(Probabilités)

On the Independence of Multiple Stochastic Integrals With Respect to a Class of Martingales

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Abstract - *We study via the chaotic calculus the independence of multiple stochastic integrals $I_n(f_n)$, $I_m(g_m)$ with respect to martingales $(M_t)_{t\in\mathbb{R}_+}$ that satisfy a deterministic structure equation. It is shown that if the integrals are independent and if a contraction denoted as $f_n \circ_1^0 g_m$ does not vanish on $A \in \mathcal{B}(\mathbb{R}_+)$, a.e., then the stochastic measure associated to $(M_t)_{t\in\mathbb{R}_+}$ is Gaussian on A. In the Poisson case, $I_n(f_n)$, $I_m(g_m)$ are independent if and only if $f_n \circ_1^0 g_m$ vanishes a.e.*

Sur l'indépendance des intégrales multiples par rapport à une classe de martingales

Résumé - On étudie par le calcul chaotique l'indépendance des intégrales stochastiques multiples $I_n(f_n)$, $I_m(g_m)$ par rapport aux martingales $(M_t)_{t\in\mathbb{R}_+}$ qui satisfont à une équation de structure déterministe. Il est montré que si les intégrales sont indépendantes et si une contraction notée $f_n \circ_1^0 g_m$ ne s'annule pas sur $A \in \mathcal{B}(\mathbb{R}_+)$, p.p., alors la mesure stochastique correspondant à $(M_t)_{t\in\mathbb{R}_+}$ est gaussienne sur A. Dans le cas poissonien, $I_n(f_n)$, $I_m(g_m)$ sont indépendantes si et seulement si $f_n \circ_1^0 g_m$ s'annule p.p.

Version française abrégée - Soit $(M_t)_{t\in\mathbb{R}_+}$ une martingale satisfaisant une équation de structure déterministe de type (2). Soit $I_n(f_n)$ l'intégrale stochastique multiple par rapport à $(M_t)_{t\in\mathbb{R}_+}$ de la fonction symétrique de carré intégrable $f_n \in L^2(\mathbb{R}_+)^{\circ n}$, définie par (3). Soit $\nabla : L^2(\Omega) \to L^2(\Omega) \otimes L^2(\mathbb{R}_+)$ l'opérateur d'annihilation défini par (5) et soit $\nabla^* : L^2(\Omega) \otimes L^2(\mathbb{R}_+) \to L^2(\Omega)$ son adjoint. Si $f_n \in L^2(\mathbb{R}_+)^{\circ n}$ et $g_m \in L^2(\mathbb{R}_+)^{\circ m}$, on définit $f_n \circ_k^l g_m$, $0 \leq l \leq k \leq n \wedge m$, comme étant la symétrisée en $n+m-k-l$ variables de

$$(x_{l+1}, \dots, x_n, y_{k+1}, \dots, y_m) \mapsto$$
$$\phi(x_{l+1}) \cdots \phi(x_k) \int_0^\infty \cdots \int_0^\infty f_n(x_1, \dots, x_n) g_m(x_1, \dots, x_k, y_{k+1}, \dots, y_m) \lambda(dx_1) \cdots \lambda(dx_l).$$

Théorème *Si les variables aléatoires $I_n(f_n)$ et $I_m(g_m)$ sont indépendantes et si pour presque tout x dans un ensemble $A \in \mathcal{B}(\mathbb{R})$,*

$$\int_0^\infty \cdots \int_0^\infty \mid f_n(x, x_2, \dots, x_n) g_m(x, y_2, \dots, y_m) \mid dx_2 \cdots dx_n dy_2 \cdots dy_n > 0,$$

alors $(M_t)_{t\in\mathbb{R}_+}$ définit une mesure aléatoire gaussienne sur $(A, \mathcal{B}(A))$. Si $(M_t)_{t\in\mathbb{R}_+}$ est un processus de Poisson compensé alors $I_n(f_n), I_m(g_m)$ sont indépendantes si et seulement si $f_n(x_1, \dots, x_n) g_m(x_1, y_2, \dots, y_m) = 0$, p.p.

Ce théorème est prouvé en utilisant le résultat suivant.

Proposition *(i) Si $I_n(f_n) I_m(g_m) \in L^2(\Omega)$, alors*

$$h_{n,m,s} = \sum_{s \leq 2i \leq 2(s\wedge n\wedge m)} i! \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} m \\ i \end{pmatrix} \begin{pmatrix} i \\ s-i \end{pmatrix} f_n \circ_i^{s-i} g_m$$

est dans $L^2(\mathbb{R}_+)^{\circ n+m-s}$, $0 \le s \le 2(n\wedge m)$, *et le développement chaotique de* $I_n(f_n)I_m(g_m)$ *est*

$$I_n(f_n)I_m(g_m) = \sum_{s=0}^{2(n\wedge m)} I_{n+m-s}(h_{n,m,s}).$$

(ii) Inversement, si $f_n \in L^2(\mathbb{R}_+)^{\circ n}$ *et* $g_m \in L^2(\mathbb{R}_+)^{\circ m}$ *sont telles que* $f_n \circ_k^l g_m \in L^2(\mathbb{R}_+)^{\circ n+m-k-l}$, $0 \le l \le k \le n \wedge m$, *alors* $I_n(f_n)I_m(g_m) \in L^2(\Omega)$ *et son développement chaotique s'écrit*

$$I_n(f_n)I_m(g_m) = \sum_{k=0}^{k=n\wedge m} k! \begin{pmatrix} n \\ k \end{pmatrix} \begin{pmatrix} m \\ k \end{pmatrix} \sum_{l=0}^{l=k} \begin{pmatrix} k \\ l \end{pmatrix} I_{n+m-k-l}(f_n \circ_k^l g_m).$$

1 Introduction and preliminaries

The problem of obtaining conditions for the independence of Wiener multiple stochastic integrals $I_n(f_n)$ and $I_m(g_m)$ in terms of their symmetric deterministic kernels $f_n \in L^2(\mathbb{R}_+)^{\circ n}$, $g_m \in L^2(\mathbb{R}_+)^{\circ m}$, has been studied in [1], [2]. Let λ denote the Lebesgue measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. A necessary and sufficient condition for this independence is that

$$\int_0^\infty f_n(x_1, \ldots, x_n) g_m(x_1, y_2, \ldots, y_m) dx_1 = 0 \quad \lambda^{\otimes n+m-2} - a.e. \tag{1}$$

In the one-dimensional case, note that it is proved in [3], [4] that for simple integrals $I_1(f)$, $I_1(g)$ with respect to a stationary stochastic measure X on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ with independent increments, the independence of the integrals is equivalent to $fg = 0$ X-a.e., except if X is Gaussian, in which case the condition is as in (1) for $n = m = 1$. Let $(M_t)_{t\in\mathbb{R}_+}$ be a martingale on $(\Omega, \mathcal{F}, P)$ satisfying the structure equation equation

$$d[M, M]_t = dt + \phi_t dM_t, \tag{2}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}$ is a measurable deterministic function. We know, cf. [5], that such martingales are normal, satisfy the chaotic representation property, and have independent increments, hence $(M_t)_{t\in\mathbb{R}_+}$ defines a stochastic measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, cf. [6]. Let $L^2(\mathbb{R}_+)^{\circ n}$ denote the subspace of $L^2(\mathbb{R}_+)^{\otimes n}$ made of symmetric functions. If $f_n \in L^2(\mathbb{R})^{\otimes n}$, the multiple stochastic integral of f_n is defined as

$$I_n(f_n) = n! \int_0^\infty \int_0^{t_n^-} \cdots \int_0^{t_2^-} f_n(t_1, \ldots, t_n) dM_{t_1} \cdots dM(t_n). \tag{3}$$

We recall the isometry formula:

$$E\left[I_n(f_n)I_m(g_m)\right] = n!(f_n, g_m)_{L^2(\mathbb{R}_+)^{\otimes n}} 1_{\{n=m\}}, \quad f_n \in L^2(\mathbb{R}_+)^{\circ n}, g_m \in L^2(\mathbb{R}_+)^{\circ m}. \tag{4}$$

Any square integrable functional $F \in L^2(\Omega, \mathcal{F}, P)$ has a chaotic decomposition, expressed as

$$F = \sum_{n\ge 0} I_n(f_n), \quad f_k \in L^2(\mathbb{R}_+)^{\circ k}, k \ge 0.$$

Denote by C_n the chaos of order $n \in \mathbb{N}$, defined as $C_n = \{I_n(f_n) \ : \ f_n \in L^2(\mathbb{R}_+)^{\circ n}\}$. An annihilation operator $\nabla : L^2(\Omega) \to L^2(\Omega) \otimes L^2(\mathbb{R}_+)$ is defined by

$$\nabla I_n(f_n) = n I_{n-1}(f_n), \tag{5}$$

$f_n \in L^2(\mathbb{R}_+)^{\circ n}$, $n \in \mathbb{N}^*$. This operator is closable, of domain $Dom(\nabla)$, and its adjoint $\nabla^* : L^2(\Omega) \otimes L^2(\mathbb{R}_+) \to L^2(\Omega)$ satisfies

$$\nabla^* I_n(f_{n+1}) = I_{n+1}(\hat{f}_{n+1}),$$

$f_{n+1} \in L^2(\mathbb{R}_+)^{\circ n} \otimes L^2(\mathbb{R}_+)$, where $\hat{f}_{n+1}$ is the symmetrization in $n+1$ variables of f_{n+1}, defined as

$$\hat{f}_{n+1}(t_1, \ldots, t_{n+1}) = \frac{1}{(n+1)!} \sum_{\sigma \in \Sigma_{n+1}} f_{n+1}(t_{\sigma(1)}, \ldots, t_{\sigma(n+1)}),$$

Σ_{n+1} being the set of all permutations of $\{1, \ldots, n+1\}$, cf. [7], [8]. Using the multiplication formula between the integrals $I_n(f_n)$ and $I_1(f_1)$, cf. [8], [9], one can easily show that

$$\nabla(FG) = F \nabla G + G \nabla F + \phi \nabla F \nabla G, \tag{6}$$

if $F, G, FG \in Dom(\nabla)$. Finally, we recall the following result, known as the Stroock formula, which is valid in general on Fock space and allows to express the chaotic decomposition of $F \in \bigcap_{n \geq 1} Dom(\nabla^n)$ using the operator ∇.

Proposition 1 *If $F \in \bigcap_{n \in \mathbb{N}} Dom(\nabla^n)$, then $F = E[F] + \sum_{n \geq 1} \frac{1}{n!} I_n(E[\nabla^n F])$.*

2 Independence of multiple stochastic integrals

We extend the results of [1], [4], to multiple stochastic integrals with respect to a process that does not necessarily have stationary increments. The result of [4] relies on the characteristic function of infinitely divisible laws, which is not available in the case of multiple stochastic integrals. We need here the following multiplication formula. If $f_n \in L^2(\mathbb{R}_+)^{\circ n}$ and $g_m \in L^2(\mathbb{R}_+)^{\circ m}$, define the function $f_n \circ_k^l g_m$, $0 \leq l \leq k$, to be the symmetrization in $n+m-k-l$ variables of the function

$$(x_{l+1}, \ldots, x_n, y_{k+1}, \ldots, y_m) \mapsto \phi(x_{l+1}) \cdots \phi(x_k) \int_0^\infty \cdots \int_0^\infty f_n(x_1, \ldots, x_n) g_m(x_1, \ldots, x_k, y_{k+1}, \ldots, y_m) \lambda(dx_1) \cdots \lambda(dx_l).$$

Proposition 2 *(i) If $I_n(f_n) I_m(g_m) \in L^2(\Omega)$, then the function*

$$h_{n,m,s} = \sum_{s \leq 2i \leq 2(s \wedge n \wedge m)} i! \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} m \\ i \end{pmatrix} \begin{pmatrix} i \\ s-i \end{pmatrix} f_n \circ_i^{s-i} g_m$$

is in $L^2(\mathbb{R}_+)^{\circ n+m-s}$, $0 \leq s \leq 2(n \wedge m)$, and the chaotic expansion of $I_n(f_n) I_m(g_m)$ is

$$I_n(f_n) I_m(g_m) = \sum_{s=0}^{2(n \wedge m)} I_{n+m-s}(h_{n,m,s}). \tag{7}$$

(ii) Conversely, if $f_n \in L^2(\mathbb{R}_+)^{\circ n}$ and $g_m \in L^2(\mathbb{R}_+)^{\circ m}$ are such that $f_n \circ_k^l g_m \in L^2(\mathbb{R}_+)^{\circ n+m-k-l}$, $0 \leq l \leq k \leq n \wedge m$, then $I_n(f_n)I_m(g_m) \in L^2(\Omega)$ and its chaotic decomposition can be written as

$$I_n(f_n)I_m(g_m) = \sum_{k=0}^{k=n\wedge m} k! \begin{pmatrix} n \\ k \end{pmatrix} \begin{pmatrix} m \\ k \end{pmatrix} \sum_{l=0}^{l=k} \begin{pmatrix} k \\ l \end{pmatrix} I_{n+m-k-l}(f_n \circ_k^l g_m). \quad (8)$$

Proof. The first part can be found in [10] under a different formulation. If $I_n(f_n)I_m(g_m) \in L^2(\Omega)$, then it is in the sum $C_0 \oplus \cdots \oplus C_{n+m}$ of the chaos of orders lower than $n+m$ since

$$E[I_n(f_n)I_m(g_m) \mid C_0 \oplus \cdots \oplus C_{n+m}] = I_n(f_n)I_m(g_m),$$

hence it belongs to $\bigcap_{n\geq 1} Dom(\nabla^n)$ and its chaotic decomposition can be obtained from Prop. 1. From (6), we have by induction for $r \geq 1$

$$\nabla_{t_1} \cdots \nabla_{t_r}(FG) = \sum_{p=0}^{p=r} \sum_{q=r-p}^{q=r} \sum_{\{k_1<\cdots<k_p\}\cup\{l_1<\cdots<l_q\}=\{1,\dots,r\}} \nabla_{t_{k_1}} \cdots \nabla_{t_{k_p}} F \nabla_{t_{l_1}} \cdots \nabla_{t_{l_q}} G \prod_{i\in\{k_1<\cdots<k_p\}\cap\{l_1<\cdots<l_q\}} \phi(t_i),$$

and $\nabla^r(FG) \in L^2(\Omega) \otimes L^2(\mathbb{R}_+)^{\circ r}$ if $FG \in Dom(\nabla^r)$. Applying this formula to $F = I_n(f_n)$ and $G = I_m(g_m)$, we obtain

$$\nabla_{t_1} \cdots \nabla_{t_r}(I_n(f_n)I_m(g_m)) = \sum_{p=0}^{p=r} \sum_{q=r-p}^{q=r} \sum_{\{k_1<\cdots<k_p\}\cup\{l_1<\cdots<l_q\}=\{1,\dots,r\}}$$

$$\frac{n!}{(n-p)!} \frac{m!}{(m-q)!} I_{n-p}(f_n(\cdot, t_{k_1}, \dots, t_{k_p})) I_{m-q}(g_m(\cdot, t_{l_1}, \dots, t_{l_q})) \prod_{i\in\{k_1<\cdots<k_p\}\cap\{l_1<\cdots<l_q\}} \phi(t_i).$$

Define a function $h_{n,m,n+m-r} \in L^2(\mathbb{R}_+)^{\circ r}$ as

$$\begin{aligned} h_{n,m,n+m-r}(t_1, \dots, t_r) &= \frac{1}{r!} E[\nabla_{t_1} \cdots \nabla_{t_r}(I_n(f_n)I_m(g_m))] \\ &= \frac{1}{r!} \sum_{p=0}^{p=r\wedge n} \sum_{q=r-p}^{q=r} 1_{\{n-p=m-q\}} \frac{n!}{(n-p)!} \frac{m!}{(m-q)!} (n-p)! a_{n,m,p,r} f_n \circ_{q+p-r}^{n-p} g_m(t_1, \dots, t_r), \\ &= \frac{1}{r!} \sum_{n-m+r\leq 2p\leq 2(n\wedge r)} \frac{n!m!}{(n-p)!} a_{n,m,p,r} f_n \circ_{m-r+p}^{n-p} g_m(t_1, \dots, t_r), \end{aligned}$$

where $a_{n,m,p,r}$ is the number of sequences $k_1 < \cdots < k_p$ and $l_1 < \cdots < l_q$ such that $\{k_1, \dots, k_p\} \cup \{l_1, \dots, l_q\} = \{1, \dots, r\}$, with exactly $m-r+p-(n-p)$ terms in common. This number is

$$a_{n,m,p,r} = \frac{r!}{(r-p)!p!} \frac{p!}{(m-n-r+2p)!(n+r-m-p)!}.$$

Hence

$$\begin{aligned} & h_{n,m,n+m-r} \\ &= \sum_{n-m+r\leq 2p\leq 2(n\wedge r)} \frac{n!m!}{(r-p)!(m-n-r+2p)!(n+r-m-p)!(n-p)!} f_n \circ_{m-r+p}^{n-p} g_m \\ &= \sum_{n+m-r\leq 2i\leq 2((n+m-r)\wedge n\wedge m)} \frac{n!}{(n-i)!} \frac{m!}{(m-i)!} \frac{1}{(2i-s)!} \frac{1}{(s-i)!} f_n \circ_i^{s-i} g_m \\ &= \sum_{s\leq 2i\leq 2(s\wedge n\wedge m)} i! \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} m \\ i \end{pmatrix} \begin{pmatrix} i \\ s-i \end{pmatrix} f_n \circ_i^{s-i} g_m, \end{aligned}$$

with $s = n + m - r$ and $i = p + m - r$. The chaotic expansion (7) follows from Prop. 1, and this proves (i) and (ii). □

Theorem 1 *Let $f_n \in L^2(\mathbb{R}_+)^{\circ n}$ and $g_m \in L^2(\mathbb{R}_+)^{\circ m}$. If the random variables $I_n(f_n)$ and $I_m(g_m)$ are independent and if for λ-a.a. x in a set $A \in \mathcal{B}(\mathbb{R}_+)$,*

$$\int_0^\infty \cdots \int_0^\infty \mid f_n(x, x_2, \ldots, x_n) g_m(x, y_2, \ldots, y_m) \mid^2 dx_2 \cdots dx_n dy_2 \cdots dy_n > 0,$$

then the stochastic measure induced by $(M_t)_{t \in \mathbb{R}_+}$ on $(A, \mathcal{B}(A))$ is Gaussian.

Proof. We follow the approach of [2]. If $I_n(f_n)$ and $I_m(g_m)$ are independent, then $I_n(f_n) I_m(g_m) \in L^2(\Omega, \mathcal{F}, P)$ and

$$\begin{aligned}
(n+m)! \mid f_n \circ g_m \mid^2_{L^2(\mathbb{R}_+)^{\otimes n+m}} \geq n!m! \mid f_n \otimes g_m \mid^2_{L^2(\mathbb{R}_+)^{\otimes n+m}} &= n!m! \mid f_n \mid^2_{L^2(\mathbb{R}_+)^{\otimes n}} \mid g_m \mid^2_{L^2(\mathbb{R}_+)^{\otimes m}} \\
&= E\left[I_n(f_n)^2\right] E\left[I_m(g_m)^2\right] = E\left[(I_n(f_n) I_m(g_m))^2\right] \\
&= \sum_{r=0}^{2(n \wedge m)} (n+m-r)! \mid h_{n,m,r} \mid^2_{L^2(\mathbb{R}_+)^{\otimes n+m-r}} \\
&\geq (n+m)! \mid h_{n,m,0} \mid^2_{L^2(\mathbb{R}_+)^{\otimes n+m}} + (n+m-1)! \mid h_{n,m,1} \mid^2_{L^2(\mathbb{R}_+)^{\otimes n+m-1}} \\
&\geq (n+m)! \mid f_n \circ g_m \mid^2_{L^2(\mathbb{R}_+)^{\otimes n+m}} + nm(n+m-1)! \mid f_n \circ_1^0 g_m \mid^2_{L^2(\mathbb{R}_+)^{\otimes n+m-1}}
\end{aligned}$$

from Prop. 2. This implies that $f_n \circ_1^0 g_m = 0$ $\lambda^{\otimes n+m-1}$-a.e., hence $\phi = 0$ λ-a.e. on A, and the random measure defined by $(M_t)_{t \in \mathbb{R}_+}$ is Gaussian on A, from [5], Prop. 4. □
Consequently, $I_n(f_n)$ and $I_m(g_m)$ are independent if and only if

$$I_n(f_n) I_m(g_m) = I_{n+m}(f_n \circ g_m) = I_n(f_n) : I_m(g_m),$$

where " : " denotes the Wick product. Until the end of this paper we assume that $\phi \neq 0$ λ-a.e., hence $(M_t)_{t \in \mathbb{R}_+}$ is written as

$$M_t = \int_0^t \phi(s) d\tilde{N}_s, \quad t \in \mathbb{R}_+,$$

where $(\tilde{N}_t)_{t \in \mathbb{R}_+}$ is a compensated Poisson process of intensity $dt/\phi(t)^2$, cf. [5].

Corollary 1 *If $\phi = 0$ a.e., then the multiple stochastic integrals $I_n(f_n), I_m(g_m)$ are independent if and only if*

$$f_n(x_1, \ldots, x_n) g_m(x_1, y_2, \ldots, y_m) = 0, \quad \lambda^{\otimes n+m-1} - a.e.$$

This corollary can also be obtained for a Poisson measure on a metric space. Proceeding as in [1], [2], we note that two arbitrary families $\{I_{n_k}(f_{n_k}) : k \in I\}$ and $\{I_{m_l}(g_{m_l}) : l \in J\}$ of Poisson multiple stochastic integrals are independent if and only if $I_{n_k}(f_{n_k})$ is independent of $I_{m_l}(g_{m_l})$ for any $k \in I$, $l \in J$, and obtain the following corollaries.

Corollary 2 *Let $f_n \in L^2(\mathbb{R}_+)^{\circ n}$, $g_m \in L^2(\mathbb{R}_+)^{\circ m}$, and*

$$S_f = \{f_n \circ_{n-1}^{n-1} h : h \in L^2(\mathbb{R}_+)^{\circ n-1}\}, \quad S_g = \{g_n \circ_{m-1}^{m-1} h : h \in L^2(\mathbb{R}_+)^{\circ m-1}\}.$$

The following statements are equivalent.
(i) $I_n(f_n)$ is independent of $I_m(g_m)$.
(ii) For any $f \in S_f$ and $g \in S_g$, $fg = 0$ λ-a.e.
(iii) The σ-algebras $\sigma(I_1(f) : f \in S_f)$ and $\sigma(I_1(g) : g \in S_g)$ are independent.

Corollary 3 *If $F \in Dom(\nabla)$ and $G \in L^2(\Omega, \mathcal{F}, P)$ with $G = \sum_{m\geq 0} I_m(g_m)$, then F is independent of G if for any $m \in \mathbb{N}$,*

$$g_m \circ_1^0 \nabla F = 0, \quad \lambda^{\otimes m} \otimes P - a.e. \tag{9}$$

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