A Malliavin calculus approach to sensitivity analysis in insurance

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Abstract

Using the Malliavin calculus on Poisson space and a method initiated by Fournié et al. (1999) for continuous financial markets, we compute the probability density of risk reserve processes and the sensitivities of probabilities of ruin at a given date for insurance portfolios under interest force. The simulation graphs provided show that this method is computationally more efficient than the standard approximation of derivatives by finite differences.

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1 Introduction

In Norberg (2002) a method based on differential equations is proposed for the computation of sensitivities of conditional expected values of reserve processes in life insurance. In this paper we present a sensitivity analysis with respect to a parameter ζ for expectations of the form $E[h(U_\zeta(T))]$, where $U_\zeta(T)$ is the value at time T

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of a risk reserve process, ζ represents the initial reserve x or the interest rate r, and h is a not necessarily smooth, arbitrary integrable function. In particular when h is the indicator function $1_{(-\infty,\zeta]}$, this corresponds to the density of ruin probabilities at a given date. Our method relies on the Malliavin calculus, which has been recently applied to numerical computations of price sensitivities of financial derivatives in continuous markets (Fournié et al., 1999) and in a market with jumps (El Khatib and Privault, 2004).

In models with interest force, probabilities of ruin at a given date have densities with respect to the Lebesgue measure, and we present a formula that allows for faster and more accurate numerical computation of this density. This method is also applied to compute the sensitivity of probabilities of ruin at a given date with respect to the initial reserve x and the interest rate parameter r. More precisely we will compute derivatives of the form

$$\frac{\partial}{\partial \zeta} E\left[h(U_\zeta(T))\right],$$

where

$$U_\zeta(T) = g(\zeta) + \int_0^T f_\zeta(t) dX(t),$$

$(X(t))_{t\in[0,T]}$ is a compound Poisson process representing the number of claims occurring in $(0,T]$ and ζ is a parameter (initial reserve x, or interest force r). Such derivatives may be expressed as

$$\frac{\partial}{\partial \zeta} E\left[h(U_\zeta(T))\right] = E\left[\left(\partial_\zeta g(\zeta) + \int_0^T \partial_\zeta f_\zeta(t) dX(t)\right) h'\left(U_\zeta(T)\right)\right].$$

However this expression makes sense only when h is differentiable, in particular h can not be an indicator function, hence the above expression can not be used for ruin probabilities. Alternatively this derivative can be estimated by finite differences:

$$\frac{1}{2\varepsilon} E\left[h(U_{\zeta+\varepsilon}(T)) - h(U_{\zeta-\varepsilon}(T))\right], \tag{1}$$

but this approximation yields poor convergence results when combined with Monte Carlo methods, as shown in the simulations of Section 5. Instead of (1) we will show that $\frac{\partial}{\partial \zeta} E\left[h(U_\zeta(T))\right]$ can be expressed as

$$\frac{\partial}{\partial \zeta} E\left[h(U_\zeta(T))\right] = E\left[W_\zeta h(U_\zeta(T))\right], \tag{2}$$

where W_ζ is a random variable called a weight which is explicitly computable and independent of h. Expression (2) above yields a substantial improvement over the finite difference method (1) in the precision and speed of Monte Carlo numerical simulations. This formula is obtained by integration by parts on the Poisson space, using a gradient operator which acts on the Poisson jump times of $(X(t))_{t\in\mathbb{R}_+}$. Our approach actually requires the considered random variable $U_\zeta(T)$ to be sufficiently smooth to be in the domain of D_w with $D_w U_\zeta(T) \neq 0$, a.s. These assumptions are linked to the existence of density with respect to the Lebesgue measure for the probability law of $U_\zeta(T)$. For example, D_w vanishes on functions of the Poisson random variable $N(T)$, which do not have a density, and this excludes in particular models without interest force from our analysis.

We proceed as follows. Section 2 contains preliminaries on the Malliavin calculus on Poisson space and on the differentiability of random functionals. In Section 3 we present the integration by parts formula which is the main tool to compute sensitivities (i.e. derivatives with respect to ζ) using a random variable called a weight. The model and explicit computations for reserve processes are presented in Section 4. In Section 5 we provide numerical simulations which demonstrate the efficiency of the Malliavin approach over finite difference methods.

2 Malliavin Calculus on Poisson space

This section gives a presentation of Malliavin calculus on Poisson space of Carlen and Pardoux (1990), Privault (1994, 1999), adapted to our framework. Let $(N(t))_{t\in[0,T]}$ be a standard Poisson process with intensity $\lambda > 0$ and jump times $(T_k)_{k\geq 1}$, on a probability space $(\Omega, \mathcal{F}_T, P)$. Let $\mathcal{C}_0([0,T])$, resp. $\mathcal{C}_0^1([0,T])$, denote the space of continuous, resp. continuously differentiable, functions on $[0,T]$, and such that $w(0) = w(T) = 0$.

Definition 2.1 *Given $T > 0$, let $\mathcal{S}_T$ denote the set of smooth Poisson functionals of the form*

$$F = f_0 \mathbf{1}_{\{N(T)=0\}} + \sum_{n=1}^{m} \mathbf{1}_{\{N(T)=n\}} f_n(T_1, \ldots, T_n), \tag{3}$$

where $f_0 \in \mathbb{R}$ and $f_n \in \mathcal{C}^1([0,T]^n)$, $1 \leq n \leq m$, is symmetric in n variables, $m \geq 1$.

Note that $\mathcal{S}_T$ is an algebra dense in $L^p(\Omega, \mathcal{F}_T, P)$, $p \geq 2$, and recall that under P we have, for all $F \in \mathcal{S}_T$ of the form (3):

$$E[F] = e^{-\lambda T} f_0 + e^{-\lambda T} \sum_{n=1}^{m} \frac{\lambda^n}{n!} \int_0^T \cdots \int_0^T f_n(t_1, \ldots, t_n) dt_1 \cdots dt_n.$$

Definition 2.2 *Given $w \in \mathcal{C}_0([0,T])$, let D_w denote the gradient operator defined on $F \in \mathcal{S}_T$ of the form (3) by*

$$D_w F = -\sum_{n=1}^{m} \mathbf{1}_{\{N(T)=n\}} \sum_{k=1}^{n} w(T_k) \partial_k f_n(T_1, \ldots, T_n),$$

where $\partial_k f_n$ denotes the partial derivative of f_n with respect to its k-th variable.

The next proposition is proved by finite dimensional integration by parts on jump times conditionally to the value of $N(T)$. It shows in particular that D_w is closable, hence D_w can be extended to the space $\mathrm{Dom}\,(D_w)$ of functionals $F \in L^2(\Omega)$ for which there exists a sequence $(F_n)_{n\in\mathbb{N}} \subset \mathcal{S}_T$ converging to F such that $(D_w F_n)_{n\in\mathbb{N}}$ converges in $L^2(\Omega)$. For all such $F \in \mathrm{Dom}\,(D_w)$ we let $D_w F = \lim_{n\to\infty} D_w F_n$, and $D_w F$ is well-defined due to the closability of D_w. Similarly, the adjoint D_w^* of D_w will be shown to have the closability property and will be extended to its closed domain $\mathrm{Dom}\,(D_w^*)$. Throughout this paper, $w'(t)$ denotes the derivative with respect to the time parameter t.

Proposition 2.1 *Let $w \in \mathcal{C}_0^1([0,T])$.*
a) The operator D_w is closable and admits a closable adjoint D_w^* such that

$$E[G D_w F] = E[F D_w^* G], \quad F, G \in \mathcal{S}_T. \tag{4}$$

b) For all $F \in \mathrm{Dom}\,(D_w) \bigcap L^4(\Omega)$ we have $F \in \mathrm{Dom}\,(D_w^*)$ and:

$$D_w^* F = F \int_0^T w'(t) dN(t) - D_w F. \tag{5}$$

Proof. By standard integration by parts we first prove (4) when $G = 1$, using the boundary condition $w(0) = w(T) = 0$:

$$E[D_w F] = -e^{-\lambda T} \sum_{n=1}^{m} \frac{\lambda^n}{n!} \int_0^T \cdots \int_0^T \sum_{k=1}^{n} w(t_k) \partial_k f_n(t_1, \ldots, t_n) dt_1 \cdots dt_n$$

$$
\begin{aligned}
&= e^{-\lambda T} \sum_{n=1}^{m} \frac{\lambda^n}{n!} \int_0^T \cdots \int_0^T f_n(t_1, \ldots, t_n) \sum_{k=1}^{n} w'(t_k) dt_1 \cdots dt_n \\
&= E\left[F \sum_{k=1}^{k=N(T)} w'(T_k)\right] = E\left[F \int_0^T w'(t) dN(t)\right].
\end{aligned}
$$

Next we define $D_w^* G$, $G \in \mathcal{S}_T$, by (5), with for all $F \in \mathcal{S}_T$:

$$
E[GD_wF] = E[D_w(FG) - FD_wG] = E\left[F\left(G\int_0^T w'(t)dN(t) - D_wG\right)\right] = E[FD_w^*G],
$$

which proves (4). The closability of D_w then follows from the integration by parts formula (4): if $(F_n)_{n\in\mathbb{N}} \subset \mathcal{S}_T$ is such that $F_n \to 0$ in $L^2(\Omega)$ and $DF_n \to U$ in $L^2(\Omega)$, then (4) implies

$$
\begin{aligned}
|E[UG]| &\leq |E[F_n D_w^* G] - E[UG]| + |E[F_n D_w^* G]| \\
&= |E[(D_w F_n - U)G]| + |E[F_n D_w^* G]| \\
&\leq \|D_w F_n - U\|_{L^2(\Omega)} \|G\|_{L^2(\Omega)} + \|F_n\|_{L^2(\Omega)} \|D_w^* G\|_{L^2(\Omega)}, \qquad n \in \mathbb{N},
\end{aligned}
$$

hence $E[UG] = 0$, $G \in \mathcal{S}_T$, i.e. $U = 0$. The proof of the closability of D_w^* is similar. Finally, (5) is extended by closability to $F \in \mathrm{Dom}\,(D_w) \bigcap L^4(\Omega)$. □

In particular, $D_w^* \mathbf{1}_\Omega$ coincides with the Poisson stochastic integral of w':

$$
D_w^* \mathbf{1}_\Omega = \int_0^T w'(t) dN(t).
$$

A conditional integration by parts formula can also be obtained, and will be used in the proof of Proposition 3.1 below.

Corollary 2.1 *Let $\mathcal{G}$ denote a sub σ-algebra of $\mathcal{F}$ such that for all $A \in \mathcal{G}$,*

$$
\mathbf{1}_A \in \mathrm{Dom}\,(D_w) \quad \text{and} \quad D_w \mathbf{1}_A = 0. \tag{6}
$$

Then we have for $w \in \mathcal{C}_0^1([0,T])$:

$$
E[GD_wF \mid \mathcal{G}] = E[FD_w^*G \mid \mathcal{G}], \quad F \in \mathrm{Dom}\,(D_w),\ G \in \mathrm{Dom}\,(D_w^*).
$$

Proof. We have from (4):

$$
E[\mathbf{1}_A G D_w F] = E[G D_w(\mathbf{1}_A F)] = E[\mathbf{1}_A F D_w^* G], \quad F, G \in \mathcal{S}_T,\ A \in \mathcal{G}, \tag{7}
$$

and this relation is extended by closability to $F \in \mathrm{Dom}\,(D_w)$ and $G \in \mathrm{Dom}\,(D_w^*)$.

□

For example, for every function $g : \mathbb{N} \to \mathbb{R}$ with support in $[0, m]$, $m \geq 1$, we have

$$g(N(T)) = \sum_{n=0}^{m} 1_{\{N(T)=n\}} g(n),$$

hence

$$D_w(g(N(T))) = 0, \quad w \in \mathcal{C}_0([0, T]),$$

and as a consequence, taking $\mathcal{G} = \sigma(N(T))$ we get:

$$E[GD_w F \mid N(T)] = E[FD_w^* G \mid N(T)], \quad F, G \in \mathcal{S}_T.$$

The following proposition provides a derivation rule for Poisson stochastic integrals.

Proposition 2.2 *Let $f(\cdot, k) \in \mathcal{C}^1([0, T])$, $k \in \mathbb{N}$, and let $w \in \mathcal{C}_0([0, T])$. Then*

$$D_w \int_0^T f(t, N(t^-)) dN(t) = -\int_0^T w(t) f'(t, N(t^-)) dN(t), \tag{8}$$

where $f'(t, k)$ denotes the derivative of $f(t, k)$ with respect to t.

Proof. We have

$$\begin{aligned} D_w \int_0^T f(t, N(t^-)) dN(t) &= D_w \left(\lim_{m \to \infty} \sum_{n=1}^{m} \mathbf{1}_{\{N(T)=n\}} \sum_{k=1}^{n} f(T_k, k-1) \right) \\ &= -\lim_{m \to \infty} \sum_{n=1}^{m} \mathbf{1}_{\{N(T)=n\}} \sum_{k=1}^{n} w(T_k) f'(T_k, k-1) \\ &= -\int_0^T w(t) f'(t, N(t^-)) dN(t). \end{aligned}$$

□

3 Computations of sensitivities

The main tool for the computation of sensitivities is presented in the next proposition. It follows from a classical Malliavin calculus argument that uses the derivation operator D_w. Let $I = (a, b)$ be an open interval of $\mathbb{R}$.

Proposition 3.1 *Let $(F^\zeta)_{\zeta \in I}$ be a family of random functionals, continuously differentiable in* $\mathrm{Dom}\,(D_w)$ *in the parameter $\zeta \in I$. Let $w \in \mathcal{C}_0^1([0, T])$, and let $A \in \mathcal{F}$ such that $\mathbf{1}_A \in \mathrm{Dom}\,(D_w)$ and $D_w \mathbf{1}_A = 0$, with*

$$D_w F^\zeta \neq 0, \quad a.s. \text{ on } A, \quad \zeta \in I,$$

and such that $\mathbf{1}_A \partial_\zeta F^\zeta / D_w F^\zeta$ *is continuous in* ζ *in* $\mathrm{Dom}\,(D_w) \bigcap L^4(\Omega)$*. We have for any function* f *such that* $f(F^\zeta) \in L^2(\Omega)$*,* $\zeta \in I$*:*

$$\frac{\partial}{\partial \zeta} E\left[\mathbf{1}_A f(F^\zeta)\right] = E\left[\mathbf{1}_A W_\zeta f(F^\zeta)\right], \tag{9}$$

where the weight W_ζ *is given on* A *by*

$$W_\zeta = \frac{\partial_\zeta F^\zeta}{D_w F^\zeta} \left(\int_0^T w'(t) dN(t) + \frac{D_w D_w F^\zeta}{D_w F^\zeta} \right) - \frac{D_w \partial_\zeta F^\zeta}{D_w F^\zeta}. \tag{10}$$

Proof. Assuming that $f \in \mathcal{C}_b^\infty(\mathbb{R})$, we have from Corollary 2.1:

$$\begin{aligned} \frac{\partial}{\partial \zeta} E\left[\mathbf{1}_A f(F^\zeta)\right] &= E\left[\mathbf{1}_A f'\left(F^\zeta\right) \frac{\partial}{\partial \zeta} F^\zeta\right] \\ &= E\left[\mathbf{1}_A \frac{\partial_\zeta F^\zeta}{D_w F^\zeta} D_w f(F^\zeta)\right] \\ &= E\left[f(F^\zeta) D_w^* \left(\mathbf{1}_A \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\right)\right]. \end{aligned}$$

Using (5), the weight $D_w^*\left(\mathbf{1}_A \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\right)$ can be computed using Poisson stochastic integrals:

$$\begin{aligned} & D_w^* \left(\mathbf{1}_A \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\right) = \mathbf{1}_A \frac{\partial_\zeta F^\zeta}{D_w F^\zeta} \int_0^T w'(t) dN(t) - D_w \left(\mathbf{1}_A \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\right) \\ &= \mathbf{1}_A \left(\frac{\partial_\zeta F^\zeta}{D_w F^\zeta} \int_0^T w'(t) dN(t) - \frac{D_w \partial_\zeta F^\zeta}{D_w F^\zeta} + \frac{\partial_\zeta F^\zeta}{(D_w F^\zeta)^2} D_w D_w F^\zeta \right). \end{aligned}$$

The extension to square-integrable f is obtained as in Fournié et al. (1999), or El Khatib and Privault (2004), using an approximating sequence $(f_n)_{n \in \mathbb{N}}$ of smooth functions and the bound

$$\begin{aligned} & \left| \frac{\partial}{\partial \zeta} E\left[f_n(F^\zeta)\right] - E\left[f(F^\zeta) D_w^* \left(\mathbf{1}_A \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\right)\right] \right| \\ &\leq \|f(F^\zeta) - f_n(F^\zeta)\|_{L^2(\Omega)} \left\| D_w^* \left(\mathbf{1}_A \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\right) \right\|_{L^2(\Omega)}. \end{aligned}$$

□

If $f \in \mathcal{C}_b^1(\mathbb{R})$ we have

$$\frac{\partial}{\partial \zeta} E\left[f(F^\zeta)\right] = E\left[\mathbf{1}_A W_\zeta f(F^\zeta)\right] + \frac{\partial}{\partial \zeta} E\left[\mathbf{1}_{A^c} f(F^\zeta)\right],$$

and if moreover F^ζ is a.s. constant and equal to $C(\zeta) \in \mathbb{R}$ on A^c, then:

$$\frac{\partial}{\partial \zeta} E\left[f(F^\zeta)\right] = E\left[\mathbf{1}_A W_\zeta f(F^\zeta)\right] + P(A^c) f'(F^\zeta) C'(\zeta).$$

When $F^y = F - y$, i.e. for the computation of probability densities, we have $\frac{\partial}{\partial y} F^y = -1$ and the weight W_y becomes on A:

$$W_y = \frac{-1}{D_w F} \left(\int_0^T w'(t) dN(t) + \frac{D_w D_w F}{D_w F} \right), \tag{11}$$

hence

$$\frac{\partial}{\partial y} E[\mathbf{1}_A h(F - y)] = -E\left[\mathbf{1}_A \frac{h(F-y)}{D_w F} \left(\int_0^T w'(t) dN(t) + \frac{D_w D_w F}{D_w F} \right) \right].$$

In particular, if h is the indicator function $h = \mathbf{1}_{[0,\infty)}$, the density of the law of F conditionally to A is given by

$$y \mapsto \frac{1}{P(A)} E\left[\mathbf{1}_A \frac{\mathbf{1}_{[y,\infty)}(F)}{D_w F} \left(\int_0^T w'(t) dN(t) + \frac{D_w D_w F}{D_w F} \right) \right], \tag{12}$$

and if moreover $F = C \in \mathbb{R}$ is constant on A^c, the law of F has a point mass $P(A^c)\delta_C$ at C.

In applications to insurance we will consider functionals F^ζ of the form

$$F^\zeta = g(\zeta) - \int_0^T f_\zeta(t, N(t^-)) dN(t),$$

where $A = \{N(T) \geq 1\}$, $g \in \mathcal{C}^1([a,b])$, and $f_\zeta(t,k)$, $k \in \mathbb{N}$, is $\mathcal{C}^2$ in (ζ, t). We have

$$\begin{aligned}
\partial_\zeta F^\zeta &= \partial_\zeta g(\zeta) - \int_0^T \partial_\zeta f_\zeta(t, N(t^-)) dN(t), \\
D_w F^\zeta &= -D_w \int_0^T f_\zeta(t, N(t^-)) dN(t) = \int_0^T w(t) f'_\zeta(t, N(t^-)) dN(t), \\
D_w \partial_\zeta F^\zeta &= -D_w \int_0^T \partial_\zeta f_\zeta(t, N(t^-)) dN(t) = \int_0^T w(t) \partial_\zeta f'_\zeta(t, N(t^-)) dN(t), \\
D_w D_w F^\zeta &= -D_w D_w \int_0^T f_\zeta(t, N(t^-)) dN(t) \\
&= -\int_0^T (w'(t) f'_\zeta(t, N(t^-)) + w(t) f''_\zeta(t, N(t^-))) dN(t).
\end{aligned}$$

Hence the weight W_ζ in the relation

$$\frac{\partial}{\partial\zeta}E[\mathbf{1}_{\{N(T)\geq 1\}}h(F^\zeta)] = E\left[\mathbf{1}_{\{N(T)\geq 1\}}W_\zeta h(F^\zeta)\right],$$

cf. Proposition 3.1, is given by

$$\begin{aligned} W_\zeta = {} & \frac{\partial_\zeta g(\zeta) - \int_0^T \partial_\zeta f_\zeta(t, N(t^-))dN(t)}{\int_0^T w(t) f'_\zeta(t, N(t^-))dN(t)} \int_0^T w'(t)dN(t) - \frac{\int_0^T w(t)\partial_\zeta f'_\zeta(t, N(t^-))dN(t)}{\int_0^T w(t) f'_\zeta(t, N(t^-))dN(t)} \\ & - \frac{\partial_\zeta g(\zeta) - \int_0^T \partial_\zeta f_\zeta(t, N(t^-))dN(t)}{\left(\int_0^T w(t) f'_\zeta(t, N(t^-))dN(t)\right)^2} \int_0^T (w'(t) f'_\zeta(t, N(t^-)) + w(t) f'_\zeta(t, N(t^-)))dN(t). \end{aligned}$$

In the particular case where $F^y = F - y$, the weight for density of F in (11) is given by:

$$\begin{aligned} W_y = {} & \frac{1}{\int_0^T w(t) f'_\zeta(t, N(t^-))dN(t)} \\ & \times \left(\int_0^T w'(t)dN(t) - \frac{\int_0^T (w'(t) f'_\zeta(t, N(t^-)) + w(t) f''_\zeta(t, N(t^-)))dN(t)}{\int_0^T w(t) f'_\zeta(t, N(t^-))dN(t)} \right). \end{aligned} \tag{13}$$

4 Application to insurance portfolios

We refer to Sundt and Teugels (1995) for the insurance framework of this section. We consider an insurance portfolio in which the accumulated amount of claims occurring in the time interval $(0, t]$ is given by

$$X(t) = \sum_{k=1}^{N(t)} X_{k-1},$$

where $(X_k)_{k\in\mathbb{N}}$ is a sequence of random variables representing claim sizes, independent of $(N(t))_{t\in\mathbb{R}_+}$, and sufficiently integrable. Since the gradient operator D_w does not act on X_i, $i \in \mathbb{N}$, these random variables may be considered as constants with respect to D_w, and defined on an auxiliary probability space which is not mentioned for the sake of simplicity. In addition to a premium income paid with constant rate $p > 0$, the company receives interests of its reserve with constant interest rate $r > 0$. The risk reserve process is then given by

$$U_r^x(T) = xe^{rT} + p\frac{e^{rT} - 1}{r} - \int_0^T e^{r(T-t)}dX(t),$$

where x is the initial reserve of the company. The discounted value at time 0 of $U_r^x(T)$ is

$$V_r^x(T) = e^{-rT} U_r^x(T) = x + p\frac{1 - e^{-rT}}{r} - \int_0^T e^{-rt} dX(t).$$

We are interested in expectations of the form

$$E[h(U_r^x(T))],$$

and in particular in the probability $P(U_r^x(T) < 0)$ of ruin at date T, obtained with $h = \mathbf{1}_{(-\infty,0)}$. We are now in a position to compute the sensitivities $\frac{\partial}{\partial x} E[h(U_r^x(T))]$ and $\frac{\partial}{\partial r} E[h(U_r^x(T))]$, as well as the probability density of $U_r^x(T)$, by applying the results of Section 3 with

$$g(x, r) = xe^{rT} + p\frac{e^{rT} - 1}{r}, \quad \text{and} \quad f_r(t, k) = e^{r(T-t)} X_k, \qquad k \in \mathbb{N}.$$

Density of probabilities of ruin at date T

The density of $U_r^x(T)$ conditionally to $A = \{N(T) \geq 1\}$ is given by

$$y \mapsto \frac{1}{1 - e^{-\lambda T}} E\left[\mathbf{1}_{\{N(T)\geq 1\}} W_y \mathbf{1}_{\{U_r^x(T) > y\}}\right],$$

with from (13):

$$W_y = \frac{1}{r\int_0^T w(t) e^{r(T-t)} dX(t)} \left(\int_0^T w'(t) dN(t) + \frac{\int_0^T e^{-rt} w(t)(rw(t) - w'(t)) dX(t)}{\int_0^T w(t) e^{-rt} dX(t)} \right).$$

Moreover the law of $U_r^x(T)$ has a Dirac mass $e^{-\lambda T}\delta_c$ at $c = xe^{rT} + p(e^{rT} - 1)/r$, which can be neglected in practice since λT is usually large.

Sensitivity with respect to the initial reserve x

We have

$$\frac{\partial}{\partial x} U_r^x(T) = e^{rT}, \qquad D_w \frac{\partial}{\partial x} U_r^x(T) = 0,$$

and

$$D_w U_r^x(T) = -r \int_0^T w(t) e^{r(T-t)} dX(t), \quad D_w D_w U_r^x(T) = r \int_0^T (w'(t) - rw(t)) e^{r(T-t)} dX(t).$$

The sensitivity with respect to x is computed as

$$\frac{\partial}{\partial x} E[\mathbf{1}_{\{N(T)\geq 1\}} h(U_r^x(T))] = E[\mathbf{1}_{\{N(T)\geq 1\}} W_x h(U_r^x(T))], \tag{14}$$

where W_x is given by a formula similar to (13):

$$W_x = -e^{rT} W_y = -\frac{e^{rT}}{r} \left(\frac{\int_0^T w'(t) dN(t)}{\int_0^T w(t) e^{r(T-t)} dX(t)} + \frac{\int_0^T (rw(t) - w'(t)) w(t) e^{-rt} dX(t)}{\left(\int_0^T w(t) e^{-rt} dX(t) \right)^2} \right).$$

Sensitivity with respect to the interest rate parameter r

We have

$$\begin{aligned}
\partial_r U_r^x(T) &= xTe^{rT} + p\frac{rTe^{rT} - e^{rT} + 1}{r^2} - \int_0^T (T-t) e^{r(T-t)} dX(t), \\
D_w U_r^x(T) &= -r \int_0^T w(t) e^{r(T-t)} dX(t), \\
D_w \partial_r U_r^x(T) &= -\int_0^T w(t) e^{r(T-t)} (1 + r(T-t)) dX(t), \\
D_w D_w U_r^x(T) &= r \int_0^T (w'(t) - rw(t)) e^{r(T-t)} dX(t),
\end{aligned}$$

hence the sensitivity with respect to r is computed as

$$\frac{\partial}{\partial r} E[\mathbf{1}_{\{N(T) \geq 1\}} h(U_r^x(T))] = E[\mathbf{1}_{\{N(T) \geq 1\}} W_r h(U_r^x(T))], \tag{15}$$

where from (10):

$$\begin{aligned}
W_r \quad = \quad & -\frac{1}{r} - \frac{\int_0^T (T-t) e^{-rt} dX(t)}{\int_0^T w(t) e^{-rt} dX(t)} \\
& + \frac{\int_0^T (rw(t) - w'(t)) e^{-rt} dX(t) - r \int_0^T w'(t) dN(t) \int_0^T w(t) e^{-rt} dX(t)}{r^2 (\int_0^T w(t) e^{-rt} dX(t))^2} \\
& \times \left(Tx + p\frac{rT - 1 + e^{-rT}}{r^2} - \int_0^T (T-t) e^{-rt} dX(t) \right).
\end{aligned}$$

Again, (15) gives a rather precise estimation of $\frac{\partial}{\partial r} E[\mathbf{1}_{\{U_r^x(T) > y\}}]$ when $P(N(T) = 0)$ is small.

5 Numerical simulations

The following graphs allow to compare the Malliavin method to the finite difference method for the estimation of the density of probabilities of ruin at a given date. The parameters of the model are set to the following values: $T = 10$, $p = 42$,

$r = 0.05$, $x = 100$, $\lambda = 50$, $\varepsilon = 0.001$, and the simulations presented use the function $w(t) = \sin(\pi t/T)$, $t \in [0, T]$. The density is estimated by finite differences as

$$y \mapsto \frac{1}{2\varepsilon y} E\left[\mathbf{1}_{[y(1-\varepsilon), y(1+\varepsilon)]}(U_r^x(T))\right]. \tag{16}$$

This quantity, as well as the expectation occurring in the Malliavin method, are evaluated via Monte Carlo simulations. Here the random variables X_i, $i \in \mathbb{N}$, are taken constant equal to 1, but the simulations can be performed with an arbitrary claim size distribution. The next graph is a simulation of the probability density of $U_r^x(T)$, the Malliavin formula being given by (12).

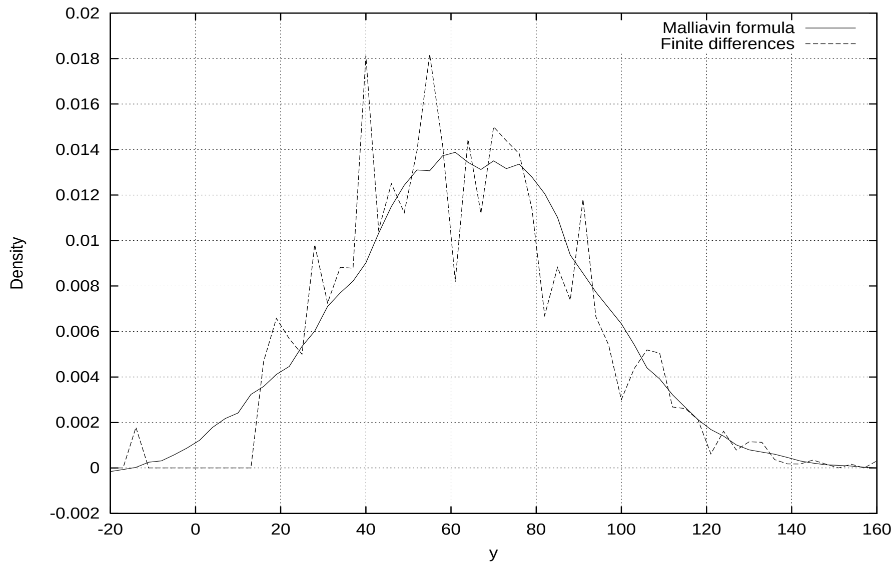

Figure 1 - Probability density of reserve process at time T (sample size: 20000)

The sensitivity graph with respect to the initial reserve is an affine transformation of the density. For the finite difference method we use (1) with $h = \mathbf{1}_{(-\infty,y)}$:

$$x \mapsto \frac{1}{2\varepsilon x} E\left[\mathbf{1}_{(-\infty,y]}(U_r^{x(1+\varepsilon)}(T)) - \mathbf{1}_{(-\infty,y]}(U_r^{x(1-\varepsilon)}(T))\right],$$

and $y = -30$, i.e. we compute the sensitivity of $P(U_x^r(T) < -30)$ with respect to x. The Malliavin formula is obtained from (14).

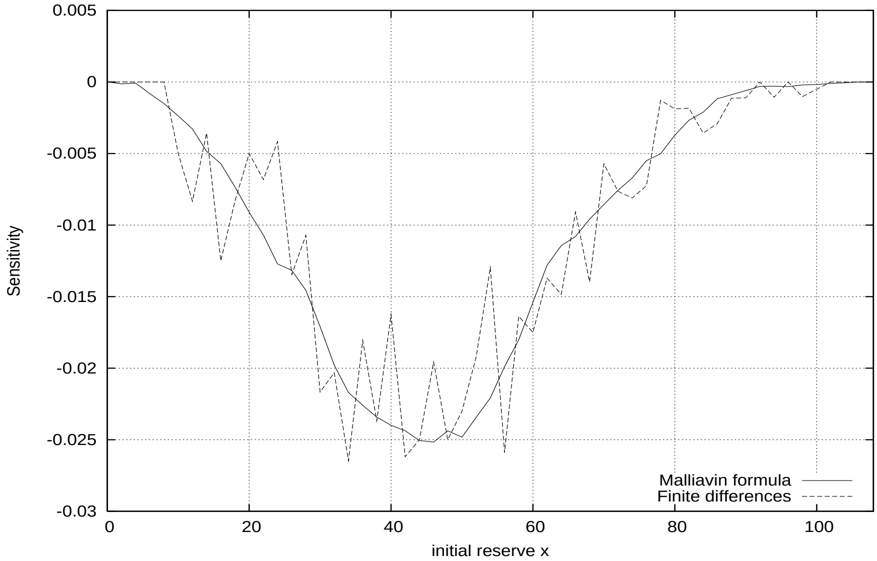

Figure 2 - Sensitivity with respect to the initial reserve x (sample size: 10000)

For the sensitivity with respect to the interest rate parameter r, the performance of the finite difference method is still degraded due to additional numerical errors in the evaluation of the integral $\int_0^T e^{-rt} dX(t)$ at $r(1 \pm \varepsilon)$. For this reason the next graph has been simulated with a sample size of 50000. The finite difference method uses the formula

$$r \mapsto \frac{1}{2\varepsilon r} E\left[\mathbf{1}_{(-\infty,0)}(U^x_{r(1+\varepsilon)}(T)) - \mathbf{1}_{(-\infty,0)}(U^x_{r(1-\varepsilon)}(T))\right],$$

i.e. we compute the sensitivity of $P(U^r_x(T) < 0)$ with respect to r, and the Malliavin formula is obtained from (15) with $h = \mathbf{1}_{(-\infty,0)}$.

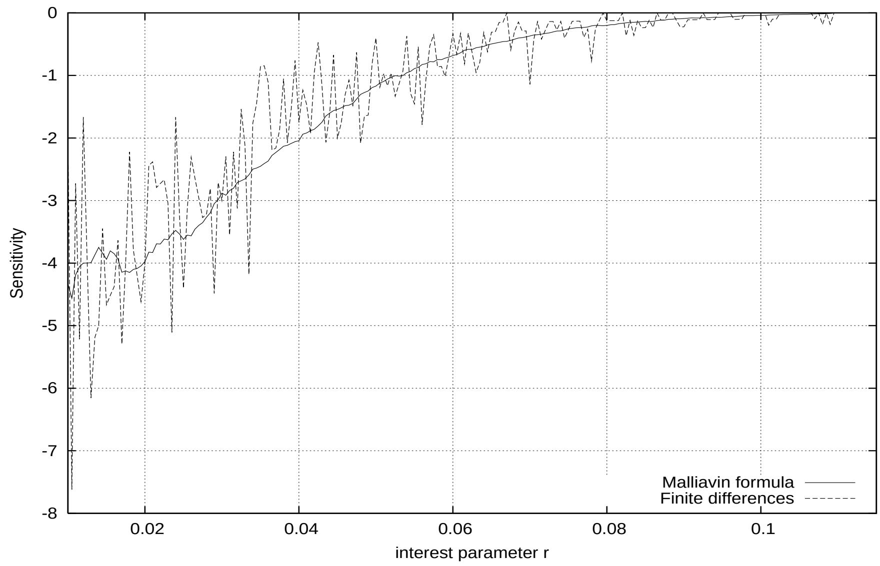

Figure 3 - Sensitivity with respect to the interest rate parameter r (sample size: 50000)

The simulation graphs show a better convergence and stability for the Monte Carlo estimation of the density via the Malliavin method on Poisson space, compared to the finite difference method.

Remark 5.1 *The values of the sensitivities are independent of the choice of the function* w *within* $\mathcal{C}_0^1([0,T])$. *Other choices for* w, *such as* $w(t) = t \wedge (T-t)$ *and* $w(t) = t(T-t)$, $t \in [0,T]$, *have been tried without notable consequences on convergence speed. In a sequel to this work we plan to address the issue of optimization of convergence regarding the choice of* w, *as done in Fournié et al. (2001) for the sensitivity analysis of options in finance.*

Remark 5.2 *The method proposed in this paper does not seem to apply to finite time ruin probabilities. For example in the simplest case of the probability*

$$P\left(\min_{k=1,\ldots,N(T)} (pT_k - \lambda k) < y\right),$$

in a Poisson model with $r = 0$ *and* $x = 0$, *the functional*

$$\min_{k=1,\ldots,N(T)} (pT_k - \lambda k)$$

belongs to the domain of D_w, *however it is not twice differentiable for* D_w *as required in Proposition 3.1.*

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