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Potential Theory in Classical Probability

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Abstract

These notes are an elementary introduction to classical potential theory and to its connection with probabilistic tools such as stochastic calculus and the Markov property. In particular we review the probabilistic interpretations of harmonicity, of the Dirichlet problem and of the Poisson equation using Brownian motion and stochastic calculus.

Key words: Potential theory, harmonic functions, Markov processes, stochastic calculus, partial differential equations.
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Contents

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1 Introduction

The origins of potential theory can be traced to the physical problem of reconstructing a repartition of electric charges inside a planar or a spatial domain, given the measurement of the electrical field created on the boundary of this domain.

In mathematical analytic terms this amounts to representing the values of a function h inside a domain given the data of the values of h on the boundary of the domain. In the simplest case of a domain empty of electric charges, the problem can be formulated as that of finding a harmonic function h on E (roughly speaking, a function with vanishing Laplacian, see § 2.2 below), given its values prescribed by a function f on the boundary ∂E, i.e. as the Dirichlet problem:

$$\begin{cases} \Delta h(y) = 0, & y \in E, \\ \\ h(y) = f(y), & y \in \partial E. \end{cases}$$

Close connections between the potential theory and the theory of Markov processes have been observed at early stages of the development of the theory, see e.g. [Doo84] and references therein. As a consequence, many potential theoretic problems have a probabilistic interpretation or can be solved by probabilistic methods.

These notes aim to gather some analytic and probabilistic aspects of potential theory into a single document. We partly follow the point of view of Chung [Chu95] with complements on analytic potential theory coming from Helms [Hel69], some additions on stochastic calculus, and probabilistic applications found in Bass [Bas98].

More precisely, we proceed as follow. In Section 2 we give a summary of classical analytic potential theory: Green kernels, Laplace and Poisson equations in particular, following mainly Brelot [Bre65], Chung [Chu95] and Helms [Hel69]. Section 3 introduces the Markovian setting of semigroups which will be the main framework for probabilistic interpretations. A sample of references used in this domain is Ethier and Kurtz [EK86], Kallenberg [Kal02], and also Chung [Chu95]. The probabilistic interpretation of potential theory also makes significant use of Brownian motion and stochastic calculus. They are summarized in Section 4, see Protter [Pro90] Ikeda and Watanabe [IW89], however our presentation of stochastic calculus is given in the framework of normal martingales due to their links with quantum stochastic calculus, cf. Biane [Bia93]. In Section 5 we present the probabilistic connection between potential theory and Markov processes, following Bass [Bas98], Dynkin [Dyn65], Kallenberg [Kal02], and Port and Stone [PS78]. Our description of the Martin boundary in discrete time follows that of Revuz [Rev75].

2 Analytic potential theory

2.1 Electrostatic interpretation

Let E denote a closed region of $\mathbb{R}^n$, more precisely a compact subset having a smooth boundary ∂E with surface measure σ. Gauss's law is the main tool for determining a repartition of electric charges inside E, given the values of the electrical field created on ∂E. It states that given a repartition of charges $q(dx)$ the flux of the electric field $\vec{U}$ across the boundary ∂E is proportional to the sum of electric charges enclosed in E. Namely we have

$$\int_E q(dx) = \epsilon_0 \int_{\partial E} \langle \vec{n}(x), \vec{U}(x) \rangle \sigma(dx), \tag{2.1}$$

where $q(dx)$ is a signed measure representing the distribution of electric charges, $\epsilon_0 > 0$ is the electrical permittivity constant, $\vec{U}(x)$ denotes the electric field at $x \in \partial E$, and $\vec{n}(x)$ represents the outer (i.e. oriented towards the exterior of E) unit vector orthogonal to the surface ∂E.

On the other hand the divergence theorem, which can be viewed as a particular case of the Stokes theorem, states that if $\vec{U} : E \to \mathbb{R}^n$ is a $\mathcal{C}^1$ vector field we have

$$\int_E \operatorname{div} \vec{U}(x) dx = \int_{\partial E} \langle \vec{n}(x), \vec{U}(x) \rangle \sigma(dx), \tag{2.2}$$

where the divergence $\operatorname{div} \vec{U}$ is defined as

$$\operatorname{div} \vec{U}(x) = \sum_{i=1}^{n} \frac{\partial \vec{U}_i}{\partial x_i}(x).$$

The divergence theorem can be seen as a mathematical formulation of the Gauss law. As a consequence, Gauss's law, associated to the divergence theorem allows us to interpret $\vec{U}(x)$

as the induced electric field on the surface ∂E, or equivalently by saying that $\operatorname{div} \vec{U}(x)$ is proportional to the density of (electric) charges in E.

This leads to the Maxwell equation

$$q(dx) = \epsilon_0 \operatorname{div} \vec{U}(x) dx, \tag{2.3}$$

where $q(dx)$ is the distribution of electric charge at x and $\vec{U}(x)$ is the electric field at x.

When $q(dx)$ has the density $q(x)$ at x, i.e. $q(dx) = q(x)dx$, and the field $\vec{U}(x)$ derives from a potential $V : \mathbb{R}^n \to \mathbb{R}_+$, i.e. when

$$\vec{U}(x) = \nabla V(x),$$

Maxwell's equation (2.3) takes the form of the Poisson equation:

$$\epsilon_0 \Delta V(x) = q(x), \tag{2.4}$$

where the Laplacian $\Delta = \operatorname{div} \nabla$ is given by

$$\Delta V(x) = \sum_{j=1}^{n} \frac{\partial^2 V}{\partial x_i^2}(x).$$

In particular, when the domain E is empty of electric charges, the potential V satisfies the Laplace equation

$$\Delta V(x) = 0.$$

As mentioned in the introduction, a typical problem in classical potential theory is to recover the values of the potential $V(x)$ in E from its values on the boundary ∂E, given that it satisfies the Poisson equation. This can be achieved in particular by representing $V(x)$, $x \in E$, as an integral with respect to the surface measure over the boundary ∂E, or by solving the Poisson equation for $V(x)$.

Consider for example the Coulomb potential

$$V(x) = \frac{q}{\epsilon_0 s_n} \frac{1}{\|x - y\|^{n-2}}, \qquad x \in \mathbb{R}^n \setminus \{y\},$$

created by a single charge q at $y \in \mathbb{R}^n$, where $s_2 = 2\pi$ and $s_3 = 4\pi$, and

$$s_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

is the surface of the unit $n-1$-dimensional sphere in $\mathbb{R}^n$. The electrical field created by V is

$$\vec{U}(x) = \nabla V(x) = \frac{q}{\epsilon_0 s_n} \frac{x - y}{\|x - y\|^{n-1}}, \qquad x \in \mathbb{R}^n \setminus \{y\}.$$

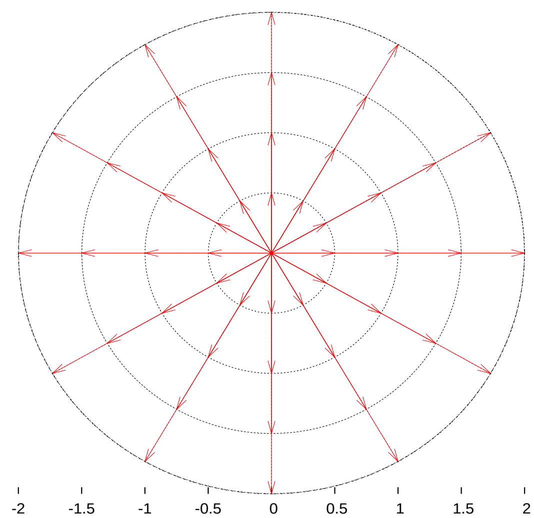

Figure 2.1: Electrical field at $y = 0$.

Letting $B(y,r)$, resp. $S(y,r)$, denote the open ball, resp. the sphere, of center $y \in \mathbb{R}^n$ and radius $r > 0$, we have

$$\begin{aligned} \int_{B(y,r)} \Delta V(x) dx &= \int_{S(y,r)} \langle \vec{n}(x), \nabla V(x) \rangle \sigma(dx) \\ &= \int_{S(y,r)} \langle \vec{n}(x), \vec{U}(x) \rangle \sigma(dx) \\ &= \frac{q}{\epsilon_0}, \end{aligned}$$

where σ denotes the surface measure on $S(y,r)$. From this we deduce

$$q(dx) = q\delta_y(dx)$$

i.e. we recover the fact that the potential V is generated by a single charge located at y. We also obtain a version of the Poisson equation (2.4) in distribution sense:

$$\Delta_x \frac{1}{\|x-y\|^{n-2}} = s_n \delta_y(dx),$$

where the Laplacian Δ_x. On the other hand, taking $E = B(0,r) \setminus B(0,\rho)$ we have $\partial E = S(0,r) \cup S(0,\rho)$ and

$$\begin{aligned} \int_E \Delta V(x) dx &= \int_{S(0,r)} \langle \vec{n}(x), \nabla V(x) \rangle \sigma(dx) + \int_{S(0,\rho)} \langle \vec{n}(x), \nabla V(x) \rangle \sigma(dx) \\ &= cs_n - cs_n = 0, \end{aligned}$$

hence

$$\Delta_x \frac{1}{\|x-y\|^{n-2}} = 0, \qquad x \in \mathbb{R}^n \setminus \{y\}.$$

The electrical permittivity ϵ_0 will be set equal to 1 in the sequel.

2.2 Harmonic functions

The notion of harmonic function will be first introduced from the mean value property. Let

$$\sigma_r^x(dy) = \frac{1}{s_n r^{n-1}} \sigma(dy)$$

denote the normalized surface measure on $S(x, r)$, and recall that

$$\int f(x)dx = s_n \int_0^\infty r^{n-1} \int_{S(y,r)} f(z)\sigma_r^y(dz)dr.$$

Definition 2.2.1. *A continuous real-valued function on an open subset O of $\mathbb{R}^n$ is said to be* harmonic, *resp. superharmonic, in O if one has*

$$f(x) = \int_{S(x,r)} f(y)\sigma_r^x(dy),$$

resp.

$$f(x) \geq \int_{S(x,r)} f(y)\sigma_r^x(dy),$$

for all $x \in O$ and $r > 0$ such that $B(x, r) \subset O$.

Next we show that the equivalence between the mean value property and the vanishing of the Laplacian.

Proposition 2.2.2. *A $\mathcal{C}^2$ function f is harmonic, resp. superharmonic, on an open subset O of $\mathbb{R}^n$ if and only if it satisfies the Laplace equation*

$$\Delta f(x) = 0, \qquad x \in O,$$

resp. the partial differential inequality

$$\Delta f(x) \leq 0, \qquad x \in O.$$

Proof. In spherical coordinates, using the divergence formula and the identity

$$\begin{aligned} \frac{d}{dr} \int_{S(0,1)} f(y + rx)\sigma_1^0(dx) &= \int_{S(0,1)} \langle x, \nabla f(y + rx)\rangle \sigma_1^0(dx) \\ &= \frac{r}{s_n} \int_{B(0,1)} \Delta f(y + rx)dx \end{aligned}$$

yields

$$\begin{aligned} \int_{B(y,r)} \Delta f(x)dx &= r^{n-1} \int_{B(0,1)} \Delta f(y + rx)dx \\ &= s_n r^{n-2} \int_{S(0,1)} \langle x, \nabla f(y + rx)\rangle \sigma_1^0(dx) \\ &= s_n r^{n-2} \frac{d}{dr} \int_{S(0,1)} f(y + rx)\sigma_1^0(dx) \end{aligned}$$

$$= \quad s_n r^{n-2} \frac{d}{dr} \int_{S(y,r)} f(x) \sigma_r^y(dx).$$

If f is harmonic, this shows that

$$\int_{B(y,r)} \Delta f(x) dx = 0,$$

for all $y \in E$ and $r > 0$ such that $B(y,r) \subset O$, hence $\Delta f = 0$ on O. Conversely, if $\Delta f = 0$ on O then

$$\int_{S(y,r)} f(x) \sigma_r^y(dx)$$

is constant in r, hence

$$f(y) = \lim_{\rho \to 0} \int_{S(y,\rho)} f(x) \sigma_\rho^y(dx) = \int_{S(y,r)} f(x) \sigma_r^y(dx), \qquad r > 0.$$

The proof is similar in the case of superharmonic functions. $\square$

The fundamental harmonic function based at $y \in \mathbb{R}^n$ are the functions which are harmonic on $\mathbb{R}^n \setminus \{y\}$ and depend only on $r = \|x - y\|$, $y \in \mathbb{R}^n$. They satisfy the Laplace equation

$$\Delta h(x) = 0, \qquad x \in \mathbb{R}^n,$$

in spherical coordinates, with

$$\Delta h(r) = \frac{d^2 h}{dr^2}(r) + \frac{(n-1)}{r} \frac{dh}{dr}(r).$$

In case $n = 2$ the fundamental harmonic functions are given by the logarithmic potential

$$h_y(x) = \begin{cases} -\dfrac{1}{s_2} \log \|x - y\|, & x \neq y, \\ \\ +\infty, & x = y, \end{cases} \tag{2.5}$$

and by the Coulomb potential in case $n \geq 3$:

$$h_y(x) = \begin{cases} \dfrac{1}{(n-2)s_n} \dfrac{1}{\|x - y\|^{n-2}}, & x \neq y, \\ \\ +\infty, & x = y. \end{cases} \tag{2.6}$$

More generally, for $a \in \mathbb{R}$ and $y \in \mathbb{R}^n$, the function

$$x \mapsto \|x - y\|^a,$$

is superharmonic on $\mathbb{R}^n$, $n \geq 3$, if and only if $a \in [2-n, 0]$, and harmonic when $a = 2 - n$.

We now focus on the Dirichlet problem on the ball $E = B(y, r)$. We consider

$$h_0(r) = -\frac{1}{s_2} \log(r), \qquad r > 0,$$

in case $n = 2$, and

$$h_0(r) = \frac{1}{(n-2)s_n r^{n-2}}, \qquad r > 0,$$

if $n \geq 3$, and let

$$x^* := y + \frac{r^2}{\|y - x\|^2}(x - y)$$

denote the inverse of $x \in B(y, r)$ with respect to the sphere $S(y, r)$. Note the relation

$$\begin{aligned} \|z - x^*\| &= \left\| z - y - \frac{r^2}{\|y - x\|^2}(x - y) \right\| \\ &= \frac{r}{\|x - y\|} \left\| \frac{\|x - y\|}{r}(z - y) - \frac{r}{\|y - x\|}(x - y) \right\|, \end{aligned}$$

hence for all $z \in S(y, r)$ and $x \in B(y, r)$,

$$\|z - x^*\| = r \frac{\|x - z\|}{\|x - y\|}, \tag{2.7}$$

since

$$\left\| \|x - y\| \frac{z - y}{\|z - y\|} - r \frac{x - y}{\|y - x\|} \right\| = \|x - z\|.$$

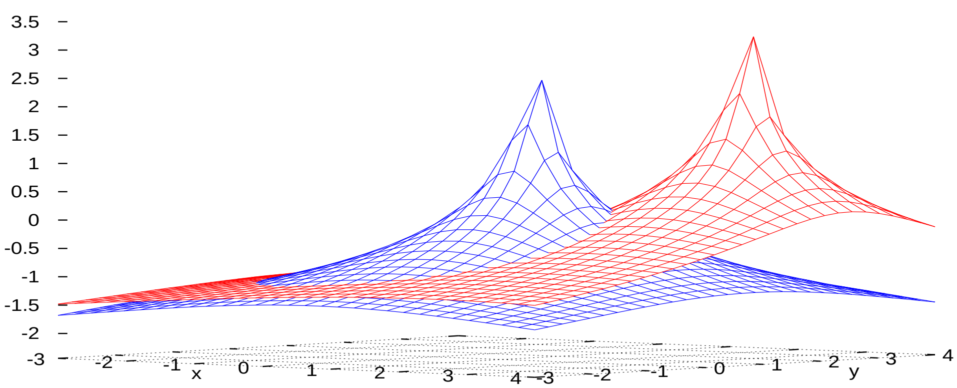

Figure 2.2: Graph of h_x and of $h_0(\|y - x\| \|z - x^*\|/r)$ when $n = 2$.

The function $z \mapsto h_x(z)$ is not $\mathcal{C}^2$ hence it is not harmonic on $\bar{B}(y,r)$. Instead of h_x we will use $z \mapsto h_{x^*}(z)$, which is harmonic on a neighborhood of $\bar{B}(y,r)$, to construct a solution of the Dirichlet problem on $B(y,r)$.

Lemma 2.2.3. *The solution of the Dirichlet problem* (2.10) *for $E = B(y,r)$ with boundary condition h_x, $x \in B(y,r)$, is given by*

$$x \mapsto h_0\left(\frac{\|y-x\|\|z-x^*\|}{r}\right), \qquad z \in B(y,r).$$

Proof. We have if $n \geq 3$:

$$\begin{aligned} h_0\left(\frac{\|y-x\|\|z-x^*\|}{r}\right) &= \frac{r^{n-2}}{(n-2)s_n\|x-y\|^{n-2}}\frac{1}{\|z-x^*\|^{n-2}} \\ &= \frac{r^{n-2}}{\|x-y\|^{n-2}}h_{x^*}(z), \end{aligned}$$

and if $n = 2$:

$$\begin{aligned} h_0\left(\frac{\|y-x\|\|z-x^*\|}{r}\right) &= -\frac{1}{s_2}\log\left(\frac{\|y-x\|\|z-x^*\|}{r}\right) \\ &= h_{x^*}(y) - \frac{1}{s_2}\log\left(\frac{\|y-x\|}{r}\right). \end{aligned}$$

This function is harmonic in $z \in B(y,r)$ and is equal to h_x on $S(y,r)$ from (2.7). □

The next figure represents the solution of the Dirichlet problem with boundary condition h_x on $S(y,r)$ for $n = 2$ as obtained after truncation of Figure 2.2.

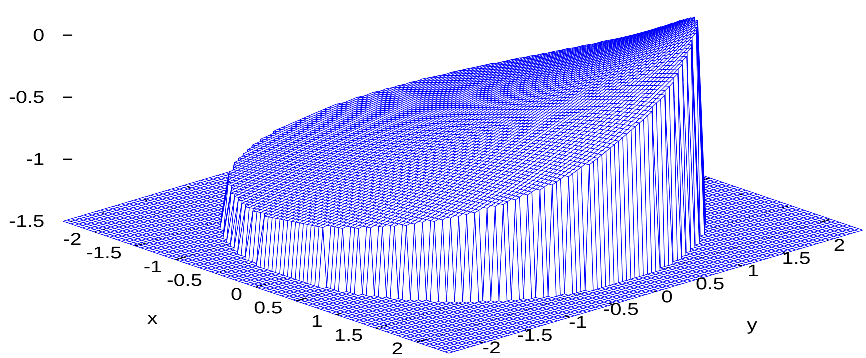

Figure 2.3: Solution of the Dirichlet problem.

2.3 Representation of a function on E from its values on ∂E

As mentioned in the introduction, it can be of interest to compute a repartition of charges inside a domain E from the values of the field generated on the boundary ∂E.

In this section we present the representation formula for the values of an harmonic function inside an arbitrary domain E as an integral over its boundary ∂E, that follows from the Green identity. This formula uses a kernel which can be explicitly computed in some special cases, e.g. $E = B(y, r)$ is an open ball, in which case it is called the Poisson formula, cf. Section 2.4 below.

Assume that E is an open domain in $\mathbb{R}^n$ with smooth boundary ∂E and let

$$\partial_n f(x) = \langle \vec{n}(x), \nabla f(x) \rangle$$

denote the normal derivative of f on ∂E.

Applying the divergence theorem (2.2) to the products $u(x)\nabla v(x)$ and $v(x)\nabla u(x)$, where u, v are $\mathcal{C}^2$ functions on E yields Green's identity:

$$\int_E (u(x)\Delta v(x) - v(x)\Delta u(x))dx = \int_{\partial E} (u(x)\partial_n v(x) - v(x)\partial_n u(x))d\sigma(x). \tag{2.8}$$

On the other hand, taking $u = 1$ in the divergence theorem yields Gauss's integral theorem

$$\int_{\partial E} \partial_n v(x) d\sigma(x) = 0, \tag{2.9}$$

provided v is harmonic on E.

In the next definition, h_x denotes the fundamental harmonic function defined in (2.5) and (2.6).

Definition 2.3.1. *The Green kernel $G^E(\cdot, \cdot)$ of E is defined as*

$$G^E(x, y) := h_x(y) - w_x(y), \qquad x, y \in E,$$

where for all $x \in \mathbb{R}^n$, w_x is a smooth solution to the Dirichlet problem

$$\begin{cases} \Delta w_x(y) = 0, & y \in E, \\ \\ w_x(y) = h_x(y), & y \in \partial E. \end{cases} \tag{2.10}$$

In the case of a general boundary ∂E, the Dirichlet problem may have no solution, even when the boundary value function f is continuous. Note that since $(x, y) \mapsto h_x(y)$ is symmetric, the Green kernel is also symmetric in two variables, i.e.

$$G^E(x, y) = G^E(y, x), \qquad x, y \in \mathbb{R}^n,$$

and $G^E(\cdot, \cdot)$ vanishes on the boundary ∂E. The next proposition provides an integral representation formula for $\mathcal{C}^2$ functions on E using the Green kernel. In the case of harmonic functions, it reduces to a representation from the values on the boundary ∂E, cf. Corollary 2.3.3 below.

Proposition 2.3.2. *For all $\mathcal{C}^2$ functions u on E we have*

$$u(x) = \int_{\partial E} u(z)\partial_n G^E(x,z)\sigma(dz) + \int_E G^E(x,z)\Delta u(z)dz, \qquad x \in E. \tag{2.11}$$

Proof. We do the proof in case $n \geq 3$, the case $n = 2$ being similar. Given $x \in E$, apply Green's identity (2.8) to the functions u and h_x, where h_x is harmonic on $E \setminus B(x,r)$ for small enough $r > 0$ to obtain

$$\begin{aligned} \int_{E\setminus B(x,r)} h_x(y)\Delta u(y)dy &- \int_{\partial E}(h_x(y)\partial_n u(y) - u(y)\partial_n h_x(y))d\sigma(y) \\ &= \frac{1}{n-2}\int_{S(x,r)}\left(\frac{1}{s_n r^{n-2}}\partial_n u(y) + \frac{n-2}{s_n r^{n-1}}u(y)\right)d\sigma(y), \end{aligned}$$

since

$$y \mapsto \partial_n \frac{1}{\|y-x\|^{n-2}} = \frac{\partial}{\partial \rho}\rho^{2-n}_{|\rho=r} = -\frac{n-2}{r^{n-1}}.$$

In case u is harmonic, from the Gauss integral theorem (2.9) and the mean value property of u we get

$$\begin{aligned} \int_{E\setminus B(x,r)} h_x(y)\Delta u(y)dy &- \int_{\partial E}(h_x(y)\partial_n u(y) - u(y)\partial_n h_x(y))d\sigma(y) \\ &= \int_{S(x,r)} u(y)\sigma_r^x(dy) \\ &= u(x). \end{aligned}$$

In the general case we need to pass to the limit as r tends to 0, which gives the same result:

$$u(x) = \int_{E\setminus B(x,r)} h_x(y)\Delta u(y)dy + \int_{\partial E}(u(y)\partial_n h_x(y) - h_x(y)\partial_n u(y))d\sigma(y). \tag{2.12}$$

Our goal is now to avoid using the values of the derivative term $\partial_n u(y)$ on ∂E in the above formula. To this end we note that from Green's identity (2.8) we have

$$\begin{aligned} \int_E w_x(y)\Delta u(y)dy &= \int_{\partial E}(w_x(y)\partial_n u(y) - u(y)\partial_n w_x(y))d\sigma(y) \\ &= \int_{\partial E}(h_x(y)\partial_n u(y) - u(y)\partial_n w_x(y))d\sigma(y) \end{aligned} \tag{2.13}$$

for any smooth solution w_x to the Dirichlet problem (2.10). Taking the difference between (2.12) and (2.13) yields

$$\begin{aligned} u(x) &= \int_E (h_x(y) - w_x(y))\Delta u(y)dy + \int_{\partial E} u(y)\partial_n(h_x(y) - w_x(y))d\sigma(y) \\ &= \int_E G^E(x,y)\Delta u(y)dy + \int_{\partial E} u(y)\partial_n G^E(x,y)d\sigma(y), \qquad x \in E. \end{aligned}$$

□

Using (2.5) and (2.6), Relation (2.11) can be reformulated as

$$\begin{aligned} u(x) &= \frac{1}{(n-2)s_n}\int_{\partial E}\left(u(z)\partial_n\frac{1}{\|x-z\|^{n-2}} - \frac{1}{\|x-z\|^{n-2}}\partial_n u(z)\right)\sigma(dz) \\ &\quad + \frac{1}{(n-2)s_n}\int_E \frac{1}{\|x-z\|^{n-2}}\Delta u(z)dz, \qquad x \in B(y,r), \end{aligned}$$

if $n \geq 3$, and if $n = 2$:

$$\begin{aligned} u(x) &= -\frac{1}{s_2}\int_{\partial E}(u(z)\partial_n \log\|x-z\| - (\log\|x-z\|)\partial_n u(z))\sigma(dz) \\ &\quad - \int_E (\log\|x-z\|)\Delta u(z)dz, \qquad x \in B(y,r). \end{aligned}$$

Corollary 2.3.3. *When u is harmonic on E we get*

$$u(x) = \int_{\partial E} u(y)\partial_n G^E(x,y)d\sigma(y), \qquad x \in E. \tag{2.14}$$

As a consequence of Lemma 2.2.3, the Green kernel $G^{B(y,r)}(\cdot,y)$ relative to the ball $B(y,r)$ is given for $x \in B(y,r)$ by

$$G^{B(y,r)}(x,z) = \begin{cases} -\dfrac{1}{s_2}\log\left(\dfrac{r}{\|y-x\|}\dfrac{\|z-x\|}{\|z-x^*\|}\right), & z \in B(y,r)\setminus\{x\}, \quad x \neq y, \\[2ex] -\dfrac{1}{s_2}\log\left(\dfrac{\|z-y\|}{r}\right), & z \in B(y,r)\setminus\{x\}, \quad x = y, \\[2ex] +\infty, & z = x, \end{cases}$$

if $n = 2$, and

$$G^{B(y,r)}(x,z) = \begin{cases} \dfrac{1}{(n-2)s_n}\left(\dfrac{1}{\|z-x\|^{n-2}} - \dfrac{r^{n-2}}{\|x-y\|^{n-2}}\dfrac{1}{\|z-x^*\|^{n-2}}\right) & z \in B(y,r)\setminus\{x\}, \quad x \neq y, \\[2ex] \dfrac{1}{(n-2)s_n}\left(\dfrac{1}{\|z-y\|^{n-2}} - \dfrac{1}{r^{n-2}},\right) & z \in B(y,r)\setminus\{x\}, \quad x = y, \\[2ex] +\infty, & z = x, \end{cases}$$

if $n \geq 3$.

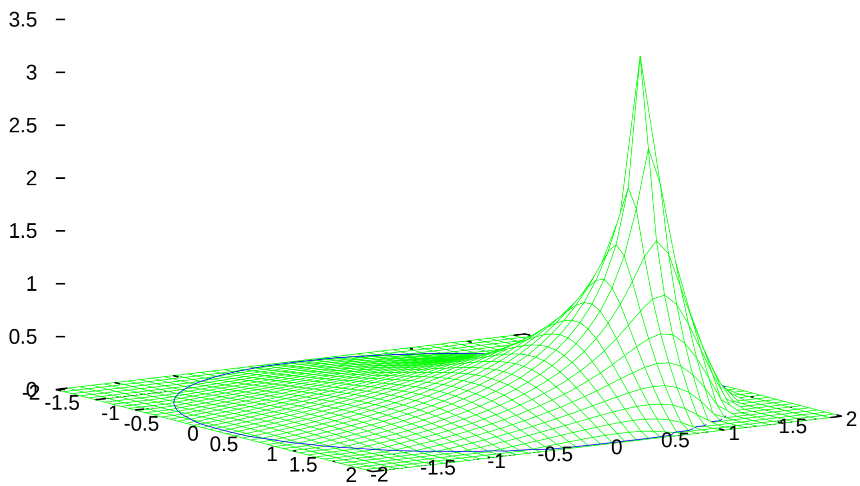

Figure 2.4: Graph of $z \mapsto G(x,z)$ with $x \in E = B(y,r)$ and $n = 2$.

The Green function on $E = \mathbb{R}^n$, $n \geq 3$, is obtained by letting r go to infinity:

$$G^{\mathbb{R}^n}(x,z) = \frac{1}{(n-2)s_n\|z-x\|^{n-2}} = h_x(z) = h_z(x), \qquad x, z \in \mathbb{R}^n. \tag{2.15}$$

2.4 Poisson formula

Given u a sufficiently integrable function on $S(y,r)$ we let $\mathcal{I}_u^{B(y,r)}(x)$ denote the Poisson integral of u over $S(y,r)$, defined as:

$$\mathcal{I}_u^{B(y,r)}(x) = \frac{1}{s_n r}\int_{S(y,r)} \frac{r^2 - \|y-x\|^2}{\|z-x\|^n} u(z)\sigma(dz), \qquad x \in B(y,r).$$

Next is the Poisson formula obtained as a consequence of Proposition 2.3.2.

Theorem 2.4.1. *Let $n \geq 2$. If u has continuous second partial derivatives on $E = \bar{B}(y,r)$ then for all $x \in B(y,r)$ we have*

$$u(x) = \mathcal{I}_u^{B(y,r)}(x) + \int_{B(y,r)} G^{B(y,r)}(x,z)\Delta u(z)dz. \tag{2.16}$$

Proof. We use the relation

$$u(x) = \int_{S(y,r)} u(z)\partial_n G^{B(y,r)}(x,z)\sigma(dz) + \int_{B(y,r)} G^{B(y,r)}(x,z)\Delta u(z)dz, \qquad x \in B(y,r),$$

and the fact that

$$z \mapsto \partial_n G^{B(y,r)}(x,z) = -\frac{1}{s_2}\partial_n \log\left(\frac{\|y-x\|}{r}\frac{\|z-x^*\|}{\|z-x\|}\right) = \frac{1}{(n-2)}\frac{r^2 - \|x-y\|^2}{s_n r\|z-x\|^2}$$

if $n = 2$, and similarly for $n \geq 3$. $\square$

When u is harmonic on a neighborhood of $\bar{B}(y,r)$ we obtain the Poisson representation formula of u on $E = \bar{B}(y,r)$ using its values on the sphere $\partial E = S(y,r)$, as a corollary of Theorem 2.4.1.

Corollary 2.4.2. *Assume that u is harmonic on a neighborhood of $\bar{B}(y,r)$. We have*

$$u(x) = \frac{1}{s_n r} \int_{S(y,r)} \frac{r^2 - \|y-x\|^2}{\|z-x\|^n} u(z)\sigma(dz), \qquad x \in B(y,r), \tag{2.17}$$

for all $n \geq 2$.

Similarly, Theorem 2.4.1 also shows that

$$u(x) \leq \mathcal{I}_u^{B(y,r)}(x), \qquad x \in B(y,r),$$

when u is superharmonic on $B(y,r)$. Note also that when $x = y$, Relation (2.17) recovers the mean value property of harmonic functions:

$$u(y) = \frac{1}{s_n r^{n-1}} \int_{S(y,r)} u(z)\sigma(dz),$$

and a similar property for superharmonic functions.

The function

$$z \mapsto \frac{r^2 - \|x-y\|^2}{\|x-z\|^n}$$

is called the Poisson kernel on $S(y,r)$ at $x \in B(y,r)$.

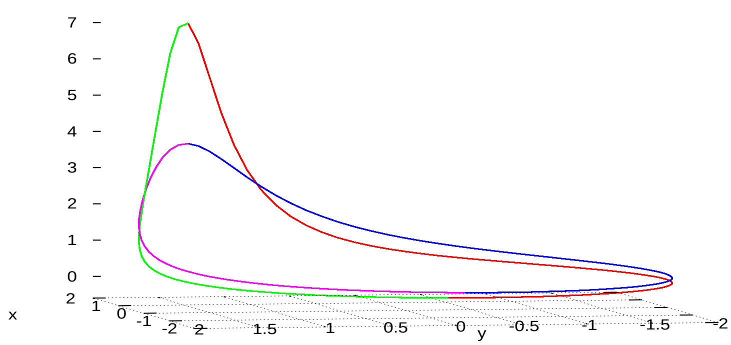

Figure 2.5: Two Poisson kernel graphs on $S(y,r)$ for $n = 2$ for two values of $x \in B(y,r)$.

A direct calculation shows that the Poisson kernel it is harmonic on $\mathbb{R}^n \setminus (S(y,r) \cup \{z\})$:

$$\Delta_x \frac{r^2 - \|y-x\|^2}{\|z-x\|^n} = 0, \tag{2.18}$$

hence all Poisson integrals are harmonic functions on $B(y,r)$. Moreover Theorem 2.17 of du Plessis [dP70] asserts that for any $z \in S(y,r)$, letting x tend to z without entering $S(y,r)$ we have

$$\lim_{x \to z} \mathcal{I}_u^{B(y,r)}(x) = u(z).$$

Hence the Poisson integral provides a solution to the Dirichlet problem (2.10) on $B(y,r)$:

Proposition 2.4.3. *Given f a continuous function on $S(y,r)$, the Poisson integral*

$$\mathcal{I}_f^{B(y,r)}(x) = \frac{1}{s_n r} \int_{S(y,r)} \frac{r^2 - \|x-y\|^2}{\|x-z\|^n} f(z)\sigma(dz), \qquad x \in B(y,r),$$

provides a solution of the Dirichlet problem

$$\begin{cases} \Delta w(x) = 0, & x \in B(y,r), \\ \\ w(x) = f(x), & x \in S(y,r), \end{cases}$$

with boundary condition f.

Recall that the Dirichlet problem on $B(y,r)$ may not have a solution when the boundary condition f is not continuous.

In particular, from Lemma 2.2.3 we have

$$\mathcal{I}_{h_x}^{B(y,r)}(z) = \frac{r^{n-2}}{\|x-y\|^{n-2}} h_{x^*}(z), \qquad x \in B(y,r),$$

for $n \geq 3$, and

$$\mathcal{I}_{h_x}^{B(y,r)}(z) = h_{x^*}(z) - \frac{1}{s_2} \log\left(\frac{\|x-y\|}{r}\right), \qquad x \in B(y,r),$$

for $n = 2$, where x^* denotes the inverse of x with respect to $S(y,r)$. This function solves the Dirichlet problem with boundary condition h_x, and the corresponding Green function satisfies

$$G^{B(y,r)}(x,z) = h_x(z) - \mathcal{I}_{h_x}^{B(y,r)}(z), \qquad x, z \in B(y,r).$$

When $x = y$ the solution $\mathcal{I}_{h_y}^{B(y,r)}(z)$ of the Dirichlet problem on $B(y,r)$ with boundary condition h_y is constant and equal to $(n-2)^{-1} s_n^{-1} r^{2-n}$ on $B(y,r)$, and we have the identity

$$\frac{1}{s_n r^{n-1}} \int_{S(y,r)} \frac{r^2 - \|y-z\|^2}{\|x-z\|^n} \sigma(dx) = r^{2-n}, \qquad z \in B(y,r),$$

for $n \geq 3$, cf. Figure 2.6 below.

2.5 Potentials and balayage

In electrostatics, the function $x \mapsto h_y(x)$ represents the Coulomb potential created at $x \in \mathbb{R}^n$ by a charge at $y \in E$, and the function

$$x \mapsto \int_E h_y(x) q(dy)$$

represents the sum of all potentials generated at $x \in \mathbb{R}^n$ by a distribution $q(dy)$ of charges inside E. In particular, when $E = \mathbb{R}^n$, $n \geq 3$, this sum equals

$$x \mapsto \int_{\mathbb{R}^n} G^{\mathbb{R}^n}(x, y) q(dy)$$

from (2.15). The definition of potentials originates from this interpretation.

Definition 2.5.1. *Given a measure μ on E, the Green potential of μ is defined as the function*

$$x \mapsto G^E \mu(x) := \int_E G^E(x, y) \mu(dy),$$

where $G^E(x, y)$ is the Green kernel on E.

Potentials will be used in the construction of superharmonic functions, cf. Proposition 5.1.1 below. Conversely, it is natural to ask whether a superharmonic function can be represented as the potential of a measure. In general, recall (cf. Proposition 2.3.2) the relation

$$u(x) = \int_{\partial E} u(z) \partial_n G^E(x, z) \sigma(dz) + \int_E G^E(x, z) \Delta u(z) dz, \qquad x \in E, \tag{2.19}$$

which yields, if $E = B(y, r)$:

$$u(x) = \mathcal{I}_u^E(x) + \int_E G^E(x, z) \Delta u(z) dz, \qquad x \in B(y, r),$$

where the Poisson integral $\mathcal{I}_u^E$ is harmonic in general from (2.18). This formula can be seen as a decomposition of u into the sum of a harmonic function on $B(y, r)$ and a potential. The question of representing superharmonic functions as potentials is examined next in Theorem 2.5.3 below, with application to the construction of Martin boundaries.

Definition 2.5.2. *Consider*

i) an open subset E of $\mathbb{R}^n$ with Green function G^E,

ii) a subset D of E, and

iii) a non-negative superharmonic function u on E.

The infimum on E over all non-negative superharmonic function on E which are (pointwise) greater than u on D is called the reduced function of u relative to D, and denoted by $\mathfrak{R}_u^D$.

In other terms, if Φ_u denotes the set of non-negative superharmonic functions v on E such that $v \geq u$ on D, we have

$$\mathfrak{R}_u^D := \inf\{v \in \Phi_u\}.$$

The lower regularization

$$\hat{\mathfrak{R}}_u^D(x) := \liminf_{y \to x} \mathfrak{R}_u^D(y), \qquad x \in E,$$

is called the *balayage* of u, and is a superharmonic function.

In case $D = \mathbb{R}^n \setminus B(y,r)$, the reduced of h_y on D is

$$\mathfrak{R}_{h_y}^D(z) = h_y(z)\mathbf{1}_{\{z \notin B(y,r)\}} + h_0(r)\mathbf{1}_{\{z \in B(y,r)\}}, \qquad z \in \mathbb{R}^n. \tag{2.20}$$

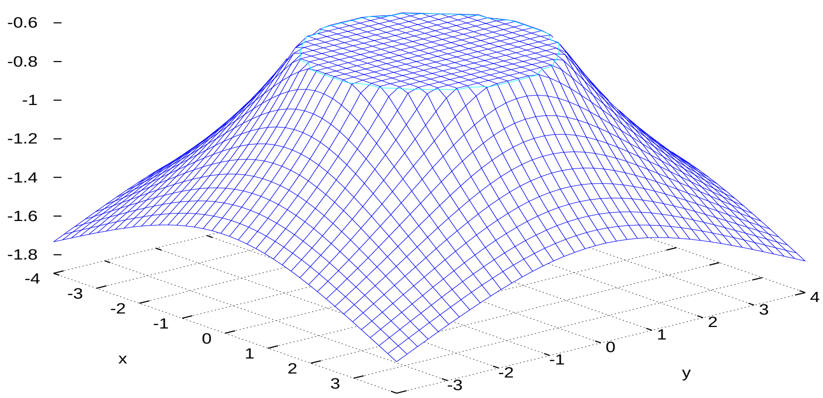

Figure 2.6: Reduced of h_y relative to $D = \mathbb{R}^2 \setminus B(y,r)$.

More generally if v is superharmonic then $\mathfrak{R}_v^D = \mathcal{I}_v^{B(y,r)}$ on $\mathbb{R}^2 \setminus D = B(y,r)$, cf. p. 62 and p. 100 of Brelot [Bre65]. Hence if $D = B^c(y,r) = \mathbb{R}^n \setminus B(y,r)$ and $x \in B(y,r)$ we have

$$\begin{aligned}
\mathfrak{R}_{h_x}^{B^c(y,r)}(z) &= \mathcal{I}_{h_x}^{B(y,r)}(z) \\
&= h_0\left(\frac{\|x-y\|\|z-x^*\|}{r}\right) \\
&= \frac{r^{n-2}}{(n-2)s_n\|x-y\|^{n-2}} \frac{1}{\|z-x^*\|^{n-2}}, \qquad z \in B(y,r).
\end{aligned}$$

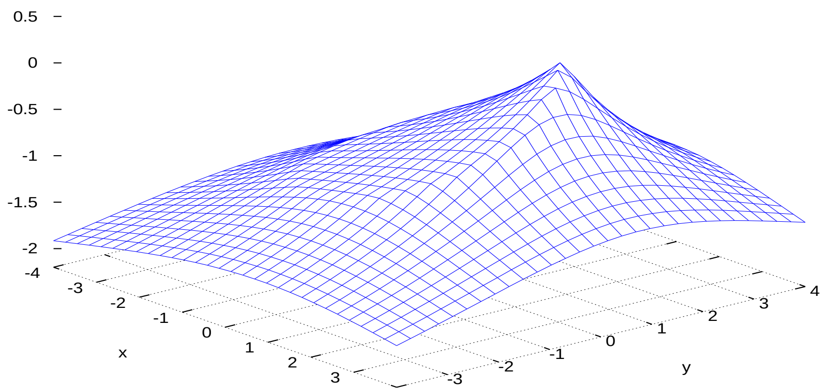

Figure 2.7: Reduced of h_x relative to $D = \mathbb{R}^2 \setminus B(y, r)$ with $x \in B(y, r) \setminus \{y\}$.

Using Proposition 2.3.2 and the identity

$$\Delta \mathfrak{R}_{h_y}^{B^c(y,r)} = \sigma_r^y,$$

in distribution sense, we can represent $\mathfrak{R}_{h_y}^{B^c(y,r)}$ on $E = B(y, R)$, $R > r$, as

$$\begin{aligned} \mathfrak{R}_{h_y}^{B^c(y,r)}(x) &= \int_{S(y,R)} \mathfrak{R}_{h_y}^{B^c(y,r)}(z) \partial_n G^{B(y,R)}(x, z) \sigma(dz) + \int_{B(y,R)} G^{B(y,R)}(x, z) \sigma_r^y(dz) \\ &= h_0(R) + \int_{S(y,r)} G^{B(y,R)}(x, z) \sigma_r^y(dz), \qquad x \in B(y, R). \end{aligned}$$

Letting R go to infinity yields

$$\mathfrak{R}_{h_y}^{B^c(y,r)}(x) = \int_{S(y,r)} G^{\mathbb{R}^n}(x, z) \sigma_r^y(dz) = \int_{S(y,r)} h_x(z) \sigma_r^y(dz),$$

and we recover (2.20) since h_x is harmonic on $\mathbb{R}^n \setminus B(y, r)$ when $x \notin B(y, r)$ and when $x \in B(y, r)$ we have

$$\mathfrak{R}_{h_y}^{B^c(y,r)}(x) = \int_{S(y,r)} h_0 \left(\frac{\|y - x\| \|z - x^*\|}{r} \right) \sigma_r^y(dz) = h_0 \left(\frac{\|y - x\| \|y - x^*\|}{r} \right) = h_0(r).$$

More generally we have for the following result, for which we refer to Theorem 7.12 of Helms [Hel69].

Theorem 2.5.3. *If D is a compact subset of E and u is a non-negative superharmonic function on E then $\hat{\mathfrak{R}}_u^D$ is a potential, i.e. there exists a measure μ on E such that*

$$\hat{\mathfrak{R}}_u^D(x) = \int_E G^E(x, y) \mu(dy), \qquad x \in E. \tag{2.21}$$

If moreover $\mathfrak{R}_v^D$ is harmonic on D then μ is supported by ∂D, cf. Theorem 6.9 in Helms [Hel69].

2.6 Martin boundary

The Martin boundary theory extends the Poisson integral representation described in Section 2.4 to arbitrary open domains with a non smooth boundary, or having no boundary. Given E a connected open subset of $\mathbb{R}^n$, the Martin boundary ΔE of E is defined in an abstract sense as $\Delta E := \hat{E} \backslash E$, where $\hat{E}$ is a suitable by constructing a compactification of E.

Next we show that every non-negative harmonic function on E admits an integral representation using its values on the boundary ΔE. For this, let u be non-negative and harmonic on E, and consider an increasing sequence $(E_n)_{n\in\mathbb{N}}$ of open sets with smooth boundaries $(\partial E_n)_{n\in\mathbb{N}}$ and compact closures, such that

$$E = \bigcup_{n=0}^{\infty} E_n.$$

Then the balayage $\mathfrak{R}_u^{E_n}$ of u on E_n coincides with u on E_n, and from Theorem 2.5.3 it can be represented as

$$\mathfrak{R}_u^{E_n}(x) = \int_{E_n} G^E(x,y) d\mu_n(y)$$

where μ_n is a measure supported by ∂E_n. Note that since

$$\mathfrak{R}_u^{E_n}(x) = u(x),$$

for all $x \in E_k$, $k \geq n$, and

$$\lim_{n\to\infty} G^E(x,y)_{|y\in\partial E_n} = 0, \qquad x \in E,$$

the total mass of μ_n increases to infinity:

$$\lim_{n\to\infty} \mu_n(E_n) = \infty.$$

For this reason one chooses to renormalize μ_n by choosing $x_0 \in E_1$ and letting

$$\tilde{\mu}_n(dy) = G^E(x_0,y)\mu_n(dy), \qquad n \in \mathbb{N},$$

so that $\tilde{\mu}_n$ has total mass

$$\tilde{\mu}_n(E_n) = u(x_0) < \infty,$$

independently of $n \in \mathbb{N}$. Next, let the kernel K_{x_0} be defined as

$$K_{x_0}(x,y) := \frac{G^E(x,y)}{G^E(x_0,y)}, \qquad x, y \in E,$$

with the relation

$$\hat{\mathfrak{R}}_u^{E_n}(x) = \int_{E_n} K_{x_0}(x,y) d\tilde{\mu}_n(y).$$

The construction of the Martin boundary of E relies on the following theorem by Constantinescu and Cornea, cf. Helms [Hel69], chapter 12.

Theorem 2.6.1. *Let E denote a locally compact and non-compact space and consider a family Φ of continuous mappings*

$$f : E \to [-\infty, \infty].$$

Then there exists a unique compact space $\hat{E}$ in which E is everywhere dense, such that:

a) every $f \in \Phi$ can be extended to a function f^* on $\hat{E}$ by continuity,

b) the extended functions separate the points of the boundary $\Delta E = \hat{E} \setminus E$ in the sense that if $x, y \in \Delta E$ with $x \neq y$, there exists $f \in \Phi$ such that $f^*(x) \neq f^*(y)$.

The Martin space is then defined as the unique compact space $\hat{E}$ in which E is everywhere dense, and such that the functions

$$\{y \mapsto K_{x_0}(x, y) \ : \ x \in E\}$$

admit continuous extensions which separate the boundary $\hat{E} \setminus E$. Such a compactification $\hat{E}$ of E exists and is unique as a consequence of the Constantinescu-Cornea Theorem 2.6.1, applied to the family

$$\Phi := \{x \mapsto K_{x_0}(x, z) \ : \ z \in E\}.$$

In this way the Martin boundary of E is defined as

$$\Delta E := \hat{E} \setminus E.$$

The Martin boundary is unique up to an homeomorphism, namely if $x_0, x_0' \in E$, then

$$K_{x_0'}(x, z) = \frac{K_{x_0}(x, z)}{K_{x_0}(x_0', z)}$$

and

$$K_{x_0}(x, z) = \frac{K_{x_0'}(x, z)}{K_{x_0'}(x_0, z)}$$

have different limits in $\hat{E}'$ which still separate the boundary ΔE. For this reason, in the sequel we will drop the index x_0 in $K_{x_0}(x, z)$.

Theorem 2.6.2. *Any non-negative harmonic function h on E can be represented as*

$$h(x) = \int_{\Delta E} K(z, x)\nu(dz), \qquad x \in E,$$

where ν is a non-negative Radon measure on ΔE.

Proof. Since $\tilde{\mu}_n(E_n)$ is bounded (actually it is constant) in $n \in \mathbb{N}$ we can extract a subsequence $(\tilde{\mu}_{n_k})_{k\in\mathbb{N}}$ converging vaguely to a measure μ on $\hat{E}$, i.e.

$$\lim_{k\to\infty} \int_{\hat{E}} f(x)\tilde{\mu}_{n_k}(dx) = \int_{\hat{E}} f(x)\mu(dx), \qquad f \in \mathcal{C}_c(E),$$

see e.g. Ex. 10, p. 81 of Hirsch and Lacombe [HL99]. The continuity of $z \mapsto K(x,z)$ in $z \in \hat{E}$ then implies

$$\begin{aligned}
h(x) &= \lim_{n\to\infty} \hat{\mathfrak{R}}_u^{E_n}(x) \\
&= \lim_{k\to\infty} \hat{\mathfrak{R}}_u^{E_{n_k}}(x) \\
&= \lim_{k\to\infty} \int_{\hat{E}} K(z,x)\nu_{n_k}(dz) \\
&= \int_{\hat{E}} K(z,x)\mu(dz), \qquad x \in E.
\end{aligned}$$

Finally, ν is supported by ΔE since for all $f \in \mathcal{C}_c(E_n)$,

$$\int_E f(x)\mu(dx) = \lim_{k\to\infty} \int_{E_k} f(x)\tilde{\mu}_{n_k}(dx) = 0.$$

□

When $E = B(y,r)$ one can check by explicit calculation that

$$\lim_{\zeta\to z} K_y(x,\zeta) = \frac{r^2 - \|x-y\|^2}{\|z-x\|^n}, \qquad z \in S(y,r),$$

is the Poisson kernel on $S(y,r)$. In this case we have

$$\mu = \sigma_r^y,$$

which is the normalized surface measure on $S(y,r)$, and the Martin boundary $\Delta B(y,r)$ of $B(y,r)$ equals the usual boundary $S(y,r)$.

3 Markov processes

3.1 Markov property

Let $\mathcal{C}_0(\mathbb{R}^n)$ denote the class of continuous functions tending to 0 at infinity. Recall that f is said to tend to 0 at infinity if for all $\varepsilon > 0$ there exists a compact subset K of $\mathbb{R}^n$ such that $|f(x)| \leq \varepsilon$ for all $x \in \mathbb{R}^n \setminus K$.

Definition 3.1.1. *An $\mathbb{R}^n$-valued stochastic process, i.e. a family $(X_t)_{t\in\mathbb{R}_+}$ of random variables on a probability space $(\Omega, \mathcal{F}, P)$, is a Markov process if for all $t \in \mathbb{R}_+$ the σ-fields*

$$\mathcal{F}_t^+ := \sigma(X_s \ : \ s \geq t)$$

and

$$\mathcal{F}_t := \sigma(X_s \ : \ 0 \leq s \leq t).$$

are conditionally independent given X_t.

This condition can be restated by saying that for all $A \in \mathcal{F}_t^+$ and $B \in \mathcal{F}_t$ we have

$$P(A \cap B \mid X_t) = P(A \mid X_t)P(B \mid X_t),$$

cf. Chung [Chu95]. This definition naturally entails that:

i) $(X_t)_{t\in\mathbb{R}_+}$ is adapted with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, i.e. X_t is $\mathcal{F}_t$-measurable, $t \in \mathbb{R}_+$, and

ii) X_u is conditionally independent of $\mathcal{F}_t$ given X_t, for all $u \geq t$, i.e.

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = \mathbb{E}[f(X_t) \mid X_s], \qquad 0 \leq s \leq t,$$

for any bounded measurable function f on $\mathbb{R}^n$.

In particular,

$$P(X_u \in A \mid \mathcal{F}_t) = \mathbb{E}[\mathbf{1}_A(X_u) \mid \mathcal{F}_t] = \mathbb{E}[\mathbf{1}_A(X_u) \mid X_t] = P(X_u \in A \mid X_t), \qquad A \in \mathcal{B}(\mathbb{R}^n).$$

Processes with independent increments provide simple examples of Markov processes. Indeed, for all bounded measurable functions f, g we have

$$\begin{aligned}
&\mathbb{E}[f(X_{t_1}, \ldots, X_{t_n})g(X_{s_1}, \ldots, X_{s_n}) \mid X_t] \\
&\quad= \mathbb{E}[f(X_{t_1} - X_t + x, \ldots, X_{t_n} - X_t + x)g(X_{s_1} - X_t + x, \ldots, X_{s_n} - X_t + x)]_{x=X_t} \\
&\quad= \mathbb{E}[f(X_{t_1} - X_t + x, \ldots, X_{t_n} - X_t + x)]_{x=X_t}\,\mathbb{E}[g(X_{s_1} - X_t + x, \ldots, X_{s_n} - X_t + x)]_{x=X_t} \\
&\quad= \mathbb{E}[f(X_{t_1}, \ldots, X_{t_n}) \mid X_t]\,\mathbb{E}[g(X_{s_1}, \ldots, X_{s_n}) \mid X_t],
\end{aligned}$$

$0 \leq s_1 < \cdots < s_n < t < t_1 < \cdots < t_n$.

In discrete time, a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables is said to be a Markov chain for all $n \in \mathbb{N}$, the σ-algebras

$$\mathcal{F}_n = \sigma(\{X_k \ : \ k \leq n\})$$

and

$$\mathcal{F}_n^+ = \sigma(\{X_k \ : \ k \geq n\})$$

are independent conditionally to X_n. In particular, for all $\mathcal{F}_n^+$-measurable bounded random variable F we have

$$\mathbb{E}[F \mid \mathcal{F}_n] = \mathbb{E}[F \mid X_n], \qquad n \in \mathbb{N}.$$

3.2 Transition kernels and semigroups

A transition kernel is a mapping $P(x, dy)$ such that

i) for every $x \in E$, $A \mapsto P(x, A)$ is a probability measure, and

ii) for every $A \in \mathcal{B}(E)$, the mapping $x \mapsto P(x, A)$ is a measurable function.

The transition kernel $\mu_{s,t}$ associated to $(X_t)_{t\in\mathbb{R}_+}$ is defined as

$$\mu_{s,t}(x,A) = P(X_t \in A \mid X_s = x) \qquad 0 \le s \le t,$$

and we have

$$\mu_{s,t}(X_s,A) = P(X_t \in A \mid X_s) = P(X_t \in A \mid \mathcal{F}_s), \qquad 0 \le s \le t.$$

The transition operator $(T_{s,t})_{0\le s\le t}$ associated to $(X_t)_{t\in\mathbb{R}_+}$ is defined as

$$T_{s,t}f(x) = \mathbb{E}[f(X_t) \mid X_s = x] = \int_{\mathbb{R}^n} f(y)\mu_{s,t}(x,dy), \qquad x \in \mathbb{R}^n.$$

Letting $p_{s,t}(x)$ denote the density of $X_t - X_s$ we have

$$\mu_{s,t}(x,A) = \int_A p_{s,t}(y-x)dy, \qquad A \in \mathcal{B}(\mathbb{R}^n),$$

and

$$T_{s,t}f(x) = \int_{\mathbb{R}^n} f(y)p_{s,t}(y-x)dy.$$

In the sequel we will assume that $(X_t)_{t\in\mathbb{R}_+}$ is time homogeneous, i.e. $\mu_{s,t}$ depends only on the difference $t-s$, and we will denote it by μ_{t-s}. In this case the family $(T_{0,t})_{t\in\mathbb{R}_+}$ is denoted by $(T_t)_{t\in\mathbb{R}_+}$ and defines a transition semigroup associated to $(X_t)_{t\in\mathbb{R}_+}$, with

$$T_tf(x) = \mathbb{E}[f(X_t) \mid X_0 = x] = \int_{\mathbb{R}^n} f(y)\mu_t(x,dy), \qquad x \in \mathbb{R}^n.$$

It satisfies the semigroup property

$$\begin{aligned}
T_tT_sf(x) &= \mathbb{E}[T_sf(X_t) \mid X_0 = x] \\
&= \mathbb{E}[\mathbb{E}[f(X_{t+s}) \mid X_s] \mid X_0 = x]] \\
&= \mathbb{E}[\mathbb{E}[f(X_{t+s}) \mid \mathcal{F}_s] \mid X_0 = x]] \\
&= \mathbb{E}[f(X_{t+s}) \mid X_0 = x] \\
&= T_{t+s}f(x),
\end{aligned}$$

which leads to the Chapman-Kolmogorov equation

$$\mu_{s+t}(x,A) = \mu_s * \mu_t(x,A) = \int_{\mathbb{R}^n} \mu_s(x,dy)\mu_t(y,A). \tag{3.1}$$

By induction we obtain

$$P_x((X_{t_1},\ldots,X_{t_n}) \in B_1 \times \cdots \times B_n) = \int_{B_1} \cdots \int_{B_n} \mu_{0,t_1}(x,dx_1)\cdots\mu_{t_{n-1},t_n}(x_{n-1},dx_n),$$

for $0 < t_1 < \cdots < t_n$ and $B_1,\ldots,B_n$ Borel subsets of $\mathbb{R}^n$.

If $(X_t)_{t\in\mathbb{R}_+}$ is a homogeneous Markov processes with independent increments then the density $p_t(x)$ of X_t satisfies the convolution property

$$p_{s+t}(x) = \int_{\mathbb{R}^n} p_s(y-x)p_t(y)dy, \qquad x \in \mathbb{R}^n,$$

which is satisfied in particular by processes with stationary and independent increments such as Lévy processes. A typical example of a probability density satisfying such a convolution property is the Gaussian density, i.e.

$$p_t(x) = \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{1}{2t}\|x\|^2_{\mathbb{R}^n}\right), \qquad x \in \mathbb{R}^n.$$

From now on we assume that $(T_t)_{t\in\mathbb{R}_+}$ is a strongly continuous Feller semigroup, i.e. a family of positive contraction operators on $\mathcal{C}_0(\mathbb{R}^n)$ such that

i) $T_{s+t} = T_sT_t$, $s,t \geq 0$,

ii) $T_t\mathcal{C}_0(\mathbb{R}^n) \subset \mathcal{C}_0(\mathbb{R}^n)$,

iii) $T_tf(x) \to f(x)$ as $t \to 0$, $f \in \mathcal{C}_0(\mathbb{R}^n)$, $x \in \mathbb{R}^n$.

The resolvent of $(T_t)_{t\in\mathbb{R}_+}$ is defined as

$$R_\lambda f(x) := \int_0^\infty e^{-\lambda t}T_tf(x)dt, \qquad x \in \mathbb{R}^n,$$

i.e.

$$R_\lambda f(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\lambda t}f(X_t)dt\right], \qquad x \in \mathbb{R}^n,$$

for sufficiently integrable f on $\mathbb{R}^n$, where $\mathbb{E}_x$ denotes the conditional expectation given that $\{X_0 = x\}$. It satisfies the resolvent equation

$$R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu, \qquad \lambda, \mu > 0.$$

We refer to [Kal02] for the following result.

Theorem 3.2.1. *Let $(T_t)_{t\in\mathbb{R}_+}$ be a Feller semigroup on $\mathcal{C}_0(\mathbb{R}^n)$ with resolvent R_λ, $\lambda > 0$. Then there exists an operator A with domain $\mathcal{D} \subset \mathcal{C}_0$ such that*

$$R_\lambda^{-1} = \lambda I - A, \qquad \lambda > 0. \tag{3.2}$$

The operator A is called the generator of $(T_t)_{t\in\mathbb{R}_+}$ and it characterizes $(T_t)_{t\in\mathbb{R}_+}$. Furthermore, the semigroup $(T_t)_{t\in\mathbb{R}_+}$ is differentiable in t for all $f \in \mathcal{D}$ and it satisfies the forward and backward Kolmogorov equations

$$\frac{dT_tf}{dt} = T_tAf, \qquad \text{and} \qquad \frac{dT_tf}{dt} = AT_tf.$$

Note that for the Gaussian transition density $p_t(x, y)$, the integral

$$\begin{aligned}\int_0^\infty p_t(x,y)dt &= \frac{\Gamma(n/2-1)}{2\pi^{n/2}\|x-y\|^{n-2}} \\ &= \frac{n-2}{2s_n\|x-y\|^{n-2}} \\ &= (n/2-1)h_y(x)\end{aligned}$$

is a Newtonian potential for $n \geq 3$. Hence the resolvent $R_0 f$ associated to the Gaussian semigroup $T_t f(x) = \int_{\mathbb{R}^n} f(y)p_t(x,y)dy$ is also a potential in the sense of Definition 2.5.1, since:

$$\begin{aligned}R_0 f(x) &= \int_0^\infty T_t f(x)dt \\ &= \mathbb{E}_x\left[\int_0^\infty f(B_t)dt\right] \\ &= \int_{\mathbb{R}^n} f(y)\int_0^\infty p_t(x,y)dtdy \\ &= \frac{n-2}{2s_n}\int_{\mathbb{R}^n}\frac{f(y)}{\|x-y\|^{n-2}}dy.\end{aligned}$$

More generally we have

$$\begin{aligned}R_\lambda f(x) &= (\lambda I - A)^{-1}f(x) \\ &= \int_0^\infty e^{-\lambda t}T_t f(x)dt \\ &= \int_0^\infty e^{-\lambda t}e^{tA}f(x)dt \\ &= \int_0^\infty e^{-\lambda t}T_t f(x)dt \\ &= \int_0^\infty \int_{\mathbb{R}^n} f(y)e^{-\lambda t}p_t(x,y)dydt \\ &= \int_{\mathbb{R}^n} f(y)g^\lambda(x,y)dy, \qquad x \in \mathbb{R}^n,\end{aligned}$$

where $g^\lambda(x,y)$ is the λ-potential kernel defined as

$$g^\lambda(x,y) := \int_0^\infty e^{-\lambda t}p_t(x,y)dt,$$

and $R_\lambda f$ is also called a λ-potential.

Recall that the Hille-Yosida theorem allows one to construct a strongly continuous semigroup from a generator A.

In another direction it is possible to associate a Markov process to any time homogeneous transition function satisfying $\mu_0(x, dy) = \delta_x(dy)$ and the Chapman-Kolmogorov equation (3.1), cf. e.g. Theorem 4.1.1 of Ethier and Kurtz [EK86].

3.3 Hitting times

Definition 3.3.1. *An a.s. non-negative random variable τ is called a stopping time with respect to a filtration $\mathcal{F}_t$ if*

$$\{\tau \le t\} \in \mathcal{F}_t, \qquad t > 0.$$

The σ-algebra $\mathcal{F}_\tau$ is defined as the collection of measurable sets A such that

$$A \cap \{\tau < t\} \in \mathcal{F}_t$$

for all $t > 0$. Note that for all $s > 0$ we have

$$\{\tau < s\},\ \{\tau \le s\},\ \{\tau > s\},\ \{\tau \ge s\} \in \mathcal{F}_\tau.$$

The strong Markov property for the process $(X_t)_{t\in\mathbb{R}_+}$ states that for any P-a.s. finite $\mathcal{F}_t$-stopping time τ we have

$$\mathbb{E}[f(X(\tau+t)) \mid \mathcal{F}_\tau] = \mathbb{E}[f(X(t)) \mid X_0 = x]_{x=X_\tau} = T_t f(X_\tau), \tag{3.3}$$

for all bounded measurable f.

The hitting time τ_B of a Borel set B is defined as

$$\tau_B = \inf\{t > 0 \ :\ \ X_t \in B\},$$

with the convention $\inf \emptyset = +\infty$. A set B such that $P_x(\tau_B < \infty)$ for all $x \in \mathbb{R}^n$ is said to be *polar*.

In discrete time it can be easily shown that hitting times are stopping times, from the relation

$$\{\tau_B \le n\}^c = \{\tau_B > n\} = \bigcap_{k=0}^{n} \{X_k \notin B\} \in \mathcal{F}_n.$$

In continuous time the situation is more complicated. From e.g. Lemma 7.6 of Kallenberg [Kal02], τ_B is a stopping time provided B is closed and $(X_t)_{t\in\mathbb{R}_+}$ is continuous, or B is open and $(X_t)_{t\in\mathbb{R}_+}$ is right-continuous.

Definition 3.3.2. *The last exit time from B is denoted by l_B and defined as*

$$l_B = \sup\{t > 0 \ :\ X_t \in B\},$$

with $l_B = 0$ if $\tau_B = +\infty$.

We say that B is recurrent if $P_x(l_B = +\infty) = 1$, $x \in \mathbb{R}^n$, and that B is transient if $l_B < \infty$, P-a.s., i.e. $P_x(l_B = +\infty) = 0$, $x \in \mathbb{R}^n$.

The λ-capacity $C^\lambda(B)$ of a Borel set B is defined as

$$C^\lambda(B) = \mathbb{E}_y\left[\int_{\mathbb{R}^n} e^{-\lambda\tau_{B^c}} dy\right],$$

and it is finite when B is bounded. A given set B is said to be polar if $C^\lambda(B) = 0$.

3.4 Dirichlet forms

Dirichlet forms provide a functional analytic approach to the Dirichlet problem. A Dirichlet form is a positive bilinear form $\mathcal{E}$ defined on a domain $I\!\!D$ dense in a real Hilbert space H, such that

i) the space $I\!\!D$, equipped with the norm

$$\|f\|_{I\!\!D} := \sqrt{\|f\|_H^2 + \mathcal{E}(f,f)},$$

is a Hilbert space, and

ii) for any $f \in I\!\!D$ we have $f \wedge 1 \in I\!\!D$ and

$$\mathcal{E}(f \wedge 1, f \wedge 1) \leq \mathcal{E}(f,f).$$

The classical example of Dirichlet form is given by $H = L^2(\mathbb{R}^n)$ and

$$\mathcal{E}(f,g) := \int_{\mathbb{R}^n} \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_i}(x) dx.$$

The generator $\mathcal{L}$ of $\mathcal{E}$ is defined by $\mathcal{L}f = g$, $f \in I\!\!D$, if for all $h \in I\!\!D$ we have

$$\mathcal{E}(f,h) = -\langle g, h \rangle_H.$$

It is known, cf. e.g. Bouleau and Hirsch [BH91], that a self-adjoint operator $\mathcal{L}$ on $H = L^2(\mathbb{R}^n)$ with domain $\mathrm{Dom}(\mathcal{L})$ is the adjoint of a Dirichlet form if and only if

$$\langle \mathcal{L}f, (f-1)^+ \rangle_{L^2(\mathbb{R}^n)} \leq 0, \qquad f \in \mathrm{Dom}(\mathcal{L}).$$

It is in turn the generator of a strongly continuous semi-group $(P_t)_{t\in\mathbb{R}_+}$ on $H = L^2(\mathbb{R}^n)$ if and only if $(P_t)_{t\in\mathbb{R}_+}$ is sub-Markovian, i.e. for all $f \in L^2(\mathbb{R}^n)$,

$$0 \leq f \leq 1 \Rightarrow 0 \leq P_t f \leq 1, \qquad t \in \mathbb{R}_+.$$

We refer to Ma and Röckner [MR92] for more details on the connection between stochastic processes and Dirichlet forms. Coming back to the Dirichlet problem

$$\begin{cases} \Delta u = 0, & x \in D, \\ \\ u(x) = 0, & x \in \partial D. \end{cases}$$

If f and g are $\mathcal{C}^1$ with compact support in D we have

$$\int_D g(x) \Delta f(x) dx = -\sum_{i=1}^n \int_D \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_i}(x) dx = -\mathcal{E}(f,g).$$

Here, $I\!D$ is the subspace of functions in $L^2(\mathbb{R}^n)$ whose derivative in distribution sense belongs to $L^2(\mathbb{R}^n)$, with norm

$$\|f\|_{I\!D}^2 = \int_{\mathbb{R}^n} |f(x)|^2 dx + \int_{\mathbb{R}^n} \sum_{i=1}^n \left| \frac{\partial f}{\partial x_i}(x) \right|^2 dx.$$

Hence the Dirichlet problem for f can be formulated as

$$\mathcal{E}(f, g) = 0,$$

for all g in the completion of $\mathcal{C}_c^\infty(D)$ with respect to the $\| \cdot \|_{I\!D}$-norm. The capacity of an open set A is defined as

$$C(A) = \inf\{\|u\|_{I\!D} \ : \ u \in I\!D \text{ and } u \geq 1 \text{ on } A\},$$

and is connected to the notion of λ-capacity C^λ, cf. Fukushima et al. [FOT94]. The notion of zero-capacity set is finer than that of zero-measure sets and gives rise to the notion of properties that hold in the quasi-everywhere sense, cf. Bouleau and Hirsch [BH91].

In discrete time, a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables is called a homogeneous Markov chain with transition kernel P if

$$\mathbb{E}[f(X_n) \mid \mathcal{F}_m] = P^{n-m} f(X_m), \qquad 0 \leq m \leq n. \tag{3.4}$$

4 Stochastic calculus

4.1 Brownian motion and the Poisson process

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ a filtration, i.e. an increasing family of sub σ-algebras of $\mathcal{F}$. We assume that $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is continuous on the right, i.e.

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s, \qquad t \in \mathbb{R}_+.$$

Recall that a process $(M_t)_{t\in\mathbb{R}_+}$ in $L^1(\Omega)$ is called an $\mathcal{F}_t$-martingale if $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$, $0 \leq s \leq t$.

For example, if $(X_t)_{t\in[0,T]}$ is a (non homogeneous) Markov process with semi-group $(P_{s,t})_{0\leq s\leq t\leq T}$ satisfying

$$P_{s,t} f(X_s) = \mathbb{E}[f(X_t) \mid X_s] = \mathbb{E}[f(X_t) \mid \mathcal{F}_s], \quad 0 \leq s \leq t \leq T,$$

on $\mathcal{C}_b^2(\mathbb{R}^n)$ functions, with

$$P_{s,t} \circ P_{t,u} = P_{s,u}, \qquad 0 \leq s \leq t \leq u \leq T,$$

then $(P_{t,T} f(X_t))_{t\in[0,T]}$ is an $\mathcal{F}_t$- martingale:

$$\mathbb{E}[P_{t,T} f(X_t) \mid \mathcal{F}_s] = \mathbb{E}[\mathbb{E}[f(X_T) \mid \mathcal{F}_t] \mid \mathcal{F}_s] = \mathbb{E}[f(X_T) \mid \mathcal{F}_s] = P_{s,T} f(X_s),$$

$0 \leq s \leq t \leq T$.

Definition 4.1.1. *A martingale* $(M_t)_{t\in\mathbb{R}_+}$ *in* $L^2(\Omega)$ *(i.e.* $\mathbb{E}[|M_t|^2]<\infty$, $t\in\mathbb{R}_+$*) and such that*

$$\mathbb{E}[(M_t-M_s)^2|\mathcal{F}_s]=t-s,\quad 0\le s<t, \tag{4.1}$$

is called a normal martingale.

Every square-integrable process $(M_t)_{t\in\mathbb{R}_+}$ with centered independent increments and generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ satisfies

$$\mathbb{E}[(M_t-M_s)^2|\mathcal{F}_s]=\mathbb{E}[(M_t-M_s)^2],\qquad 0\le s\le t,$$

hence the following remark.

Remark 4.1.2. *A square-integrable process* $(M_t)_{t\in\mathbb{R}_+}$ *with centered independent increments is a normal martingale if and only if*

$$\mathbb{E}[(M_t-M_s)^2]=t-s,\qquad 0\le s\le t.$$

In our presentation of stochastic integration we will restrict ourselves to normal martingales. As will be seen in the next sections, this family contains Brownian motion and the standard Poisson process as particular cases.

Remark 4.1.3. *A martingale* $(M_t)_{t\in\mathbb{R}_+}$ *is normal if and only if* $(M_t^2-t)_{t\in\mathbb{R}_+}$ *is a martingale, i.e.*

$$\mathbb{E}[M_t^2-t|\mathcal{F}_s]=M_s^2-s,\quad 0\le s<t.$$

Proof. This follows from the equalities

$$\begin{aligned}\mathbb{E}[(M_t-M_s)^2|\mathcal{F}_s]-(t-s) &= \mathbb{E}[M_t^2-M_s^2-2(M_t-M_s)M_s|\mathcal{F}_s]-(t-s)\\ &= \mathbb{E}[M_t^2-M_s^2|\mathcal{F}_s]-2M_s\,\mathbb{E}[M_t-M_s|\mathcal{F}_s]-(t-s)\\ &= \mathbb{E}[M_t^2|\mathcal{F}_s]-t-(\mathbb{E}[M_s^2|\mathcal{F}_s]-s).\end{aligned}$$

□

Throughout the remainder of this chapter, $(M_t)_{t\in\mathbb{R}_+}$ will be a normal martingale.

We now turn to the Brownian motion and the compensated Poisson process as the fundamental examples of normal martingales. Our starting point is now a family $(\xi_n)_{n\in\mathbb{N}}$ of independent standard (i.e. centered and with unit variance) Gaussian random variables under $\gamma_{\mathbb{N}}$, constructed as the canonical projections from $(\mathbb{R}^{\mathbb{N}},\mathcal{B}_{\mathbb{R}^{\mathbb{N}}},\gamma_{\mathbb{N}})$ into $\mathbb{R}$. The measure $\gamma_{\mathbb{N}}$ is characterized by its Fourier transform

$$\alpha\mapsto\mathbb{E}\left[\mathrm{e}^{i\langle\xi,\alpha\rangle_{\ell^2(\mathbb{N})}}\right]=\mathbb{E}\left[\mathrm{e}^{i\sum_{n=0}^{\infty}\xi_n\alpha_n}\right]=\prod_{n=0}^{\infty}\mathrm{e}^{-\alpha_n^2/2}=\mathrm{e}^{-\frac{1}{2}\|\alpha\|_{\ell^2(\mathbb{N})}^2},\quad \alpha\in\ell^2(\mathbb{N}),$$

i.e. $\langle\xi,\alpha\rangle_{\ell^2(\mathbb{N})}$ is a centered Gaussian random variable with variance $\|\alpha\|_{\ell^2(\mathbb{N})}^2$. Let $(e_n)_{n\in\mathbb{N}}$ be of $L^2(\mathbb{R}_+)$ denote an orthonormal basis$(e_n)_{n\in\mathbb{N}}$ be of $L^2(\mathbb{R}_+)$.

Definition 4.1.4. *Given $u \in L^2(\mathbb{R}_+)$ with decomposition*

$$u = \sum_{n=0}^{\infty} \langle u, e_n \rangle e_n,$$

we let $J_1 : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}^{\mathbb{N}}, \gamma^{\mathbb{N}})$ be defined as

$$J_1(u) = \sum_{n=0}^{\infty} \xi_n \langle u, e_n \rangle.$$

We have the isometry property

$$\mathbb{E}[|J_1(u)|^2] = \sum_{k=0}^{\infty} |\langle u, e_n \rangle|^2 \, \mathbb{E}[|\xi_n|^2] = \sum_{k=0}^{\infty} |\langle u, e_n \rangle|^2 = \|u\|^2_{L^2(\mathbb{R}_+)}. \tag{4.2}$$

We have

$$\mathbb{E}\left[\mathrm{e}^{iJ_1(u)} \right] = \prod_{n=0}^{\infty} \mathbb{E}\left[\mathrm{e}^{i\xi_n \langle u, e_n \rangle} \right] = \prod_{n=0}^{\infty} \mathrm{e}^{-\frac{1}{2}\langle u, e_n \rangle^2_{L^2(\mathbb{R}_+)}} = \exp\left(-\frac{1}{2} \|u\|^2_{L^2(\mathbb{R}_+)} \right),$$

hence $J_1(u)$ is a centered Gaussian random variable with variance $\|u\|^2_{L^2(\mathbb{R}_+)}$. Next is a constructive approach to the definition of Brownian motion, using the decomposition

$$\mathbf{1}_{[0,t]} = \sum_{n=0}^{\infty} e_n \int_0^t e_n(s) ds.$$

Definition 4.1.5. *For all $t \in \mathbb{R}_+$, let*

$$B_t(\omega) = J_1(\mathbf{1}_{[0,t]}) = \sum_{n=0}^{\infty} \xi_n(\omega) \int_0^t e_n(s) ds. \tag{4.3}$$

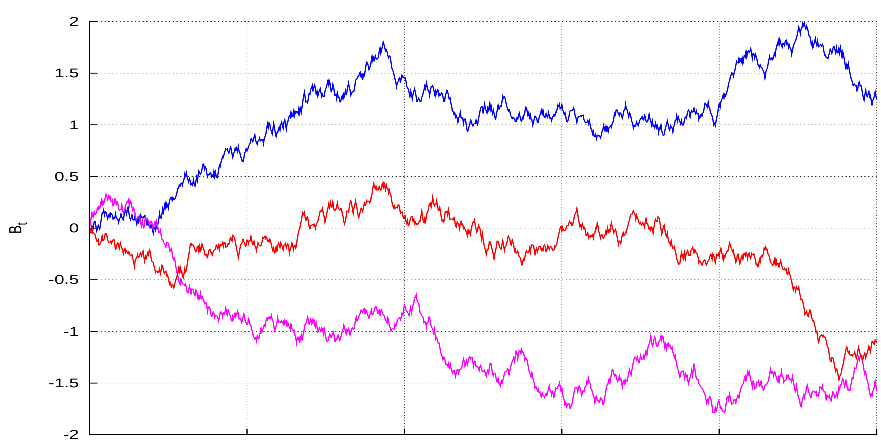

Figure 4.1: Sample paths of one-dimensional Brownian motion.

Clearly, $B_t - B_s = J_1(\mathbf{1}_{[s,t]})$ is a Gaussian centered random variable with variance:

$$\mathbb{E}[(B_t - B_s)^2] = \mathbb{E}[|J_1(\mathbf{1}_{[s,t]})|^2] = \|\mathbf{1}_{[s,t]}\|^2_{L^2(\mathbb{R}_+)} = t - s. \tag{4.4}$$

Moreover, the isometry formula (4.2) shows that if $u_1, \ldots, u_n$ are orthogonal in $L^2(\mathbb{R}_+)$ then $J_1(u_1), \ldots, J_1(u_n)$ are also mutually orthogonal in $L^2(\Omega)$, hence from Corollary 16.1 of Jacod and Protter [JP00], we get the following.

Proposition 4.1.6. *Let $u_1, \ldots, u_n$ be an orthogonal family in $L^2(\mathbb{R}_+)$, i.e.*

$$\langle u_i, u_j \rangle_{L^2(\mathbb{R}_+)} = 0, \qquad 1 \le i \ne j \le n.$$

Then $(J_1(u_1), \ldots, J_1(u_n))$ is a vector of independent Gaussian centered random variables with respective variances $\|u_1\|^2_{L^2(\mathbb{R}_+)}, \ldots, \|u_1\|^2_{L^2(\mathbb{R}_+)}$.

As a consequence of Proposition 4.1.6, $(B_t)_{t\in\mathbb{R}_+}$ has centered independent increments hence it is a martingale.

Moreover, from (4.4) and Remark 4.1.2 we deduce the following proposition.

Proposition 4.1.7. *The Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ is a* normal *martingale.*

The n-dimensional Brownian motion will be constructed as $(B_t^1, \ldots, B_t^n)_{t\in\mathbb{R}_+}$ where $(B_t^1)_{t\in\mathbb{R}_+}$, $\ldots$,$(B_t^n)_{t\in\mathbb{R}_+}$ are independent copies of $(B_t)_{t\in\mathbb{R}_+}$.

The compensated Poisson process will provide a second example of normal martingale. Let now $(\tau_n)_{n\ge 1}$ denote a sequence of independent and identically exponentially distributed random variables, with parameter $\lambda > 0$, i.e.

$$\mathbb{E}[f(\tau_1, \ldots, \tau_n)] = \lambda^n \int_0^\infty \cdots \int_0^\infty \mathrm{e}^{-\lambda(s_1 + \cdots + s_n)} f(s_1, \ldots, s_n) ds_1 \cdots ds_n,$$

for all sufficiently integrable measurable $f : \mathbb{R}_+^n \to \mathbb{R}$. Let now

$$T_n = \tau_1 + \cdots + \tau_n, \qquad n \ge 1.$$

We now consider the canonical point process associated to $(T_k)_{k\ge 1}$.

Definition 4.1.8. *The point process $(N_t)_{t\in\mathbb{R}_+}$ defined by*

$$N_t = \sum_{k=1}^\infty \mathbf{1}_{[T_k,\infty)}(t), \qquad t \in \mathbb{R}_+ \tag{4.5}$$

is called the standard Poisson point process.

The process $(N_t)_{t\in\mathbb{R}_+}$ has independent increments which are distributed according to the Poisson law, i.e. for all $0 \le t_0 \le t_1 < \cdots < t_n$,

$$(N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}})$$

is a vector of independent Poisson random variables with respective parameters

$$(\lambda(t_1 - t_0), \ldots, \lambda(t_n - t_{n-1})).$$

Compound Poisson processes provide other examples of normal martingales. Given $(Y_k)_{k\geq 1}$ a sequence of independent identically distributed random variables, define the compound Poisson process as

$$X_t = \sum_{k=1}^{N_t} Y_k, \qquad t \in \mathbb{R}_+.$$

The compensated compound Poisson martingale defined as

$$M_t := \frac{X_t - \lambda t \, \mathbb{E}[Y_1]}{\sqrt{\lambda \mathrm{Var}[Y_1]}}, \qquad t \in \mathbb{R}_+,$$

is a normal martingale.

4.2 Stochastic integration

In this section we construct the Itô stochastic integral of square-integrable adapted processes with respect to normal martingales. The filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is generated by $(M_t)_{t\in\mathbb{R}_+}$:

$$\mathcal{F}_t = \sigma(M_s \ : \ 0 \leq s \leq t), \qquad t \in \mathbb{R}_+.$$

A process $(X_t)_{t\in\mathbb{R}_+}$ is said to be $\mathcal{F}_t$-adapted if X_t is $\mathcal{F}_t$-measurable for all $t \in \mathbb{R}_+$.

Definition 4.2.1. *Let $L^p_{ad}(\Omega \times \mathbb{R}_+)$, $p \in [1, \infty]$, denote the space of $\mathcal{F}_t$-adapted processes in $L^p(\Omega \times \mathbb{R}_+)$.*

Stochastic integrals will be first constructed as integrals of simple predictable processes.

Definition 4.2.2. *Let $\mathcal{S}$ be a space of random variables dense in $L^2(\Omega, \mathcal{F}, P)$. Consider the following spaces of simple processes: let $\mathcal{P}$ denote the space of simple predictable processes $(u_t)_{t\in\mathbb{R}_+}$ of the form*

$$u_t = \sum_{i=1}^{n} F_i \mathbf{1}_{(t^n_{i-1}, t^n_i]}(t), \qquad t \in \mathbb{R}_+, \tag{4.6}$$

where F_i is $\mathcal{F}_{t^n_{i-1}}$-measurable, $i = 1, \ldots, n$.

One easily checks that the set $\mathcal{P}$ of simple predictable processes forms a linear space. Part (ii) of the next proposition also follows from Lemma 1.1 of Ikeda and Watanabe [IW89], p. 22 and p. 46. For any $p \geq 1$, the space $\mathcal{P}$ of simple predictable processes are dense in $L^p(\Omega \times \mathbb{R}_+)$ and $L^p_{ad}(\Omega \times \mathbb{R}_+)$ respectively.

Proposition 4.2.3. *The stochastic integral with respect to the normal martingale $(M_t)_{t\in\mathbb{R}_+}$, defined on simple predictable processes $(u_t)_{t\in\mathbb{R}_+}$ of the form (4.6) by*

$$\int_0^\infty u_t dM_t := \sum_{i=1}^{n} F_i (M_{t_i} - M_{t_{i-1}}), \tag{4.7}$$

extends to $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ via the isometry formula

$$\mathbb{E}\left[\int_0^\infty u_t dM_t \int_0^\infty v_t dM_t\right] = \mathbb{E}\left[\int_0^\infty u_t v_t dt\right]. \tag{4.8}$$

Proof. We start by showing that the isometry (4.8) holds for the simple predictable process $u = \sum_{i=1}^n G_i \mathbf{1}_{(t_{i-1},t_i]}$, with $0 = t_0 < t_1 < \cdots t_n$:

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^\infty u_t dM_t\right)^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^n G_i (M_{t_i} - M_{t_{i-1}})\right)^2\right] \\
&= \mathbb{E}\left[\sum_{i=1}^n |G_i|^2 (M_{t_i} - M_{t_{i-1}})^2\right] \\
&\quad +2\,\mathbb{E}\left[\sum_{1\le i<j\le n} G_i G_j (M_{t_i} - M_{t_{i-1}})(M_{t_j} - M_{t_{j-1}})\right] \\
&= \sum_{i=1}^n \mathbb{E}[\mathbb{E}[|G_i|^2 (M_{t_i} - M_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}]] \\
&\quad +2 \sum_{1\le i<j\le n} \mathbb{E}[\mathbb{E}[G_i G_j (M_{t_i} - M_{t_{i-1}})(M_{t_j} - M_{t_{j-1}}) | \mathcal{F}_{t_{j-1}}]] \\
&= \sum_{i=1}^n \mathbb{E}[|G_i|^2\, \mathbb{E}[(M_{t_i} - M_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}]] \\
&\quad +2 \sum_{1\le i<j\le n} \mathbb{E}[G_i G_j (M_{t_i} - M_{t_{i-1}})\, \mathbb{E}[(M_{t_j} - M_{t_{j-1}}) | \mathcal{F}_{t_{j-1}}]] \\
&= \mathbb{E}\left[\sum_{i=1}^n |G_i|^2 (t_i - t_{i-1})\right] = \mathbb{E}[\|u\|^2_{L^2(\mathbb{R}_+)}].
\end{aligned}$$

The stochastic integral operator extends to $L^2_{ad}(\Omega \times \mathbb{R}_+)$ by density and a Cauchy sequence argument, applying the isometry (4.8) with $s = 0$. □

Proposition 4.2.4. *For any $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ we have*

$$\mathbb{E}\left[\int_0^\infty u_s dM_s \Big| \mathcal{F}_t\right] = \int_0^t u_s dM_s, \qquad t \in \mathbb{R}_+.$$

In particular, $\int_0^t u_s dM_s$ is $\mathcal{F}_t$-measurable, $t \in \mathbb{R}_+$.

Proof. Let $u \in \mathcal{P}$ have the form $u = G\mathbf{1}_{(a,b]}$, where G is bounded and $\mathcal{F}_a$-measurable.

i) If $0 \le a \le t$ we have

$$\mathbb{E}\left[\int_0^\infty u_s dM_s \Big| \mathcal{F}_t\right] = \mathbb{E}\left[G(M_b - M_a) | \mathcal{F}_t\right]$$

$$
\begin{aligned}
&= G\,\mathbb{E}\left[(M_b - M_a)|\mathcal{F}_t\right] \\
&= G\,\mathbb{E}\left[(M_b - M_t)|\mathcal{F}_t\right] + G\,\mathbb{E}\left[(M_t - M_a)|\mathcal{F}_t\right] \\
&= G(M_t - M_a) \\
&= \int_0^\infty \mathbf{1}_{[0,t]}(s) u_s dM_s.
\end{aligned}
$$

ii) If $0 \leq t \leq a$ we have for all bounded $\mathcal{F}_t$-measurable random variable F:

$$
\mathbb{E}\left[F \int_0^\infty u_s dM_s\right] = \mathbb{E}\left[FG(M_b - M_a)\right] = 0,
$$

hence

$$
\mathbb{E}\left[\int_0^\infty u_s dM_s \Big| \mathcal{F}_t\right] = \mathbb{E}\left[G(M_b - M_a)|\mathcal{F}_t\right] = 0 = \int_0^\infty \mathbf{1}_{[0,t]}(s) u_s dM_s.
$$

This statement is extended by linearity and density, since from the continuity of the conditional expectation on L^2 we have:

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t u_s dM_s - \mathbb{E}\left[\int_0^\infty u_s dM_s \Big| \mathcal{F}_t\right]\right)^2\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^t u_s^n dM_s - \mathbb{E}\left[\int_0^\infty u_s dM_s \Big| \mathcal{F}_t\right]\right)^2\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\left(\mathbb{E}\left[\int_0^\infty u_s^n dM_s - \int_0^\infty u_s dM_s \Big| \mathcal{F}_t\right]\right)^2\right] \\
&\leq \lim_{n\to\infty} \mathbb{E}\left[\mathbb{E}\left[\left(\int_0^\infty u_s^n dM_s - \int_0^\infty u_s dM_s\right)^2 \Big| \mathcal{F}_t\right]\right] \\
&\leq \lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^\infty (u_s^n - u_s) dM_s\right)^2\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\int_0^\infty |u_s^n - u_s|^2 ds\right] \\
&= 0.
\end{aligned}
$$

$\square$

In particular, since $\mathcal{F}_0 = \{\emptyset, \Omega\}$, the Itô integral is a centered random variable:

$$
\mathbb{E}\left[\int_0^\infty u_s dM_s\right] = 0. \tag{4.9}
$$

The following is an immediate corollary of Proposition 4.2.4.

Corollary 4.2.5. *The indefinite stochastic integral* $\left(\int_0^t u_s dM_s\right)_{t\in\mathbb{R}_+}$ *of* $u \in L^2_{ad}(\Omega\times\mathbb{R}_+)$ *is a martingale, i.e.:*

$$\mathbb{E}\left[\int_0^t u_\tau dM_\tau \Big| \mathcal{F}_s\right] = \int_0^s u_\tau dM_\tau, \quad 0 \le s \le t.$$

Recall that since the Poisson martingale $(M_t)_{t\in\mathbb{R}_+} = (N_t - t)_{t\in\mathbb{R}_+}$ is a normal martingale,

$$\int_0^T u_t dM_t$$

is defined in Itô sense as an $L^2(\Omega)$-limit of stochastic integrals of simple adapted processes.

4.3 Quadratic variation

We now introduce the notion of quadratic variation for normal martingales.

Definition 4.3.1. *The quadratic variation of* $(M_t)_{t\in\mathbb{R}_+}$ *is the process* $([M,M]_t)_{t\in\mathbb{R}_+}$ *defined as*

$$[M,M]_t = M_t^2 - 2\int_0^t M_s dM_s, \quad t \in \mathbb{R}_+. \tag{4.10}$$

Let now

$$\pi^n = \{0 = t_0^n < t_1^n < \cdots < t_{n-1}^n < t_n^n = t\}$$

denote a family of subdivision of $[0,t]$, such that $|\pi^n| := \max_{i=1,\dots,n} |t_i^n - t_{i-1}^n|$ converges to 0 as n goes to infinity.

Proposition 4.3.2. *We have*

$$[M,M]_t \quad = \quad \lim_{n\to\infty} \sum_{i=1}^n (M_{t_i^n} - M_{t_{i-1}^n})^2, \quad t \ge 0,$$

where the limit exists in $L^2(\Omega)$ *and is independent of the sequence* $(\pi^n)_{n\in\mathbb{N}}$ *of subdivisions chosen.*

Proof. As an immediate consequence of the definition 4.7 of the stochastic integral we have

$$M_s(M_t - M_s) = \int_s^t M_s dM_\tau, \qquad 0 \le s \le t,$$

hence

$$\begin{aligned} [M,M]_{t_i^n} - [M,M]_{t_{i-1}^n} &= M_{t_i^n}^2 - M_{t_{i-1}^n}^2 - 2\int_{t_{i-1}^n}^{t_i^n} M_s dM_s \\ &= (M_{t_i^n} - M_{t_{i-1}^n})^2 + 2\int_{t_{i-1}^n}^{t_i^n} (M_{t_{i-1}^n} - M_s) dM_s, \end{aligned}$$

hence

$$\begin{aligned}
&\mathbb{E}\left[\left([M,M]_t - \sum_{i=1}^n (M_{t_i^n} - M_{t_{i-1}^n})^2\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^n [M,M]_{t_i^n} - [M,M]_{t_{i-1}^n} - (M_{t_i^n} - M_{t_{i-1}^n})^2\right)^2\right] \\
&= 4\,\mathbb{E}\left[\left(\sum_{i=1}^n \int_0^t \mathbf{1}_{(t_{i-1}^n, t_i^n]}(s)(M_s - M_{t_{i-1}^n})dM_s\right)^2\right] \\
&= 4\,\mathbb{E}\left[\sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (M_s - M_{t_{i-1}^n})^2 ds\right] \\
&= 4\,\mathbb{E}\left[\sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (s - t_{i-1}^n)^2 ds\right] \\
&\leq 4t|\pi|.
\end{aligned}$$

□

Clearly, from the definition (4.3), $J_1(u)$ coincides with the single stochastic integral $I_1(u)$ with respect to $(B_t)_{t\in\mathbb{R}_+}$.

Proposition 4.3.3. *The quadratic variation of Brownian motion* $(B_t)_{t\in\mathbb{R}_+}$ *is*

$$[B,B]_t = t, \quad t \in \mathbb{R}_+.$$

Proof. (cf. e.g. Protter [Pro90], Theorem I-28). For every subdivision $\{0 = t_0^n < \cdots < t_n^n = t = t\}$ we have, by independence of the increments of Brownian motion:

$$\begin{aligned}
\mathbb{E}\left[\left(t - \sum_{i=1}^n (B_{t_i^n} - B_{t_{i-1}^n})^2\right)^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^n (B_{t_i^n} - B_{t_{i-1}^n})^2 - (t_i^n - t_{i-1}^n)\right)^2\right] \\
&= \sum_{i=1}^n (t_i^n - t_{i-1}^n)^2\,\mathbb{E}\left[\left(\frac{(B_{t_i^n} - B_{t_{i-1}^n})^2}{t_i^n - t_{i-1}^n} - 1\right)^2\right] \\
&= \mathbb{E}[(Z^2-1)^2]\sum_{i=0}^n (t_i^n - t_{i-1}^n)^2 \\
&\leq t|\pi|\,\mathbb{E}[(Z^2-1)^2],
\end{aligned}$$

where Z is a standard Gaussian random variable. □

A simple analysis of the Poisson paths shows that the quadratic variation of the compensated Poisson process $(M_t)_{t\in\mathbb{R}_+} = (N_t - t)_{t\in\mathbb{R}_+}$ is

$$[M,M]_t = N_t, \qquad t \in \mathbb{R}_+.$$

Similarly for the compensated compound Poisson martingale

$$M_t := \frac{X_t - \lambda t\, \mathbb{E}[Y_1]}{\sqrt{\lambda \mathrm{Var}[Y_1]}}, \qquad t \in \mathbb{R}_+,$$

we have

$$[M, M]_t = \sum_{k=1}^{N_t} |Y_k|^2, \qquad t \in \mathbb{R}_+.$$

Definition 4.3.4. *An equation of the form*

$$[M, M]_t = t + \int_0^t \phi_s dM_s, \quad t \in \mathbb{R}_+, \tag{4.11}$$

where $(\phi_t)_{t\in\mathbb{R}_+}$ is a square-integrable adapted process, is called a structure equation, cf. Emery [É90].

Definition 4.3.5. *The angle bracket $\langle M, M\rangle_t$ is the unique increasing process such that*

$$M_t^2 - \langle M, M\rangle_t, \qquad t \in \mathbb{R}_+,$$

is a martingale.

As a consequence of Remark 4.1.3 we have

$$\langle M, M\rangle_t = t, \qquad t \in \mathbb{R}_+,$$

for every normal martingale. Moreover,

$$[M, M]_t - \langle M, M\rangle_t, \qquad t \in \mathbb{R}_+,$$

is also a martingale as a consequence of Remark 4.1.3 and Proposition 4.2.4, since by Definition 4.3.1 we have

$$[M, M]_t - \langle M, M\rangle_t = [M, M]_t - t = M_t^2 - t - 2\int_0^t M_s dM_s, \qquad t \in \mathbb{R}_+. \tag{4.12}$$

We say that the martingale $(M_t)_{t\in\mathbb{R}_+}$ has the predictable representation property if any square-integrable martingale $(X_t)_{t\in\mathbb{R}_+}$ with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ can be represented as

$$X_t = X_0 + \int_0^t u_s dM_s, \quad t \in \mathbb{R}_+, \tag{4.13}$$

where $(u_t)_{t\in\mathbb{R}_+} \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ is an adapted process such that $u\mathbf{1}_{[0,T]} \in L^2(\Omega \times \mathbb{R}_+)$ for all $T > 0$. It is known that Brownian motion and the compensated Poisson process have the predictable representation property. This is however not true of compound Poisson processes in general.

As a consequence of (4.12) and (4.13) we have the following proposition.

Proposition 4.3.6. *Assume that $(M_t)_{t\in\mathbb{R}_+}$ is in $L^4(\Omega)$ and has the predictable representation property. Then $(M_t)_{t\in\mathbb{R}_+}$ satisfies the structure equation (4.11), i.e. there exists a square-integrable adapted process $(\phi_t)_{t\in\mathbb{R}_+}$ such that*

$$[M,M]_t = t + \int_0^t \phi_s dM_s, \qquad t \in \mathbb{R}_+.$$

Proof. Since $([M,M]_t - t)_{t\in\mathbb{R}_+}$ is a martingale, the predictable representation property shows the existence of a square-integrable adapted process $(\phi_t)_{t\in\mathbb{R}_+}$ such that

$$[M,M]_t - t = \int_0^t \phi_s dM_s, \qquad t \in \mathbb{R}_+.$$

□

In particular,

a) the Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ satisfies the structure equation (4.11) with $\phi_t = 0$, since the quadratic variation of $(B_t)_{t\in\mathbb{R}_+}$ is $[B,B]_t = t$, $t \in \mathbb{R}_+$. Informally we have $\Delta B_t = \pm\sqrt{\Delta t}$ with equal probabilities 1/2.

b) The compensated Poisson martingale $(M_t)_{t\in\mathbb{R}_+} = \lambda(N_t - t/\lambda^2)_{t\in\mathbb{R}_+}$, where $(N_t)_{t\in\mathbb{R}_+}$ is a standard Poisson process with intensity $1/\lambda^2$ satisfies the structure equation (4.11) with $\phi_t = \lambda \in \mathbb{R}$, $t \in \mathbb{R}_+$, since

$$[M,M]_t = \lambda^2 N_t = t + \lambda M_t, \qquad t \in \mathbb{R}_+.$$

In this case, $\Delta M_t \in \{0, \lambda\}$ with respective probabilities $1 - \lambda^{-2}\Delta t$ and $\lambda^{-2}\Delta t$.

The Azéma martingales correspond to $\phi_t = \beta M_t$, $\beta \in [-2, 0)$, and provide other examples of processes having the chaos representation property, but whose increments are not independent, cf. Emery [É90]. Note that not all normal martingales satisfy a structure equation and have the predictable representation property. For instance the compensated compound Poisson martingale does not satisfy a structure equation and does not have the predictable representation property.

4.4 Itô's formula

We consider a normal martingale $(M_t)_{t\in\mathbb{R}_+}$ which satisfies the structure equation

$$d[M,M]_t = dt + \phi_t dM_t.$$

Such an equation is satisfied in particular if $(M_t)_{t\in\mathbb{R}_+}$ has the predictable representation property, cf. Proposition 4.3.6.

The following is a statement of Itô's formula for normal martingales, cf. Emery [É90], Proposition 2, p. 70.

Proposition 4.4.1. *Assume that* $\phi \in L^\infty_{ad}(\mathbb{R}_+ \times \Omega)$. *Let* $(X_t)_{t\in\mathbb{R}_+}$ *be a process given by*

$$X_t = X_0 + \int_0^t u_s dM_s + \int_0^t v_s ds, \tag{4.14}$$

where $(u_s)_{s\in\mathbb{R}_+}, (v_s)_{s\in\mathbb{R}_+}$ *are adapted processes in* $L^2_{ad}([0,t] \times \Omega)$ *for all* $t > 0$. *We have for* $f \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$:

$$\begin{aligned} f(t, X_t) - f(0, X_0) &= \int_0^t \frac{f(s, X_{s^-} + \phi_s u_s) - f(s, X_{s^-})}{\phi_s} dM_s \qquad (4.15) \\ &+ \int_0^t \frac{f(s, X_s + \phi_s u_s) - f(s, X_s) - \phi_s u_s \frac{\partial f}{\partial x}(s, X_s)}{\phi_s^2} ds \\ &+ \int_0^t \frac{\partial f}{\partial x}(s, X_s) v_s ds + \int_0^t \frac{\partial f}{\partial s}(s, X_s) ds. \end{aligned}$$

If $\phi_s = 0$, *the terms*

$$\frac{f(X_{s^-} + \phi_s u_s) - f(X_{s^-})}{\phi_s}$$

and

$$\frac{f(X_s + \phi_s u_s) - f(X_s) - \phi_s u_s f'(X_s)}{\phi_s^2}$$

have to be replaced by their respective limits $u_s f'(X_{s^-})$ *and* $\frac{1}{2} u_s^2 f''(X_{s^-})$ *as* $\phi_s \to 0$.

Examples

i) For the d-dimensional Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, $\phi = 0$ and the Itô formula reads

$$f(B_t) = f(B_0) + \int_0^t \langle \nabla f(B_s), dB_s \rangle_H + \frac{1}{2} \int_0^t \Delta f(B_s) ds,$$

for all $\mathcal{C}^2$ functions f, hence

$$\begin{aligned} T_t f(x) &= \mathbb{E}_x[f(B_t)] \\ &= \mathbb{E}_x\left[f(x) + \int_0^t \langle \nabla f(B_s), dB_s \rangle_H + \frac{1}{2} \int_0^t \Delta f(B_s) ds \right] \\ &= \mathbb{E}_x\left[f(x) + \frac{1}{2} \int_0^t \Delta f(B_s) ds \right] \\ &= f(x) + \frac{1}{2} \int_0^t \mathbb{E}_x[\Delta f(B_s)] ds \\ &= f(x) + \frac{1}{2} \int_0^t T_s \Delta f(x) ds. \end{aligned}$$

hence $(B_t)_{t\in\mathbb{R}_+}$ has generator $\dfrac{1}{2}\Delta$.

ii) For the compensated Poisson process $(N_t - t)_{t\in\mathbb{R}_+}$ we have $\phi_s = 1$, $s \in \mathbb{R}_+$, hence

$$\begin{aligned} f(N_t - t) &= f(0) + \int_0^t (f(1 + N_{s^-} - s) - f(N_{s^-} - s)) d(N_s - s) \\ &\quad + \int_0^t (f(1 + N_s - s) - f(N_s - s) - f'(N_s - s)) ds. \end{aligned}$$

This formula can actually be recovered by elementary calculus. Hence the generator of the compensated Poisson process is

$$\mathcal{L} f(x) = f(x + 1) - f(x) - f'(x).$$

We will use the following multidimensional version of the change of variable formula.

Proposition 4.4.2. *Let $(X_t)_{t\in\mathbb{R}_+}$ be a $\mathbb{R}^n$-valued process satisfying*

$$dX_t = Y_t dt + Z_t dM_t, \quad X_0 > 0,$$

where $(Y_t)_{t\in\mathbb{R}_+}$ and $(Z_t)_{t\in\mathbb{R}_+}$ are predictable square-integrable $\mathbb{R}^n$-valued processes. For any function $\mathbb{R}_+ \times \mathbb{R}^n \ni (t, x) \to f_t(x)$ in $\mathcal{C}_b^2(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R})$ we have

$$f_t(X_t) = f_0(X_0) + \int_0^t L_s f_s(X_s) dM_s + \int_0^t U_s f_s(X_s) ds + \int_0^t \frac{\partial f_s}{\partial s}(X_s) ds, \tag{4.16}$$

where

$$L_s f_s(X_s) = i_s \langle Z_s, \nabla f_s(X_s)\rangle + \frac{j_s}{\phi_s}(f_s(X_{s^-} + \phi_s Z_{s^-}) - f_s(X_{s^-})),$$

and

$$\begin{aligned} U_s f_s(X_s) &= \langle Y_s, \nabla f_s(X_s)\rangle \\ &\quad + \alpha_s^2 \left(\frac{1}{2} i_s \langle \nabla\nabla f_s(X_s), Z_s \otimes Z_s\rangle \right. \\ &\quad \left. + \frac{j_s}{\phi_s^2}(f_s(X_{s^-} + \phi_s Z_{s^-}) - f_s(X_{s^-}) - \phi_s \langle Z_s, \nabla f_s(X_s)\rangle) \right), \end{aligned}$$

with the convention $0/0 = 0$.

From Itô's formula we have for any stopping time τ and $\mathcal{C}^2$ function u, under suitable integrability conditions:

$$\begin{aligned} \mathbb{E}_x[u(B_\tau)] &= u(x) + \mathbb{E}_x\left[\int_0^\tau u(B_s) dB_s\right] + \frac{1}{2}\,\mathbb{E}_x\left[\int_0^\tau \Delta u(B_s) ds\right] \\ &= u(x) + \mathbb{E}_x\left[\int_0^\infty \mathbf{1}_{\{s\le\tau\}} u(B_s) dB_s\right] + \frac{1}{2}\,\mathbb{E}_x\left[\int_0^\tau \Delta u(B_s) ds\right] \\ &= u(x) + \frac{1}{2}\,\mathbb{E}_x\left[\int_0^\tau \Delta u(B_s) ds\right], \end{aligned} \tag{4.17}$$

which is called Dynkin's formula, cf. Dynkin [Dyn65], Theorem 5.1. Let now

$$\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^d \otimes \mathbb{R}^n$$

and

$$b : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$$

satisfy the global Lipschitz condition

$$\|\sigma(t,x) - \sigma(t,y)\|^2 + \|b(t,x) - b(t,y)\|^2 \leq K^2 \|x-y\|^2, \qquad t \in \mathbb{R}_+, \ x, y \in \mathbb{R}^n.$$

Then there exists a unique strong solution to the stochastic differential equation

$$X_t = X_0 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds,$$

and $(X_t)_{t \in \mathbb{R}_+}$ is a Markov process with generator

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^n a_{i,j}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(t,x) \frac{\partial}{\partial x_i},$$

where $a = \sigma^T \sigma$.

4.5 Killed Brownian motion

The transition operator for the Brownian motion $(B_t^D)_{t \in [0, \tau_{\partial D}]}$ killed on ∂D is defined as

$$q_t^D(x, A) = P_x(B_t^D \in A, \ \tau_{\partial D} > t),$$

where

$$\tau_{\partial D} = \inf\{t > 0 \ : \ B_t \in \partial D\}$$

is the first hitting time of ∂D by $(B_t)_{t \in \mathbb{R}_+}$. By the strong Markov property we have

$$\begin{aligned}
P_x(B_t \in A) &= P_x(B_t \in A, t < \tau_{\partial D}) + P_x(B_t \in A, t \geq \tau_{\partial D}) \\
&= q_t^D(x, A) + P_x(B_t \in A, t \geq \tau_{\partial D}) \\
&= q_t^D(x, A) + \mathbb{E}_x[\mathbf{1}_{\{B_t \in A\}} \mathbf{1}_{\{t \geq \tau_{\partial D}\}}] \\
&= q_t^D(x, A) + \mathbb{E}_x[\mathbb{E}[\mathbf{1}_{\{B_t \in A\}} \mathbf{1}_{\{t \geq \tau_{\partial D}\}} \mid \mathcal{F}_{\tau_{\partial D}}]] \\
&= q_t^D(x, A) + \mathbb{E}_x[\mathbf{1}_{\{t \geq \tau_{\partial D}\}} \, \mathbb{E}[\mathbf{1}_{\{B_t \in A\}} \mathbf{1}_{\{t \geq \tau_{\partial D}\}} \mid \mathcal{F}_{\tau_{\partial D}}]] \\
&= q_t^D(x, A) + \mathbb{E}_x[\mathbf{1}_{\{t \geq \tau_{\partial D}\}} \, \mathbb{E}[\mathbf{1}_{\{B_{t - \tau_{\partial D}} \in A\}} \mid B_0 = x]_{x = B_{\tau_{\partial D}}}] \\
&= q_t^D(x, A) + \mathbb{E}_x[T_{t - \tau_{\partial D}} \mathbf{1}_A(B_{\tau_{\partial D}}^D) \mathbf{1}_{\{t \geq \tau_{\partial D}\}}],
\end{aligned}$$

hence

$$q_t^D(x, A) = P_x(B_t \in A) - \mathbb{E}_x[T_{t - \tau_{\partial D}} \mathbf{1}_A(B_{\tau_{\partial D}}^D) \mathbf{1}_{\{t \geq \tau_{\partial D}\}}],$$

and the killed process has transition the densities

$$p_t^D(x, y) := p_t(x, y) - \mathbb{E}_x[p_{t - \tau_{\partial D}}(B_{\tau_{\partial D}}^D, y) \mathbf{1}_{\{t \geq \tau_{\partial D}\}}], \qquad x, y \in D, \ t > 0. \tag{4.18}$$

The Green function is defined as

$$g_D(x,y)=\int_0^\infty p_t^D(x,y)dt,$$

and the associated Green potential is

$$G_D\mu(x)=\int_0^\infty g_D(x,y)\mu(dy),$$

with

$$G_Df(x)=\mathbb{E}\left[\int_0^{\tau_{\partial D}} f(B_t)dt\right].$$

When $D=\mathbb{R}^n$ we have $\tau_{\partial D}=\infty$ a.s. hence $G_{\mathbb{R}^n}=G$.

Theorem 4.5.1. *The function g_D is symmetric and continuous on D^2, and $x\mapsto g_D(x,y)$ is harmonic on $D\setminus\{y\}$, $y\in D$.*

Proof. The function $G_D\mathbf{1}_A$ defined as

$$G_D\mathbf{1}_A(x)=\int_{\mathbb{R}^n} g_D(x,y)\mathbf{1}_A(y)dy=\mathbb{E}_x\left[\int_0^{\tau_{\partial D}}\mathbf{1}_A(B_t)dt\right]$$

has the mean value property in $D\setminus\bar{A}$ for all bounded domains A in $\mathbb{R}^n$, and the property extends to g_D. $\square$

From (4.18), the associated λ-potential kernel is given by

$$g^\lambda(x,y)=\int_0^\infty e^{-\lambda t}p_t(x,y)dt=g_D^\lambda(x,y)+\int g^\lambda(z,y)h_D^\lambda(x,dz), \tag{4.19}$$

where

$$g_D^\lambda(x,y):=\int_0^\infty e^{-\lambda t}q_t^D(x,y)dt,$$

and

$$h_D^\lambda(x,A)=\mathbb{E}_x[e^{-\lambda\tau_{\partial D}}\mathbf{1}_{\{B^D_{\tau_{\partial D}}\in A\}}\mathbf{1}_{\{\tau_{\partial D}<\infty\}}].$$

5 Probabilistic interpretations

5.1 Harmonicity

We start by two simple examples on the connection between harmonic functions and stochastic calculus. First, we show that Proposition 2.2.2, i.e. the fact that harmonic functions satisfy the mean-value property, can be recovered using stochastic calculus. Let

$$\tau_r=\inf\{t\in\mathbb{R}_+\ :\ B_t\in S(y,r)\}$$

denote the first exit time of $(B_t)_{t\in\mathbb{R}_+}$ from the open ball $B(y,r)$.

Due to the symmetry of Brownian motion, B_τ is uniformly distributed on $S(x,r)$ hence

$$\mathbb{E}_x[u(B_\tau)] = \int_{S(x,r)} u(y)\sigma_y^r(dy).$$

On the other hand, from Dynkin's formula (4.17) we have

$$\mathbb{E}_x[u(B_{\tau_r})] = u(x) + \frac{1}{2}\,\mathbb{E}_x\left[\int_0^{\tau_r} \Delta u(B_s)ds\right].$$

Hence the condition $\Delta u = 0$ implies the mean value property

$$u(x) = \int_{S(x,r)} u(y)\sigma_y^r(dy), \tag{5.1}$$

and similarly the condition $\Delta u \leq 0$ implies

$$u(x) \geq \int_{S(x,r)} u(y)\sigma_y^r(dy). \tag{5.2}$$

From Remark 3, page 134 of Dynkin [Dyn65], for all $n \geq 1$ we have

$$\frac{1}{2}\Delta u(x) = \lim_{n\to\infty} \frac{\mathbb{E}_x[u(B_{\tau_{1/n}})] - u(x)}{\mathbb{E}_x[\tau_{1/n}]}, \tag{5.3}$$

which shows conversely that (5.1), resp. (5.2), implies $\Delta u \leq 0$, resp $\Delta u = 0$, which recovers Proposition 2.2.2.

Next we recover the superharmonicity property of the potential

$$\begin{aligned} R_0 f(x) &= \int_0^\infty T_t f(x)dt \qquad (5.4)\\ &= \mathbb{E}_x\left[\int_0^\infty f(B_t)dt\right] \\ &= \int_0^\infty \int_{\mathbb{R}^n} f(y)p_t(x,y)dtdy \\ &= \frac{n-2}{2}\int_{\mathbb{R}^n} h_x(y)f(y)dy, \qquad x \in \mathbb{R}^n. \end{aligned}$$

Proposition 5.1.1. *Let f be a non-negative function on $\mathbb{R}^n$. Then the potential $R_0 f$ given in (5.4) is a superharmonic function provided it is $\mathcal{C}^2$ on $\mathbb{R}^n$.*

Proof. For all $r > 0$, using the strong Markov property (3.3) we have

$$\begin{aligned} g(x) &= \mathbb{E}_x\left[\int_0^{\tau_r} f(B_t)dt\right] + \mathbb{E}_x\left[\int_{\tau_r}^\infty f(B_t)dt\right] \\ &= \mathbb{E}_x\left[\int_0^{\tau_r} f(B_t)dt\right] + \mathbb{E}_x\left[E\left[\int_{\tau_r}^\infty f(B_t)dt \Big| B_{\tau_r}\right]\right] \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_x\left[\int_0^{\tau_r} f(B_t)dt\right] + \mathbb{E}_x[g(B_{\tau_r})] \\
&\geq \mathbb{E}_x[g(B_{\tau_r})] \\
&= \int_{S(x,r)} g(y)\sigma_x^r(dy),
\end{aligned}
$$

which shows that g is Δ-superharmonic from Proposition 2.2.2. □

Other non-negative superharmonic functionals can also be constructed by convolution, i.e. if f is Δ-superharmonic and g is non-negative and sufficiently integrable, then

$$
x \mapsto \int_{\mathbb{R}^n} g(y)f(x-y)dy
$$

is non-negative and Δ-superharmonic.

We now turn to an example in discrete time, with the notation of (3.4). Here a function f is called harmonic when $(I-P)f = 0$, and superharmonic if $(I-P)f \geq 0$.

Proposition 5.1.2. *A function f is superharmonic if and only if the sequence $(f(X_n))_{n\in\mathbb{N}}$ is a supermartingale.*

Proof. We have

$$
\begin{aligned}
\mathbb{E}[f(X_m) \mid \mathcal{F}_n] &= \mathbb{E}[f(X_{m-n}) \mid X_0 = x]_{x=X_n} \\
&= P_{m-n} f(X_n) \\
&\leq f(X_n).
\end{aligned}
$$

□

5.2 Dirichlet problem

In this section we revisit the Dirichlet problem using probabilistic tools.

Theorem 5.2.1. *Consider an open domain in $\mathbb{R}^n$ and f a function on $\mathbb{R}^n$, and assume that the Dirichlet problem*

$$
\begin{cases}
\Delta u = 0, & x \in D, \\
\\
u(x) = f(x), & x \in \partial D,
\end{cases}
$$

has a $\mathcal{C}^2$ solution u. Then we have

$$
u(x) = \mathbb{E}_x[f(B_{\tau_{\partial D}})], \qquad x \in \bar{D},
$$

where

$$
\tau_{\partial D} = \inf\{t > 0 \ : \ B_t \in \partial D\}
$$

is the first hitting time of ∂D by $(B_t)_{t\in\mathbb{R}_+}$.

Proof. For all $r > 0$ such that $B(x,r) \subset D$ we have

$$\begin{aligned}
u(x) &= \mathbb{E}_x[f(B_{\tau_{\partial D}})] \\
&= \mathbb{E}_x[\mathbb{E}[f(B_{\tau_{\partial D}}) \mid B_{\tau_r}]] \\
&= \mathbb{E}_x[u(B_{\tau_r})] \\
&= \int_{S(x,r)} u(y)\sigma_x^r(dy), \qquad x \in \bar{D},
\end{aligned}$$

hence u has the mean value property, thus $\Delta u = 0$ Proposition 2.2.2. □

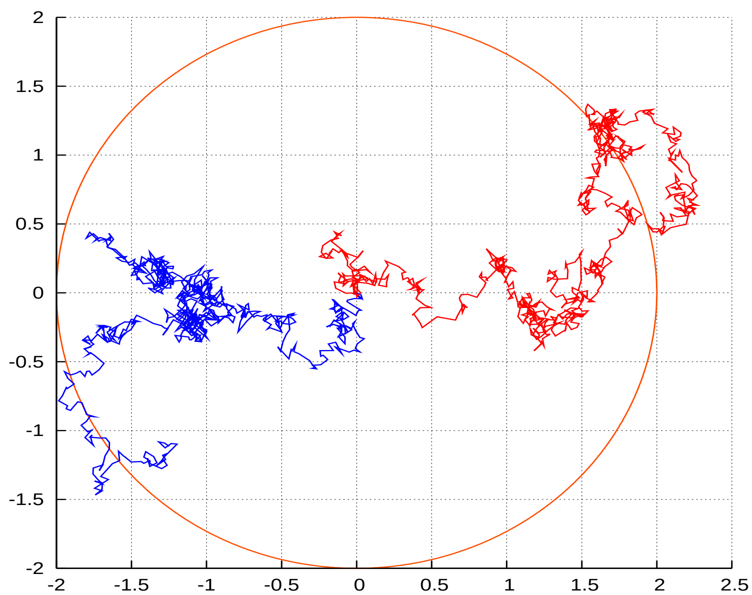

Figure 5.1: Sample paths of a two-dimensional Brownian motion.

5.3 Poisson equation

In the next theorem we show that the resolvent

$$\begin{aligned}
R_\lambda f(x) &= \mathbb{E}_x\left[\int_0^\infty e^{-\lambda t} f(B_t)dt\right] \\
&= \int_0^\infty e^{-\lambda t} T_t f(x)dt, \qquad x \in \mathbb{R}^n,
\end{aligned}$$

solves the Poisson equation. This is consistent with the fact that $R_\lambda = (\lambda I - \Delta/2)^{-1}$, cf. Relation (3.2) in Theorem 3.2.1.

Theorem 5.3.1. *Let $\lambda > 0$ and f a non-negative function on $\mathbb{R}^n$, and assume that the Poisson equation*

$$\frac{1}{2}\Delta u(x) - \lambda u(x) = -f(x), \qquad x \in \mathbb{R}^n, \tag{5.5}$$

has a $\mathcal{C}_b^2$ solution u. Then we have

$$u(x) = \mathbb{E}_x \left[\int_0^\infty e^{-\lambda t} f(B_t) dt \right], \qquad x \in \mathbb{R}^n.$$

Proof. By Itô's formula we have:

$$\begin{aligned} e^{-\lambda t} u(B_t) &= u(B_0) + \int_0^t e^{-\lambda s} \langle \nabla f(B_s), dB_s \rangle + \frac{1}{2} \int_0^t e^{-\lambda s} \Delta u(B_s) ds - \lambda \int_0^t e^{-\lambda s} u(B_s) ds \\ &= u(B_0) + \int_0^t e^{-\lambda s} \langle \nabla f(B_s), dB_s \rangle - \frac{1}{2} \int_0^t e^{-\lambda s} f(B_s) ds, \end{aligned}$$

hence

$$e^{-\lambda t} \mathbb{E}_x[u(B_t)] = u(x) - \frac{1}{2} \mathbb{E}_x \left[\int_0^t e^{-\lambda s} f(B_s) ds \right],$$

and letting t tend to infinity we get

$$0 = u(x) - \frac{1}{2} \mathbb{E}_x \left[\int_0^\infty e^{-\lambda s} f(B_s) ds \right].$$

□

When $\lambda = 0$ a similar result holds adding a boundary condition on a smooth domain D.

Theorem 5.3.2. *Let f a non-negative function on $\mathbb{R}^n$, and assume that the Poisson equation*

$$\begin{cases} \frac{1}{2} \Delta u(x) = -f(x), & x \in D, \\ u(x) = 0, & x \in \partial D, \end{cases} \tag{5.6}$$

has a $\mathcal{C}_b^2$ solution u. Then we have

$$u(x) = \mathbb{E}_x \left[\int_0^{\tau_{\partial D}} f(B_t) dt \right], \qquad x \in D.$$

Proof. Similarly to the proof of Theorem 5.3.1 we have

$$\begin{aligned} u(B_t) &= u(B_0) + \int_0^t \langle \nabla f(B_s), dB_s \rangle + \frac{1}{2} \int_0^t \Delta u(B_s) ds \\ &= u(B_0) + \int_0^t \langle \nabla f(B_s), dB_s \rangle - \int_0^t f(B_s) ds, \end{aligned}$$

hence

$$0 = \mathbb{E}_x[u(B_{\tau_{\partial D}})] = u(x) - \mathbb{E}_x \left[\int_0^{\tau_{\partial D}} f(B_s) ds \right].$$

We easily check that the boundary condition $u(x) = 0$ is satisfied on $x \in \partial D$ since $\tau_{\partial D} = 0$ a.s. given that $B_0 \in \partial D$. □

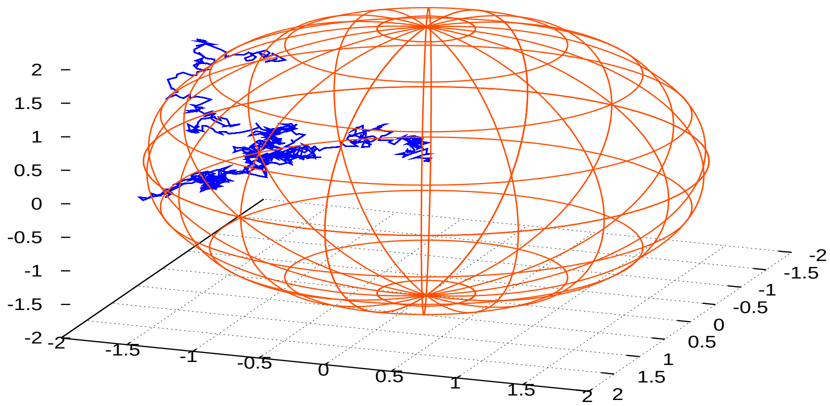

Figure 5.2: Sample paths of a three-dimensional Brownian motion.

The probabilistic interpretation of the solution of the Poisson equation can also be formulated in terms of a Brownian motion killed at the boundary ∂D.

Proposition 5.3.3. *The solution u of* (5.6) *is given by*

$$u(x) = G_D f(x) = \int_{\mathbb{R}^n} g_D(x,y) f(y) dy$$

where g_D and $G_D f$ are the Green function and the associated Green potential of Brownian motion killed on ∂D.

Proof. We need to show that the function g_D is the fundamental solution of the Poisson equation, i.e.

$$\begin{aligned}
\mathbb{E}_x \left[\int_0^{\tau_{\partial D}} \mathbf{1}_A(B_t) dt \right] &= \mathbb{E}_x \left[\int_0^{\tau_{\partial D}} \mathbf{1}_A(B_t^D) dt \right] \\
&= \mathbb{E}_x \left[\int_0^{\tau_{\partial D}} \mathbf{1}_{\{B_t^D \in A\}} dt \right] \\
&= \mathbb{E}_x \left[\int_0^{\infty} \mathbf{1}_{\{B_t^D \in A\}} dt \right] \\
&= \int_0^\infty P_x(B_t^D \in A) dt \\
&= \int_0^\infty q_t^D(x, A) dt \\
&= \int_0^\infty \int_A p_t^D(x,y) dy dt \\
&= G_D \mathbf{1}_A(x) \\
&= \int_{\mathbb{R}^n} g_D(x,y) \mathbf{1}_A(y) dy,
\end{aligned}$$

hence

$$u(x) = \mathbb{E}_x\left[\int_0^{\tau_{\partial D}} f(B_t^D)dt\right] = G_D f(x) = \int_{\mathbb{R}^n} g_D(x,y)f(y)dy.$$

$\square$

Using the λ-potential kernel g^λ we get the following expression for the solution of (5.6).

Proposition 5.3.4. *The solution u of (5.5) can be represented as*

$$\begin{aligned} u(x) &= \int_0^\infty e^{-\lambda t}\int_{\mathbb{R}^n} q_t^D(x,y)f(y)dydt \\ &\quad + \int_{\mathbb{R}^n} f(y)\int_{\mathbb{R}^n} g^\lambda(z,y)\,\mathbb{E}_x[e^{-\lambda\tau_{\partial D}}\mathbf{1}_{\{B^D_{\tau_{\partial D}}\in dz\}}\mathbf{1}_{\{\tau_{\partial D}<\infty\}}]dy. \end{aligned}$$

Proof. From (4.19) we have

$$\begin{aligned} u(x) &= R_\lambda f(x) \\ &= \int_{\mathbb{R}^n} g^\lambda(x,y)f(y)dy \\ &= \int_{\mathbb{R}^n} g_D^\lambda(x,y)f(y)dy + \int_{\mathbb{R}^n} f(y)\int_{\mathbb{R}^n} g^\lambda(z,y)h_D^\lambda(x,dz)dy \\ &= \int_0^\infty e^{-\lambda t}\int_{\mathbb{R}^n} q_t^D(x,y)f(y)dydt \\ &\quad + \int_{\mathbb{R}^n} f(y)\int_{\mathbb{R}^n} g^\lambda(z,y)\,\mathbb{E}_x[e^{-\lambda\tau_{\partial D}}\mathbf{1}_{\{B^D_{\tau_{\partial D}}\in dz\}}\mathbf{1}_{\{\tau_{\partial D}<\infty\}}]dy. \end{aligned}$$

$\square$

In discrete time, the potential kernel of P is defined as

$$G = \sum_{n=0}^\infty P^n,$$

and satisfies

$$Gf(x) = \mathbb{E}_x\left[\sum_{n=0}^\infty f(X_n)\right],$$

i.e.

$$G\mathbf{1}_A(x) = \sum_{n=0}^\infty P_x(\{X_n\in A\}).$$

The Poisson equation with second member f is here the equation

$$(I-P)u = f.$$

Let

$$\tau_D = \inf\{n\geq 1 \ : \ X_n\in D\}$$

denote the hitting time of $D \subset E$. Then the function

$$u(x) := \mathbb{E}_x \left[\sum_{k=1}^{\tau_D} f(X_k) \right], \qquad x \in E,$$

solves the Poisson equation

$$\begin{cases} (I - P)u(x) = f(x), & x \in E, \\ \\ u(x) = 0, & x \in D. \end{cases}$$

5.4 Cauchy problem

This section presents a version of the Feynman-Kac formula. Consider the PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) - V(x)u(t,x) \\ \\ u(0,x) = f(x). \end{cases} \tag{5.7}$$

Proposition 5.4.1. *Assume that $f, V \in \mathcal{C}_b(\mathbb{R}^n)$ and V is non-negative. Then the solution of (5.7) is given by*

$$u(t,x) = \mathbb{E}_x \left[\exp\left(-\int_0^t V(B_s)ds \right) f(B_t) \right], \qquad t \in \mathbb{R}_+, \quad x \in \mathbb{R}^n.$$

Proof. Let

$$\tilde{T}_t f(x) = \mathbb{E}_x \left[\exp\left(-\int_0^t V(B_s)ds \right) f(B_t) \right], \qquad t \in \mathbb{R}_+, \quad x \in \mathbb{R}^n.$$

We have

$$\begin{aligned} \tilde{T}_{t+s} f(x) &= \mathbb{E}_x \left[\exp\left(-\int_0^{t+s} V(B_u)du \right) f(B_{t+s}) \right] \\ &= \mathbb{E}_x \left[\exp\left(-\int_0^t V(B_u)du \right) \exp\left(-\int_t^{t+s} V(B_u)du \right) f(B_{t+s}) \right] \\ &= \mathbb{E}_x \left[E\left[\exp\left(-\int_0^{t+s} V(B_u)du \right) f(B_{t+s}) \Big| \mathcal{F}_t \right] \right] \\ &= \mathbb{E}_x \left[\exp\left(-\int_0^t V(B_u)du \right) E\left[\exp\left(-\int_t^{t+s} V(B_u)du \right) f(B_{t+s}) \Big| \mathcal{F}_t \right] \right] \\ &= \mathbb{E}_x \left[\exp\left(-\int_0^t V(B_u)du \right) E\left[\exp\left(-\int_0^s V(B_u)du \right) f(B_s) \Big| B_0 = x \right]_{x=B_t} \right] \\ &= \mathbb{E}_x \left[\exp\left(-\int_0^t V(B_u)du \right) \tilde{T}_s f(B_t) \right] \\ &= \tilde{T}_t \tilde{T}_s f(x), \end{aligned}$$

hence $(\tilde{T}_t)_{t\in\mathbb{R}_+}$ has the semigroup property. Next we have

$$\begin{aligned} \frac{\tilde{T}_t f(x) - f(x)}{t} &= \frac{1}{t}\left(\mathbb{E}_x\left[\exp\left(-\int_0^t V(B_u)du\right) f(B_t)\right] - f(x)\right) \\ &= \frac{1}{t}(\mathbb{E}_x[f(B_t)] - f(x)) + \frac{1}{t}\,\mathbb{E}_x\left[\exp\left(-\int_0^t V(B_u)du\right) f(B_t)\right] + o(t), \end{aligned}$$

hence

$$\frac{d\tilde{T}_t}{dt}|_{t=0} = \frac{1}{2}\Delta - V$$

and

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{d\tilde{T}_t}{dt}(t,x)f(x) \\ &= \frac{d\tilde{T}_t}{dt}|_{t=0}\tilde{T}_t f(x) \\ &= \left(\frac{1}{2}\Delta - V(x)\right)\tilde{T}_t f(x) \\ &= \frac{1}{2}\Delta u(t,x) - V(x)u(t,x). \end{aligned}$$

□

Moreover this yields

$$\begin{aligned} \int_{\mathbb{R}^n} f(y)p_t(x,y)dy &= f(x) + \frac{1}{2}\int_0^t \int_{\mathbb{R}^n} \Delta_x f(y)p_s(x,y)dyds \\ &= f(x) + \frac{1}{2}\int_0^t \int_{\mathbb{R}^n} f(y)\Delta_x p_s(x,y)dyds, \end{aligned}$$

for all sufficiently regular functions f on $\mathbb{R}^n$, hence after differentiation with respect to t,

$$\frac{\partial p_t}{\partial t}(x,y) = \frac{1}{2}\Delta_x p_t(x,y). \tag{5.8}$$

From (5.8) we recover the harmonicity of h_y on $\mathbb{R}^n \setminus \{y\}$:

$$\begin{aligned} \Delta_x h_y(x) &= \Delta_x \int_0^\infty p_t(x,y)dt \\ &= \int_0^\infty \Delta_x p_t(x,y)dt \\ &= 2\int_0^\infty \frac{\partial p_t}{\partial t}(x,y)dt \\ &= 2\lim_{t\to\infty} p_t(x,y) - 2\lim_{t\to 0} p_t(x,y) \\ &= 0, \end{aligned}$$

provided $x \neq y$. The backward Kolmogorov partial differential equation

$$\begin{cases} -\dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) - V(x)u(t,x), \\ \\ u(T,x) = f(x), \end{cases} \tag{5.9}$$

can be similarly solved as

$$u(t,x) = \mathbb{E}_x \left[\exp\left(-\int_t^T V(B_s)ds \right) f(B_T) \Big| B_t = x \right], \qquad t \in \mathbb{R}_+, \quad x \in \mathbb{R}^n.$$

Indeed we have

$$\begin{aligned} &\exp\left(-\int_0^t V(B_s)ds \right) u(t,B_t) \\ &= \exp\left(-\int_0^t V(B_s)ds \right) \mathbb{E}_x \left[\exp\left(-\int_t^T V(B_s)ds \right) f(B_T) \Big| B_t \right] \\ &= \exp\left(-\int_0^t V(B_s)ds \right) \mathbb{E}_x \left[\exp\left(-\int_t^T V(B_s)ds \right) f(B_T) \Big| \mathcal{F}_t \right] \\ &= \mathbb{E}_x \left[\exp\left(-\int_0^T V(B_s)ds \right) f(B_T) \Big| \mathcal{F}_t \right], \end{aligned}$$

$t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, which is a martingale by construction. Applying Itô's formula to this process we get

$$\begin{aligned} \exp\left(-\int_0^t V(B_s)ds \right) u(t,B_t) = u(0,B_0) &+ \int_0^t \exp\left(-\int_0^s V(B_\tau)d\tau \right) \langle \nabla_x u(s,B_s), dB_s \rangle \\ &+ \frac{1}{2} \int_0^t \exp\left(-\int_0^s V(B_\tau)d\tau \right) \Delta_x u(s,B_s)ds \\ &- \int_0^t V(B_s) \exp\left(-\int_0^s V(B_\tau)d\tau \right) u(s,B_s)ds \\ &+ \int_0^t \exp\left(-\int_0^s V(B_\tau)d\tau \right) \frac{\partial u}{\partial s} u(s,B_s)ds. \end{aligned}$$

The martingale property shows that the absolutely continuous finite variation terms vanish (see e.g. Cor. 1, p. 64 of Protter [Pro90]), hence

$$\frac{1}{2}\Delta_x u(s,B_s) - V(B_s)u(s,B_s) + \frac{\partial u}{\partial s}(s,B_s) = 0, \qquad s \in \mathbb{R}_+.$$

5.5 Martin boundary

Our aim is now to provide a probabilistic interpretation of the Martin boundary in discrete time. Namely we use the Martin boundary theory to study the way a Markov chain with transition operator P leaves the space E, in particular when E is a union

$$E = \bigcup_{n=0}^{\infty} E_n,$$

of transient sets E_n, $n \in \mathbb{N}$. We assume that E is a metric space with distance δ such that the Cauchy completion of E coincides with its Alexandrov compactification $(E, \hat{x})$.

For simplicity we will assume that the transition operator P is self-adjoint with respect to a reference measure m. Let now G denote the potential

$$Gf(x) = \sum_{n=0}^{\infty} P^n f(x), \qquad x \in E.$$

with kernel $g(\cdot,\cdot)$, i.e.

$$Gf(x) = \int_E f(y) g(x,y) m(dy), \qquad x \in E,$$

for a given reference measure m. Fix $x_0 \in E_1$. For f in the space $\mathcal{C}_c(E)$ of compactly supported function on E, define the kernel

$$k_{x_0}(x,z) = \frac{g(x,z)}{g(x,x_0)}, \qquad x, z \in E,$$

and the operator

$$K_{x_0} f(x) = \frac{Gf(x)}{g(x,x_0)} = \int_E k_{x_0}(x,z) f(z) m(dz), \qquad x \in E.$$

Consider a sequence $(f_n)_{n\in\mathbb{N}}$ dense in $\mathcal{C}_c(E)$ and the metric defined by

$$d(x,y) := \sum_{n=1}^{\infty} \zeta_n |K_{x_0} f_n(x) - K_{x_0} f_n(y)|,$$

where $(\zeta_n)_{n\in\mathbb{N}}$ is a sequence of non-negative numbers such that

$$\sum_{n=1}^{\infty} \zeta_n \|K_{x_0} f_n\|_\infty < \infty.$$

The Martin space $\hat{E}$ for X started with distribution rm is constructed as the Cauchy completion of $(E, \delta + d)$. Then $K_{x_0} f$, $f \in \mathcal{C}_c(E)$, can be extended by continuity to $\hat{E}$ since for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $x, y \in E$,

$$\begin{aligned}
&|K_{x_0} f(x) - K_{x_0} f(y)| \\
&\quad \leq \ |K_{x_0} f(x) - K_{x_0} f_n(x)| + |K_{x_0} f_n(x) - K_{x_0} f_n(y)| + |K_{x_0} f_n(y) - K_{x_0} f(y)| \\
&\quad \leq \ \varepsilon + \zeta_n d(x,y),
\end{aligned}$$

and, from Proposition 2.3 of Revuz [Rev75], the sequence $(K_{x_0} f_n)_{n\in\mathbb{N}}$ is dense in $\{K_{x_0} f \ : f \in \mathcal{C}_c(E)\}$.

If $x \in \Delta E$, a sequence $(x_n)_{n\in\mathbb{N}} \subset E$ converges to x if and only if it converges to $\hat{x}$ for the metric δ and $(K_{x_0} f(x_n))_{n\in\mathbb{N}}$ converges to $K_{x_0} f(x)$ for every $f \in \mathcal{C}_c(E)$.

Theorem 5.5.1. *([Rev75]) The sequence $(X_n)_{n\in\mathbb{N}}$ converges P_{x_0}-a.s. in $\hat{E}$ and the law of X_∞ under P_{x_0} is carried by ΔE.*

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