Random Hermite polynomials and Girsanov identities on the Wiener space

Nicolas Privault*
Division of Mathematical Sciences
School of Physical and Mathematical Sciences
Nanyang Technological University
SPMS-MAS-05-43, 21 Nanyang Link
Singapore 637371

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Abstract

In this paper we derive a formula for the expectation of random Hermite polynomials in Skorohod integrals, extending classical results in the adapted case. As an application we recover, under simple conditions and with short proofs, the anticipative Girsanov identity and quasi-invariance results obtained in [6] for quasi-nilpotent shifts on the Wiener space.

Key words: Malliavin calculus, Skorohod integral, Wiener measure, quasi-invariance, Girsanov identity, Euclidean motions.
Mathematics Subject Classification: 60H07, 60G30.

1 Introduction

It is well known that the Hermite polynomial

$$H_n(x,\mu)=\sum_{0\le 2k\le n}\frac{n!(-\mu/2)^k}{k!(n-2k)!}x^{n-2k},\qquad x\in\mathbb{R},\tag{1.1}$$

with parameter $\mu\in\mathbb{R}$ and generating function

$$e^{tx-t^2\mu/2}=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x,\mu),\qquad x,t\in\mathbb{R},\tag{1.2}$$

*nprivault@ntu.edu.sg

satisfies the identity

$$E[H_n(X,\sigma^2)] = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} H_n(x,\sigma^2)e^{-x^2/(2\sigma^2)}dx = 0, \qquad n \geq 1, \tag{1.3}$$

where $X \simeq \mathcal{N}(0,\sigma^2)$ is a centered Gaussian random variable with variance $\sigma^2 \geq 0$, since

$$\begin{aligned} E[H_{2n}(X,\sigma^2)] &= \sum_{k=0}^{n} \frac{(2n)!(-\sigma^2/2)^k}{k!(2n-2k)!} E[X^{2n-2k}] \\ &= \frac{(2n)!}{n!}\sum_{k=0}^{n}\binom{n}{k}(-\sigma^2/2)^k(\sigma^2/2)^{n-k} \\ &= 0, \end{aligned}$$

from the even Gaussian moment $E[X^{2m}] = (\sigma^2/2)^m(2m)!/m!$, $m \geq 0$.

The identity (1.3) holds in particular when X is written as the stochastic integral

$$X = \int_0^\infty f(s)dB_s$$

of a deterministic real-valued function f with respect to the standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ and σ^2 is the constant $\sigma^2 = \int_0^\infty |f(s)|^2 ds$.

It is well known, however, that the Gaussianity of X is not required for $E[H_n(X,\sigma^2)]$ to vanish when σ^2 is allowed to be random. Indeed, such an identity also holds in the random adapted case under the form

$$E\left[H_n\left(\int_0^\infty u_t dB_t, \int_0^\infty |u_t|^2 dt\right)\right] = 0, \tag{1.4}$$

where $(u_t)_{t\in\mathbb{R}_+}$ is a square-integrable process adapted to the filtration generated by $(B_t)_{t\in\mathbb{R}_+}$, due to the fact that

$$H_n\left(\int_0^\infty u_t dB_t, \int_0^\infty |u_t|^2 dt\right) = n!\int_0^\infty u_{t_n}\int_0^{t_n} u_{t_{n-1}}\cdots\int_0^{t_2} u_{t_1} dB_{t_1}\cdots dB_{t_n},$$

is the n-th order iterated multiple stochastic integral of $u_{t_1}\cdots u_{t_n}$ with respect to $(B_t)_{t\in\mathbb{R}_+}$, cf. [4] and [2] page 319.

In this paper we prove an extension of (1.4) to the random case, by computing in Theorem 3.1 the expectation

$$E[H_n(\delta(u), \|u\|^2)], \qquad n \geq 1,$$

of the random Hermite polynomial $H_n(\delta(u), \|u\|^2)$, where $\delta(u)$ is the Skorohod integral of a possibly anticipating process $(u_t)_{t\in\mathbb{R}_+}$. In particular we provide conditions on the process $(u_t)_{t\in\mathbb{R}_+}$ for the expectation $E[H_n(\delta(u), \|u\|^2)]$, $n \geq 1$, to vanish. Such conditions cover the quasi-nilpotence condition of [5] and include the adaptedness of $(u_t)_{t\in\mathbb{R}_+}$, which recovers (1.4) as a particular case since $\delta(u)$ coincides with the Itô integral when $(u_t)_{t\in\mathbb{R}_+}$ is adapted.

This type of argument has been applied in [3] to the computation of moments and to the invariance of the Skorohod integral under random rotations, however the case of Hermite polynomials is more complicated and it leads to Girsanov identities as an additional application.

Indeed, it is well known that in the adapted case, (1.4) and (1.2) can be used for the proof of the (adapted) Girsanov identity

$$E\left[\exp\left(\int_0^\infty u_t dB_t - \frac{1}{2}\int_0^\infty |u_t|^2 dt\right)\right] = 1,$$

under the Novikov type condition

$$E\left[\exp\left(\frac{1}{2}\int_0^T |u_t|^2 dt\right)\right] < \infty.$$

Similarly we recover, under simple conditions and with short proofs, the anticipating Girsanov identity obtained in [5] for quasi-nilpotent anticipative shifts of Brownian motion. This also simplifies the proof of classical results on the quasi-invariance of Euclidean motions [6], cf. Section 4, and on the invariance of random rotations.

The results of this paper can be formally summarized by the derivation formula

$$\frac{\partial}{\partial t} E\left[e^{t\delta(u) - \frac{t^2}{2}\|u\|^2}\right] = tE\left[e^{t\delta(u) - t^2\langle u,u\rangle/2}\langle D^*u, D(I_H - tDu)^{-1}u\rangle\right], \qquad (1.5)$$

for t in a neighborhood of 0, cf. Relation (4.3) below, where D and δ respectively denote the Malliavin gradient and Skorohod integral, showing that

$$E\left[\exp\left(\delta(u)-\frac{1}{2}\|u\|^2\right)\right]=1,$$

provided

$$\langle D^*u, D(I_H-tDu)^{-1}u\rangle = 0,$$

for t in a neighborhood of 0, cf. Corollary 3.3 below for a formal statement.

We proceed as follows. In Section 2 we introduce the notation on the Malliavin derivative and Skorohod integral used in this paper. In Section 3 we derive our main formula for the expectation of Hermite polynomials composed with a Skorohod anticipative integral. Finally in Section 4 we study the applications of Theorem 3.1 to anticipative Girsanov identities on the Wiener space. These results can be similarly derived for the Hitsuda-Skorohod integral in the white noise framework, cf. [1].

2 Notation

We refer to [7] and to Appendix B in [6] for the notation recalled in this section. Let $(B_t)_{t\in\mathbb{R}_+}$ denote a standard $\mathbb{R}^d$-valued Brownian motion on the Wiener space (W,P) with $W=\mathcal{C}_0(\mathbb{R}_+,\mathbb{R}^d)$. For any separable Hilbert space X, consider the Malliavin derivative D with values in $H=L^2(\mathbb{R}_+,X\otimes\mathbb{R}^d)$, defined by

$$D_tF=\sum_{i=1}^n \mathbf{1}_{[0,t_i]}(t)\partial_i f(B_{t_1},\dots,B_{t_n}),\qquad t\in\mathbb{R}_+,$$

for F of the form

$$F=f(B_{t_1},\dots,B_{t_n}), \tag{2.1}$$

$f\in\mathcal{C}_b^\infty(\mathbb{R}^n,X)$, $t_1,\dots,t_n\in\mathbb{R}_+$, $n\geq 1$. Let $\mathbb{D}_{p,k}(X)$ denote the completion of the space of smooth X-valued random variables under the norm

$$\|u\|_{\mathbb{D}_{p,k}(X)}=\sum_{l=0}^k\|D^lu\|_{L^p(W,X\otimes H^{\otimes l})},\qquad p>1,$$

where $X \otimes H$ denotes the completed symmetric tensor product of X and H. For all $p, q > 1$ such that $p^{-1} + q^{-1} = 1$ and $k \geq 1$, let

$$\delta : I\!D_{p,k}(X \otimes H) \to I\!D_{q,k-1}(X)$$

denote the Skorohod integral operator adjoint of

$$D : I\!D_{p,k}(X) \to I\!D_{q,k-1}(X \otimes H),$$

with

$$E[\langle F, \delta(u) \rangle_X] = E[\langle DF, u \rangle_{X \otimes H}], \qquad F \in I\!D_{p,k}(X), \quad u \in I\!D_{q,k}(X \otimes H).$$

For $u \in I\!D_{2,1}(H)$ we identify $Du = (D_t u_s)_{s,t \in \mathbb{R}_+}$ to the random operator $Du : H \to H$ almost surely defined by the relation

$$(Du)v(s) = \int_0^\infty (D_t u_s) v_t dt, \quad s \in \mathbb{R}_+, \quad v \in L^2(W; H),$$

in which $a \otimes b \in X \otimes H$ is identified to a linear operator $a \otimes b : H \to X$ via

$$(a \otimes b)c = a \langle b, c \rangle_H, \qquad a \otimes b \in X \otimes H, \quad c \in H.$$

The adjoint $D^* u$ of Du on $H \otimes H$ is given by

$$(D^* u)v(s) = \int_0^\infty (D_s^\dagger u_t) v_t dt, \qquad s \in \mathbb{R}_+, \quad v \in L^2(W; H),$$

where $D_s^\dagger u_t$ denotes the transpose matrix of $D_s u_t$ in $\mathbb{R}^d \otimes \mathbb{R}^d$. We will use the commutation relation

$$D\delta(u) = u + \delta(D^* u), \qquad u \in I\!D_{2,2}(H). \tag{2.2}$$

Finally, recall that $Du : H \to H$ is a quasi-nilpotent operator if

$$\text{trace}\,(Du)^k = 0, \qquad k \geq 2, \tag{2.3}$$

where the trace of $(Du)^k$ is a.s. given for all $k \geq 2$ by

$$\text{trace}\,(Du)^k = \int_0^\infty \cdots \int_0^\infty \langle D_{t_{k-1}}^\dagger u_{t_k}, D_{t_{k-2}} u_{t_{k-1}} \cdots D_{t_1} u_{t_1} D_{t_k} u_{t_1} \rangle_{\mathbb{R}^d \otimes \mathbb{R}^d} dt_1 \cdots dt_k.$$

In the sequel we will drop the indices in the scalar products and norms in $\mathbb{R}^d \otimes \mathbb{R}^d$, H, and $H \otimes H$, letting in particular $\|u\| = \|u\|_H$.

3 Random Hermite polynomials

In Theorem 3.1 below we extend Relations (1.3) and (1.4) by computing the expectation of the random Hermite polynomial $H_n(\delta(u), \|u\|^2)$ in the Skorohod integral $\delta(u)$, $n \geq 1$. This result will be applied in Section 4 to anticipating Girsanov identities on the Wiener space. In the sequel, all scalar products in H and in $H \otimes H$ will be simply denoted by $\langle \cdot, \cdot \rangle$, with $\|h\|^2 = \langle h, h \rangle_H$, $h \in H$.

Theorem 3.1 *For any $n \geq 0$ and $u \in \mathbb{D}_{n+1,2}(H)$ we have*

$$E[H_{n+1}(\delta(u), \|u\|^2)] = \sum_{l=0}^{n-1} \frac{n!}{l!} E\left[\delta(u)^l \sum_{0 \leq 2k \leq n-1-l} \frac{(-1)^k}{k!} \frac{\|u\|^{2k}}{2^k} \langle D^* u, D((Du)^{n-2k-l-1} u) \rangle \right].$$

Clearly it follows from Theorem 3.1 that if $u \in \mathbb{D}_{n,2}(H)$ and

$$\langle D^* u, D((Du)^k u) \rangle = 0, \qquad 0 \leq k \leq n-2, \tag{3.1}$$

then we have

$$E[H_n(\delta(u), \|u\|^2)] = 0, \qquad n \geq 1, \tag{3.2}$$

which extends Relation (1.4) to the anticipating case.

Lemma 3.2 *For all $k \geq 0$ and $u \in \mathbb{D}_{k+1,2}(H)$ we have*

$$\langle D^* u, D((Du)^k u) \rangle = \operatorname{trace}(Du)^{k+2} + \sum_{i=0}^{k-1} \frac{1}{k+1-i} \langle (Du)^i u, D \operatorname{trace}(Du)^{k+1-i} \rangle.$$

Proof. From [3] pages 120-121 we have

$$\begin{aligned} \langle D^* u, D((Du)^k u) \rangle &= \langle D^* u, (Du)^{k+1} \rangle + \langle D^* u, D(Du)^k u \rangle \\ &= \operatorname{trace}(Du)^{k+2} + \sum_{i=0}^{k-1} \frac{1}{k+1-i} \langle (Du)^i u, D \operatorname{trace}(Du)^{k+1-i} \rangle. \end{aligned}$$

□

As a consequence of Lemma 3.2, if $Du : H \to H$ is a.s. quasi-nilpotent in the sense of (2.3) then it satisfies (3.1). This leads to the following corollary of Theorem 3.1.

Corollary 3.3 *Let $u \in \mathbb{D}_{n,2}(H)$ for some $n \geq 1$, such that $Du : H \to H$ is a.s. quasi-nilpotent or satisfies* (3.1). *Then we have*

$$E[H_n(\delta(u), \|u\|^2)] = 0.$$

Recall that when the process $(u_t)_{t\in\mathbb{R}_+}$ is adapted with respect to the Brownian filtration we have trace $(Du)^k = 0$, $k \geq 2$, cf. [6], and therefore Condition (3.1) is satisfied. This recovers (1.4) in the setting of adapted processes since in this case $\delta(u)$ coincides with the Itô integral of $u \in L^2(W; H)$ with respect to Brownian motion, i.e.

$$\delta(u) = \int_0^\infty u_t dB_t. \tag{3.3}$$

Proof of Theorem 3.1. Step 1. We show that for any $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$ we have

$$\begin{aligned} E[H_{n+1}(\delta(u), \|u\|^2)] &= \sum_{0\leq 2k\leq n-1} (-1)^k \frac{n!}{k!2^k(n-2k-1)!} E[\delta(u)^{n-2k-1}\langle u,u\rangle^k \langle u, \delta(D^*u)\rangle] \\ &+ \sum_{1\leq 2k\leq n} (-1)^k \frac{n!}{k!2^k(n-2k)!} E[\delta(u)^{n-2k}\langle u, D\langle u,u\rangle^k\rangle]. \end{aligned} \tag{3.4}$$

For $F \in \mathbb{D}_{2,1}$ and $l, k \geq 1$ we have

$$\begin{aligned} E[F\delta(u)^{l+1}] &= \frac{l+2k+1}{2k} E[F\delta(u)^{l+1}] - \frac{l+1}{2k} E[F\delta(u)^{l+1}] \\ &= \frac{l+2k+1}{2k} E[F\delta(u)^{l+1}] - \frac{l+1}{2k} E[\langle u, D(\delta(u)^l F)\rangle] \\ &= \frac{l+2k+1}{2k} E[F\delta(u)^{l+1}] - \frac{l(l+1)}{2k} E[F\delta(u)^{l-1}\langle u, D\delta(u)\rangle] - \frac{l+1}{2k} E[\delta(u)^l\langle u, DF\rangle] \\ &= \frac{l+2k+1}{2k} E[F\delta(u)^{l+1}] - \frac{l(l+1)}{2k} E[F\delta(u)^{l-1}\langle u, u\rangle] \\ &\quad - \frac{l(l+1)}{2k} E[F\delta(u)^{l-1}\langle u, \delta(D^*u)\rangle] - \frac{l+1}{2k} E[\delta(u)^l\langle u, DF\rangle], \end{aligned}$$

i.e.

$$\begin{aligned} &E[F\delta(u)^{n-2k+1}] + \frac{(n-2k)(n-2k+1)}{2k} E[F\delta(u)^{n-2k-1}\langle u,u\rangle] \\ &= \frac{n+1}{2k} E[F\delta(u)^{n-2k+1}] - \frac{(n-2k)(n-2k+1)}{2k} E[F\delta(u)^{n-2k-1}\langle u, \delta(D^*u)\rangle] \\ &\quad - \frac{n-2k+1}{2k} E[\delta(u)^{n-2k}\langle u, DF\rangle]. \end{aligned}$$

Hence, taking $F = \langle u, u\rangle^k$, we get

$$
\begin{aligned}
E[\delta(u)^{n+1}] &= E[\langle u, D\delta(u)^n\rangle] \\
&= nE[\delta(u)^{n-1}\langle u, D\delta(u)\rangle] \\
&= nE[\delta(u)^{n-1}\langle u, u\rangle] + nE[\delta(u)^{n-1}\langle u, \delta(D^*u)\rangle] \\
&= nE[\delta(u)^{n-1}\langle u, \delta(D^*u)\rangle] \\
&\quad - \sum_{1\le 2k\le n+1} (-1)^k \frac{n!}{(k-1)!2^{k-1}(n+1-2k)!} \big(E[\delta(u)^{n-2k+1}\langle u, u\rangle^k] \\
&\quad + \frac{(n-2k+1)(n-2k)}{2k} E[\delta(u)^{n-2k-1}\langle u, u\rangle^{k+1}]\Big) \\
&= nE[\delta(u)^{n-1}\langle u, \delta(D^*u)\rangle] \\
&\quad - \sum_{1\le 2k\le n+1} (-1)^k \frac{n!}{(k-1)!2^{k-1}(n+1-2k)!} \left(\frac{n+1}{2k} E[\delta(u)^{n-2k+1}\langle u, u\rangle^k]\right. \\
&\quad - \frac{(n-2k)(n-2k+1)}{2k} E[\delta(u)^{n-2k-1}\langle u, u\rangle^k\langle u, \delta(D^*u)\rangle] \\
&\quad \left. - \frac{n-2k+1}{2k} E[\delta(u)^{n-2k}\langle u, D\langle u, u\rangle^k\rangle]\right) \\
&= - \sum_{1\le 2k\le n+1} (-1)^k \frac{(n+1)!}{k!2^k(n+1-2k)!} E[\delta(u)^{n-2k+1}\langle u, u\rangle^k] \\
&\quad + \sum_{0\le 2k\le n-1} (-1)^k \frac{n!}{k!2^k(n-2k-1)!} E[\delta(u)^{n-2k-1}\langle u, u\rangle^k\langle u, \delta(D^*u)\rangle] \\
&\quad + \sum_{1\le 2k\le n} (-1)^k \frac{n!}{k!2^k(n-2k)!} E[\delta(u)^{n-2k}\langle u, D\langle u, u\rangle^k\rangle],
\end{aligned}
$$

which yields (3.4) after using (1.1).

Step 2. For $F \in \mathbb{D}_{2,1}$ and $0 \le i \le l$ we have

$$
\begin{aligned}
& E[F\delta(u)^l\langle (Du)^i u, \delta(D^*u)\rangle] - lE[F\delta(u)^{l-1}\langle (D^*u)^{i+1}u, \delta(D^*u)\rangle] \\
&= E[\langle D^*u, D(F\delta(u)^l(Du)^i u)\rangle] - lE[F\delta(u)^{l-1}\langle (D^*u)^{i+1}u, \delta(D^*u)\rangle] \\
&= lE[F\delta(u)^{l-1}\langle D^*u, (Du)^i u \otimes D\delta(u)\rangle] - lE[F\delta(u)^{l-1}\langle (D^*u)^{i+1}u, \delta(D^*u)\rangle] \\
&\quad + E[\delta(u)^l\langle D^*u, D(F(Du)^i u)\rangle] \\
&= lE[F\delta(u)^{l-1}\langle D^*u, (Du)^i u \otimes D\delta(u)\rangle] + lE[F\delta(u)^{l-1}\langle D^*u, (Du)^i u \otimes \delta(D^*u)\rangle] \\
&\quad - lE[F\delta(u)^{l-1}\langle (D^*u)^{i+1}u, \delta(D^*u)\rangle] + E[\delta(u)^l\langle D^*u, D(F(Du)^i u)\rangle]
\end{aligned}
$$

$$= \quad lE[F\delta(u)^{l-1}\langle (Du)^{i+1}u, u\rangle] + E[\delta(u)^l\langle (Du)^{i+1}u, DF\rangle]$$
$$+E[F\delta(u)^l\langle D^*u, D((Du)^iu)\rangle].$$

Hence, replacing l above with $l-i$, we get

$$\begin{aligned}
E[F\delta(u)^l\langle u, \delta(D^*u)\rangle] &= l!E[F\langle (Du)^lu, \delta(D^*u)\rangle] \\
&\quad + \sum_{i=0}^{l-1}\frac{l!}{(l-i)!}\left(E[F\delta(u)^{l-i}\langle (Du)^iu, \delta(D^*u)\rangle] - (l-i)E[F\delta(u)^{l-i-1}\langle (D^*u)^{i+1}u, \delta(D^*u)\rangle]\right) \\
&= l!E[F\langle (Du)^lu, \delta(D^*u)\rangle] + \sum_{i=0}^{l-1}\frac{l!}{(l-i-1)!}E[F\delta(u)^{l-i-1}\langle (Du)^{i+1}u, u\rangle] \\
&\quad + \sum_{i=0}^{l-1}\frac{l!}{(l-i)!}E[\delta(u)^{l-i}\langle (Du)^{i+1}u, DF\rangle] + \sum_{i=0}^{l-1}\frac{l!}{(l-i)!}E[F\delta(u)^{l-i}\langle D^*u, D((Du)^iu)\rangle] \\
&= l!E[\langle (Du)^{l+1}u, DF\rangle] + \sum_{i=0}^{l-1}\frac{l!}{(l-i-1)!}E[F\delta(u)^{l-i-1}\langle (Du)^{i+1}u, u\rangle] \\
&\quad + \sum_{i=0}^{l-1}\frac{l!}{(l-i)!}E[\delta(u)^{l-i}\langle (Du)^{i+1}u, DF\rangle] + \sum_{i=0}^{l}\frac{l!}{(l-i)!}E[F\delta(u)^{l-i}\langle D^*u, D((Du)^iu)\rangle] \\
&= l!E[\langle (Du)^{l+1}u, DF\rangle] + \sum_{i=0}^{l-1}\frac{l!}{(l-i-1)!}E[F\delta(u)^{l-i-1}\langle (Du)^{i+1}u, u\rangle] \\
&\quad + \sum_{i=1}^{l}\frac{l!}{(l-i+1)!}E[\delta(u)^{l-i+1}\langle (Du)^iu, DF\rangle] + \sum_{i=0}^{l}\frac{l!}{(l-i)!}E[F\delta(u)^{l-i}\langle D^*u, D((Du)^iu)\rangle],
\end{aligned}$$

thus letting $F = \langle u, u\rangle^k$ and $l = n-2k-1$ above, and using (3.4) in Step 1, we get

$$\begin{aligned}
E[H_{n+1}(\delta(u), \|u\|^2)] &= \sum_{0\le 2k\le n-1}(-1)^k\frac{n!}{k!2^k(n-2k-1)!}E[\delta(u)^{n-2k-1}\langle u, u\rangle^k\langle u, \delta(D^*u)\rangle] \\
&\quad + \sum_{1\le 2k\le n}(-1)^k\frac{n!}{k!2^k(n-2k)!}E[\delta(u)^{n-2k}\langle u, D\langle u, u\rangle^k\rangle] \\
&= \sum_{0\le 2k\le n}(-1)^k\frac{n!}{k!2^k}E[\langle (Du)^{n-2k}u, D\langle u, u\rangle^k\rangle] \\
&\quad + \sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!2^k}\sum_{i=0}^{n-2k-2}\frac{n!}{(n-2(k+1)-i)!}E[\langle u, u\rangle^k\delta(u)^{n-2(k+1)-i}\langle (Du)^{i+1}u, u\rangle] \\
&\quad + \sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!2^k}\sum_{i=1}^{n-2k-1}\frac{n!}{(n-2k-i)!}E[\delta(u)^{n-2k-i}\langle (Du)^iu, D\langle u, u\rangle^k\rangle]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!2^k}\sum_{i=0}^{n-2k-1}\frac{n!}{(n-2k-1-i)!}E[\langle u,u\rangle^k\delta(u)^{n-2k-i-1}\langle D^*u,D((Du)^iu)\rangle]\\
&+\sum_{0\le 2k\le n-1}(-1)^k\frac{n!}{k!2^k(n-2k)!}E[\delta(u)^{n-2k}\langle u,D\langle u,u\rangle^k\rangle]\\
=&\sum_{0\le 2k\le n}(-1)^k\frac{n!}{k!2^k}E[\langle (Du)^{n-2k}u,D\langle u,u\rangle^k\rangle]\\
&-\sum_{0\le 2k\le n-1}\frac{(-1)^{k+1}}{(k+1)!2^{k+1}}\sum_{i=0}^{n-2k-2}\frac{n!}{(n-2(k+1)-i)!}E[\delta(u)^{n-2(k+1)-i}\langle (Du)^iu,D\langle u,u\rangle^{k+1}\rangle]\\
&+\sum_{0\le 2k\le n}\frac{(-1)^k}{k!2^k}\sum_{i=0}^{n-2k-1}\frac{n!}{(n-2k-i)!}E[\delta(u)^{n-2k-i}\langle (Du)^iu,D\langle u,u\rangle^k\rangle]\\
&+\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!2^k}\sum_{i=0}^{n-2k-1}\frac{n!}{(n-2k-1-i)!}E[\langle u,u\rangle^k\delta(u)^{n-2k-i-1}\langle D^*u,D((Du)^iu)\rangle]\\
=&\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!2^k}\sum_{i=0}^{n-2k-1}\frac{n!}{(n-2k-1-i)!}E[\langle u,u\rangle^k\delta(u)^{n-2k-i-1}\langle D^*u,D((Du)^iu)\rangle],
\end{aligned}
$$

where we applied the relation

$$
\begin{aligned}
\langle u,u\rangle^k\langle (Du)^{i+1}u,u\rangle &= \frac{1}{2}\langle u,u\rangle^k\langle (Du)^iu,D\langle u,u\rangle\rangle\\
&= \frac{1}{2(k+1)}\langle (Du)^iu,D\langle u,u\rangle^{k+1}\rangle\\
&= \frac{1}{2(k+1)}\langle (Du)^iu,D\langle u,u\rangle^{k+1}\rangle,
\end{aligned}
$$

which follows from $D\langle u,u\rangle = 2(D^*u)u$ and the derivation property of the gradient operator D. □

4 Anticipative Girsanov identities

The next proposition is an immediate consequence of (3.2), using the generating function (1.2). In comparison with Proposition 8.2.1 of [6] we do not require assumptions on the inverse mapping $(I_H - Du)^{-1}$ and we show that quasi-nilpotence of Du can be replaced by the weaker condition (3.1), while working under a stronger integrability condition. Let $\mathbb{D}_{\infty,2}(H) = \bigcap_{n\ge 1}\mathbb{D}_{n,2}(H)$.

Corollary 4.1 *Assume that $u \in \mathbb{D}_{\infty,2}(H)$ with $E[e^{|\delta(u)|+\|u\|^2/2}] < \infty$, and that $Du : H \to H$ is a.s. quasi-nilpotent, or more generally that* (3.1) *holds. Then we have*

$$E\left[\exp\left(\delta(u) - \frac{1}{2}\|u\|^2\right)\right] = 1. \tag{4.1}$$

Proof. From (1.1) we have the bound

$$|H_n(x,\sigma^2)| \leq \sum_{0\leq 2k\leq n} \frac{(-1)^k}{k!2^k} \frac{n!}{(n-2k)!} |x|^{n-2k}(-\sigma^2)^k = H_n(|x|, -\sigma^2),$$

hence

$$\begin{aligned} E\left[\sum_{n=0}^{\infty} \frac{1}{n!}|H_n(\delta(u), \|u\|^2)|\right] &\leq E\left[\sum_{n=0}^{\infty} \frac{1}{n!} H_n(|\delta(u)|, -\|u\|^2)\right] \\ &= E\left[e^{|\delta(u)|+\|u\|^2/2}\right] < \infty. \end{aligned}$$

Consequently, by Theorem 3.1 and the Fubini theorem we have

$$\begin{aligned} E\left[\exp\left(\delta(u) - \frac{1}{2}\|u\|^2\right)\right] &= 1 + E\left[\sum_{n=0}^{\infty} \frac{1}{(n+1)!} H_{n+1}(\delta(u), \|u\|^2)\right] \\ &= 1 + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} E\left[H_{n+1}(\delta(u), \|u\|^2)\right] = 1. \end{aligned}$$

□

This shows in particular that if $u \in \mathbb{D}_{\infty,2}(H)$ is such that $\|u\|$ is deterministic and $Du : H \to H$ is a.s. quasi-nilpotent, or more generally (3.1) holds, then we have

$$E\left[e^{\delta(u)}\right] = e^{\frac{1}{2}\|u\|^2},$$

i.e. $\delta(u)$ has a centered Gaussian distribution with variance $\|u\|^2$, cf. Theorem 2.1-b) of [5] and Corollary 2.2 of [3].

More generally, Corollary 4.2 below states an anticipative Girsanov identity (4.2) that recovers Proposition 8.2.1 of [6] under simpler hypotheses, namely without requirements on the smoothness and integrability of $(I_H - Du)^{-1}$. In the sequel, for $u \in \mathbb{D}_{2,1}(H)$ we let

$$\Lambda_u = \exp\left(\delta(u) - \frac{1}{2}\|u\|^2\right),$$

and we denote by T_u the transformation of W defined by

$$T_u\omega(t) = \omega(t) + \int_0^t u_s(\omega)ds, \qquad t \in \mathbb{R}_+, \quad \omega \in W.$$

Corollary 4.2 *Assume that $u \in \mathbb{D}_{\infty,2}(H)$ with $E[e^{\varepsilon(|\delta(u)|+\|u\|^2/2)}] < \infty$ for some $\varepsilon > 1$, and that $Du : H \to H$ is a.s. quasi-nilpotent, or more generally that* (3.1) *holds. Then the transformation $T_u : W \to W$ satisfies the Girsanov type identity*

$$E[F \circ T_u \ \Lambda_u] = E[F], \tag{4.2}$$

for all bounded random variables F.

Proof. For all exponential vectors $\Lambda_f = e^{\int_0^\infty f(t)dB_t - \frac{1}{2}\|f\|^2}$, $f \in L^2(\mathbb{R}_+)$, we have

$$\begin{aligned} \Lambda_f \circ T_u \ \Lambda_u &= e^{\int_0^\infty f(t)dB_t - \int_0^\infty f(t)u(t)dt - \frac{1}{2}\|f\|^2} \Lambda_u \\ &= e^{\delta(f+u) - \frac{1}{2}\|f\|^2 - \frac{1}{2}\|u\|^2 - \langle f,u\rangle} \\ &= \Lambda_{u+f}, \end{aligned}$$

hence by Corollary 4.2 we have

$$E\left[\Lambda_f \circ T_u \ \Lambda_u\right] = E\left[\Lambda_{u+f}\right] = 1,$$

and we conclude by density of the linear combination of exponential vectors Λ_f, $f \in L^2(\mathbb{R}_+)$, in $L^2(W)$. □

In particular, if $T_u : W \to W$ is invertible, then by Corollary 4.2 it is absolutely continuous with respect to the Wiener measure, and

$$\frac{dT_u^* P}{dP} = \Lambda_u.$$

We refer to Corollary 8.4.1 of [6] for sufficient conditions for the invertibility of $T_u : W \to W$.

The conditions imposed to obtain the Girsanov identity for Euclidean motions written as the sum of a rotation and a quasi-nilpotent shifts as in Theorem 8.6.1 of [6] can be simplified similarly.

Finally we sketch the proof of the formal identity (1.5) stated in the introduction, i.e.

$$\frac{\partial}{\partial t} E\left[e^{t\delta(u)-\frac{t^2}{2}\|u\|^2}\right] = tE\left[e^{t\delta(u)-t^2\langle u,u\rangle/2}\langle D^*u, D(I_H - tDu)^{-1}u\rangle\right]. \tag{4.3}$$

Proof. We have

$$\begin{aligned}
&\frac{\partial}{\partial t} E[e^{t\delta(u)-\frac{t^2}{2}\|u\|^2}] = \sum_{n=0}^{\infty} \frac{t^n}{n!} E[H_{n+1}(\delta(u), \|u\|^2)] \\
&= \sum_{n=0}^{\infty} t^n E\left[\sum_{l=0}^{n-1} \frac{\delta(u)^l}{l!} \sum_{0\le 2k\le n-1-l} \frac{(-1)^k}{k!} \frac{\|u\|^{2k}}{2^k} \langle D^*u, D((Du)^{n-2k-l-1}u)\rangle\right] \\
&= tE\left[\sum_{l=0}^{\infty} \frac{t^l\delta(u)^l}{l!} \sum_{n=0}^{\infty} t^n \sum_{0\le 2k\le n} \frac{(-1)^k}{k!} \frac{\|u\|^{2k}}{2^k} \langle D^*u, D((Du)^{n-2k}u)\rangle\right] \\
&= tE\left[e^{t\delta(u)} \sum_{n=0}^{\infty} t^n \sum_{0\le 2k\le n} \frac{(-1)^k}{k!} \frac{\|u\|^{2k}}{2^k} \langle D^*u, D((Du)^{n-2k}u)\rangle\right] \\
&= tE\left[e^{t\delta(u)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\|u\|^{2k}}{2^k} \sum_{n=0}^{\infty} t^n \langle D^*u, D((Du)^n u)\rangle\right] \\
&= tE\left[e^{t\delta(u)-t^2\langle u,u\rangle/2}\langle D^*u, D(I_H - tDu)^{-1}u\rangle\right].
\end{aligned}$$

□

In a similar way, from Theorem 2.1 of [3] we get

$$\begin{aligned}
&\frac{\partial}{\partial t} E[e^{t\delta(u)}] = \sum_{n=0}^{\infty} \frac{t^n}{n!} E[(\delta(u))^{n+1}] \\
&= \sum_{n=0}^{\infty} t^n \sum_{k=1}^{n} \frac{1}{(n-k)!} E\left[(\delta(u))^{n-k} \left(\langle (Du)^{k-1}u, u\rangle + \langle D^*u, D((Du)^{k-1}u)\rangle\right)\right] \\
&= t\sum_{n=0}^{\infty} \frac{t^n}{n!} E\left[(\delta(u))^n \sum_{k=0}^{\infty} t^k \left(\langle (Du)^k u, u\rangle + \langle D^*u, D((Du)^k u)\rangle\right)\right] \\
&= tE\left[e^{t\delta(u)} \left(\langle u, (I_H - tDu)^{-1}u\rangle + \langle D^*u, D((I_H - tDu)^{-1}u)\rangle\right)\right],
\end{aligned}$$

hence

$$\frac{\partial}{\partial t} E[e^{t\delta(u)}] = tE\left[e^{t\delta(u)} \left(\langle u, (I_H - tDu)^{-1}u\rangle + \langle D^*u, D((I_H - tDu)^{-1}u)\rangle\right)\right].$$

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