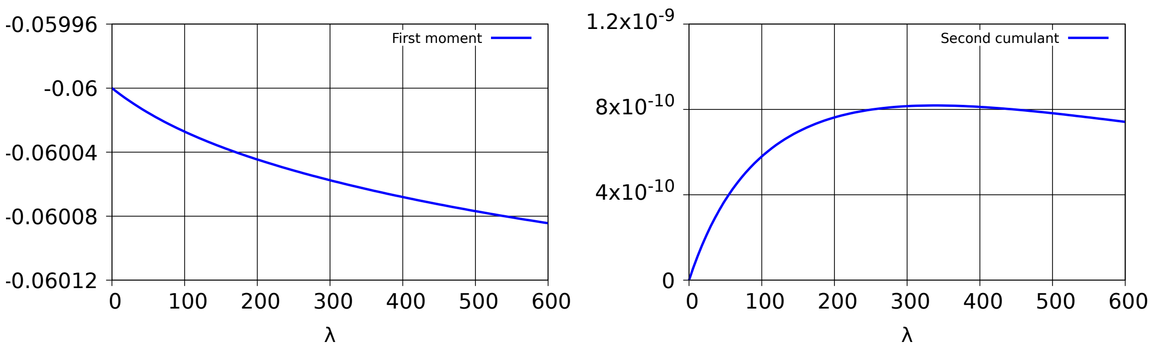
$$\mathbb{E}\left[\mathrm{e}^{-\sum_{l=1}^{n}\int_{z_l}^{t_l}Q_0(u,\xi)du}\right]$$

$$= \mathrm{e}^{-\frac{1}{\tau}\sum_{l=1}^{n}(t_l - z_l)} \exp\left(\int_{(-\infty, \max(t_1,\dots,t_n)]\times S} \big(\mathrm{e}^{-\frac{1}{\tau}\sum_{l=1}^{n}\int_{z_l}^{t_l} f(u-s,\theta)du} - 1 \big) \sigma(ds, d\theta) \right).$$

Figures 9, 10 and 11 present the evolutions of the mean κ_1, variance κ_2, third and fourth cumulants κ_3, κ_4, and skewness and excess kurtosis

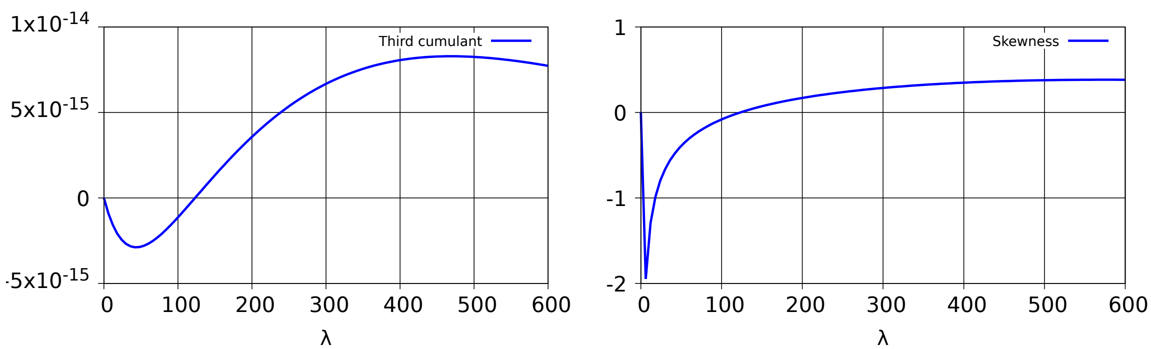
$$\frac{\kappa_3}{(\kappa_2)^{3/2}} = \frac{\mathbb{E}[(V_2 - \mathbb{E}[V_2])^3]}{(\mathbb{E}[(V_2 - \mathbb{E}[V_2])^2])^{3/2}} \quad \text{and} \quad \frac{\kappa_4}{(\kappa_2)^2} = \frac{\mathbb{E}[(V_2 - \mathbb{E}[V_2])^4]}{(\mathbb{E}[(V_2 - \mathbb{E}[V_2])^2])^2} - 3 \qquad (5.3)$$

of the potential $V_2(t, \xi)$, computed from Proposition 5.1 as functions of the arrival intensity parameter λ at $t = 0.2$.



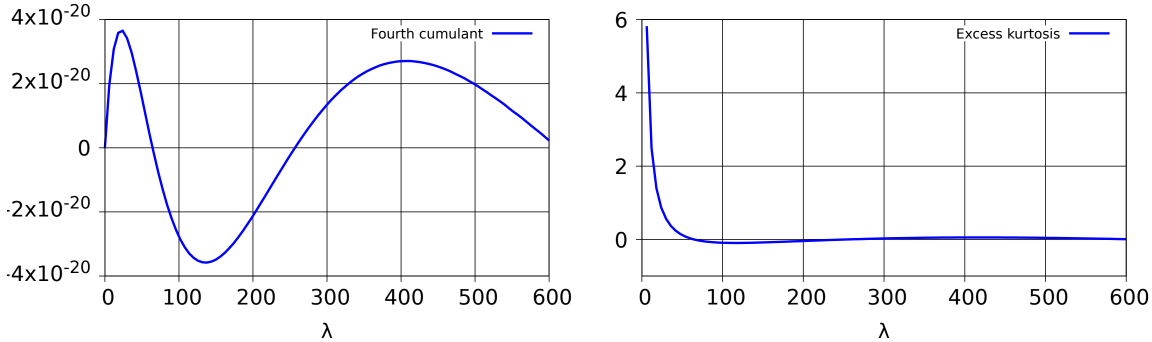
(a) First moment of $V_2(t, \xi)$. (b) Second cumulant of $V_2(t, \xi)$.

Figure 9: First and second cumulants of $V_2(t, \xi)$ at $t = 0.2$.



(a) Third cumulant of $V_2(t, \xi)$. (b) Skewness of $V_2(t, \xi)$.

Figure 10: Third cumulant and skewness of $V_2(t, \xi)$ at $t = 0.2$.



(a) Fourth cumulant of $V_2(t,\xi)$. (b) Excess kurtosis of $V_2(t,\xi)$.

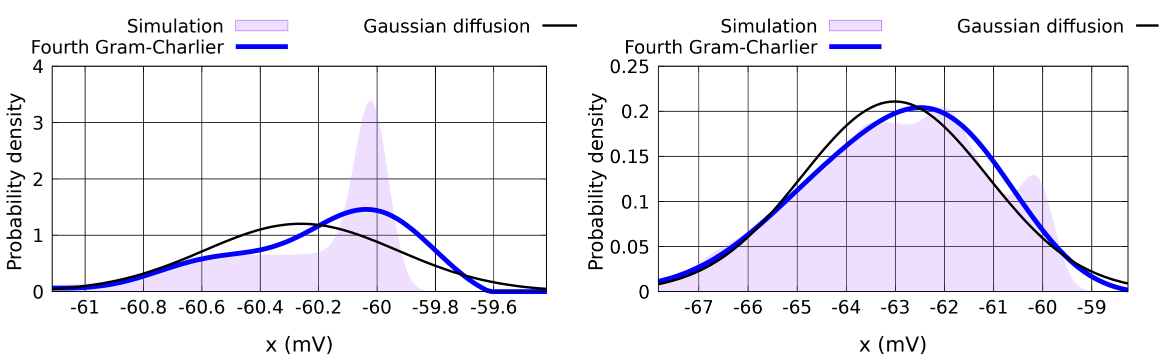
Figure 11: Fourth cumulant and excess kurtosis of $V_2(t,\xi)$ at $t=0.2$.

Gram-Charlier expansions

The Gram-Charlier expansion of the continuous probability density function $\phi_X(x)$ of a random variable X, see § 17.6 of [Cra46], is given by

$$\phi_X(x) = \frac{1}{\sqrt{\kappa_2}}\varphi\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right) + \frac{1}{\sqrt{\kappa_2}}\sum_{n=3}^{\infty} c_n H_n\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right)\varphi\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right), \tag{5.4}$$

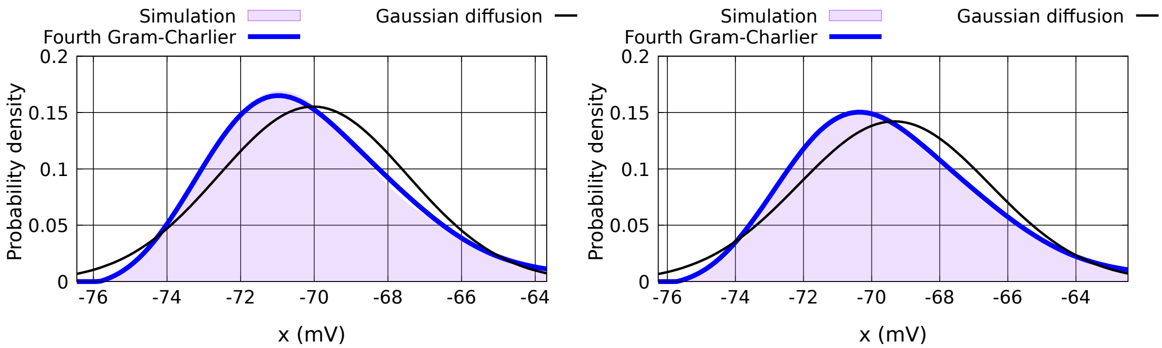
where $\varphi(x)$ is the standard normal density function, $H_n(x)$ denotes the Hermite polynomial of degree n, and the sequence $(c_n)_{n\geq 3}$ is given from the cumulants $(\kappa_n)_{n\geq 1}$ of X. In particular, the coefficients c_3 and c_4 can be expressed from the skewness $\kappa_3/\kappa_2^{3/2}$ and the excess kurtosis κ_4/κ_2^2 as $c_3 = \kappa_3/(3!\kappa_2^{3/2})$ and $c_4 = \kappa_4/(4!\kappa_2^2)$, which are computed from (5.3) Figures 12 and 13 present the Gram-Charlier density expansions (5.4) at different times for the estimation of the probability density function of the membrane potential $V_2(t,\xi)$ in the double source model (5.1) of Figure 8, see [Pri20] for details.



(a) t=1 ms. (b) t=5 ms.

Figure 12: Gram-Charlier density expansions vs simulated densities.

In comparison with the Gaussian diffusion approximation with matching mean and variance, the fourth-order Gram-Charlier approximations provide a better fit of the actual probability densities obtained by Monte Carlo simulation of (5.2) (purple areas), which show time-varying skewness.



(a) t=30 ms.

(b) t=55 ms.

Figure 13: Gram-Charlier density expansions vs simulated densities.

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