Distribution-valued iterated gradient and chaotic decompositions of Poisson jump times functionals

Nicolas Privault

Département de Mathématiques, Université de La Rochelle
Avenue Michel Crépeau, 17042 La Rochelle Cedex 1, France
e-mail: nprivaul@univ-lr.fr

Abstract

We define a class of distributions on Poisson space which allows to iterate a modification of the gradient of [1]. As an application we obtain, with relatively short calculations, a formula for the chaos expansion of functionals of jump times of the Poisson process.

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1 Introduction

Let $(N_t)_{t\in\mathbb{R}_+}$ be a standard Poisson process with jump times $(T_k)_{k\geq 1}$, and $T_0 = 0$. The underlying probability space is denoted by $(\Omega, \mathcal{F}, P)$, so that $L^2(\Omega, \mathcal{F}, P)$ is the space of square-integrable functionals of $(N_t)_{t\in\mathbb{R}_+}$. Any $F \in L^2(\Omega, \mathcal{F}, P)$ can be expanded into the series

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \frac{1}{n!} I_n(f_n) \tag{1}$$

where $I_n(f_n)$ is the iterated stochastic integral

$$I_n(f_n) = n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) d(N_{t_1} - t_1) \cdots d(N_{t_n} - t_n)$$

of the symmetric function $f_n \in L^2(\mathbb{R}_+^{\circ n})$ (stochastic integrals are taken in the Itô sense, thus diagonal terms have no influence in the above expression), with the isometry

$$\langle I_n(f_n), I_m(g_m)\rangle_{L^2(\Omega)} = n! \mathbf{1}_{\{n=m\}} \langle f_n, g_m\rangle_{L^2(\mathbb{R}_+, dt)^{\circ n}}, \quad f_n \in L^2(\mathbb{R}_+, dt)^{\circ n},\ g_m \in L^2(\mathbb{R}_+, dt)^{\circ m}.$$

If f_n is not symmetric we let $I_n(f_n) = I_n(\tilde{f}_n)$, where $\tilde{f}_n$ denotes the symmetrization of f_n in n variables, hence (1) can be written as

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n \mathbf{1}_{\Delta_n})$$

where

$$\Delta_n = \{(t_1, \ldots, t_n) \in \mathbb{R}_+^n \ : \ 0 \leq t_1 < \cdots < t_n\}.$$

Let $D : L^2(\Omega) \longrightarrow L^2(\Omega \times \mathbb{R}_+)$ denote the linear unbounded operator defined on multiple stochastic integrals as

$$D_t I_n(f_n) = n I_{n-1}(f_n(*, t)), \quad a.e. \ t \in \mathbb{R}_+.$$

The formula of Y. Ito [3] (Relations (7.4) and (7.5), pp. 26-27), allows in principle to compute f_n as

$$f_n(t_1, \ldots, t_n) = \mathbb{E}[D_{t_1} \cdots D_{t_n} F], \quad a.e. \ t_1, \ldots, t_n \in \mathbb{R}_+.$$

Given the probabilistic interpretation of D as a finite difference operator (cf. [3] and [6]), we have for $F = f(T_1, \ldots, T_d)$:

$$D_t F = \sum_{k=1}^{k=d} \mathbf{1}_{]T_{k-1}, T_k]}(t)(f(T_1, \ldots, T_{k-1}, t, T_k, \ldots, T_{d-1}) - f(T_1, \ldots, T_d)), \quad t \in \mathbb{R}_+,$$

thus $D_{t_1} \cdots D_{t_n} F$ is well defined and explicit computations can be carried out but may be complicated due to the recursive application of a finite difference operator, cf. [4]. See [8] for an elementary approach using only orthogonal expansions in Charlier polynomials.

On the other hand, the gradient $\tilde{D}$ of [1] (see also [2]), defined as

$$\tilde{D}_t = -\sum_{k=1}^{k=d} \mathbf{1}_{[0, T_k]}(t) \partial_k f(T_1, \ldots, T_d),$$

has some properties in common with D, namely its adapted projection coincides with that of D, and in particular we have

$$\mathbb{E}[D_t F] = \mathbb{E}[\tilde{D}_t F], \quad t \in \mathbb{R}_+.$$

Since the operator $\tilde{D}$ has the derivation property it is easier to manipulate than the finite difference operator D in recursive computations. Its disadvantage is that it can not be iterated in L^2 due to the non-differentiability of $\mathbf{1}_{[0,T_k]}(t)$ in T_k, thus an expression such as $\mathbb{E}[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F]$ makes a priori no sense, moreover $\mathbb{E}[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F]$ may differ from $\mathbb{E}[D_{t_1} \cdots D_{t_n} F]$ for $n \geq 2$ (see Relations (13) and (14) below).

In [7] the combined use of D^n and $\tilde{D}$ in L^2 sense has led to the computation of the expansion of the jump time T_d, $d \geq 1$. A direct calculation using only the operator D can be found in [5], concerning a Poisson process on a bounded interval.

In this paper we show that the gradient $\tilde{D}$ can be iterated in a precise sense of distributions on Poisson space, to be introduced in Sect. 3. For example we have for $(t_1, \ldots, t_n) \in \Delta_n$:

$$\begin{aligned} \tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_d) &= (-1)^n f^{(n)}(T_d) \mathbf{1}_{[0,T_d]}(t_1 \vee \cdots \vee t_n) \\ &\quad + (-1)^n \mathbf{1}_{\{0 \leq t_1, \ldots, t_{n-1} < t_n\}} \sum_{j=1}^{n-1} f^{(n-j)}(t_n) \delta_{t_n}^{(j-1)}(T_d), \end{aligned}$$

where $\delta_{t_n}(T_d)$ is a generalized functional, i.e. the composition of the Dirac distribution δ_{t_n} at t_n with the jump time T_d, cf. Prop. 3, $f^{(n)}$ denotes the n-th derivative of the function $f \in \mathcal{C}_b^\infty(\mathbb{R}_+)$, and $t_1 \vee \cdots \vee t_n = \max(t_1, \ldots, t_n)$. Moreover we obtain the equality

$$\mathbb{E}[D_{t_1} \cdots D_{t_n} F \mid \mathcal{F}_a] = \mathbb{E}[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F \mid \mathcal{F}_a], \quad 0 \leq a < t_1 < \cdots < t_n, \ n \geq 2,$$

where we make sense of the conditional expectation $\mathbb{E}[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F \mid \mathcal{F}_a]$ using the pairing $\langle \cdot, \cdot \rangle$ between distributions and test functions. This implies

$$f_n(t_1, \ldots, t_n) = \mathbb{E}[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F], \quad 0 < t_1 < \cdots < t_n.$$

This gives an expression for the decomposition of $f(T_1, \ldots, T_n)$, $f \in \mathcal{C}_b^\infty(\Delta_d)$, with relatively short computations, cf. Prop. 6, for example

$$f(T_d) = \sum_{n=0}^{\infty} I_n(h_n \mathbf{1}_{\Delta_n}),$$

with

$$h_n(t_1, \ldots, t_n) = \mathbb{E}[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_d)]$$

$$= \quad (-1)^n \int_{t_n}^{\infty} f^{(n)}(t) p_{d-1}(t) dt + (-1)^n \sum_{j=1}^{j=n-1} (-1)^{j-1} f^{(n-j)}(t_n) p_{d-1}^{(j-1)}(t_n),$$

$0 < t_1 < \cdots < t_n$, where $p_{d-1}(t) = \frac{t^{d-1}}{(d-1)!} e^{-t}$, $t \in \mathbb{R}_+$, $d \geq 1$.

2 Integration by parts

In this section we review the definition of the three main gradient operators on Poisson space, and present an elementary derivation of integration by parts formulas. All $\mathcal{C}^\infty$ functions on Δ_d are extended by continuity to the closure of Δ_d.

Definition 1 *Let $a \geq 0$. Let $\mathcal{S}_d(\Omega \times [a,\infty[^l)$ denote the test function space*

$$\mathcal{S}_d(\Omega \times [a,\infty[^l) = \{h_1 \otimes \cdots \otimes h_l \otimes f(T_1,\ldots,T_d) \ : \quad f \in \mathcal{C}_b^\infty(\Delta_d),\ h_1,\ldots,h_l \in \mathcal{C}_b([a,\infty[)\},$$

with $\mathcal{S}_d(\Omega) = \mathcal{S}_d(\Omega \times \mathbb{R}_+^0)$ for $l = 0$.

We recall that if $f \in L^2(\Delta_d, e^{-t_d} dt_1 \cdots dt_d)$ then

$$\mathbb{E}[f(T_1,\ldots,T_d)] = \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} f(t_1,\ldots,t_d) dt_1 \cdots dt_d,$$

which follows e.g. from the fact that $(\tau_n)_{n\geq 1} = (T_n - T_{n-1})_{n\geq 1}$ is a family of independent exponential random variables.

2.1 Intrinsic gradient

The intrinsic gradient $\hat{D}$ on Poisson space is defined on $\mathcal{S}_d(\Omega)$ as

$$\hat{D}_t F = \sum_{k=1}^{k=d} \mathbf{1}_{\{T_k\}}(t) \partial_k f(T_1,\ldots,T_d), \quad dN_t - a.e.,$$

with $F = f(T_1,\ldots,T_d)$, $f \in \mathcal{C}_b^\infty(\Delta_d)$, where $\partial_k f$ represents the partial derivative of f with respect to its k-th variable, $1 \leq k \leq d$.

Lemma 1 *Let $F \in \mathcal{S}_d(\Omega)$ and $h \in \mathcal{C}_b^1(\mathbb{R}_+)$ with $h(0) = 0$. We have the integration by parts formula*

$$\mathbb{E}[\langle \hat{D}F, h\rangle_{L^2(\mathbb{R}_+,dN_t)}] = \mathbb{E}\left[F U_{h'}^d\right] = -\mathbb{E}\left[F \int_0^\infty h'(t) d(N_t - t)\right]. \qquad (2)$$

where $U_{h'}^d = -\left(\sum_{k=1}^{k=d} h'(T_k) - \int_0^{T_d} h'(t) dt\right) \in \mathcal{S}_d(\Omega)$.

Proof. We have by integration by parts on Δ_d:

$$\mathbb{E}[\langle \hat{D}F, h\rangle_{L^2(\mathbb{R}_+, dN_t)}] = \sum_{k=1}^{k=d} \int_0^\infty \int_0^{t_d} \cdots \int_0^{t_2} e^{-t_d} h(t_k) \partial_k f(t_1, \ldots, t_d) dt_1 \cdots dt_d$$

$$\begin{aligned}
= & \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h(t_1) \partial_1 f(t_1, \ldots, t_d) dt_1 \cdots dt_d \\
& + \sum_{k=2}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_{k+1}} h(t_k) \frac{\partial}{\partial t_k} \int_0^{t_k} \cdots \int_0^{t_2} f(t_1, \ldots, t_d) dt_1 \cdots dt_d \\
& - \sum_{k=2}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdot \int_0^{t_{k+1}} h(t_k) \int_0^{t_k} \int_0^{t_{k-2}} \cdot \int_0^{t_2} f(t_1, \ldots, t_{k-2}, t_k, t_k, \ldots, t_d) dt_1 \cdot d\hat{t}_{k-1} \cdot dt_d \\
= & - \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h'(t_1) f(t_1, \ldots, t_d) dt_1 \cdots dt_d \\
& + \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_3} h(t_2) f(t_2, t_2, \ldots, t_d) dt_2 \cdots dt_d \\
& - \sum_{k=2}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdot \int_0^{t_{k+1}} h'(t_k) \int_0^{t_k} \cdots \int_0^{t_2} f(t_1, \ldots, t_d) dt_1 \cdots dt_d \\
& + \int_0^\infty e^{-t_d} h(t_d) \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d) dt_1 \cdots dt_d \\
& + \sum_{k=2}^{k=d-1} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdot \int_0^{t_{k+2}} h(t_{k+1}) \int_0^{t_{k+1}} \int_0^{t_{k-1}} \cdot \int_0^{t_2} f(t_1, ., t_{k-1}, t_{k+1}, t_{k+1}, ., t_d) dt_1 \cdot d\hat{t}_k \cdot dt_d \\
& - \sum_{k=2}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdot \int_0^{t_{k+1}} h(t_k) \int_0^{t_k} \int_0^{t_{k-2}} \cdot \int_0^{t_2} f(t_1, ., t_{k-2}, t_k, t_k, ., t_d) dt_1 \cdot dt_d \\
= & - \sum_{k=1}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_{k+1}} h'(t_k) \int_0^{t_k} \cdots \int_0^{t_2} f(t_1, \ldots, t_d) dt_1 \cdots dt_d \\
& + \int_0^\infty e^{-t_d} h(t_d) \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d) dt_1 \cdots dt_d \\
= & -\mathbb{E}\left[F \left(\sum_{k=1}^{k=d} h'(T_k) - \int_0^{T_d} h'(t) dt \right) \right],
\end{aligned}$$

where $d\hat{t}_k$ denotes the absence of dt_k. Concerning the second part of the equality it suffices to notice that if $k > d$,

$$\begin{aligned}
\mathbb{E}[Fh'(T_k)] &= \int_0^\infty e^{-t_k} h'(t_k) \int_0^{t_k} \cdots \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d) dt_1 \cdots dt_k \\
&= \int_0^\infty e^{-t_k} h(t_k) \int_0^{t_k} \cdots \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d) dt_1 \cdots dt_k
\end{aligned}$$

$$
\begin{aligned}
& - \int_0^\infty e^{-t_{k-1}} h(t_{k-1}) \int_0^{t_{k-1}} \cdots \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d) dt_1 \cdots dt_{k-1} \\
= \ & \mathbb{E}[F(h(T_k) - h(T_{k-1}))] = \mathbb{E}\left[F \int_{T_{k-1}}^{T_k} h'(t) dt \right].
\end{aligned}
$$

□

Relation (2) implies immediately for $F, G \in \mathcal{S}_d(\Omega)$:

$$
\begin{aligned}
\mathbb{E}[\langle \hat{D}F, hG \rangle_{L^2(\mathbb{R}_+, dN_t)}] &= \mathbb{E}\left[\langle \hat{D}(FG), h \rangle_{L^2(\mathbb{R}_+, dN_t)} - F \langle \hat{D}G, h \rangle_{L^2(\mathbb{R}_+, dN_t)} \right] \\
&= \mathbb{E}\left[F(GU^d_{h'} - \langle h, \hat{D}G \rangle_{L^2(\mathbb{R}_+, dN_t)}) \right] \\
&= -\mathbb{E}\left[F \left(G \int_0^\infty h'(t) d(N_t - t) + \langle h, \hat{D}G \rangle_{L^2(\mathbb{R}_+, dN_t)} \right) \right].
\end{aligned}
$$

2.2 Damped gradient

Let $r(s,t) = -s \vee t$ denote the Green function associated to the Laplacian $\mathcal{L}$ on $\mathbb{R}_+$:

$$
\mathcal{L} f = -f'', \quad f \in \mathcal{C}_c^\infty(]0, \infty[),
$$

i.e. we have, with $g = -f''$:

$$
\int_0^\infty r(s,t) g(t) dt = - \int_0^s \int_0^t g(u) du dt, \quad s \in \mathbb{R}_+.
$$

Definition 2 *Given $F \in \mathcal{S}_d(\Omega)$, $F = f(T_1, \ldots, T_d)$, we let*

$$
r^{(1)}(s,t) = \frac{\partial}{\partial s} r(s,t) = -\mathbf{1}_{]-\infty, s]}(t), \quad s, t \in \mathbb{R}_+,
$$

and

$$
\tilde{D}_t F = \int_0^\infty r^{(1)}(s,t) \hat{D}_s F dN_s.
$$

We have

$$
\tilde{D}_t F = \sum_{k=1}^{k=d} r^{(1)}(T_k, t) \partial_k f(T_1, \ldots, T_d) = - \sum_{k=1}^{k=d} \mathbf{1}_{[0, T_k]}(t) \partial_k f(T_1, \ldots, T_d).
$$

In fact $\tilde{D}$ is (up to a minor modification) the gradient introduced in [1]. This presentation of $\tilde{D}$ using the Green function $r(s,t)$ is motivated by [9].

Proposition 1 *We have for $F \in \mathcal{S}_d(\Omega)$ and $h \in \mathcal{C}_c(\mathbb{R}_+)$:*

$$\mathbb{E}[\langle \tilde{D}F, h\rangle_{L^2(\mathbb{R}_+,dt)}] = \mathbb{E}\left[FU_h^d\right] = \mathbb{E}\left[F\int_0^\infty h(t)d(N_t - t)\right], \tag{3}$$

where $U_h^d = \sum_{k=1}^{k=d} h(T_k) - \int_0^{T_d} h(t)dt$.

Proof. We have

$$\begin{aligned}\mathbb{E}\left[\langle \tilde{D}F, h\rangle_{L^2(\mathbb{R}_+,dt)}\right] &= \mathbb{E}\left[\int_0^\infty \int_0^\infty r^{(1)}(s,t)\hat{D}_s F h(t) dN_s dt\right] \\ &= -\mathbb{E}\left[\left\langle \hat{D}F, \int_0^\cdot h(t)dt\right\rangle_{L^2(\mathbb{R}_+,dN_t)}\right] = \mathbb{E}\left[F\int_0^\infty h(t)d(N_t - t).\right]\end{aligned}$$

□

Relation (3) also implies that for $F, G \in \mathcal{S}_d(\Omega)$,

$$\begin{aligned}\mathbb{E}[\langle \tilde{D}F, hG\rangle_{L^2(\mathbb{R}_+,dt)}] &= \mathbb{E}\left[\langle \tilde{D}(FG), h\rangle_{L^2(\mathbb{R}_+,dt)} - F\langle \tilde{D}G, h\rangle_{L^2(\mathbb{R}_+,dt)}\right] \\ &= \mathbb{E}\left[F(GU_h^d - \langle h, \tilde{D}G\rangle_{L^2(\mathbb{R}_+,dt)})\right] \\ &= \mathbb{E}\left[F\left(G\int_0^\infty h(t)d(N_t - t) - \langle h, \tilde{D}G\rangle_{L^2(\mathbb{R}_+,dt)}\right)\right].\end{aligned} \tag{4}$$

2.3 Finite difference gradient

For completeness we mention the gradient D which is associated to the Fock space structure, and whose properties have been discussed in the introduction.

3 Distribution-valued gradient

For $n \geq 2$ and $F \in \mathcal{S}_d(\Omega)$ we let $dN_{t_1} \otimes \cdots \otimes dN_{t_n} - a.e.$:

$$\hat{D}^n_{t_1,\ldots,t_n} F = \sum_{1 \leq j_1,\ldots,j_n \leq d} \mathbf{1}_{\{T_{j_1}\}}(t_1) \cdots \mathbf{1}_{\{T_{j_n}\}}(t_n) \partial_{j_1} \cdots \partial_{j_n} f(T_1, \ldots, T_d).$$

This is not the n-th iteration of $\hat{D}$, in fact we have

$$\|\hat{D}^n F\|^2_{L^2(\mathbb{R}_+,dN_t)^{\otimes n}} = \sum_{1 \leq j_1,\ldots,j_n \leq d} (\partial_{j_1} \cdots \partial_{j_n} f(T_1, \ldots, T_d))^2.$$

Definition 3 *Let $a \in \mathbb{R}_+$ and $l \in \mathbb{N}$.*

i) We denote by $\mathcal{S}'_d(\Omega\times[a,\infty[^l)$ *the space of continuous linear forms (distributions) on* $\mathcal{S}_d(\Omega\times[a,\infty[^l)$*, i.e.* $F\in\mathcal{S}'_d(\Omega\times[a,\infty[^l)$ *if there exists* $k\geq 0$ *and* $C>0$ *such that*

$$|\langle F, h_1\otimes\cdots\otimes h_l\otimes G\rangle|\leq C\sum_{i=0}^{i=k}\|h_1\otimes\cdots\otimes h_l\|_\infty\|\hat{D}^i G\|_{L^\infty(\Omega,L^2(\mathbb{R}_+,dN_t)^{\otimes i})},$$

$G\in\mathcal{S}_d(\Omega\times\mathbb{R}_+^l)$, $h_1,\dots,h_l\in\mathcal{C}_c([a,\infty[)$.

ii) A sequence $(F_n)_{n\in\mathbb{N}}\subset\mathcal{S}'_d(\Omega\times[a,\infty[^l)$ *is said to converge in* $\mathcal{S}'_d(\Omega\times[a,\infty[^l)$ *if the sequence* $(\langle F_n,G\rangle)_{n\in\mathbb{N}}$ *converges to* $\langle F,G\rangle$ *for all* $G\in\mathcal{S}_d(\Omega\times[a,\infty[^l)$.

The notation $\langle\cdot,\cdot\rangle$ will be used to denote the pairing between $\mathcal{S}_d(\Omega\times[a,\infty[^l)$ and $\mathcal{S}'_d(\Omega\times[a,\infty[^l)$, for all values of $l\in\mathbb{N}$. Every $F\in\mathcal{S}_d(\Omega\times[a,\infty[^l)$ is identified to an element of $\mathcal{S}'_d(\Omega\times[a,\infty[^l)$ by letting

$$\langle F,G\rangle=\int_\Omega\langle F(\omega,\cdot),G(\omega,\cdot)\rangle_{L^2([a,\infty[,dt)^{\otimes l}}P(d\omega),\qquad G\in\mathcal{S}_d(\Omega\times[a,\infty[^l).$$

The closability property in L^2 of the operator $\tilde{D}$ is a well-known statement which extends to distributions in $\mathcal{S}'_d(\Omega\times[a,\infty[^l)$.

Proposition 2 *Let* $(F_n)_{n\in\mathbb{N}}\subset\mathcal{S}_d(\Omega\times[a,\infty[^l)$ *such that*

i) $(F_n)_{n\in\mathbb{N}}$ *converges to* 0 *in* $\mathcal{S}'_d(\Omega\times[a,\infty[^l)$,

ii) $(\tilde{D}F_n)_{n\in\mathbb{N}}$ *converges in* $\mathcal{S}'_d(\Omega\times[a,\infty[^{l+1})$.

Then $(\tilde{D}F_n)_{n\in\mathbb{N}}$ *converges to* 0 *in* $\mathcal{S}'_d(\Omega\times[a,\infty[^{l+1})$.

Proof. For $l=0$ this is a direct consequence of the integration by parts formula (4), which shows that

$$\langle\tilde{D}F_n,hG\rangle=\mathbb{E}[F_n(GU_h^d-\langle\tilde{D}G,h\rangle_{L^2(\mathbb{R}_+,dt)})],\quad h\in\mathcal{C}_c([a,\infty[),\ G\in\mathcal{S}_d(\Omega),$$

with $GU_h^d-\langle\tilde{D}G,h\rangle_{L^2(\mathbb{R}_+,dt)}\in\mathcal{S}_d(\Omega)$. The generalization to $l\geq 1$ is straightforward. □

This proposition justifies the following extension of $\tilde{D}$ to generalized functionals.

Definition 4 *Let $a \in \mathbb{R}_+$. We let $\mathbb{D}_l([a,\infty[)$ denote the subspace of $F \in \mathcal{S}'_d(\Omega \times [a,\infty[^l)$ such that*

i) there exists $(F_n)_{n\in\mathbb{N}} \subset \mathcal{S}_d(\Omega \times [a,\infty[^l)$ that converges to F in $\mathcal{S}'_d(\Omega \times [a,\infty[^l)$,

ii) $(\tilde{D}F_n)_{n\in\mathbb{N}}$ converges in $\mathcal{S}'_d(\Omega \times [a,\infty[^{l+1})$.

Given F as above we define $\tilde{D}F$ as the limit in $\mathcal{S}'_d(\Omega \times [a,\infty[^{l+1})$

$$\tilde{D}F = \lim_{n\to\infty} \tilde{D}F_n, \quad F \in \mathbb{D}_l([a,\infty[) \subset \mathcal{S}'_d(\Omega \times [a,\infty[^l).$$

4 Iterated gradient in distribution sense

We let for $n \geq 2$:

$$r^{(n)}(T_k,t) = \partial_1^n r(T_k,t) = -\delta_t^{(n-2)}(T_k),$$

in distribution sense, i.e. $r^{(n)}(T_k,t)$ belongs to $\mathcal{S}'_d(\Omega)$ with for $k = 1,\ldots,d$, and $f \in \mathcal{C}_b^\infty(\Delta_d)$:

$$\begin{aligned}
&\langle r^{(n)}(T_k,t), f(T_1,\ldots,T_d)\rangle \\
&= (-1)^{n+1}\mathbf{1}_{\{k<d\}} \int_0^\infty e^{-s_d} \int_0^{s_d} \cdots \int_0^{s_{k+2}} \\
&\qquad \left(\frac{\partial^{n-2}}{\partial s_k^{n-2}} \int_0^{s_k} \cdots \int_0^{s_2} f(s_1,\ldots,s_d) ds_1 \cdots ds_{k-1} \right)_{|s_k=t} ds_{k+1} \cdots ds_d \\
&\quad +(-1)^{n+1}\mathbf{1}_{\{k=d\}} \frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^t \int_0^{s_{d-1}} \cdots \int_0^{s_2} e^{-t} f(s_1,\ldots,s_{d-1},t) ds_1 \cdots ds_{d-1},
\end{aligned}$$

and for $n = 1$:

$$\langle r^{(1)}(T_k,t), f(T_1,\ldots,T_d)\rangle = -\int_0^\infty \int_0^{s_d} \cdots \int_0^{s_2} e^{-s_d} \mathbf{1}_{[t,\infty[}(s_k) f(s_1,\ldots,s_d) ds_1 \cdots ds_d.$$

Let $\phi \in \mathcal{C}_c^\infty([-1,1])$, $\phi \geq 0$, such that $\int_{-1}^1 \phi(t)dt = 1$, and let

$$\phi_\varepsilon(t) = \varepsilon^{-1}\phi(\varepsilon^{-1}t), \quad t \in \mathbb{R}, \quad \varepsilon > 0.$$

Let $\phi_\varepsilon * r^{(n)}(T_k,t)$, $n \geq 1$, denote the convolution of ϕ_ε with $r^{(n)}(T_k,t)$ in the first variable, i.e. for $n=1$:

$$\phi_\varepsilon * r^{(1)}(T_k,t) = -\int_{-\infty}^{\infty} \phi_\varepsilon(u)\mathbf{1}_{]-\infty,T_k-u]}(t)du = -\int_{-\infty}^{\infty} \phi_\varepsilon(u+T_k)\mathbf{1}_{]-\infty,u]}(t)du, \quad s,t \in \mathbb{R}_+,$$

and for $n \geq 2$, $l \in \mathbb{N}$:

$$\phi_\varepsilon^{(l)} * r^{(n)}(T_k,t) = \phi_\varepsilon * r^{(n+l)}(T_k,t) = -\phi_\varepsilon^{(n+l-2)}(T_k-t),$$

which converges in $\mathcal{S}_d'(\Omega)$ to $r^{(n+l)}(T_k,t)$ if $n+l \geq 1$ (i.e. to $-\delta_t^{(n+l-2)}(T_k)$ if $n+l \geq 2$), $k=1,\ldots,d$. Let $t_1 \vee \cdots \vee t_n = \max(t_1,\ldots,t_n)$, $t_1,\ldots,t_n \in \mathbb{R}_+$.

Proposition 3 *Let $k \geq 1$ and $f \in \mathcal{C}_b^\infty(\mathbb{R}_+)$. Then for all $n \geq 1$, $\tilde{D}_{t_1}\cdots\tilde{D}_{t_n} f(T_k) \in \mathbb{D}_0(\mathbb{R}_+)$ for a.a. $(t_1,\ldots,t_n) \in \mathbb{R}_+^n$, and*

$$\begin{aligned}\tilde{D}_{t_1}\cdots\tilde{D}_{t_n} f(T_k) &= (-1)^n f^{(n)}(T_k)\mathbf{1}_{[0,T_k]}(t_1 \vee \cdots \vee t_n) \\ &+(-1)^n \sum_{j=1}^{n-1} \mathbf{1}_{\{t_1\vee\cdots\vee t_j < t_{j+1}\vee\cdots\vee t_n\}} f^{(n-j)}(t_{j+1}\vee\cdots\vee t_n)\delta^{(j-1)}_{t_{j+1}\vee\cdots\vee t_n}(T_k). \qquad (5)\end{aligned}$$

Proof. For $n=1$ this is the definition of $\tilde{D}$. We proceed by induction, assuming that $\tilde{D}_{t_3}\cdots\tilde{D}_{t_{n+1}} f(T_k) \in \mathbb{D}_0(\mathbb{R}_+)$ for some $n \geq 1$, and

$$\begin{aligned}\tilde{D}_{t_2}\cdots\tilde{D}_{t_{n+1}} f(T_k) &= (-1)^n f^{(n)}(T_k)\mathbf{1}_{[0,T_k]}(t_2 \vee \cdots \vee t_{n+1}) \\ &+(-1)^n \sum_{j=1}^{n-1} \mathbf{1}_{\{t_2\vee\cdots\vee t_{j+1} < t_{j+2}\vee\cdots\vee t_{n+1}\}} f^{(n-j)}(t_{j+2}\vee\cdots\vee t_{n+1})\delta^{(j-1)}_{t_{j+2}\vee\cdots\vee t_{n+1}}(T_k).\end{aligned}$$

Let for $\varepsilon > 0$. We define a smooth approximation of $\tilde{D}_{t_2}\cdots\tilde{D}_{t_{n+1}} f(T_k)$ by letting

$$\begin{aligned}&F_\varepsilon(t_2,\ldots,t_{n+1}) = (-1)^n f^{(n)}(T_k)\phi_\varepsilon * r^{(1)}(T_k, t_2 \vee \cdots \vee t_{n+1}) \\ &+(-1)^n \sum_{j=1}^{n-1} \mathbf{1}_{\{t_2\vee\cdots\vee t_{j+1} < t_{j+2}\vee\cdots\vee t_{n+1}\}} f^{(n-j)}(t_{j+2}\vee\cdots\vee t_{n+1})\phi_\varepsilon * r^{(j+1)}(T_k, t_{j+2}\vee\cdots\vee t_{n+1}).\end{aligned}$$

Then $F_\varepsilon(t_2,\ldots,t_{n+1}) \in \mathcal{S}_d(\Omega)$ and

$$\begin{aligned}\tilde{D}_{t_1} F_\varepsilon(t_2,\ldots,t_{n+1}) &= (-1)^{n+1}\mathbf{1}_{[0,T_k]}(t_1)\phi_\varepsilon * r^{(1)}(T_k, t_2\vee\cdots\vee t_{n+1}) f^{(n+1)}(T_k) \\ &\quad +(-1)^{n+1}\mathbf{1}_{[0,T_k]}(t_1) f^{(n)}(T_k)\phi'_\varepsilon * r^{(1)}(T_k, t_2\vee\cdots\vee t_{n+1})\end{aligned}$$

$$
\begin{aligned}
&+(-1)^{n+1}\mathbf{1}_{[0,T_k]}(t_1)\sum_{j=1}^{n-1}\mathbf{1}_{\{t_2\vee\cdots\vee t_{j+1}<t_{j+2}\vee\cdots\vee t_{n+1}\}}f^{(n-j)}(t_{j+2}\vee\cdots\vee t_{n+1})\\
&\qquad\times\phi'_\varepsilon * r^{(j+1)}(T_k,t_{j+2}\vee\cdots\vee t_{n+1})\\
&= (-1)^{n+1}\mathbf{1}_{[0,T_k]}(t_1)\phi_\varepsilon * r^{(1)}(T_k,t_2\vee\cdots\vee t_{n+1})f^{(n+1)}(T_k)\\
&\qquad+(-1)^{n+1}\mathbf{1}_{[0,T_k]}(t_1)f^{(n)}(T_k)\phi_\varepsilon * r^{(2)}(T_k,t_2\vee\cdots\vee t_{n+1})\\
&+(-1)^{n+1}\mathbf{1}_{[0,T_k]}(t_1)\sum_{j=2}^{n}\mathbf{1}_{\{t_2\vee\cdots\vee t_{j}<t_{j+1}\vee\cdots\vee t_{n+1}\}}f^{(n+1-j)}(t_{j+1}\vee\cdots\vee t_{n+1})\\
&\qquad\times\phi_\varepsilon * r^{(j+1)}(T_k,t_{j+1}\vee\cdots\vee t_{n+1})\\
&= (-1)^{n+1}\mathbf{1}_{[0,T_k]}(t_1)\phi_\varepsilon * r^{(1)}(T_k,t_2\vee\cdots\vee t_{n+1})f^{(n+1)}(T_k)\\
&+(-1)^{n+1}\mathbf{1}_{[0,T_k]}(t_1)\sum_{j=1}^{n}\mathbf{1}_{\{t_2\vee\cdots\vee t_{j}<t_{j+1}\vee\cdots\vee t_{n+1}\}}f^{(n+1-j)}(t_{j+1}\vee\cdots\vee t_{n+1})\\
&\qquad\times\phi_\varepsilon * r^{(j+1)}(T_k,t_{j+1}\vee\cdots\vee t_{n+1}),
\end{aligned}
$$

where we used the relation $\phi'_\varepsilon * r^{(j+1)} = \phi_\varepsilon * r^{(j+2)}$. As $\varepsilon \to 0$, $\tilde{D}_{t_1}F_\varepsilon(t_2,\ldots,t_{n+1})$ converges in $\mathcal{S}'_d(\Omega)$ to

$$
\begin{aligned}
&(-1)^{n+1}f^{(n+1)}(T_k)\mathbf{1}_{[0,T_k]}(t_1\vee\cdots\vee t_{n+1})\\
&+(-1)^{n+1}\sum_{j=1}^{n}\mathbf{1}_{\{t_1\vee\cdots\vee t_j<t_{j+1}\vee\cdots\vee t_{n+1}\}}f^{(n+1-j)}(t_{j+1}\vee\cdots\vee t_{n+1})r^{(j+1)}(T_k,t_{j+1}\vee\cdots\vee t_{n+1}).
\end{aligned}
$$

□

In particular, for $n \geq 2$:

$$
\tilde{D}_{t_1}\cdots\tilde{D}_{t_n}T_k = (-1)^{n+1}\mathbf{1}_{\{0\leq t_1,\ldots,t_{n-1}<t_n\}}r^{(n)}(T_k,t_n) = (-1)^n\mathbf{1}_{\{0\leq t_1,\ldots,t_{n-1}<t_n\}}\delta_{t_n}^{(n-2)}(T_k).
$$

We note that since $f\in\mathcal{C}_b^\infty(\mathbb{R}_+)$, there exists $C>0$ such that

$$
|\langle\tilde{D}_{t_1}\cdots\tilde{D}_{t_n}f(T_k),h_1\otimes\cdots\otimes h_l\otimes G\rangle| \leq C\sum_{i=0}^{i=n}\|h_1\otimes\cdots\otimes h_l\|_\infty\|\hat{D}^iG\|_{L^\infty(\Omega,L^2(\mathbb{R}_+,dN_t)^{\otimes i})},
$$

$dt_1\cdots dt_n - a.e.$, for all $G\in\mathcal{S}_d(\Omega)$ and $h_1,\ldots,h_l\in\mathcal{C}_c(\mathbb{R}_+)$. Hence $\tilde{D}^nf(T_k)\in\mathbb{D}_n(\mathbb{R}_+)\subset\mathcal{S}'_d(\Omega\times\mathbb{R}_+^n)$, and (5) can be written as

$$
\tilde{D}_{t_1}\cdots\tilde{D}_{t_n}f(T_k) = (-1)^n\sum_{j=0}^{j=n}\mathbf{1}_{\{t_1\vee\cdots\vee t_j<t_{j+1}\vee\cdots\vee t_n\}}f^{(n-j)}(T_{k,j})(-r^{(j+1)}(T_k,t_{j+1}\vee\cdots\vee t_n))
$$

with $t_0 = 0$ and

$$T_{k,j} = \begin{cases} T_k & \text{if } j = 0, \\ t_{j+1} \vee \cdots \vee t_n & \text{if } j \geq 1, \end{cases}$$

i.e. if $0 \leq t_1 < \cdots < t_n$:

$$\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_k) = (-1)^n \sum_{j=0}^{j=n} f^{(n-j)}(T_{k,j})(-r^{(j+1)}(T_k, t_n)).$$

In case $t_1 > \cdots > t_n$ and $F = f(T_1, \ldots, T_d)$ with $f \in \mathcal{C}_b^\infty(\Delta_n)$, it is shown in [10] that

$$\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F = (-1)^n \sum_{1 \leq j_1, \ldots, j_n \leq d} \mathbf{1}_{[0,T_{j_1}]}(t_1) \cdots \mathbf{1}_{[0,T_{j_n}]}(t_n) \partial_{j_1} \cdots \partial_{j_n} f(T_1, \ldots, T_d).$$

Given $j_1, \ldots, j_n \in \{1, \ldots, d\}$ and $i \in \{1, \ldots, d\}$, let

$$a_i(j_1, \ldots, j_n) = \text{Card}\{l \in \{1, \ldots n\} \ : \ j_l = i\},$$

$$c_i(j_1, \ldots, j_n) = \max\{l : j_l = i\}.$$

With this notation we obtain the following formula, in which the indices $j_1, \ldots, j_n$ are omitted in $a_i(j_1, \ldots, j_n)$ and $c_i(j_1, \ldots, j_n)$.

Theorem 1 *Let $F = f(T_1, \ldots, T_d)$, with $f \in \mathcal{C}_b^\infty(\Delta_d)$. We have $\tilde{D}^n F \in \mathbb{D}_n(\mathbb{R}_+)$ for all $n \geq 0$, and:*

$$\begin{aligned} \tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F \;=\; & (-1)^n \sum_{1 \leq j_1, \ldots, j_n \leq d} \; \sum_{0 \leq i_1 \leq (a_1 - 1) \vee 0} \cdots \sum_{0 \leq i_d \leq (a_d - 1) \vee 0} \\ & \partial_1^{a_1 - i_1} \cdots \partial_d^{a_d - i_d} f(T_{1,i_1}, \ldots, T_{d,i_d}) \prod_{\substack{l=1 \\ a_l \neq 0}}^{l=d} (-r^{(1+i_l)}(T_l, t_{c_l})), \end{aligned} \tag{6}$$

$(t_1, \ldots, t_n) \in \Delta_n$, $n \geq 2$, where

$$T_{l,i_l} = \begin{cases} T_l & \textit{if } i_l = 0, \\ t_{c_l} & \textit{if } i_l \geq 1. \end{cases}$$

Proof. Let $\tilde{D}_{t,k}$ denote the partial gradient with respect to the k-th variable, i.e.

$$\tilde{D}_{t,k} f(T_1, \ldots, T_d) = -\mathbf{1}_{[0,T_k]}(t) \partial_k f(T_1, \ldots, T_d), \quad 1 \leq k \leq d.$$

Then $\tilde{D}_{t,k}$ and $\tilde{D}_{s,l}$, are commuting operators, $1 \leq k < l \leq d$, and for $0 < t_1 < \cdots < t_l$,

$$\tilde{D}_{t_1,k} \cdots \tilde{D}_{t_l,k} F = (\tilde{D}_{t_l,k})^l F.$$

Consequently,

$$\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F = \sum_{1 \leq j_1,\ldots,j_n \leq d} \tilde{D}_{t_1,j_1} \cdots \tilde{D}_{t_n,j_n} F = \sum_{1 \leq j_1,\ldots,j_n \leq d} (\tilde{D}_{t_{c_1},1})^{a_1} \cdots (\tilde{D}_{t_{c_d},d})^{a_d} F.$$

It remains to apply Prop. 3 under the form

$$(\tilde{D}_{t_{c_l},l})^{a_l} F \quad = \quad (-1)^{a_l} \sum_{0 \leq i_l \leq (a_l-1)\vee 0} \partial_l^{a_l-i_l} f(T_1, \ldots, T_{l-1}, T_{l,i_l}, T_{l+1}, \ldots, T_d)(-r^{(i_l+1)}(T_l, t_{c_l})),$$

if $a_l \geq 1$, and

$$(\tilde{D}_{t_{c_l},l})^{a_l} F \quad = \quad (-1)^{a_l} \sum_{0 \leq i_l \leq (a_l-1)\vee 0} \partial_l^{a_l-i_l} f(T_1, \ldots, T_{l-1}, T_{l,i_l}, T_{l+1}, \ldots, T_d)$$

if $a_l = 0$. □

5 Equality of adapted projections in distribution sense

We recall that the adjoint of D extends the compensated Poisson stochastic integral, cf. [3], Th. 6.9, p. 23, i.e. for all adapted square-integrable process $u \in L^2(\Omega \times \mathbb{R}_+)$ we have

$$\mathbb{E}[\langle DF, u \rangle_{L^2(\mathbb{R}_+,dt)}] = \mathbb{E}\left[F \int_0^\infty u(t) d(N_t - t)\right]. \tag{7}$$

Since the adapted projections of $\tilde{D}$ and D coincide, cf. [7], Prop. 20:

$$\mathbb{E}[\tilde{D}_t F \mid \mathcal{F}_a] = \mathbb{E}[D_t F \mid \mathcal{F}_a], \quad 0 < a < t, \tag{8}$$

the same property hold for $\tilde{D}$:

$$\mathbb{E}[\langle \tilde{D}F, u \rangle_{L^2(\mathbb{R}_+,dt)}] = \mathbb{E}\left[F \int_0^\infty u(t) d(N_t - t)\right].$$

We now show that Relation (8) can be extended in distribution sense to $\tilde{D}^n$ and D^n, $n \geq 2$. The next proposition will be interpreted in terms of generalized conditional expectations as

$$\mathbb{E}[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F \mid \mathcal{F}_a] = \mathbb{E}[D_{t_1} \cdots D_{t_n} F \mid \mathcal{F}_a], \quad (t_1, \ldots, t_n) \in \Delta_n \cap [a, \infty[^n.$$

Proposition 4 *Let $F \in \mathcal{S}_d(\Omega)$ and $G \in \mathcal{S}_d(\Omega)$ be $\mathcal{F}_a$-measurable. We have*

$$\langle G\tilde{D}_{t_1}\cdots\tilde{D}_{t_n}F, 1\rangle = \mathbb{E}[GD_{t_1}\cdots D_{t_n}F], \qquad (t_1,\ldots,t_n) \in \Delta_n \cap [a,\infty[^n. \tag{9}$$

Proof. The proposition holds for $n = 1$. We assume that it holds for some $n \geq 1$. Let $F_\varepsilon(t_2,\ldots,t_{n+1})$ denote the regularization of $\tilde{D}_{t_2}\cdots\tilde{D}_{t_{n+1}}F$ constructed as in the proof of Prop. 3:

$$\begin{aligned} F_\varepsilon(t_2,\ldots,t_{n+1}) \;&=\; (-1)^n \sum_{1\leq j_1,\ldots,j_n\leq d} \;\sum_{0\leq i_1\leq(a_1-1)\vee 0} \cdots \sum_{0\leq i_d\leq(a_d-1)\vee 0} \\ &\partial_1^{a_1-i_1}\cdots\partial_d^{a_d-i_d} f(T_{1,i_1},\ldots,T_{d,i_d}) \prod_{\substack{l=1\\ a_l\neq 0}}^{l=d} (-\phi^\varepsilon * r^{(1+i_l)}(T_l, t_{1+c_l})). \end{aligned}$$

Let $f_{n+1} \in \mathcal{C}_c^\infty(\Delta_{n+1}\cap[a,\infty[^{n+1})$. Since G is $\mathcal{F}_a$ measurable we have $\tilde{D}_{t_1}G = 0$, $t_1 > a$, hence from Prop. 1:

$$\begin{aligned} &\mathbb{E}\left[\int_a^{t_2} \tilde{D}_{t_1}F_\varepsilon(t_2,\ldots,t_{n+1})f_{n+1}(t_1,\ldots,t_{n+1})dt_1 G\right] \\ &\quad= \mathbb{E}\left[\int_a^{t_2} \tilde{D}_{t_1}(GF_\varepsilon(t_2,\ldots,t_{n+1}))f_{n+1}(t_1,\ldots,t_{n+1})dt_1\right] \\ &\quad= \mathbb{E}\left[GF_\varepsilon(t_2,\ldots,t_{n+1})\left(\sum_{k=1}^{k=d} f_{n+1}(T_k,t_2,\ldots,t_{n+1}) - \int_0^{T_d} f_{n+1}(t,t_2,\ldots,t_{n+1})dt\right)\right], \end{aligned}$$

and

$$\begin{aligned} &\langle \tilde{D}^{n+1}F, \mathbf{1}_{\Delta_{n+1}} f_{n+1}G\rangle \\ &\quad= \lim_{\varepsilon\to 0}\left\langle \int_a^\infty\int_a^{t_{n+1}}\cdots\int_a^{t_2} \tilde{D}_{t_1}F_\varepsilon(t_2,\ldots,t_{n+1})f_{n+1}(t_1,\ldots,t_{n+1})dt_1\cdots dt_{n+1}, G\right\rangle \\ &\quad= \lim_{\varepsilon\to 0}\mathbb{E}\left[\int_a^\infty\int_a^{t_{n+1}}\cdots\int_a^{t_2} \tilde{D}_{t_1}F_\varepsilon(t_2,\ldots,t_{n+1})f_{n+1}(t_1,\ldots,t_{n+1})dt_1\cdots dt_{n+1}G\right] \\ &\quad= \lim_{\varepsilon\to 0}\mathbb{E}\left[G\int_a^\infty\int_a^{t_{n+1}}\cdots\int_a^{t_3} F_\varepsilon(t_2,\ldots,t_{n+1})\right. \\ &\qquad\left.\left(\sum_{k=1}^{k=d} f_{n+1}(T_k,t_2,\ldots,t_{n+1}) - \int_0^{T_d} f_{n+1}(t,t_2,\ldots,t_{n+1})dt\right)dt_2\cdots dt_{n+1}\right] \\ &\quad= \mathbb{E}\left[G\int_a^\infty\int_a^{t_{n+1}}\cdots\int_a^{t_3}\right. \\ &\qquad\left.\left(\sum_{k=1}^{k=d} f_{n+1}(T_k,t_2,\ldots,t_{n+1}) - \int_0^{T_d} f_{n+1}(t_1,t_2,\ldots,t_{n+1})dt_1\right)\tilde{D}_{t_2}\cdots\tilde{D}_{t_{n+1}}F dt_2\cdots dt_{n+1}\right] \end{aligned}$$

$$= \mathbb{E}\left[G \int_a^\infty \int_a^{t_{n+1}} \cdots \int_a^{t_3} \left(\sum_{k=1}^{k=d} f_{n+1}(T_k, t_2, \ldots, t_{n+1}) - \int_0^{T_d} f_{n+1}(t_1, t_2, \ldots, t_{n+1}) dt_1 \right) D_{t_2} \cdots D_{t_{n+1}} F dt_2 \cdots dt_{n+1} \right],$$

where on the last step we used the induction hypothesis with $a = t_2$. This is possible because the functional $\sum_{k=1}^{k=d} f_{n+1}(T_k, t_2, \ldots, t_{n+1}) - \int_0^{T_d} f_{n+1}(t_1, t_2, \ldots, t_{n+1}) dt_1$ is $\mathcal{F}_{t_2}$-measurable since it depends on T_k only when $T_k \leq t_2$, $1 \leq k \leq d$, due to the fact that $f_{n+1} \in \mathcal{C}_c^\infty(\Delta_{n+1})$.

The proof of Prop. 1 also shows that

$$\begin{aligned}
&\mathbb{E}\left[G \left(\sum_{k=1}^{k=d} f_{n+1}(T_k, t_2, \ldots, t_{n+1}) - \int_0^{T_d} f_{n+1}(t_1, t_2, \ldots, t_{n+1}) dt_1 \right) D_{t_2} \cdots D_{t_{n+1}} F \right] \\
&= \mathbb{E}\left[G \left(\sum_{k=1}^{k=\infty} f_{n+1}(T_k, t_2, \ldots, t_{n+1}) - \int_0^{\infty} f_{n+1}(t, t_2, \ldots, t_{n+1}) dt \right) D_{t_2} \cdots D_{t_{n+1}} F \right] \\
&= \mathbb{E}\left[G \int_a^{t_2} f_{n+1}(t_1, t_2, \ldots, t_{n+1}) d(N_{t_1} - t_1) D_{t_2} \cdots D_{t_{n+1}} F \right] \\
&= \mathbb{E}\left[G \int_a^{t_2} f_{n+1}(t_1, \ldots, t_{n+1}) D_{t_1} \cdots D_{t_{n+1}} F dt_1 \right],
\end{aligned}$$

where on the last line we used the duality (7) between D and the Poisson compensated integral on the adapted processes. Hence

$$\begin{aligned}
&\langle \tilde{D}^{n+1} F, \mathbf{1}_{\Delta_{n+1}} f_{n+1} G \rangle \\
&\quad = \mathbb{E}\left[G \int_a^\infty \int_a^{t_{n+1}} \cdots \int_a^{t_2} f_{n+1}(t_1, \ldots, t_{n+1}) D_{t_1} \cdots D_{t_{n+1}} F dt_1 \cdots dt_{n+1} \right].
\end{aligned}$$

This shows the almost-sure equality

$$\langle G \tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F, 1 \rangle = \mathbb{E}[G D_{t_1} \cdots D_{t_n} F], \qquad \text{a.e. } (t_1, \ldots, t_n) \in \Delta_n \cap [a, \infty[^n,$$

which becomes an equality for all $(t_1, \ldots, t_n) \in \Delta_n \cap [a, \infty[^n$ since $(t_1, \ldots, t_n) \mapsto \langle G \tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F, 1 \rangle$ and $(t_1, \ldots, t_n) \mapsto \mathbb{E}[G D_{t_1} \cdots D_{t_n} F]$ are clearly continuous functions on Δ_n when $F, G \in \mathcal{S}_d(\Omega)$. □

Note that Relation (9) does not hold if $(t_1, \ldots, t_n) \notin \Delta_n$, see Relation (14) below.

6 Chaos expansions of jump times functionals

Our result is stated for smooth functions $f(T_1, \ldots, T_d)$ of a finite number of jump times. We start with the simple case of $f(T_d)$. For $n \in \mathbb{Z}$ and $t \in \mathbb{R}_+$, let

$$p_n(t) = P(N_t = n) = \frac{t^n}{n!} e^{-t} = e^{-t} \int_0^t \int_0^{s_{n-1}} \cdots \int_0^{s_2} ds_1 \cdots ds_{n-1},$$

if $n \geq 0$, and $p_n(t) = P(N_t = n) = 0$ if $n < 0$, i.e. $p_{n-1} : \mathbb{R}_+ \to \mathbb{R}_+$, $n \geq 1$, is the density function of T_n, and

$$p_n^{(k)}(t) = \frac{\partial^k}{\partial t^k} p_n(t) = (-\Delta)^k p_n(t) = (-1)^k \frac{p_n(t)}{t^k} C_k^t(n),$$

where Δ is the finite difference operator $\Delta f(n) = f(n) - f(n-1)$ and C_k^t is the Charlier polynomial of order $k \in \mathbb{N}$ and parameter $t \in \mathbb{R}_+$.

Proposition 5 *The decomposition*

$$f(T_d) = \sum_{n=0}^{\infty} I_n(h_n \mathbf{1}_{\Delta_n}),$$

is given for $(t_1, \ldots, t_n) \in \Delta_n$ *as:*

$$h_n(t_1, \ldots, t_n) = (-1)^n \int_{t_n}^{\infty} f^{(n)}(t) p_{d-1}(t) dt + (-1)^n \sum_{j=1}^{j=n-1} (-1)^{j-1} f^{(n-j)}(t_n) p_{d-1}^{(j-1)}(t_n).$$

Proof. From Relation (9) of Prop. 4 and Relation (5) of Prop. 3 we have

$$\begin{aligned}
h_n(t_1, \ldots, t_n) &= \mathbb{E}[D_{t_1} \cdots D_{t_n} f(T_d)] =< \tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_d), 1 > \\
&= (-1)^n \left\langle f^{(n)}(T_d) \mathbf{1}_{[0,T_d]}(t_n) + \sum_{j=1}^{n-1} f^{(n-j)}(t_n) \delta_{t_n}^{(j-1)}(T_d), 1 \right\rangle \\
&= (-1)^n \int_0^{\infty} \mathbf{1}_{[t_n,\infty[}(s_d) e^{-s_d} f^{(n)}(s_d) \int_0^{s_d} \cdots \int_0^{s_2} ds_1 \cdots ds_d \\
&\quad + (-1)^n \sum_{j=1}^{n-1} f^{(n-j)}(t_n) \int_0^{\infty} e^{-s_d} \int_0^{s_d} \cdots \int_0^{s_2} ds_1 \cdots ds_{d-1} \delta_{t_n}^{(j-1)}(ds_d) \\
&= (-1)^n \int_{t_n}^{\infty} e^{-s_d} f^{(n)}(s_d) \frac{{s_d}^{d-1}}{(d-1)!} ds_d \\
&\quad + (-1)^n \sum_{j=1}^{n-1} f^{(n-j)}(t_n) \int_0^{\infty} e^{-s_d} \frac{{s_d}^{d-1}}{(d-1)!} \delta_{t_n}^{(j-1)}(ds_d) \\
&= (-1)^n \int_{t_n}^{\infty} f^{(n)}(t) p_{d-1}(t) dt + (-1)^n \sum_{j=1}^{j=n-1} (-1)^{j-1} f^{(n-j)}(t_n) p_{d-1}^{(j-1)}(t_n).
\end{aligned}$$

□

By induction this gives for $l = 0, \ldots, n-1$:

$$h_n(t_1, \ldots, t_n) = (-1)^{n+l} \int_{t_n}^{\infty} f^{(n-l)}(t) p_{d-1}^{(l)}(t) dt + (-1)^n \sum_{j=l+1}^{j=n-1} (-1)^{j-1} f^{(n-j)}(t_n) p_{d-1}^{(j-1)}(t_n),$$

and in particular for $l = n-1$:

$$h_n(t_1, \ldots, t_n) = -\int_{t_n}^{\infty} f'(s) p_{d-1}^{(n-1)}(s) ds = f(t_n) p_{d-1}^{(n-1)}(t_n) + \int_{t_n}^{\infty} f(s) p_{d-1}^{(n)}(s) ds,$$

hence

$$f(T_d) = \sum_{n \geq 0} \frac{1}{n!} I_n \left(f(t_1 \vee \cdots \vee t_n) p_{d-1}^{(n-1)}(t_1 \vee \cdots \vee t_n) + \int_{t_1 \vee \cdots \vee t_n}^{\infty} f(s) p_{d-1}^{(n)}(s) ds \right), \tag{10}$$

with the convention $t_1 \vee t_0 = 0$. In order to treat the case of d variables we recall the notation

$$a_i(j_1, \ldots, j_n) = \text{Card}\{l \ : \ j_l = i\}, \quad \text{and} \quad c_i(j_1, \ldots, j_n) = \max\{l : j_l = i\}.$$

Proposition 6 *Let $f \in \mathcal{C}_b^{\infty}(\Delta_d)$. We have*

$$f(T_1, \ldots, T_d) = \sum_{n=0}^{\infty} I_n(h_n \mathbf{1}_{\Delta_n}),$$

with

$$\begin{aligned} h(t_1, \ldots, t_n) &= (-1)^n \sum_{1 \leq j_1, \ldots, j_n \leq d} \ \sum_{0 \leq i_1 \leq (a_1 - 1) \vee 0} \cdots \sum_{0 \leq i_d \leq (a_d - 1) \vee 0} \end{aligned} \tag{11}$$

$$\int_0^{\infty} \int_0^{s_d} \cdots \int_0^{s_2} e^{-s_d} \partial_1^{a_1 - i_1} \cdots \partial_d^{a_d - i_d} f(s_{1,i_1}, \ldots, s_{d,i_d}) \prod_{\substack{l=1 \\ a_l \neq 0}}^{l=d} (-r^{(1+i_l)}(ds_l, t_{c_l})),$$

where $r^{(1)}(ds, t) = \mathbf{1}_{[t, \infty[}(s) ds$, and

$$s_{l,i_l} = \begin{cases} s_l & \text{if } i_l = 0, \\ t_{c_l} & \text{if } i_l \geq 1, \quad l = 1, \ldots, d. \end{cases}$$

Proof. We apply Th. 1 and Relation (9) of Prop. 4. □

This expression can be made more explicit by evaluation of the action of $r^{(1+i_l)}(ds_{j_l}, t_{c_l})$, either as an indicator function or as a derivative in t_{c_l}. However this will not be done here in order to keep formula (11) to a reasonable size.

In [8], another expression (different from (11)) has been obtained using elementary orthogonal decompositions in Charlier polynomials. Let $n_1, \dots, n_l \in \mathbb{N}$ with $1 \le n_1 < \cdots < n_l$, and let $f \in \mathcal{C}_b^d(\Delta_l)$. As a convention, if $k_1 \ge 0, \dots, k_d \ge 0$ satisfy $k_1 + \cdots + k_d = n$, we let for $(t_1, \dots, t_n) \in \Delta_n$:

$$(t_1^1, \dots, t_{k_1}^1, t_1^2, \dots, t_{k_2}^2, \dots, t_1^d, \dots, t_{k_d}^d) = (t_1, \dots, t_n),$$

with $t_{k_i}^i = 0$ if $k_i = 0$, and $(t_1^i, t_0^i) = ()$. We have

$$f(T_{n_1}, \dots, T_{n_l}) \;=\; \sum_{n=0}^{\infty} I_n(\mathbf{1}_{\Delta_n} h_n),$$

where

$$h_n(t_1, \dots, t_n) = \quad (12)$$
$$(-1)^l \sum_{k_1+\cdots+k_l=n} \int_{t_{k_l}^l}^{\infty} \cdots \int_{t_{k_i}^i}^{t_1^{i+1}} \cdots \int_{t_{k_1}^1}^{t_1^2} \partial_1 \cdots \partial_l f(s_1, \dots, s_l) K_{s_1,\dots,s_l}^{k_1,\dots,k_l} ds_1 \cdots ds_l,$$

with, for $0 \le s_1 \le \cdots \le s_l$ and $k_1 \ge 0, \dots, k_l \ge 0$:

$$K_{s_1,\dots,s_l}^{k_1,\dots,k_l} = \sum_{\substack{m_1 \ge n_1, \dots, m_l \ge n_l \\ m_1 \le \cdots \le m_l}} p_{m_1-m_0}^{(k_1)}(s_1 - s_0) \cdots p_{m_l-m_{l-1}}^{(k_l)}(s_l - s_{l-1}), \qquad m_0 = 0, \;\; s_0 = 0,$$

cf. Th. 1 of [8]. For $l = 1$, i.e. for $f(T_d)$, $n_1 = d$, we have

$$K_s^k \;=\; \sum_{m=d}^{\infty} p_m^{(k)}(s) = \frac{\partial^{k-1}}{\partial s^{k-1}} \sum_{m=d}^{\infty} p_m^{(1)}(s) = \frac{\partial^{k-1}}{\partial s^{k-1}} \sum_{m=d}^{\infty} p_{m-1}(s) - p_m(s) = p_{d-1}^{(k-1)}(s)$$

hence

$$h_n(t_1, \dots, t_n) = -\int_{t_n}^{\infty} f'(s) p_{d-1}^{(n-1)}(s) ds,$$

which coincides with (10).

Remarks

i) All expressions obtained above for $f(T_1, \dots, T_d)$, $f \in \mathcal{C}_b^\infty(\Delta_d)$, extend to $f \in L^2(\Delta_d, e^{-s_d} ds_1 \cdots ds_d)$, i.e. to square-integrable $f(T_1, \dots, T_d)$, by repeated integrations by parts.

ii) Chaotic decompositions on the Poisson space on the compact interval $[0,1]$ as in [4] or [5] can be obtained by considering the functional $f(1 \wedge T_1, \dots, 1 \wedge T_d)$ instead of $f(T_1, \dots, T_d)$.

iii) If $t_1 > \cdots > t_n$ then Relation (9) does not hold, for example we have

$$\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_d) = (-1)^n \mathbf{1}_{[0,T_d]}(t_1) f^{(n)}(T_d),$$

and

$$\mathbb{E}[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_d)] = (-1)^n \mathbb{E}[\mathbf{1}_{[0,T_d]}(t_1) f^{(n)}(T_d)] = (-1)^n \int_{t_1}^\infty f^{(n)}(s) p_{d-1}(s) ds, \tag{13}$$

which differs (if $n \geq 2$) from

$$\mathbb{E}[D_{t_1} \cdots D_{t_n} f(T_d)] = -\int_{t_1}^\infty f'(s) p_{d-1}^{(n-1)}(s) ds. \tag{14}$$

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