INDEPENDENCE OF A CLASS OF MULTIPLE STOCHASTIC INTEGRALS

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Abstract

We show that two multiple stochastic integrals $I_n(f_n)$, $I_m(g_m)$ with respect to the solution $(M_t)_{t\in\mathbf{R}_+}$ of a deterministic structure equation are independent if and only if two contractions of f_n and g_m, denoted as $f_n \circ_1^0 g_m$, $f_n \circ_1^1 g_m$, vanish almost everywhere.

1 Introduction

This paper aims to extend the necessary and sufficient conditions for the independence of single or multiple stochastic integrals of [12], [14], [15], [16], [17], cf. also [6], [7], proving and extending results that have been partially announced in [9]. Let $(M_t)_{t\in\mathbf{R}_+}$ be a martingale satisfying the structure equation

$$d[M,M]_t = dt + \phi_t dM_t, \tag{1}$$

where $\phi : \mathbf{R}_+ \to \mathbf{R}$ is a measurable deterministic function. Such martingales are normal in the sense of [2], i.e. $d < M, M >_t = dt$, $t \in \mathbf{R}_+$ and they satisfy the chaos representation property, cf. [3]. Moreover, they have independent increments, and if $(B_t)_{t\in\mathbf{R}_+}$, $(N_t)_{t\in\mathbf{R}_+}$ are independent standard Brownian motion and Poisson process of intensity ds/ϕ_s^2, then $(M_t)_{t\in\mathbf{R}_+}$ can be represented as

$$M_t = \int_0^t 1_{\{\phi_s=0\}} dB_s + \int_0^t \phi_s \left(dN_s - \frac{ds}{\phi_s^2} \right), \quad t \in \mathbf{R}_+. \tag{2}$$

We choose to construct the processes $(B_t)_{t\in\mathbf{R}_+}$ on the classical Wiener space $(\Omega_1, \mathcal{F}_1, P_1)$, where Ω_1 is the space of cadlag functions starting at zero. We denote by $(\Omega_2, \mathcal{F}_2, P_2)$ the space

$$\Omega_2 = \left\{ \sum_{i=1}^{i=N} \delta_{t_i} \ : \ (t_i)_{i=1,\ldots,N} \in \mathbf{R}_+, \ N \in \mathbf{N} \cup \{\infty\} \right\},$$

with the σ-algebra and probability measure $\mathcal{F}_2$, P_2 under which the canonical random measure is Poisson with mean measure μ on $(\mathbf{R}_+, \mathcal{B}(\mathbf{R}_+))$ defined as

$$\mu(A) = \int_{A\cap\{\phi\neq 0\}} \frac{1}{\phi_s^2} ds, \qquad A \in \mathcal{B}(\mathbf{R}_+).$$

Mathematics Subject Classification (1991): 60H05, 60G44, 60H07.
Keywords and phrases. Multiple stochastic integrals, Independence, Martingales.

With this notation, $(N_t)_{t\in\mathbf{R}_+}$ is written as $N_t(\omega_2) = \omega_2([0,t])$, and $(B_t)_{t\in\mathbf{R}_+}$ satisfies $B_t(\omega_1) = \omega_1(t)$, $t \in \mathbf{R}_+$. For $A \in \mathcal{B}(\mathbf{R}_+)$ we call $\mathcal{F}_2^A$ the σ-algebra on Ω_2 generated by all random variables $\omega_2 \to \omega_2(A \cap B)$, $B \in \mathcal{B}(\mathbf{R}_+)$. The martingale M is then explicitly constructed as $M_t(\omega_1,\omega_2) = X_t(\omega_2) + B_t(\omega_1)$, $t \in \mathbf{R}_+$, on $(\Omega,\mathcal{F},P) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, P_1 \otimes P_2)$, where

$$X_t = \int_0^t \phi_s(dN_s - ds/\phi_s^2), \quad t \in \mathbf{R}_+.$$

If $f_n \in L^2(\mathbf{R})^{\otimes n}$, the multiple stochastic integral with respect to M, X, and B of f_n are respectively defined as

$$I_n(f_n) = n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} \hat{f}_n(t_1,\ldots,t_n) dM_{t_1} \cdots dM_{t_n}, \tag{3}$$

$$I_n^X(f_n) = n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} \hat{f}_n(t_1,\ldots,t_n) dX_{t_1} \cdots dX_{t_n}, \tag{4}$$

$$I_n^B(f_n) = n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} \hat{f}_n(t_1,\ldots,t_n) dB_{t_1} \cdots dB_{t_n}, \tag{5}$$

where $\hat{f}_n$ is the symmetrization in n variables of f_n. We note the relation

$$I_n(f_n) = \sum_{i=0}^{i=n} \begin{pmatrix} n \\ k \end{pmatrix} I_{n-k}^X(I_k^B(\hat{f}_n)) = \sum_{i=0}^{i=n} \begin{pmatrix} n \\ k \end{pmatrix} I_{n-k}^B(I_k^X(\hat{f}_n)). \tag{6}$$

Let $L^2(\mathbf{R}_+)^{\circ n}$ denote the subspace of $L^2(\mathbf{R}_+)^{\otimes n}$ made of symmetric functions. Let $f_n \otimes g_m$ denote the completed tensor product of two functions $f_n \in L^2(\mathbf{R}_+^n)$ and $g_m \in L^2(\mathbf{R}_+^m)$, and let $f_n \circ g_m$ denote the symmetrization of $f_n \otimes g_m$, $n,m \in \mathbf{N}$. Since $d < M, M >_t = dt$, we have

$$E\left[I_n(f_n)I_m(g_m)\right] = n!(f_n,g_m)_{L^2(\mathbf{R}_+)^{\otimes n}} 1_{\{n=m\}}, \quad f_n \in L^2(\mathbf{R}_+)^{\circ n}, g_m \in L^2(\mathbf{R}_+)^{\circ m}. \tag{7}$$

Since $(M_t)_{t\in\mathbf{R}_+}$ has the chaos representation property, any square integrable functional $F \in L^2(\Omega,\mathcal{F},P)$ has a chaos expansion

$$F = \sum_{n\geq 0} I_n(f_n), \qquad f_k \in L^2(\mathbf{R}_+)^{\circ k}, k \geq 0.$$

A linear operator $\nabla : L^2(\Omega) \to L^2(\Omega) \otimes L^2(\mathbf{R}_+)$ is defined by annihilation as

$$\nabla_t I_n(f_n) = nI_{n-1}(f_n(\cdot,t)), \quad t \in \mathbf{R}_+, \tag{8}$$

$f_n \in L^2(\mathbf{R}_+)^{\circ n}$, $n \in \mathbf{N}^*$, cf. e.g. [5]. This operator is closable, of L^2-domain $Dom_2(\nabla)$, and its closed adjoint $\nabla^* : L^2(\Omega) \otimes L^2(\mathbf{R}_+) \to L^2(\Omega)$ satisfies

$$\nabla^* I_n(f_{n+1}) = I_{n+1}(\hat{f}_{n+1}),$$

$f_{n+1} \in L^2(\mathbf{R}_+)^{\circ n} \otimes L^2(\mathbf{R}_+)$. We denote by $Dom_1(\nabla)$ the set of functionals $F \in L^2(\Omega)$ such that there exists a sequence $(F_n)_{n\in\mathbf{N}} \subset Dom_2(\nabla)$ converging to F in $L^2(\Omega)$ and such that $(\nabla F_n)_{n\in\mathbf{N}}$ converges in $L^1(\Omega\times\mathbf{R}_+)$. The limit of the sequence $(\nabla F_n)_{n\in\mathbf{N}}$ is denoted ∇F which is well-defined, due to the relation

$$E[(\nabla F_n, u)_{L^2(\mathbf{R}_+)}] = E[F_n \nabla^*(u)], \qquad n \in \mathbf{N},$$

$u \in Dom(\nabla^*) \cap L^\infty(\Omega \times \mathbf{R}_+)$, and since $Dom(\nabla^*) \cap L^\infty(\Omega \times \mathbf{R}_+)$ is dense in $L^1(\Omega \times \mathbf{R}_+)$. For $f_n \in L^2(\mathbf{R}_+)^{\circ n}$ and $g_m \in L^2(\mathbf{R}_+)^{\circ m}$, we define $f_n \otimes_k^l g_m$, $0 \leq l \leq k$, to be the function

$$(x_{l+1}, \ldots, x_n, y_{k+1}, \ldots, y_m) \mapsto$$
$$\phi(x_{l+1}) \cdots \phi(x_k) \int_{\mathbf{R}^l} f_n(x_1, \ldots, x_n) g_m(x_1, \ldots, x_k, y_{k+1}, \ldots, y_m) dx_1 \cdots dx_l$$

of $n+m-k-l$ variables. We denote by $f_n \circ_k^l g_m$ the symmetrization in $n+m-k-l$ variables of $f_n \otimes_k^l g_m$, $0 \leq l \leq k$.

Definition 1 *Let $\mathcal{S}$ denote the vector space in $L^2(\Omega)$ generated by*

$$\{I_n(f_1 \circ \cdots \circ f_n) \ : \ f_1, \ldots, f_n \in \mathcal{C}_c(\mathbf{R}_+), \ n \geq 1\}.$$

The vector space $\mathcal{S}$ is dense in $L^2(\Omega)$. For $F \in \mathcal{S}$ and $f \in L^2(\mathbf{R}_+)$, we have from a general result in quantum stochastic calculus, cf. for example Th. II.1 of [1]:

$$F \int_0^\infty f(s) dM_s = \int_0^\infty f(s) \nabla_s F ds + \nabla^*(fF) + \nabla^*(\phi f \nabla F). \tag{9}$$

This formula is usually stated under the form

$$\int_0^\infty f(s) dM_s = \int_0^\infty f(s) da_s^- + \int_0^\infty f(s) da_s^+ + \int_0^\infty \phi_s f(s) da_s^\circ$$

by quantum probabilists, where $\int_0^\infty f(s) dM_s$ is identified to a multiplication operator. The identity (9) can be easily rewritten into a multiplication formula between first and nth order stochastic integrals:

$$I_1(h) I_n(f_n) = I_{n+1}(f_n \circ h) + n \int_0^\infty h_t I_{n-1}(f_n(\cdot, t)) dt + n I_n(f_n \circ_1^0 (\phi h)). \tag{10}$$

We note that as a consequence of this formula, every element of $\mathcal{S}$ has a unique expression as a polynomial in single stochastic integrals and conversely, any polynomial in stochastic integrals has a finite chaos expansion.

Remark 1 *This implies that each element of $\mathcal{S}$ has a version which is defined for every $\omega = (\omega_1, \omega_2) \in \Omega$, since $I_1(f) \in \mathcal{S}$ can be written as*

$$I_1(f) = -\int_0^\infty f'(s) B_s 1_{\{\phi_s = 0\}} ds + \sum_{\{t \ : \ dN_t = 1\}} \phi_t f(t) - \int_0^\infty 1_{\{\phi_s \neq 0\}} f(s) \frac{1}{\phi_s} ds.$$

Throughout this paper, $F \in \mathcal{S}$ will always refer to the version of F defined via the above identity.

From (10), one can prove the following result which shows that the function ϕ accounts for the perturbation of the usual derivation rule for the Malliavin derivative on Wiener space.

Proposition 1 *For any $F, G \in \mathcal{S}$ we have*

$$\nabla_t(FG) = F\nabla_t G + G\nabla_t F + \phi_t \nabla_t F \nabla_t G, \qquad t \in \mathbf{R}_+. \tag{11}$$

If $\phi \in L^\infty(\mathbf{R}_+)$ then for any $F, G \in Dom_2(\nabla)$, we have $FG \in Dom_1(\nabla)$ and the above relation holds.

Proof. We first notice that for $F = I_1(h)$ and $G = I_n(f_n)$, this formula is a consequence of the multiplication formula (10), since

$$\begin{aligned}
&\nabla_t(I_1(h)I_n(f_n)) \\
&= \nabla_t\left(I_{n+1}(f_n \circ h) + n\int_0^\infty h_s I_{n-1}(f_n(\cdot, s)ds + nI_n(f_n \circ_1^0 (\phi h))\right) \\
&= I_n(f_n)\nabla_t I_1(h) + nI_n(f_n(\cdot, t)\circ h) + n(n-1)\int_0^\infty h_s I_{n-2}(f_n(\cdot, t, s))ds \\
&\quad + n(n-1)I_n(f_n(\cdot, t)\circ_1^0 (\phi h)) + \phi_t \nabla_t I_1(h)\nabla_t I_n(f_n) \\
&= I_n(f_n)\nabla_t I_1(h) + I_1 \nabla_t I_n(f_n) + \phi_t \nabla_t I_1(h)\nabla_t I_n(f_n).
\end{aligned}$$

Next, we prove by induction on $k \geq 1$ that

$$\nabla_t(I_n(f_n)I_1(h)^k) = I_1(h)^k \nabla_t I_n(f_n) + I_n(f_n)\nabla_t I_1(h)^k + \phi_t \nabla_t I_1(h)^k \nabla_t I_n(f_n).$$

We have

$$\begin{aligned}
&\nabla_t(I_n(f_n)I_1(h)^{k+1}) \\
&= I_1(h)^k \nabla_t(I_n(f_n)I_1(h)) + I_n(f_n)I_1(h)\nabla_t I_1(h)^k \\
&\quad + \phi_t \nabla_t I_1(h)^k \nabla_t(I_n(f_n)I_1(h)) \\
&= I_1(h)^{k+1}\nabla_t I_n(f_n) + I_n(f_n)I_1(h)\nabla_t I_1(h)^k + I_n(f_n)I_1(h)^k \nabla_t I_1(h) \\
&\quad + \phi_t I_n(f_n)\nabla_t I_1(h)\nabla_t I_1(h)^k + \phi_t I_1(h)\nabla_t I_1(h)^k \nabla_t I_n(f_n) \\
&\quad + \phi_t I_1(h)^k \nabla_t I_1(h)\nabla_t I_n(f_n) + \phi_t^2 \nabla_t I_1(h)\nabla_t I_1(h)^k \nabla_t I_n(f_n) \\
&= I_1(h)^{k+1}\nabla_t I_n(f_n) + I_n(f_n)\nabla_t I_1(h)^{k+1} + \phi_t \nabla_t I_1(h)^{k+1}\nabla_t I_n(f_n).
\end{aligned}$$

Consequently, (11) holds for any polynomial in single stochastic integrals, hence it holds for any $F, G \in \mathcal{S}$. In order to prove the second part of the proposition, we assume that $F, G \in Dom_2(\nabla)$ and choose two sequences $(F_n)_{n\in\mathbf{N}}$ and $(G_n)_{n\in\mathbf{N}}$ contained in $\mathcal{S}$, converging respectively to F and G in $L^2(\Omega)$ and such that $(\nabla F_n)_{n\in\mathbf{N}}$ and $(\nabla G_n)_{n\in\mathbf{N}}$ converge to ∇F and ∇G in $L^2(\Omega \times \mathbf{R}_+)$. Then $(\phi \nabla F_n \nabla G_n)_{n\in\mathbf{N}}$ converges in $L^1(\Omega \times \mathbf{R}_+)$ to $\phi \nabla F \nabla G$, hence $(\nabla(F_n G_n))_{n\in\mathbf{N}}$ converges in $L^1(\Omega \times \mathbf{R}_+)$ to $F\nabla G + G\nabla F + \phi \nabla F \nabla G$, and $FG \in Dom_1(\nabla)$. □

The product rule for ∇ unifies the chain rule of derivation of the Wiener space Malliavin derivative and the finite difference rule of the Poisson space gradient of [8].

Proposition 2 *For any $F \in \mathcal{S}$ we have*

$$\nabla_t F = \lim_{\varepsilon \to 0} \frac{F\big(M_\cdot + (\varepsilon + \phi_t) 1_{[t,\infty[}(\cdot)\big) - F(M_\cdot)}{\varepsilon + \phi_t}, \quad t \in \mathbf{R}_+. \tag{12}$$

Proof. The statement (12) can be more precisely formulated as

$$\nabla_t F(\omega_1, \omega_2) = \lim_{\varepsilon \to 0} \frac{F(\omega_1 + \varepsilon 1_{[t,\infty[}, \omega_2 + \phi_t \delta_t) - F(\omega_1, \omega_2)}{\varepsilon + \phi_t},$$

where the notation F refers to the version defined in Remark 1. We first show that (12) holds for $F = I_1(f)$:

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{F(\omega_1 + \varepsilon 1_{[t,\infty[}, \omega_2 + \phi_t \delta_t) - F(\omega_1, \omega_2)}{\varepsilon + \phi_t} \\
&= 1_{\{\phi_t \neq 0\}} \frac{1}{\phi_t} \left(\sum_{\{s \ : \ dN_s = 1\}} \phi_s f(s) - \int_0^\infty 1_{\{\phi_s \neq 0\}} f(s) \frac{1}{\phi_s} ds + \phi_t f(t) \right. \\
&\quad \left. - \sum_{\{s \ : \ dN_s = 1\}} \phi_s f(s) - \int_0^\infty 1_{\{\phi_s \neq 0\}} f(s) \frac{1}{\phi_s} ds \right) \\
&\quad + 1_{\{\phi_t = 0\}} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(- \int_0^\infty f'(s)(B_s + \varepsilon) 1_{[t,\infty[}(s) 1_{\{\phi_s = 0\}} ds \right. \\
&\quad \left. + \int_0^\infty f'(s) B_s 1_{\{\phi_s = 0\}} ds \right) \\
&= 1_{\{\phi_t = 0\}} f(t) + 1_{\{\phi_t \neq 0\}} f(t) = f(t), \quad t \in \mathbf{R}_+.
\end{aligned}$$

Moreover, the limit (12) satisfies the product rule (11), hence if $F, G \in \mathcal{S}$ are of the form $F = I_1(f)$ and $G = I_1(g)$, we have

$$\begin{aligned}
\lim_{\varepsilon \to 0} \frac{(FG)\big(M_\cdot + (\varepsilon + \phi_t) 1_{[t,\infty[}(\cdot)\big) - (FG)(M_\cdot)}{\varepsilon + \phi_t} &= F \nabla_t G + G \nabla_t F + \phi_t \nabla_t (FG) \\
&= \nabla_t (FG), \quad t \in \mathbf{R}_+.
\end{aligned}$$

Thus by induction, (12) holds for any polynomial in single stochastic integrals, and for any element of $\mathcal{S}$. □

With help of Prop. 11, the following multiplication formula has been proved in [9], as a generalization of (10). We refer to p. 216 of [2], and to [4], [13], [14], for different versions of this formula in the Poisson case. In [11] a more general result is proven, allowing to represent the product $I_n(f_n) I_m(g_m)$ as a sum of $n \wedge m$ terms that are not necessarily linear combinations of multiple stochastic integrals with respect to $(M_t)_{t \in \mathbf{R}_+}$, except if $d[M, M]_t$ is a linear deterministic combination of dt and dM_t, cf. [10].

Proposition 3 *The product $I_n(f_n)I_m(g_m) \in L^2(\Omega)$ is in $L^2(B)$ if and only if the function*

$$h_{n,m,s} = \sum_{s \le 2i \le 2(s\wedge n\wedge m)} i! \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} m \\ i \end{pmatrix} \begin{pmatrix} i \\ s-i \end{pmatrix} f_n \circ_i^{s-i} g_m$$

is in $L^2(\mathbf{R}_+)^{\circ n+m-s}$, $0 \le s \le 2(n \wedge m)$, and in this case the chaotic expansion of $I_n(f_n)I_m(g_m)$ is

$$I_n(f_n)I_m(g_m) = \sum_{s=0}^{2(n\wedge m)} I_{n+m-s}(h_{n,m,s}). \tag{13}$$

The fact that $I_n(f_n)I_m(g_m)$ can be expanded as a sum of multiple stochastic integrals with respect to $(M_t)_{t\in\mathrm{R}_+}$ is essential in the proof of independence, cf. Th. 1.

2 Independence of multiple stochastic integrals

In the case of single stochastic integrals, the following proposition extends the result of [15] to a process that does not have stationary increments. In the case of multiple stochastic integrals, it extends the result of [17] since it includes a Poisson component in the martingale $(M_t)_{t\in\mathrm{R}_+}$.

Theorem 1 *Let $f_n \in L^2(\mathbf{R}_+)^{\circ n}$ and $g_m \in L^2(\mathbf{R}_+)^{\circ m}$. The random variables $I_n(f_n)$ and $I_m(g_m)$ are independent and if and only if $f_n \circ_1^1 g_m = 0$ and $f_n \circ_1^0 g_m = 0$ a.e., i.e.*

$$\int_0^\infty f_n(t, x_2, \ldots, x_n) g_m(t, x_{n+1}, \ldots, x_{n+m-2}) dt = 0, \quad dx_2 \cdots dx_{n+m-2} \ a.e. \tag{14}$$

and

$$f_n(x_1, x_2, \ldots, x_n) g_m(x_1, x_{n+1}, \ldots, x_{n+m-1}) = 0, \quad |\, \phi_{x_1} \,|\, dx_1 dx_2 \cdots dx_{n+m-1} \ a.e. \tag{15}$$

Proof. If $I_n(f_n)$ and $I_m(g_m)$ are independent, then $I_n(f_n)I_m(g_m) \in L^2(\Omega, \mathcal{F}, P)$ and following [16],

$$\begin{aligned} |\, f_n \circ g_m \,|^2_{L^2(\mathrm{R}_+)^{\circ(m+m)}} &= (n+m)! \,|\, f_n \otimes g_m \,|^2_{L^2(\mathrm{R}_+)^{\otimes(n+m)}} \\ &\ge n!m! \,|\, f_n \,|^2_{L^2(\mathrm{R}_+)^{\otimes n}} |\, g_m \,|^2_{L^2(\mathrm{R}_+)^{\otimes m}} \\ &= E\left[I_n(f_n)^2\right] E\left[I_m(g_m)^2\right] = E\left[(I_n(f_n)I_m(g_m))^2\right] \\ &= \sum_{r=0}^{2(n\wedge m)} (n+m-r)! \,|\, h_{n,m,r} \,|^2_{L^2(\mathrm{R}_+)^{\otimes(n+m-r)}} \end{aligned}$$

$$\begin{aligned}
&\geq (n+m)! \mid h_{n,m,0} \mid^2_{L^2(\mathrm{R}_+)^{\otimes(n+m)}} +(n+m-1)! \mid h_{n,m,1} \mid^2_{L^2(\mathrm{R}_+)^{\otimes(n+m-1)}} \\
&\quad +(n+m-2)! \mid h_{n,m,2} \mid^2_{L^2(\mathrm{R}_+)^{\otimes(n+m-2)}} \\
&\geq (n+m)! \mid f_n \otimes g_m \mid^2_{L^2(\mathrm{R}_+)^{\otimes(n+m)}} +nm(n+m-1)! \mid f_n \circ_1^0 g_m \mid^2_{L^2(\mathrm{R}_+)^{\otimes(n+m-1)}} \\
&\quad +(n+m-2)! \mid nm f_n \circ_1^1 g_m + n(n-1)\frac{m(m-1)}{2} f_n \circ_2^0 g_m \mid^2_{L^2(\mathrm{R}_+)^{\otimes(n+m-2)}} .
\end{aligned}$$

We obtain $f_n \circ_1^0 g_m = 0$ a.e., and $f_n \circ_1^1 g_m = 0$ a.e.
Conversely, if (14) is satisfied, then $dP_2(\omega_2)$ almost surely, $I_n(f_n)(\cdot,\omega_2)$ and $I_m(g_m)(\cdot,\omega_2)$ are Wiener integrals of square-integrable functions that also satisfy (14), hence $I_n(f_n)(\cdot,\omega_2)$ is independent of $I_m(g_m)(\cdot,\omega_2)$ under P_1 from [16], and for any $u,v \in \mathcal{C}_b(\mathbf{R})$,

$$\int_{\Omega_1} u(I_n(f_n))v(I_m(g_m))dP_1 = \int_{\Omega_1} u(I_n(f_n))dP_1 \int_{\Omega_1} v(I_m(g_m))dP_1, \quad dP_2(\omega_2)-a.s.$$

If further (15) is satisfied, we choose two version $\bar{f}_n$ and $\bar{g}_m$ of f_n, g_m and let

$$A = \left\{ s \ : \| \bar{f}_n(s,\cdot) \|_{L^2(\mathrm{R}_+)^{\circ(n-1)}} \neq 0 \ and \ \phi_s \neq 0 \right\},$$

and

$$B = \left\{ s \ : \| \bar{g}_m(s,\cdot) \|_{L^2(\mathrm{R}_+)^{\circ(m-1)}} \neq 0 \ and \ \phi_s \neq 0 \right\}.$$

Then $\int_{\Omega_1} u(I_n(f_n))dP_1$ and $\int_{\Omega_1} v(I_m(g_m))dP_1$ are respectively $\mathcal{F}_1^A$-measurable and $\mathcal{F}_1^B$-measurable. Moreover,

$$\begin{aligned}
0 &= \int_0^\infty \| \bar{f}_n(s,\cdot) \|_{L^2(\mathrm{R}_+)^{\circ(n-1)}} \| \bar{g}_m(s,\cdot) \|_{L^2(\mathrm{R}_+)^{\circ(m-1)}} \mid \phi_s \mid ds \\
&= \int_{A\cap B} \| \bar{f}_n(s,\cdot) \|_{L^2(\mathrm{R}_+)^{\circ(n-1)}} \| \bar{g}_m(s,\cdot) \|_{L^2(\mathrm{R}_+)^{\circ(m-1)}} \mid \phi_s \mid ds,
\end{aligned}$$

hence $\mu(A\cap B) = 0$ and $\mathcal{F}_1^A$, $\mathcal{F}_2^B$ are independent σ-algebras because $(N_t)_{t\in\mathrm{R}_+}$ has independent increments, and

$$\int_\Omega u(I_n(f_n))v(I_m(g_m))dP = \int_\Omega u(I_n(f_n))dP \int_\Omega v(I_m(g_m))dP, \quad u,v \in \mathcal{C}_b(\mathbf{R}),$$

proving the independence of $I_n(f_n)$ and $I_m(g_m)$. □
The following corollaries, cf. [16], [17], can be extended from the Wiener case to the martingale $(M_t)_{t\in\mathrm{R}_+}$.

Proposition 4 *Two arbitrary families $\{I_{n_k}(f_{n_k}) \ : \ k \in I\}$ and $\{I_{m_l}(g_{m_l}) \ : \ l \in J\}$ of Poisson multiple stochastic integrals are independent if and only if $I_{n_k}(f_{n_k})$ is independent of $I_{m_l}(g_{m_l})$ for any $k \in I$, $l \in J$.*

Proof. We start by considering families of the form $\{I_n(f_n)\}$, $\{I_k(g_k), I_m(h_m)\}$. If $I_n(f_n)$ is independent of $I_k(g_k)$ and $I_n(f_n)$ is independent of $I_m(h_m)$, then (14) is satisfied for f_n, g_k and for f_n, g_m. Moreover, $dP_2(\omega_2)$ almost surely, $I_n(f_n)(\cdot,\omega_2)$, $I_k(g_k)(\cdot,\omega_2)$ and $I_m(h_m)(\cdot,\omega_2)$ are multiple Wiener integrals of square-integrable functions that also satisfy (14), hence for $u \in \mathcal{C}_b(\mathbf{R})$ and $v \in \mathcal{C}_b(\mathbf{R}^2)$, $u(I_n(f_n))(\cdot,\omega_2)$ is independent of $v(I_k(g_k), I_m(h_m))(\cdot,\omega_2)$ under P_1 from the analog of this proposition in [16], and

$$\int_{\Omega_1} u(I_n(f_n))v(I_k(g_k), I_m(h_m))dP_1 = \int_{\Omega_1} u(I_n(f_n))dP_1 \int_{\Omega_1} v(I_k(g_k), I_m(h_m))dP_1,$$

$dP_2(\omega_2)$-a.s.
We choose three versions $\bar{f}_n$, $\bar{g}_k$, and $\bar{h}_m$ of of f_n, g_k, h_m and let

$$A = \left\{s \ : \| \bar{f}_n(s,\cdot) \|_{L^2(\mathbf{R}_+)^{\circ(n-1)}} \neq 0 \ and \ \phi_s \neq 0\right\},$$

$$B = \left\{s \ : \| \bar{g}_k(s,\cdot) \|_{L^2(\mathbf{R}_+)^{\circ(k-1)}} \neq 0 \ and \ \phi_s \neq 0\right\},$$

and

$$C = \left\{s \ : \| \bar{f}_m(s,\cdot) \|_{L^2(\mathbf{R}_+)^{\circ(m-1)}} \neq 0 \ and \ \phi_s \neq 0\right\}.$$

Since $I_n(f_n)$ is independent of $I_k(g_k)$ and $I_n(f_n)$ is independent of $I_m(h_m)$, (15) holds for f_n, g_k and f_n, h_m. This implies

$$\begin{aligned} 0 &= \int_0^\infty \| \bar{f}_n(s,\cdot) \|_{L^2(\mathbf{R}_+)^{\circ(n-1)}} \| \bar{g}_k(s,\cdot) \|_{L^2(\mathbf{R}_+)^{\circ(k-1)}} | \phi_s | \, ds \\ &= \int_{A\cap B} \| \bar{f}_n(s,\cdot) \|_{L^2(\mathbf{R}_+)^{\circ(n-1)}} \| \bar{g}_k(s,\cdot) \|_{L^2(\mathbf{R}_+)^{\circ(k-1)}} | \phi_s | \, ds, \end{aligned}$$

hence $\mu(A\cap B) = 0$ and in the same way we get $\mu(A\cap C) = 0$, hence $\mu(A\cap(B\cup C)) = 0$. Consequently, $\mathcal{F}_1^A$ is independent of $\mathcal{F}_2^{B\cup C}$ since $(N_t)_{t\in\mathbf{R}_+}$ has independent increments. Moreover, $\int_{\Omega_1} u(I_n(f_n))dP_1$ and $\int_{\Omega_1} v(I_k(g_k), I_m(h_m))dP_1$ are respectively $\mathcal{F}_2^A$ and $\mathcal{F}_2^{B\cup C}$-measurable, hence

$$\int_\Omega u(I_n(f_n))v(I_k(g_k), I_m(h_m))dP = \int_\Omega u(I_n(f_n))dP \int_\Omega v(I_k(g_k), I_m(h_m))dP,$$

$u \in \mathcal{C}_b(\mathbf{R})$, $v \in \mathcal{C}_b(\mathbf{R}^2)$, and $u(I_n(f_n))$ is independent of $v(I_k(g_k), I_m(h_m))$. The above proof generalizes to arbitrary families of multiple stochastic integrals. □

Corollary 1 *Let $f_n \in L^2(\mathbf{R}_+)^{\circ n}$, $g_m \in L^2(\mathbf{R}_+)^{\circ m}$, and*

$$S_{f_n} = \{f_n \circ_{n-1}^{n-1} h \ : \ h \in L^2(\mathbf{R}_+)^{\circ n-1}\}, \quad S_{g_m} = \{g_n \circ_{m-1}^{m-1} h \ : \ h \in L^2(\mathbf{R}_+)^{\circ m-1}\}.$$

The following statements are equivalent.
(i) $I_n(f_n)$ is independent of $I_m(g_m)$.
(ii) For any $f \in S_{f_n}$ and $g \in S_{g_m}$ we have $fg = 0$, $| \phi_t | dt$-a.e. and $(f,g)_{L^2(\mathbf{R}_+)} = 0$
(iii) The σ-algebras $\sigma(I_1(f) \ : \ f \in S_{f_n})$ and $\sigma(I_1(g) \ : \ g \in S_{g_m})$ are independent.

Proof. $(i) \Leftrightarrow (ii)$ relies on the fact that any $f \in S_{f_n}$ and $g \in S_{g_m}$ can be written as $f = f_n \circ_{n-1}^{n-1} h$, $g = g_m \circ_{m-1}^{m-1} k$ with $h \in L^2(\mathbf{R}_+)^{\circ n-1}$, $k \in L^2(\mathbf{R}_+)^{\circ m-1}$, and that $\phi_t f(t)g(t) = (f_n \otimes_1^0 g_m(t,\cdot), h \otimes k)_{L^2(\mathbf{R}_+)^{\circ n+m-2}}$, $t \in \mathbf{R}_+$, and $(f,g)_{L^2(\mathbf{R}_+)} = (f_n \circ_1^1 g_m, h \otimes k)_{L^2(\mathbf{R}_+)^{\circ n+m-2}}$. $(ii) \Leftrightarrow (iii)$ is a consequence of Prop. 4. □

Let $(h_k)_{k\in\mathbf{N}^*}$ be an orthonormal basis of $L^2(\mathbf{R}_+)$. For simplicity, we denote by

$$\sigma(I_n(f_n), \nabla I_n(f_n), \ldots, \nabla^{n-1} I_n(f_n))$$

the σ-algebra

$$\sigma\left(I_n(f_n), \left(\nabla I_n(f_n), h_{k_1^1}\right)_{L^2(\mathbf{R}_+)}, \ldots, \right.$$
$$\left. \left(\nabla^{n-1} I_n(f_n), h_{k_1^{n-1}} \circ \cdots \circ h_{k_{n-1}^{n-1}}\right)_{L^2(\mathbf{R}_+)^{\circ n-1}}, \quad k_j^i \in \mathbf{N}^*, \ 1 \le i \le j \right).$$

Corollary 2 *The multiple stochastic integrals $I_n(f_n)$ and $I_m(g_m)$ are independent if and only if the σ-algebras*

$$\sigma(I_n(f_n), \nabla I_n(f_n), \ldots, \nabla^{n-1} I_n(f_n))$$

and

$$\sigma(I_m(g_m), \nabla I_m(g_m), \ldots, \nabla^{m-1} I_m(g_m))$$

are independent.

Proof. This is a consequence of Th. 1, Prop. 4, and the definition (8) of ∇. □

Let λ denote the Lebesgue measure on $(\mathbf{R}_+, \mathcal{B}(\mathbf{R}_+))$.

Corollary 3 *If $F \in Dom_2(\nabla)$ and $G \in L^2(\Omega, \mathcal{F}, P)$ with $G = \sum_{m\ge 0} I_m(g_m)$, then F is independent of G if for any $m \ge 1$,*

$$g_m \circ_1^1 \nabla F = 0 \quad \lambda^{\otimes(m-1)} \otimes P - a.e. \quad and \quad g_m \circ_1^0 \nabla F = 0, \quad \lambda^{\otimes m} \otimes P - a.e. \tag{16}$$

Proof. Assume that $F = \sum_{n\ge 0} I_n(f_n)$. Condition (16) is equivalent to $g_m \circ_1^1 f_n = 0$ and $g_m \circ_1^0 f_n = 0$ a.e. for any $n, m \in \mathbf{N}$, since the decomposition $\nabla F = \sum_{n\ge 0} n I_{n-1}(f_n)$ is orthogonal in $L^2(\Omega) \otimes L^2(\mathbf{R}_+)$. The result follows then from Th. 1 and Prop. 4. □

Remarks. *a)* In the Poisson case, the results of this paper can also be obtained for a Poisson measure on a metric space with a σ-finite diffuse measure.
b) The independence criterion also means that $I_n(f_n)$ and $I_m(g_m)$ are independent if and only if their Wick product coincides with their ordinary product:

$$I_n(f_n) I_m(g_m) = I_{n+m}(f_n \circ g_m) = I_n(f_n) : I_m(g_m).$$

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