

Option pricing and implied volatilities in a 2-hypergeometric stochastic volatility model

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Abstract

We derive closed-form analytical approximations in terms of series expansions for option prices and implied volatilities in a 2-hypergeometric stochastic volatility model with correlated Brownian motions. As in [4], these expansions allow us to recover the well-known skew and smile phenomena on implied volatility surfaces, depending on the values of the correlation parameter.

Key words: Stochastic volatility; 2-hypergeometric model; implied volatility; series expansions.

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1 Introduction

Stochastic volatility models have been introduced as realistic models for the motion of asset prices in financial markets. The most well-known of such models is the Heston [7] model, which however has one major drawback as its stochastic volatility may reach zero in finite time unless one imposes the Feller condition, and this poses potential

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problems in model calibration, cf. e.g. § 1.3.3 of Janek et al. [8] and § 6.5.2 of Henry-Labordère [6]. In view of this, the α -hypergeometric stochastic volatility model has been introduced by Da Fonseca and Martini [2] to ensure strict positivity of volatility. In the α -hypergeometric model the dynamics of the asset price S_t at time t and the volatility V_t are governed by

$$dS_t = S_t e^{V_t} dW_t^1, \quad dV_t = \left(a - \frac{c}{2} e^{\alpha V_t} \right) dt + \eta dW_t^2, \quad (1)$$

$c > 0$, $\eta > 0$, $a \in \mathbb{R}$, $\alpha > 0$, and W_t^1 and W_t^2 are correlated Brownian motions satisfying $\langle W^1, W^2 \rangle_t = \rho t$. In this model the risk free rate r is taken to be equal to 0 and the value of c can be used to set the price of volatility risk.

Stochastic volatility models generally do not admit explicit solutions, and this has motivated the development of approximate expansions. In Fouque et al. [3] a method to obtain series expansions for European option prices has been proposed in the Heston model. The first and second order terms in this expansion do not depend on the value of stochastic volatility which is a key quantity in the Heston model, and as a consequence it cannot be used to reproduce the smile effect in model calibration. A more accurate approximation has been proposed in Han et al. [4] for European option prices in the Heston model via a series expansion that involves the underlying stochastic volatility, allowing the authors to recover the smile effect and to avoid the secular effect and terminal layer problems posed by the third term in the expansion of [3], see also Kim [9] under stochastic interest rates.

In this paper we extend the method of [4], see also [10], in order to derive series expansions based on approximations of the 2-hypergeometric model of [2]. In particular, our analytical approximate solution depends on the underlying stochastic volatility. We check that our approximate solutions agree with Monte Carlo simulations, including in the case of first order approximations. We also derive implied volatility estimates which display the well known phenomena of skew and smile.

2 Stochastic volatility

We start with a general class of stochastic volatility models in which the dynamics of the asset price and volatility processes are given by

$$dS_t^\varepsilon = S_t^\varepsilon p(t, V_t^\varepsilon) dW_t^1, \quad dV_t^\varepsilon = u(t, V_t^\varepsilon) dt + \varepsilon h(t, V_t^\varepsilon) dW_t^2,$$

where $\varepsilon > 0$. Note that by Brownian rescaling, small volatility coefficients can be used to derive small time asymptotics, cf. e.g. Section 2.1.8 of [2]. Recall that under absence of arbitrage, the vanilla option price of an option with payoff $g(S_T^\varepsilon)$ takes the form

$$f(t, S_t^\varepsilon, V_t^\varepsilon) := \mathbb{E}[g(S_T^\varepsilon) | \mathcal{F}_t]$$

where $(\mathcal{F}_t)_{t \in [0, T]}$ is the filtration generated by $(W_t^1, W_t^2)_{t \in [0, T]}$, and the function $f(t, x, v)$ solves the PDE

$$\frac{\partial f}{\partial t} + u(t, v) \frac{\partial f}{\partial v} + \frac{x^2}{2} p^2(t, v) \frac{\partial f}{\partial x^2} + \varepsilon \rho x p(t, v) h(t, v) \frac{\partial^2 f}{\partial x \partial v} + \frac{\varepsilon^2}{2} h^2(t, v) \frac{\partial^2 f}{\partial v^2} = 0, \quad (2)$$

cf. e.g. (2.17) in [3], with the terminal condition $f(T, x, v) = g(x)$. We start by expanding $f(t, x, v)$ as

$$f(t, x, v) = f_0(t, x, v) + \varepsilon f_1(t, x, v) + o(\varepsilon). \quad (3)$$

By plugging in the expansion (3) into the pricing PDE (2) we get the system of equations

$$\frac{\partial f_n}{\partial t} + \mathcal{L}_0 f_n + \mathcal{L}_1 f_{n-1} + \mathcal{L}_2 f_{n-2} = 0, \quad n \in \mathbb{N},$$

with $f_n = 0$, $n \leq -1$, $f_0(T, x, v) = g(x)$ and $f_n(T, x, v) = 0$, $n \geq 1$. In particular the operators \mathcal{L}_0 , \mathcal{L}_1 and \mathcal{L}_2 are given by

$$\mathcal{L}_0 = u(t, v) \frac{\partial}{\partial v} + \frac{x^2}{2} p^2(t, v) \frac{\partial^2}{\partial x^2}, \quad \mathcal{L}_1 = \rho x p(t, v) h(t, v) \frac{\partial^2}{\partial x \partial v}, \quad \mathcal{L}_2 = \frac{1}{2} h^2(t, v) \frac{\partial^2}{\partial v^2}. \quad (4)$$

3 Deterministic volatility

When $n = 0$ we have $\frac{\partial f_0}{\partial t} + \mathcal{L}_0 f_0 = 0$, $(S_t^0)_{t \in [0, T]}$ and $(V_t^0)_{t \in [0, T]}$ are given by

$$dS_t^0 = S_t^0 p(t, V_t^0) dW_t^1, \quad dV_t^0 = u(t, V_t^0) dt$$

and the vanilla option price $f_0(t, S_t^0, V_t^0) := \mathbb{E}[g(S_T^0) | \mathcal{F}_t]$ can be computed by the Black-Scholes formula as

$$f_0(t, S_t^0, V_t^0) = \mathbb{E} \left[(S_T^0 - K)^+ \middle| \mathcal{F}_t \right] = \mathbb{E} \left[\left(S_t^0 \exp \left(Z\gamma(t, V_t^0) - \frac{1}{2}\gamma^2(t, V_t^0) \right) - K \right)^+ \middle| \mathcal{F}_t \right],$$

where $Z \simeq \mathcal{N}(0, 1)$ is independent of \mathcal{F}_t and $\gamma^2(t, V_t^0) := \int_t^T p^2(u, V_u^0) du$, $t \in [0, T]$.

We note that in the α -hypergeometric model (1) with $\eta = 0$ the integral $\int_t^T e^{\alpha V_u^0} du$ can be computed in closed form as

$$\int_t^T e^{\alpha V_u^0} du = \frac{2}{\alpha c} \log \left(1 + \frac{\alpha c}{2} e^{\alpha V_t^0} \int_0^{T-t} e^{\alpha a s} ds \right) = \frac{2}{\alpha c} \log \left(1 + \frac{\alpha c}{2} e^{\alpha V_t^0} \frac{e^{\alpha a(T-t)} - 1}{\alpha a} \right),$$

cf. § 2.1.1 of [2], and this yields the following proposition.

Proposition 1. *In the 2-hypergeometric model (1) with $\eta = 0$ the European call price*

$$f_0(t, S_t^0, V_t^0) = \mathbb{E} \left[(S_T^0 - K)^+ \middle| \mathcal{F}_t \right]$$

under the terminal condition $f_0(T, x, v) = (x - K)^+$ is given by

$$f_0(t, x, v) = x\Phi(d_+(t, x, v)) - K\Phi(d_-(t, x, v)),$$

where Φ is the standard Gaussian cumulative distribution function,

$$d_{\pm}(t, x, v) = \frac{1}{\gamma(t, v)} \left(\log \left(\frac{x}{K} \right) \pm \frac{\gamma^2(t, v)}{2} \right), \text{ and } \gamma^2(t, v) = \frac{1}{c} \log \left(1 + ce^{2v} \frac{e^{2a(T-t)} - 1}{2a} \right). \quad (5)$$

In the case of a put option the function $f_0(t, x, v)$ can be obtained as

$$f_0(t, x, v) = -x\Phi(-d_+(t, x, v)) + K\Phi(-d_-(t, x, v)), \quad t \in [0, T],$$

by a standard call-put parity argument. In the remainder of this paper we work in the 2-hypergeometric model with $\alpha = 2$.

4 First order expansion

In this section we consider small values of the volatility of volatility by replacing η in (1) with $\varepsilon\eta e^{V_t^\varepsilon}$, $\varepsilon > 0$, i.e. we have

$$dS_t^\varepsilon = S_t^\varepsilon e^{V_t^\varepsilon} dW_t^1, \quad dV_t^\varepsilon = \left(a - \frac{c}{2} e^{2V_t^\varepsilon} \right) dt + \varepsilon\eta e^{V_t^\varepsilon} dW_t^2, \quad (6)$$

and from (4) the operators \mathcal{L}_0 , \mathcal{L}_1 and \mathcal{L}_2 are given by

$$\mathcal{L}_0 = \left(a - \frac{c}{2} e^{2v} \right) \frac{\partial}{\partial v} + \frac{x^2}{2} e^{2v} \frac{\partial^2}{\partial x^2}, \quad \mathcal{L}_1 = \eta\rho x e^{2v} \frac{\partial^2}{\partial x \partial v}, \quad \mathcal{L}_2 = \frac{\eta^2}{2} e^{2v} \frac{\partial^2}{\partial v^2}.$$

In particular when $n = 1$ we get

$$\frac{\partial f_1}{\partial t} + \mathcal{L}_0 f_1 + \mathcal{L}_1 f_0 = 0, \quad \text{with} \quad f_1(T, x, v) = 0.$$

Note that our approximation $(S_t^\varepsilon, V_t^\varepsilon)_{t \in [0, T]}$ does not lie within the class of 2-hypergeometric models.

Proposition 2. *The solution of $\frac{\partial f_1}{\partial t} + \mathcal{L}_0 f_1 + \mathcal{L}_1 f_0 = 0$ under the terminal condition $f_1(T, x, v) = 0$ is given by*

$$f_1(t, x, v) = -\rho K \frac{\eta}{c} d_-(t, x, v) \phi(d_-(t, x, v)) \frac{e^{-c\gamma^2(t, v)} + c\gamma^2(t, v) - 1}{c\gamma^2(t, v)}, \quad t \in [0, T],$$

where $\phi(x)$ is the standard Gaussian probability density function and $\gamma(t, v)$ is defined in (5).

Proof. From the relation $\phi(d_+(t, x, v)) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(d_+(t, x, v))^2\right) = \frac{K}{x} \phi(d_-(t, x, v))$ and using the Feynman-Kac formula with locally Lipschitz coefficients as in e.g. Theorem 1 of Heath and Schweizer [5], we have

$$\begin{aligned} f_1(t, S_t^0, V_t^0) &= \int_t^T \mathbb{E} [\mathcal{L}_1 f_0(r, S_r^0, V_r^0) | \mathcal{F}_t] dr \\ &= -\eta \int_t^T \frac{\rho K e^{2V_r^0}}{\gamma(r, V_r^0)} \frac{\partial \gamma}{\partial v}(r, V_r^0) \mathbb{E} [d_-(r, S_r^0, V_r^0) \phi(d_-(r, S_r^0, V_r^0)) | \mathcal{F}_t] dr \\ &= -\eta \rho K \frac{d_-(t, S_t^0, V_t^0)}{\gamma^2(t, V_t^0)} \phi(d_-(t, S_t^0, V_t^0)) \int_t^T e^{2V_r^0} \gamma(r, V_r^0) \frac{\partial \gamma}{\partial v}(r, V_r^0) dr, \end{aligned}$$

by a standard computation based on the Gaussian distribution

$$d_-(r, S_r^0, V_r^0) \sim \mathcal{N}\left(\frac{1}{\gamma(r, V_r^0)}\left(\log\left(\frac{S_t^0}{K}\right) - \frac{\gamma^2(t, V_t^0)}{2}\right), \frac{\gamma^2(t, V_t^0)}{\gamma^2(r, V_r^0)} - 1\right), \quad r \in [t, T],$$

given \mathcal{F}_t . Finally we note that from (5) we have

$$\begin{aligned} \int_t^T e^{2V_r^0} \gamma(r, V_r^0) \frac{\partial \gamma}{\partial v}(r, V_r^0) dr &= \frac{1}{c} \int_t^T e^{2V_r^0} \left(1 - e^{-c\gamma^2(r, V_r^0)}\right) dr \\ &= \frac{1}{c^2} \left(e^{-c\gamma^2(t, V_t^0)} + c\gamma^2(t, V_t^0) - 1\right). \end{aligned} \quad (7)$$

□

5 Second order expansion

The computation of a second order correction term $\tilde{f}_2(t, x, v)$ requires us to replace η in (1) with $\varepsilon\eta e^{V_t} \gamma^4(t, V_t)$ in order to involve only even powers of $\gamma(r, v)$ when extending the computation of (7) above. In this case, \mathcal{L}_1 and \mathcal{L}_2 are replaced by the operators

$$\tilde{\mathcal{L}}_1 = \eta\rho x e^{2v} \gamma^4(t, v) \frac{\partial^2}{\partial x \partial v}, \quad \tilde{\mathcal{L}}_2 = \frac{\eta^2}{2} e^{2v} \gamma^8(t, v) \frac{\partial^2}{\partial v^2},$$

and we look for an expansion of the form

$$f(t, x, v) = f_0(t, x, v) + \varepsilon \tilde{f}_1(t, x, v) + \varepsilon^2 \tilde{f}_2(t, x, v) + o(\varepsilon^2), \quad (8)$$

where $f_0(T, x, v) = (x - K)^+$, $\tilde{f}_1(T, x, v) = 0$, $\tilde{f}_2(T, x, v) = 0$, and

$$\frac{\partial f_0}{\partial t} + \mathcal{L}_0 f_0 = 0, \quad \frac{\partial \tilde{f}_1}{\partial t} + \mathcal{L}_0 \tilde{f}_1 + \tilde{\mathcal{L}}_1 f_0 = 0, \quad \frac{\partial \tilde{f}_2}{\partial t} + \mathcal{L}_0 \tilde{f}_2 + \tilde{\mathcal{L}}_1 \tilde{f}_1 + \tilde{\mathcal{L}}_2 f_0 = 0,$$

Proposition 3. *The first and second order coefficients appearing in the expansion (8) are given by*

$$\begin{aligned} \tilde{f}_1(t, x, v) &= -\eta\rho K \frac{d_-(t, x, v)}{c^3 \gamma^2(t, v)} \phi(d_-(t, x, v)) \left(e^{-c\gamma^2(t, v)} (c^2 \gamma^4(t, v) + c^2 \gamma^2(t, v) + 2) - 2 + \frac{c^3}{3} \gamma^6(t, v) \right), \\ \tilde{f}_2(t, x, v) &= \frac{\eta^2}{c} K \phi(d_-(t, x, v)) \left(\frac{A_3(t, v)}{\gamma(t, v)} + d_-(t, x, v) B_3(t, v) + \frac{(d_-(t, x, v))^2}{\gamma(t, v)} B_3(t, v) \right) \\ &\quad + \eta^2 \rho^2 K \phi(d_-(t, x, v)) (C_3(t, v) \\ &\quad + \frac{2D(t, v)}{3c^7 \gamma^4(t, v)} \left(\frac{(d_-(t, x, v))^4}{3\gamma(t, v)} - d_-(t, x, v) + \frac{(d_-(t, x, v))^3}{3} \right) + \frac{(d_-(t, x, v))^2}{\gamma^5(t, v)} E_3(t, v) \right), \quad t \in [0, T], \end{aligned}$$

where the functions A_i, B_i, C_i, D, E_i are given below for $i = 1, 2, 3$.

Proof. The expression of \tilde{f}_1 is obtained by the same argument as in the proof of Proposition 2. For \tilde{f}_2 we have $\frac{\partial \tilde{f}_2}{\partial t} + \mathcal{L}_0 \tilde{f}_2 + \tilde{\mathcal{L}}_1 \tilde{f}_1 + \tilde{\mathcal{L}}_2 f_0 = 0$ with $\tilde{f}_2(T, x, v) = 0$, hence \tilde{f}_2 can be computed by similar arguments from the Feynman-Kac formula and the expected value

$$\tilde{f}_2(t, S_t, V_t) = \int_t^T \mathbb{E} \left[\tilde{\mathcal{L}}_1 \tilde{f}_1(r, S_r, V_r) + \tilde{\mathcal{L}}_2 f_0(r, S_r, V_r) \mid \mathcal{F}_t \right] dr.$$

For simplicity of exposition we skip the corresponding computations, which are significantly longer than in the proof of Proposition 2. \square

We have

$$\begin{aligned} A_1(t, v) &= \frac{\gamma^8(t, v)}{2c} + \frac{5\gamma^6(t, v)}{4c^2} + \frac{2\gamma^4(t, v)}{c^3} + \frac{9\gamma^2(t, v)}{4c^4} + \frac{3}{2c^5} + \frac{3c^{-6}}{8\gamma^2(t, v)}, \\ A_2(t, v) &= -\frac{\gamma^8(t, v)}{c} - \frac{5\gamma^6(t, v)}{c^2} - \frac{16\gamma^4(t, v)}{c^3} - \frac{24}{c^4}\gamma^2(t, v) - \frac{48}{c^5} - \frac{24c^{-6}}{\gamma^2(t, v)}, \\ A_3(t, v) &= -\frac{\gamma^8(t, v)}{10c} + e^{-2c\gamma^2(t, v)}A_1(t, v) + A_2(t, v)e^{-c\gamma^2(t, v)} + \frac{93}{4c^5} + \frac{189c^{-6}}{8\gamma^2(t, v)}, \\ B_1(t, v) &= -\frac{\gamma^6(t, v)}{4c^2} - \frac{\gamma^4(t, v)}{2c^3} - \frac{3\gamma^2(t, v)}{4c^4} - \frac{3}{4c^5} - \frac{3c^{-6}}{8\gamma^2(t, v)}, \\ B_2(t, v) &= \frac{\gamma^6(t, v)}{c^2} + \frac{4\gamma^4(t, v)}{c^3} + \frac{12\gamma^2(t, v)}{c^4} + \frac{24}{c^5} + \frac{24c^{-6}}{\gamma^2(t, v)}, \\ B_3(t, v) &= \frac{\gamma^8(t, v)}{10c} + e^{-2c\gamma^2(t, v)}B_1(t, v) + B_2(t, v)e^{-c\gamma^2(t, v)} - \frac{189}{8c^6\gamma^2(t, v)}, \\ C_1(t, v) &= -\frac{\gamma^5(t, v)}{c^2} + \frac{\gamma^3(t, v)}{2c^3} + \frac{3\gamma^2(t, v)}{c^4} + \frac{4\gamma(t, v)}{c^4} + \frac{9c^{-5}}{2} + \frac{21c^{-5}}{2\gamma(t, v)} + \frac{9c^{-6}}{2\gamma^2(t, v)} \\ &\quad + \frac{9c^{-6}}{\gamma^3(t, v)} + \frac{9c^{-7}}{4\gamma^4(t, v)} + \frac{15c^{-7}}{4\gamma^5(t, v)}, \\ C_2(t, v) &= -\frac{3\gamma^5(t, v)}{c^2} - \frac{9\gamma^3(t, v)}{c^3} - \frac{6\gamma^2(t, v)}{c^4} - \frac{64\gamma(t, v)}{c^4} - \frac{36}{c^5} - \frac{120c^{-5}}{\gamma(t, v)} - \frac{36c^{-6}}{\gamma^2(t, v)} \\ &\quad + \frac{24c^{-6}}{\gamma^3(t, v)} - \frac{36c^{-7}}{\gamma^4(t, v)} + \frac{24c^{-7}}{\gamma^5(t, v)}, \\ C_3(t, v) &= -\frac{7\gamma^7(t, v)}{30c} - \frac{2\gamma(t, v)}{c^4} + C_1(t, v)e^{-2c\gamma^2(t, v)} + C_2(t, v)e^{-c\gamma^2(t, v)} + \frac{189c^{-6}}{2\gamma^3(t, v)} \\ &\quad + \frac{135c^{-7}}{4\gamma^4(t, v)} - \frac{111c^{-7}}{4\gamma^5(t, v)}, \\ D(t, v) &= e^{-2c\gamma^2(t, v)} \left(e^{c\gamma^2(t, v)} (c^3\gamma^6(t, v) - 3) + 3c\gamma^2(t, v) \left(\frac{c}{2}\gamma^2(t, v) + 1 \right) + 3 \right)^2, \end{aligned}$$

$$\begin{aligned}
E_1(t, v) &= \frac{\gamma^{10}(t, v)}{c^2} + \frac{\gamma^8(t, v)}{c^3} - \frac{15\gamma^6(t, v)}{c^4} - \frac{27\gamma^4(t, v)}{c^5} - \frac{51\gamma^2(t, v)}{2c^6} - \frac{12}{c^7}, \\
E_2(t, v) &= \frac{2\gamma^{10}(t, v)}{c^2} + \frac{2\gamma^8(t, v)}{c^3} + \frac{17\gamma^6(t, v)}{c^4} + \frac{216\gamma^4(t, v)}{c^5} + \frac{24\gamma^2(t, v)}{c^6} + \frac{24}{c^7}, \\
E_3(t, v) &= \frac{\gamma^{12}(t, v)}{15c} + \frac{4\gamma^6(t, v)}{c^4} - \frac{189\gamma^2(t, v)}{2c^6} + E_1(t, v)e^{-2c\gamma^2(t, v)} + e^{-c\gamma^2(t, v)}E_2(t, v) - \frac{492}{c^7}.
\end{aligned}$$

Note that in the case of put options, only the function $f_0(t, x, v)$ is modified by the standard call-put parity argument, while higher order terms such as $f_1(t, x, v)$, $\tilde{f}_1(t, x, v)$ and $\tilde{f}_2(t, x, v)$ remain unchanged. In Figure 1 we plot the option price against the stochastic variance v with correlation $\rho = -0.5$ and parameters $x = K$, $T = 0.1$, $t = 0$, $a = c/2 = 1$, $\eta = 2$ and $\varepsilon = 0.01$. The Monte Carlo curve required 300,000 samples based on 30,000 time steps.

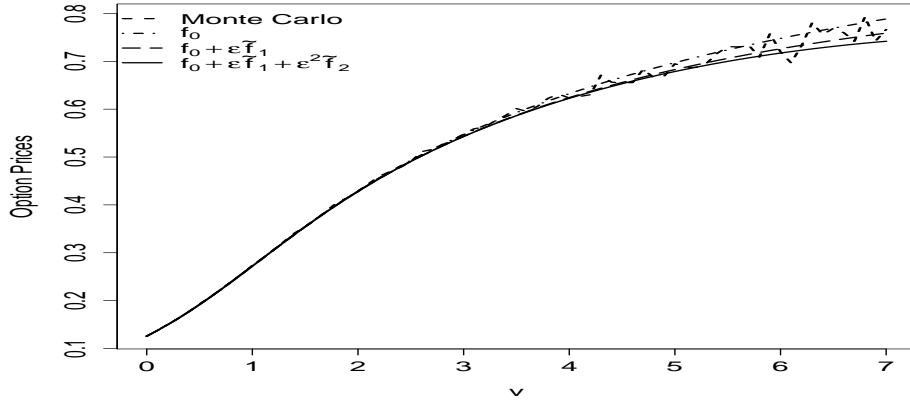


Figure 1: Option price f plotted against v with $\rho = -0.5$.

Figure 1 shows that our asymptotic solutions are in agreement with the Monte Carlo solution, even if we only use f_0 of our analytical approximate solutions when v is in the interval $[0, 4]$, while the Monte Carlo estimate hovers around the analytical approximate solutions for larger values of v . For larger v , further increasing the numbers of simulations and time steps would yield a smoother Monte Carlo graph. In the next Table 1 we present the approximated values obtained from f_0 , $f_0 + \epsilon\tilde{f}_1$, $f_0 + \epsilon\tilde{f}_1 + \epsilon^2\tilde{f}_2$, with the parameters $S = K$, $T = 0.1$, $t = 0$, $a = c/2 = 1$, $\eta = 2$, $\epsilon = 0.01$ and $\rho = -0.5$. The corresponding Monte Carlo estimates required 1,000,000 samples based on 100,000 time steps, while the evaluation of the approximations is instantaneous. The large number of time steps is due to instabilities in the solution of stochastic differential equations (SDEs) with non-Lipschitz (here exponential)

coefficients such as (6).

v	f_0	$f_0 + \epsilon \tilde{f}_1$	$f_0 + \epsilon \tilde{f}_1 + \epsilon^2 \tilde{f}_2$	Monte Carlo
5.5	0.7239	0.7060	0.6998	0.6827
5.6	0.7289	0.7103	0.7036	0.6975
5.7	0.7338	0.7144	0.7072	0.7045
5.8	0.7386	0.7185	0.7107	0.7089
5.9	0.7433	0.7224	0.7140	0.7078
6.0	0.7478	0.7262	0.7172	0.7097

Table 1: Values of f_0 , $f_0 + \epsilon \tilde{f}_1$, $f_0 + \epsilon \tilde{f}_1 + \epsilon^2 \tilde{f}_2$ compared to Monte Carlo estimates.

We check that our approximate solutions using up to the second correction terms gives values closer to the Monte Carlo estimates.

6 Implied volatility

In this section we provide an estimation of the implied volatility. σ^{imp} which is determined by the equation

$$f^{\text{BS}}(t, x, T, K, \sigma^{\text{imp}}) = f(t, x, v),$$

where $f^{\text{BS}}(t, x, T, K, \sigma^{\text{imp}})$ is the classical Black-Scholes function, cf. e.g. Da Fonseca and Grasselli [1] in multi-factor models.

Theorem 4. *The implied volatility σ^{imp} admits the series expansion*

$$\sigma^{\text{imp}}(t, x, v) = \sigma_0(t, x, v) + \epsilon \sigma_1(t, x, v) + \epsilon^2 \sigma_2(t, x, v) + o(\epsilon^2),$$

where $\sigma_0(t, x, v) := \gamma(t, v)/\sqrt{T-t}$,

$$\sigma_1(t, x, v) := \frac{\tilde{f}_1(t, x, v)}{K\sqrt{T-t}\phi(d_-(t, x, v))},$$

and

$$\sigma_2(t, x, v) := \frac{\tilde{f}_2(t, x, v)}{K\sqrt{T-t}\phi(d_-(t, x, v))} - d_+(t, x, v)d_-(t, x, v)\frac{\sigma_1^2(t, x, v)}{2\sigma_0(t, x, v)}.$$

Proof. The implied volatility σ^{imp} is determined by equating

$$f^{\text{BS}}(t, x, T, K, \sigma^{\text{imp}}) = f(t, x, v) = f_0(t, x, v) + \varepsilon \tilde{f}_1(t, x, v) + \varepsilon^2 \tilde{f}_2(t, x, v) + o(\varepsilon^2),$$

where f^{BS} is the classical Black-Scholes function with implied volatility σ^{imp} . Expressing the implied volatility as a power series

$$\sigma^{\text{imp}}(t, x, v) = \sigma_0(t, x, v) + \varepsilon \sigma_1(t, x, v) + \varepsilon^2 \sigma_2(t, x, v) + o(\varepsilon^2)$$

in ε , we expand $f^{\text{BS}}(t, x, T, K, \sigma^{\text{imp}})$ and using a Taylor expansion in terms of ε to obtain

$$\begin{aligned} f^{\text{BS}}(t, x, T, K, \sigma^{\text{imp}}) &= f^{\text{BS}}(t, x, T, K, \sigma_0(t, x, v)) \\ &\quad + (\varepsilon \sigma_1(t, x, v) + \varepsilon^2 \sigma_2(t, x, v)) \frac{\partial f^{\text{BS}}}{\partial \sigma}(t, x, T, K, \sigma_0(t, x, v)) \\ &\quad + \frac{1}{2} \varepsilon^2 \sigma_1^2(t, x, v) \frac{\partial^2 f^{\text{BS}}}{\partial \sigma^2}(t, x, T, K, \sigma_0(t, x, v)) + \dots \end{aligned}$$

The first three terms of the implied volatility expansion are obtained by identification of coefficients in the above expressions. \square

In Figure 2 we plot the estimation of implied volatility against the ratio (moneyness) K/x of the strike price to the asset price, with the parameters $T = 1$, $t = 0$, $a = c/2 = v = 1$, $\eta = 3.5$ and $\varepsilon = 0.1$.

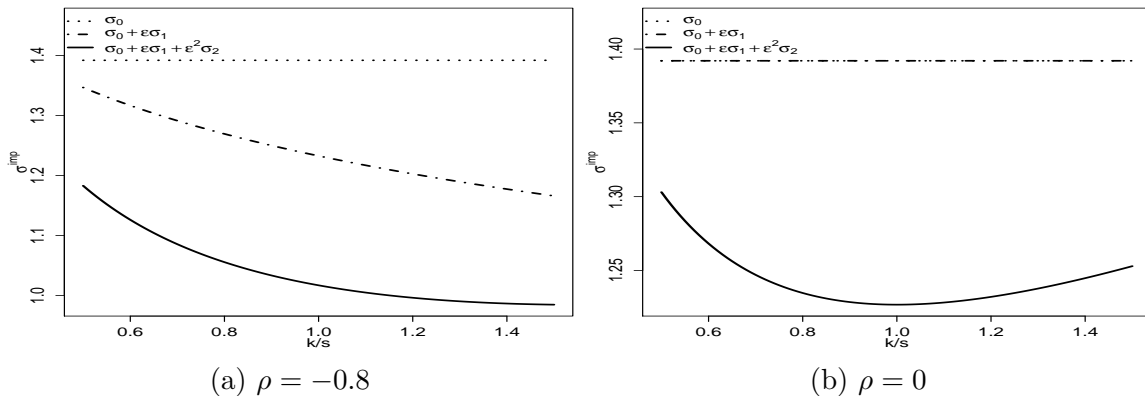


Figure 2: Implied volatility σ^{imp} plotted against the moneyness K/x .

Our implied volatility estimate $\sigma_0(t, x, v) + \varepsilon \sigma_1(t, x, v) + \varepsilon^2 \sigma_2(t, x, v)$ exhibits the well-known skew and smile phenomena. In addition they show that it can be necessary to

take into account the correction terms \tilde{f}_1 and \tilde{f}_2 for improved calibration. Monte Carlo estimates of volatility are not available due to the instabilities observed in Figure 1 for the numerical solution of SDEs such as (6).

References

- [1] J. Da Fonseca and M. Grasselli. Riding on the smiles. *Quant. Finance*, 11(11):1609–1632, 2011.
- [2] J. Da Fonseca and C. Martini. The α -hypergeometric stochastic volatility model. Preprint arXiv:1409.5142, 2014.
- [3] J.P. Fouque, G. Papanicolaou, K.R. Sircar, and K. Sølna. *Multiscale Stochastic Volatility for Equity, Interest Rate Derivatives, and Credit Derivatives*. Cambridge University Press, Cambridge, 2011.
- [4] J. Han, M. Gao, Q. Zhang, and Y. Li. Option prices under stochastic volatility. *Appl. Math. Lett.*, 26(1):1–4, 2013.
- [5] D. Heath and M. Schweizer. Martingales versus PDEs in finance: an equivalence result with examples. *J. Appl. Probab.*, 37(4):947–957, 2000.
- [6] P. Henry-Labordère. *Analysis, Geometry, and Modeling in Finance*. Chapman & Hall/CRC Financial Mathematics Series. CRC Press, Boca Raton, FL, 2009.
- [7] S.L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies*, 6(2):327–343, 1993.
- [8] A. Janek, T. Kluge, R. Weron, and U. Wystup. FX smile in the Heston model. SFB 649 Discussion Paper 047, 2010.
- [9] Y.J. Kim. Option pricing under stochastic interest rate: An empirical investigation. *Asia-Pacific Financial Markets*, 9:23–44, 2002.
- [10] Q. Zhang, J. Han, and M. Gao. Option price with stochastic volatility for both fast and slow mean-reverting regimes. *C. R. Math. Acad. Sci. Paris*, 351(9-10):411–414, 2013.