Girsanov identities for Poisson measures under quasi-nilpotent transformations

Nicolas Privault

Division of Mathematical Sciences

School of Physical and Mathematical Sciences

Nanyang Technological University

SPMS-MAS, 21 Nanyang Link

Singapore 637371

E-mail: nprivault@ntu.edu.sg

December 15, 2010

Abstract

We prove a Girsanov identity on the Poisson space for anticipating transformations that satisfy a strong quasi-nilpotence condition. Applications are given to the Girsanov theorem and to the invariance of Poisson measures under random transformations. The proofs use combinatorial identities for the central moments of Poisson stochastic integrals.

Key words: Poisson measures, random transformations, Girsanov identities, quasi-invariance, invariance, Skorohod integral, moment identities, Stirling numbers.
Mathematics Subject Classification: 60G57, 60G30, 60H07, 28D05, 28C20, 11B73.

1 Introduction

The Wiener and Poisson measures are well known to be quasi-invariant under adapted shifts. This quasi-invariance property has been extended to anticipative shifts by several authors; cf. [23], [9] and [26] and references therein in the Wiener case, and, for example, [2], [16], [17], [18], in the Poisson case.

In the anticipative case the corresponding Radon-Nikodym density is usually written as the product

$$|\det_2(I + \nabla u)| \exp\left(-\delta(u) - \frac{1}{2}\|u\|^2\right)$$

of a Skorohod-Doléans exponential with the Carleman-Fredholm determinant of the Malliavin gradient ∇u of the shift u; cf. [23], [9], [26]. A similar formula can be obtained for Poisson random measures; cf. Section 8.

It has been noted in [27] that the standard Doléans form of the density for anticipative shifts $u : W \rightarrow H$ on the Wiener space W with Cameron-Martin space H can be conserved (i.e. the Carleman-Fredholm determinant $\det_2(I + \nabla u)$ equals one) when the gradient ∇u of the shift u is quasi-nilpotent, that is,

$$\lim_{n \to \infty} \|(\nabla u)^n\|_{HS}^{1/n} = 0, \quad \text{or equivalently} \quad \text{trace}(\nabla u)^n = 0, \quad n \geq 2, \tag{1.1}$$

cf. [27] or Theorem 3.6.1 of [26]. In particular, when ∇u is quasi-nilpotent and $\|u\|$ is constant, it has been shown in [25] that $\delta(u)$ has a centered Gaussian law with variance $\|u\|^2$, cf. [20] for a simplified proof.

In this paper we consider the Poisson space Ω^X over a metric space X with σ-finite intensity measure $\sigma(dx)$, and investigate the quasi-invariance of random transformations $\tau(\omega, \cdot)$ which are assumed to be quasi-nilpotent in the sense that the finite difference gradient $D_s\tau(\omega, t)$ satisfies the cyclic finite difference condition (2.3) below, which is a strenghtened version of (1.1). We show in particular that such anticipating quasi-nilpotent transformations are quasi-invariant, and their Radon-Nikodym densities are given by Doléans stochastic exponentials with jumps. This also extends and recovers other results on the invariance of random transformations of Poisson measures; cf. [22].

Our starting point is the classical Girsanov identity for Poisson random measures which states that

$$E_\sigma\left[\exp\left(-\int_X g(x)\sigma(dx)\right)\prod_{x \in \omega}(1 + g(x))\right] = 1, \tag{1.2}$$

2

and rewrites when $g = \mathbf{1}_A$ as

$$E_\sigma \left[e^{-r\sigma(A)}(1+r)^{\omega(A)} \right] = 1, \qquad r \in \mathbb{R},$$

which is equivalent to the vanishing of the expectation

$$E[C_n(Z, \lambda)] = 0, \qquad n \geq 1,$$

for $Z = \omega(A)$ a Poisson random variable with intensity $\lambda = \sigma(A)$, where $C_n(x, \lambda)$ is the Charlier polynomials of degree $n \in \mathbb{N}$, with generating function

$$e^{-r\lambda}(1+r)^x = \sum_{n=0}^{\infty} \frac{r^n}{n!} C_n(x, \lambda), \qquad r > -1.$$

It is well known, however, that Z need not have a Poisson distribution for $E[C_n(Z, \lambda)]$ to vanish when λ is allowed to be random. Indeed, such an identity also holds in the random adapted case under the form

$$E[C_n(N_{\tau^{-1}(t)}, \tau^{-1}(t))] = 0, \qquad n \geq 1, \tag{1.3}$$

where $(N_t)_{t \in \mathbb{R}_+}$ is a standard Poisson process generating a filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ and $\tau(t)$ is an $\mathcal{F}_t$-adapted time change, due to the fact that

$$C_n\left(N_{\tau^{-1}(t)}, \tau^{-1}(t)\right) = n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} d(N_{\tau^{-1}(t_1)} - d\tau^{-1}(t_1)) \cdots d(N_{\tau^{-1}(t_n)} - d\tau^{-1}(t_n)),$$

is an adapted nth order iterated multiple stochastic integral with respect to the compensated point martingale $(N_{\tau^{-1}(t)} - \tau^{-1}(t))_{t \in \mathbb{R}_+}$; cf. [24] and [12] page 320. In this case we also have

$$E_\sigma \left[e^{-r\tau^{-1}(t)}(1+r)^{N_{\tau^{-1}(t)}} \right] = 1, \qquad r \in \mathbb{R},$$

and more generally

$$E_\sigma \left[\exp\left(-\int_0^\infty g(\tau(s)) ds \right) \prod_{\substack{\Delta N_s = 1 \\ 0 < s < \infty}} (1 + g(\tau(s))) \right] = 1, \tag{1.4}$$

under a Novikov-type integrability condition on $g : \mathbb{R} \to \mathbb{R}$; cf., for example, [10].

In Corollary 2.2 below we will extend the Girsanov identity (1.4) to random anticipating processes indexed by an abstract space X, by computing the expectation

$$E_\sigma[C_n(\omega(A), \sigma(A))], \qquad n \geq 1,$$

of the random Charlier polynomial $C_n(\omega(A), \sigma(A))$, where $A(\omega)$ is a random, possibly anticipating set. In particular we provide conditions on $A(\omega)$ for the expectation $E_\sigma[C_n(\omega(A), \sigma(A))]$, $n \geq 1$, to vanish, cf. Proposition 7.1 below. Such conditions are satisfied, in particular, under the quasi-nilpotence condition (2.3) below and include the adaptedness of $(\tau(t))_{t \in \mathbb{R}_+}$ above, which recovers the classical adapted Girsanov identity (1.4) as a particular case; cf. Proposition 2.1. As a consequence we will obtain a Girsanov theorem for random transformations of Poisson samples on an arbitrary measure space.

The above results will be proved using the Skorohod integral and integration by parts on the Poisson space. This type of argument has been applied in [22] to the inductive computation of moments of Poisson stochastic integrals and to the invariance of the Skorohod integral under random intensity preserving transformations. However, the case of Charlier polynomials is more complicated, and it leads to Girsanov identities and a Girsanov theorem as additional applications.

Since our use of integration by parts formulas and moment identities relies on compensated Poisson stochastic integrals, we will need to work with a family $B_n(y, \lambda)$ of polynomials such that

$$B_n(y, -\lambda) = E_\lambda[(Z + y - \lambda)^n],$$

where Z is a Poisson random variable with intensity $\lambda > 0$, and which are related to the Charlier polynomials by the relation

$$C_n(y, \lambda) = \sum_{k=0}^n s(n, k) B_k(y - \lambda, \lambda),$$

where $s(k, l)$ is the Stirling number of the first kind, that is, $(-1)^{k-l} s(k, l)$ is the number of permutations of k elements which contain exactly l permutation cycles, $n \in \mathbb{N}$;

4

cf. Proposition 6.1 below.

The outline of this paper is as follows. Section 2 contains our main results on antic-ipative Girsanov identities and applications to the Girsanov theorem. In Section 3 we consider some examples of anticipating transformations to which this theorem can be applied; this includes the adapted case as well as transformations that act inside the convex hull generated by Poisson random measures, given the positions of the ex-tremal vertices. In Section 4 we show that those results are consequences of identities for multiple integrals and stochastic exponentials. In Section 5 we review some results of [22] (cf. also [19]) on the computation of moments of Poisson stochastic integrals, and we derive some of their corollaries to be applied in this paper. In Section 6 we de-rive some combinatorial identities that allow us, in particular, to rewrite the Charlier polynomials into a form suitable to the use of moment identities. Finally in Section 7 we prove the results of Section 4, and in Section 8 we make some remarks on how the results of this paper can be connected to the Carleman-Fredholm determinant.

2 Main results

Let Ω^X denote the configuration space on a σ-compact metric space X with Borel σ-algebra $\mathcal{B}(X)$, that is,

$$\Omega^X = \left\{ \omega = (x_i)_{i=1}^N \subset X, \ x_i \neq x_j \ \forall i \neq j, \ N \in \mathbf{N} \cup \{\infty\} \right\}$$

is the space of at most countable locally finite subsets of X, endowed with the Poisson probability measure π_σ with σ-finite diffuse intensity $\sigma(dx)$ on X, which is character-ized by its Laplace transform

$$\psi_\sigma(f) = E_\sigma \left[\exp \left(\int_X f(x)(\omega(dx) - \sigma(dx)) \right) \right] = \exp \left(\int_X (e^{f(x)} - f(x) - 1)\sigma(dx) \right),$$
(2.1)

$f \in L^2_\sigma(X)$, or by the Girsanov identity (1.2) by taking $f(x) = \log(1 + g(x))$, $x \in X$, $g \in \mathcal{C}_c(X)$, where E_σ denotes the expectation under π_σ, and $\mathcal{C}_c(X)$ is the space of continuous functions with compact support in X.

Each element ω of Ω^X is identified to the Radon point measure

$$\omega = \sum_{i=1}^{\omega(X)} \epsilon_{x_i},$$

where ϵ_x denotes the Dirac measure at $x \in X$, and $\omega(X) \in \mathbb{N} \cup \{\infty\}$ denotes the cardinality of $\omega \in \Omega^X$.

Consider a measurable random transformation

$$\tau : \Omega^X \times X \to X,$$

of X, let $\tau_*(\omega)$, $\omega \in \Omega^X$, denote the image measure of $\omega(dx)$ by $\tau(\omega, \cdot) : X \to X$, that is,

$$\tau_* : \Omega^X \to \Omega^X \tag{2.2}$$

maps

$$\omega = \sum_{i=1}^{\omega(X)} \epsilon_{x_i} \qquad \text{to} \qquad \tau_*(\omega) = \sum_{i=1}^{\omega(X)} \epsilon_{\tau(\omega, x_i)}.$$

In other words, the random mapping $\tau_* : \Omega^X \to \Omega^X$ shifts each configuration point $x \in \omega$ according to $x \mapsto \tau(\omega, x)$.

Let D denote the finite difference gradient defined on any random variable $F : \Omega^X \to \mathbb{R}$ as

$$D_x F(\omega) = F(\omega \cup \{x\}) - F(\omega), \qquad \omega \in \Omega^X, \quad x \in X,$$

for any random variable $F : \Omega^X \to \mathbb{R}$; cf. [7], [8], [15]. The operator D is continuous on the space $\mathbb{D}_{2,1}$ defined by the norm

$$\|F\|_{2,1}^2 = \|F\|_{L^2(\Omega^X, \pi_\sigma)}^2 + \|DF\|_{L^2(\Omega^X \times X, \pi_\sigma \otimes \sigma)}^2, \qquad F \in \mathbb{D}_{2,1}.$$

The next result is a Girsanov identity for random, non-adapted shifts of Poisson configuration points, obtained as a consequence of Proposition 4.1 below which is proved at the end of Section 4. Here we let Y denote another metric space with Borel σ-algebra $\mathcal{B}(Y)$.

Proposition 2.1 *Assume that $\tau : \Omega^X \times X \to Y$ satisfies the cyclic condition*

$$D_{t_1}\tau(\omega, t_2) \cdots D_{t_k}\tau(\omega, t_1) = 0, \qquad \sigma(dt_1), \ldots, \sigma(dt_k) - a.e., \quad \omega \in \Omega^X, \qquad (2.3)$$

for all $k \geq 2$, and let $g : Y \to \mathbb{R}$ be a measurable function such that

$$E_\sigma \left[e^{\int_X |g(\tau(\omega,x))|\sigma(dx)} \prod_{x \in \omega} (1 + |g(\tau(\omega,x))|) \right] < \infty. \qquad (2.4)$$

Then we have

$$E_\sigma \left[e^{- \int_X g(\tau(\omega,x))\sigma(dx)} \prod_{x \in \omega} (1 + g(\tau(\omega,x))) \right] = 1.$$

As a consequence of Proposition 2.1, if $\tau : \Omega^X \times X \to X$ satisfies (2.3) and $\tau(\omega, \cdot) : X \to Y$ maps σ to a fixed measure μ on $(Y, \mathcal{B}(Y))$ for all $\omega \in \Omega^X$, then we have

$$
\begin{aligned}
E_\sigma \left[\prod_{x \in \omega} (1 + g(\tau(\omega,x))) \right] &= e^{\int_X g(\tau(\omega,x))\sigma(dx)} \\
&= e^{\int_Y g(y)\mu(dy)}, \qquad g \in \mathcal{C}_c(Y);
\end{aligned}
$$

hence $\tau_* : \Omega^X \to \Omega^X$ maps π_σ to π_μ, which recovers Corollary 3.3 of [22].

Proposition 2.1 then implies the following anticipating Girsanov theorem, in which the Radon-Nikodym density is given by a Doléans exponential.

Corollary 2.2 *Assume that for all $\omega \in \Omega^X$, $\tau(\omega, \cdot) : X \to X$ is invertible on X and that for all $t_0, \ldots, t_k \in X$, $k \geq 1$, there exists $i \in \{0, \ldots, k\}$ such that*

$$D_{t_i}\tau(\omega, x) = 0, \qquad (2.5)$$

for all x in a neighborhood of $t_{i+1 \bmod k}$, and that the density

$$\phi(\omega, x) := \frac{d\tau_*^{-1}(\omega, \cdot)\sigma}{d\sigma}(x) - 1, \qquad x \in X,$$

exists for all $\omega \in \Omega^X$, with

$$E_\sigma \left[e^{(1+\varepsilon) \int_X \phi(\omega,x)\sigma(dx)} \prod_{x \in \omega} (1 + \phi(\omega,x))^{1+\varepsilon} \right] < \infty, \qquad (2.6)$$

7

for some $\varepsilon > 0$. Then we have the Girsanov identity

$$E_\sigma \left[F(\tau_*(\omega)) e^{-\int_X \phi(\omega,x)\sigma(dx)} \prod_{x\in\omega}(1 + \phi(\omega,x)) \right] = E_\sigma[F],$$

for all $F \in L^1(\Omega^X)$.

Proof. First we note that from (2.5), for all $\omega \in \Omega^X$ and $t_0, \ldots, t_k \in X$, $k \geq 1$, there exists $i \in \{0, \ldots, k\}$ such that

$$D_{t_i}\tau(\omega, t_{i+1 \mod k}) = D_{t_i}\phi(\omega, t_{i+1 \mod k}) = 0. \tag{2.7}$$

Next from Proposition 2.1, for all $f \in \mathcal{C}_c(X)$ we have

$$E_\sigma \left[e^{-\int_X f(x)\sigma(dx) - \int_X \phi(\omega,x)\sigma(dx)} \prod_{x\in\omega}(1 + f(\tau(\omega,x)))(1 + \phi(\omega,x)) \right]$$

$$= E_\sigma \left[e^{-\int_X f(\tau(\omega,x))(1+\phi(\omega,x))\sigma(dx) - \int_X \phi(\omega,x)\sigma(dx)} \right.$$

$$\left. \times \prod_{x\in\omega}(1 + f(\tau(\omega,x)) + \phi(\omega,x) + f(\tau(\omega,x))\phi(\omega,x)) \right]$$

$$= 1,$$

by Proposition 2.1, since

$$x \mapsto f(\tau(\omega,x)) + \phi(\omega,x) + f(\tau(\omega,x))\phi(\omega,x)$$

satisfies Condition (2.3) by (2.7). We conclude by the density in $L^1(\Omega^X)$ of linear combinations of F of the form

$$F = \exp\left(-\int_X f(x)\sigma(dx) \right) \prod_{x\in\omega}(1 + f(x)), \qquad f \in \mathcal{C}_c(X).$$

$\square$

Under the hypotheses of Corollary 2.2, if $\tau_* : \Omega^X \to \Omega^X$ is invertible then the random transformation $\tau_*^{-1} : \Omega^X \to \Omega^X$ is absolutely continuous with respect to π_σ, with density

$$\frac{d\tau_*^{-1}\pi_\sigma}{d\pi_\sigma} = e^{-\int_X \phi(\omega,x)\sigma(dx)} \prod_{x\in\omega}(1 + \phi(\omega,x)). \tag{2.8}$$

In Corollary 2.2, Condition (2.6) actually requires $\sigma(\tau(X))$ to be a.s. finite.

3 Examples

In this section we present an example of a random non-adapted transformation satisfying the hypotheses of Corollary 2.2. First we note that Condition (2.3) is an extension of the usual adaptedness condition, as it holds when $\tau : X \to X$ is adapted to a given total binary relation $\preceq$ on X. Indeed, if $\tau : \Omega^X \times X \to X$ satisfies

$$D_x \tau(\omega, y) = 0, \qquad y \preceq x,$$

then Condition (2.3) is satisfied since for all $t_1, \ldots, t_k \in X$ there exists $i \in \{1, \ldots, k\}$ such that $t_j \preceq t_i$, for all $1 \le j \le k$; hence $D_{t_i} \tau(\omega, t_j) = 0$, $1 \le j \le k$. In this case, Corollary 2.2 recovers a classical result in the case where $\tau : X \to X$ is deterministic or adapted; cf., for example, Theorem 3.10.21 of [4].

Next, let $X = \bar{B}(0, 1)$ denote the closed unit ball in $\mathbb{R}^d$, with $\sigma(dx)$ the Lebesgue measure. For all $\omega \in \Omega^X$, let $\mathcal{C}(\omega)$ denote the convex hull of ω in X with interior $\dot{\mathcal{C}}(\omega)$, and let $\omega_e = \omega \cap (\mathcal{C}(\omega) \setminus \dot{\mathcal{C}}(\omega))$ denote the extremal vertices of $\mathcal{C}(\omega)$. Consider a measurable mapping $\tau : \Omega^X \times X \to X$ such that for all $\omega \in \Omega^X$, $\tau(\omega, \cdot)$ is measure preserving, maps $\dot{\mathcal{C}}(\omega)$ to $\dot{\mathcal{C}}(\omega)$, and for all $\omega \in \Omega^X$,

$$\tau(\omega, x) = \begin{cases} \tau(\omega_e, x), & x \in \dot{\mathcal{C}}(\omega), \\ x, & x \in X \setminus \dot{\mathcal{C}}(\omega), \end{cases} \tag{3.1}$$

that is, $\tau(\omega, \cdot) : X \to X$ modifies only the inside points of the convex hull of ω, depending on the positions of its extremal vertices, which are left invariant by $\tau(\omega, \cdot)$, as illustrated in the next figure.
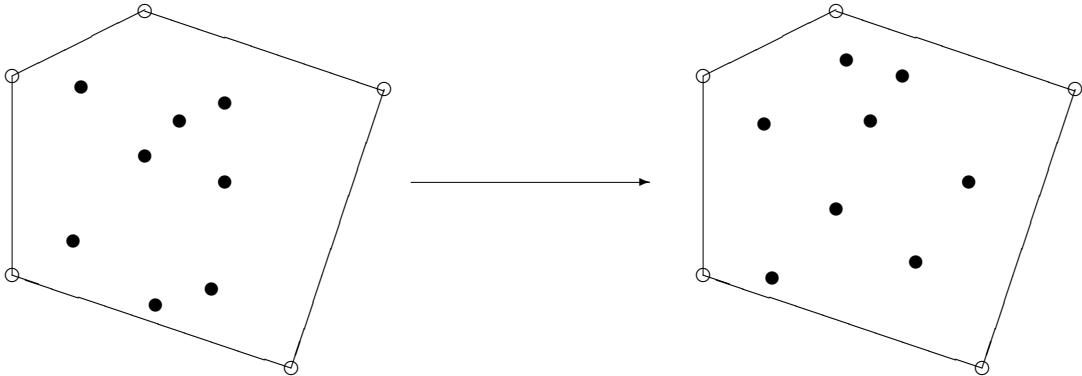
Next, assume that $\tau(\omega, \cdot) : X \to X$ in (3.1) has the form

$$\tau(\omega, x) = x + \psi(\omega_e, x), \qquad x \in X,$$

for fixed $\omega \in \Omega^X$, where $\psi(\omega_e, \cdot) : X \to X$ is a diffeomorphism such that $\tau(\omega, \cdot) : X \to X$ is invertible for all $\omega \in \Omega^X$; for example,

$$\psi(\omega_e, x) = u \mathbf{1}_{\mathcal{C}(\omega)}(x) \frac{(d(x, \mathcal{C}(\omega) \setminus \dot{\mathcal{C}}(\omega)))^2}{1 + (d(x, \mathcal{C}(\omega) \setminus \dot{\mathcal{C}}(\omega)))^2}, \qquad x \in X, \tag{3.2}$$

with $u \in \mathbb{R}^d$ such that $\|u\|_d < 1/4$, where $d(x, A)$ denotes the Euclidean distance from $x \in \mathbb{R}^d$ to the closed set $A \subset \mathbb{R}^d$. Then the transformation $\tau : \Omega^X \times X \to X$ satisfies the hypotheses of Corollary 2.2 by Proposition 3.1 below, and $\tau^* : \Omega^X \to \Omega^X$ is invertible with

$$(\tau^*)^{-1}(\omega) = \omega_e \cup \bigcup_{x \in \omega \cap \dot{\mathcal{C}}(\omega)} \{\tau^{-1}(\omega_e, x)\}, \qquad \omega \in \Omega^X,$$

thus the associated Radon-Nikodym density (2.8) is given by taking

$$\phi(\omega, x) = \det \left(I_{\mathbb{R}^d} + \nabla_x \psi(\omega_e, x) \right) - 1, \qquad \omega \in X, \quad x \in X.$$

This quasi-invariance property is related to the intuitive fact that a Poisson random measure remains Poisson within its convex hull when its configuration points are shifted given to the position of its extremal vertices, cf., for example, [6].

Proposition 3.1 *Assume that the random transformation $\tau : \Omega^X \times X \to X$ satisfies Condition (3.1). Then τ satisfies the cyclic Condition (2.5) of Corollary 2.2.*

Proof. Let $t_1, \ldots, t_k \in X$. First, if there exists $i \in \{1, \ldots, k\}$ such that $t_i \in \mathcal{C}(\omega)$, then for all $x \in X$ we have $t_i \in \mathcal{C}(\omega \cup \{x\})$, and by Lemma 3.2 below we get

$$D_{t_i} \tau(\omega, x) = 0, \qquad x \in X;$$

thus (2.5) holds, and we may assume that $t_i \notin \mathcal{C}(\omega)$ for all $i = 1, \ldots, k$. In this case, if $t_{i+1 \bmod k} \notin \mathcal{C}(\omega \cup \{t_i\})$ for some $i = 1, \ldots, k$, then by Lemma 3.2 we have

$$D_{t_i} \tau(\omega, t_{i+1 \bmod k}) = 0;$$

hence (2.5) holds since the set $\mathcal{C}(\omega \cup \{t_i\})$ is closed. Next, if $t_1 \in \mathcal{C}(\omega \cup \{t_k\})$, $t_k \in \mathcal{C}(\omega \cup \{t_{k-1}\})$, ..., $t_2 \in \mathcal{C}(\omega \cup \{t_1\})$, then we have $t_1 \in \mathcal{C}(\omega \cup \{t_k\})$ and $t_k \in \mathcal{C}(\omega \cup \{t_1\})$, which implies $t_1 = t_k \notin \mathcal{C}(\omega)$, and we check that $D_{t_k} \tau(\omega, t_1) = 0$. $\square$

Next we present the proof of Lemma 3.2; cf. [22].

Lemma 3.2 *[22] For all $x, y \in X$ and $\omega \in \Omega^X$ we have*

$$x \in \mathcal{C}(\omega \cup \{y\}) \implies D_x \tau(\omega, y) = 0 \tag{3.3}$$

and

$$y \in \mathcal{C}(\omega \cup \{x\}) \implies D_x \tau(\omega, y) = 0. \tag{3.4}$$

Proof. Let $x, y \in X$ and $\omega \in \Omega^X$. First, if $y \notin \mathcal{C}(\omega \cup \{x\})$ we have $\tau(\omega \cup \{x\}, y) = \tau(\omega, y) = y$. Next, if $x \in \mathcal{C}(\omega \cup \{y\})$, we can distinguish two cases:

a) $x \in \mathcal{C}(\omega)$. In this case we have $\mathcal{C}(\omega \cup \{x\}) = \mathcal{C}(\omega)$; hence $\tau(\omega \cup \{x\}, y) = \tau(\omega, y)$ for all $y \in X$.

b) $x \in \mathcal{C}(\omega \cup \{y\}) \setminus \mathcal{C}(\omega)$. If $y \in \mathcal{C}(\omega \cup \{x\})$, then $x = y \notin \dot{\mathcal{C}}(\omega \cup \{x\})$; hence $\tau(\omega \cup \{x\}, y) = \tau(\omega, y)$. On the other hand if $y \notin \mathcal{C}(\omega \cup \{x\})$, then $\tau(\omega \cup \{x\}, y) = \tau(\omega, y) = y$ as above.

We conclude that $D_x \tau(\omega, y) = 0$ in both cases. $\square$

We refer to [22] for more examples of transformations $\tau : \Omega^X \times X \to X$ that satisfy the cyclic condition (2.3).

4   Multiple integrals and stochastic exponentials

The proofs of the above results will use properties of stochastic exponentials and multiple stochastic integrals which are introduced and proved in this section. Let now

$$I_n(f_n)(\omega) = \int_{\Delta_n} f_n(x_1, \ldots, x_n)(\omega(dx_1) - \sigma(dx_1)) \cdots (\omega(dx_n) - \sigma(dx_n))$$

denote the multiple Poisson stochastic integral of the symmetric function $f_n \in L^2_\sigma(X^n)$, where

$$\Delta_n = \{(x_1, \ldots, x_n) \in X^n \ : \ x_i \neq x_j, \ \forall i \neq j\},$$

with

$$e^{-\int_X g(x)\sigma(dx)} \prod_{x \in \omega} (1 + g(x)) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(g^{\otimes n}),$$

11

for $g \in L^2_\sigma(X)$ with bounded support, where "$\otimes$" denotes the tensor product of functions in $L^2_\sigma(X)$. For all (possibly random) disjoint subsets $A_1, \ldots, A_n$ of X with finite measure, we have the relation

$$I_N(\mathbf{1}_{A_1^{k_1}} \circ \cdots \circ \mathbf{1}_{A_n^{k_n}}) = \prod_{i=1}^n C_{k_i}(\omega(A_i), \sigma(A_i)) \tag{4.1}$$

between the multiple Poisson integrals and the Charlier polynomials, where "$\circ$" denotes the symmetric tensor product of functions in $L^2_\sigma(X)$ and $N = k_1 + \cdots + k_n$; cf., for example, Proposition 6.2.9 in [21].

Proposition 2.1 will be proved using the following Proposition 4.1 which is a restatement of Corollary 7.2 below. It provides a formula for the expectation of a multiple stochastic integral of a time-changed function.

Proposition 4.1 *Assume that $\tau : \Omega^X \times X \to Y$ satisfies*

$$D_t \tau(\omega, t) = 0, \qquad \omega \in \Omega^X, \quad t \in X. \tag{4.2}$$

Then for all symmetric step functions $g : Y^N \to \mathbb{R}$ of the form

$$g = \sum_{\substack{k_1 + \cdots + k_n = N \\ 1 \leq n \leq N}} c_{k_1, \ldots, k_n} \mathbf{1}_{B_{1,k_1}}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{B_{n,k_n}}^{\otimes k_n}$$

where $N \geq 1$ and $B_{1,k_1}, \ldots, B_{n,k_n}$ are deterministic disjoint Borel subsets of Y and $c_{k_1, \ldots, k_n} \in \mathbb{R}$, we have

$$E_\sigma \left[I_N(\mathbf{1}_{A^N}(\cdot)\tau^{\otimes N}(\omega, \cdot)) \right] = E_\sigma \left[\int_{A^N} D_{t_1} \cdots D_{t_N} g(\tau(\omega, t_1), \ldots, \tau(\omega, t_N)) \sigma(dt_1) \cdots \sigma(dt_N) \right],$$

for all compact subset $A \in \mathcal{B}(X)$ of X.

Proof. It suffices to prove that for all deterministic disjoint Borel subsets $B_1, \ldots, B_n$ of Y we have

$$E_\sigma \left[I_N(\mathbf{1}_{A^N} \mathbf{1}_{\tau^{-1}(B_1)}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{\tau^{-1}(B_n)}^{\otimes k_n}) \right] = E_\sigma \left[I_N(\mathbf{1}_{A \cap \tau^{-1}(B_1)}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{A \cap \tau^{-1}(B_n)}^{\otimes k_n}) \right]$$

$$= E_\sigma \left[\int_{A^N} \left(\prod_{i=1}^N D_{t_i} \right) \left(\mathbf{1}_{B_1^{k_1}} \otimes \cdots \otimes \mathbf{1}_{B_n^{k_n}}(\tau(\omega, t_1), \ldots, \tau(\omega, t_N)) \right) \sigma(dt_1) \cdots \sigma(dt_N) \right],$$

with $N = k_1 + \cdots + k_n$, and this is a direct consequence of Relation (4.1) above and Corollary 7.2 below applied to the random sets $A \cap \tau^{-1}(B_1), \ldots, A \cap \tau^{-1}(B_n)$. $\qquad \square$

As a particular case of Proposition 4.1, for $g = \mathbf{1}_B$ and $B \in \mathcal{B}(Y)$ such that $\tau^{-1}(B) \subset A$ a.s., where A is a fixed compact subset of X, we have

$$E_\sigma \left[C_n(\tau_*\omega(B), \tau_*\sigma(B)) \right] = E_\sigma \left[\int_{A^n} D_{s_1} \cdots D_{s_n} \prod_{p=1}^{n} \mathbf{1}_B(\tau(\omega, s_p)) \sigma(ds_1) \cdots \sigma(ds_n) \right],$$
$$(4.3)$$

under Condition (4.2). When $D_s \mathbf{1}_B(\tau(\omega, t))$ is quasi-nilpotent in the sense of Condition (2.3) above for all $k \geq 2$, $\omega \in \Omega^X$, Relation (4.3) and Lemma 4.3 below show that

$$E_\sigma \left[C_n(\tau_*\omega(B), \tau_*\sigma(B)) \right] = 0,$$

and this extends (1.3) as a particular case since when $X = \mathbb{R}_+$, Condition (2.3) holds in particular when either

$$D_s g(\tau(\omega, t)) = 0, \qquad 0 \leq s \leq t,$$

or

$$D_t g(\tau(\omega, s)) = 0, \qquad 0 \leq s \leq t,$$

that is, when the process $\tau(\omega, t)$ is forward or backward adapted with respect to the filtration generated by the standard Poisson process $(N_t)_{t \in [0,T]}$.

Proof of Proposition 2.1. We take $g : Y \to \mathbb{R}$ to be the step function

$$g(t) = \sum_{i=1}^{m} c_i \mathbf{1}_{B_i}(t), \qquad t \in Y,$$

where $c_1, \ldots, c_m \in \mathbb{R}$ and $B_1, \ldots, B_m \in \mathcal{B}(Y)$ are disjoint Borel subsets of Y. Then the expression

$$C_n(x, \lambda) = \sum_{k=0}^{n} x^k \sum_{l=0}^{k} \binom{n}{l} (-\lambda)^{n-l} s(k, l), \qquad x, \lambda \in \mathbb{R},$$

for the Charlier polynomial of order $n \in \mathbb{N}$, shows that

$$|C_n(x, \lambda)| \leq \sum_{k=0}^{n} x^k \sum_{l=0}^{k} \binom{n}{l} \lambda^{n-l} s(k, l) = C_n(x, -\lambda), \qquad x, \lambda \geq 0;$$

13

hence

$$\sum_{n=0}^{\infty} \frac{|r|^n}{n!} |C_n(x,\lambda)| \leq e^{|r\lambda|}(1+|r|)^x, \qquad r \in \mathbb{R},$$

and letting $A \in \mathcal{B}(X)$ be a compact subset of X we have

$$
E_\sigma \left[\sum_{n=0}^{\infty} \frac{1}{n!} |I_n(\mathbf{1}_{A^n}(\cdot) g^{\otimes n}(\tau^{\otimes n}(\omega,\cdot)))| \right]
$$

$$
= E_\sigma \left[\sum_{N=0}^{\infty} \left| \sum_{\substack{k_1+\cdots+k_n=N \\ n \geq 0}} \left(\prod_{l=1}^{n} \frac{c_i^{k_i}}{k_i!} \right) I_N \left(\mathbf{1}_{A \cap \tau^{-1}(B_1)}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{A \cap \tau^{-1}(B_n)}^{\otimes k_n} \right) \right| \right]
$$

$$
= E_\sigma \left[\sum_{N=0}^{\infty} \left| \sum_{\substack{k_1+\cdots+k_n=N \\ n \geq 0}} \prod_{i=1}^{n} \frac{c_i^{k_i}}{k_i!} C_{k_i}(\omega(A \cap \tau^{-1}(B_i)), \sigma(A \cap \tau^{-1}(B_i))) \right| \right]
$$

$$
\leq E_\sigma \left[\sum_{N=0}^{\infty} \sum_{\substack{k_1+\cdots+k_n=N \\ n \geq 0}} \prod_{i=1}^{n} \frac{|c_i|^{k_i}}{k_i!} |C_{k_i}(\omega(A \cap \tau^{-1}(B_i)), \sigma(A \cap \tau^{-1}(B_i)))| \right]
$$

$$
\leq E_\sigma \left[\sum_{N=0}^{\infty} \sum_{\substack{k_1+\cdots+k_n=N \\ n \geq 0}} \prod_{i=1}^{n} \frac{|c_i|^{k_i}}{k_i!} C_{k_i}(\omega(A \cap \tau^{-1}(B_i)), -\sigma(A \cap \tau^{-1}(B_i))) \right]
$$

$$
= E_\sigma \left[\prod_{i=1}^{n} \sum_{k_i=0}^{\infty} \frac{|c_i|^{k_i}}{k_i!} C_{k_i}(\omega(A \cap \tau^{-1}(B_i)), -\sigma(A \cap \tau^{-1}(B_i))) \right]
$$

$$
= E_\sigma \left[\prod_{i=1}^{n} \exp\left(|c_i| \sigma(A \cap \tau^{-1}(B_i)) \right) (1 + |c_i|)^{\omega(A \cap \tau^{-1}(B_i))} \right]
$$

$$
\leq E_\sigma \left[\prod_{i=1}^{n} \exp\left(|c_i| \tau_* \sigma(B_i) \right) (1 + |c_i|)^{\tau_* \omega(B_i)} \right]
$$

$$
= E_\sigma \left[e^{\int_X |g(\tau(\omega,x))| \sigma(dx)} \prod_{x \in \omega} (1 + |g(\tau(\omega,x))|) \right]
$$

$$
< \infty. \tag{4.4}
$$

Consequently we can apply the Fubini theorem, which shows that

$$
E_\sigma \left[e^{-\int_A g(\tau(\omega,x)) \sigma(dx)} \prod_{x \in A \cap \omega} (1 + g(\tau(\omega,x))) \right] = E_\sigma \left[\sum_{n=0}^{\infty} \frac{1}{n!} I_n(\mathbf{1}_{A^n}(\cdot) g^{\otimes n}(\tau^{\otimes n}(\omega,\cdot))) \right]
$$

$$
= \sum_{n=0}^{\infty} \frac{1}{n!} E_\sigma \left[I_n(\mathbf{1}_{A^n}(\cdot) g^{\otimes n}(\tau^{\otimes n}(\omega,\cdot))) \right]
$$

14

$$= \sum_{n=0}^{\infty} \frac{1}{n!} E_\sigma \left[\int_{A^n} D_{s_1} \cdots D_{s_n} \prod_{p=1}^{n} g(\tau(\omega, s_p)) \sigma(ds_1) \cdots \sigma(ds_n) \right]$$

$$= 0,$$

by Proposition 4.1, provided

$$\int_{A^n} D_{s_1} \cdots D_{s_n} \prod_{p=1}^{n} g(\tau(\omega, s_p)) \sigma(ds_1) \cdots \sigma(ds_n) = 0, \qquad n \geq 1, \qquad (4.5)$$

$\pi_\sigma(d\omega)$-a.s., which holds by Lemma 4.3 below since $D_s \tau(\omega, t)$ is quasi-nilpotent in the sense of (2.3). The extension from A to X, and then from g, a step function, to a measurable function satisfying (2.4) can be done by dominated convergence using bound (4.4) above. $\qquad \square$

The above results can also be summarized in the following general statement which is also proved in Section 7 by the same argument as in the proof of Proposition 2.1.

Proposition 4.2 *Assume that $\tau : \Omega^X \times X \to Y$ satisfies*

$$D_t \tau(\omega, t) = 0, \qquad \omega \in \Omega^X, \quad t \in X.$$

Then for all bounded measurable functions $g : Y \to \mathbb{R}$ satisfying (2.4) we have

$$E_\sigma \left[e^{-\int_X g(\tau(\omega, x)) \sigma(dx)} \prod_{x \in \omega} (1 + g(\tau(\omega, x))) \right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} E_\sigma \left[\int_{X^n} D_{s_1} \cdots D_{s_n} \prod_{p=1}^{n} g(\tau(\omega, s_p)) \sigma(ds_1) \cdots \sigma(ds_n) \right], \qquad (4.6)$$

provided

$$\sum_{n=0}^{\infty} \frac{1}{n!} E_\sigma \left[\int_{X^n} \left| D_{s_1} \cdots D_{s_n} \prod_{p=1}^{n} g(\tau(\omega, s_p)) \sigma(ds_1) \cdots \sigma(ds_n) \right| \right] < \infty. \qquad (4.7)$$

In the next lemma we show that Relation (4.5) is satisfied provided $D_s \tau(\omega, t)$ satisfies the cyclic condition (4.8) below.

Lemma 4.3 *Let $N \geq 1$, and assume that $\tau : \Omega^X \times X \to X$ satisfies the cyclic condition*

$$D_{t_0} \tau(\omega, t_1) \cdots D_{t_k} \tau(\omega, t_0) = 0, \qquad \omega \in \Omega^X, \quad t_0, \ldots, t_k \in X, \qquad (4.8)$$

for $k = 1, \ldots, N$. *Then we have*

$$D_{t_0} \cdots D_{t_k} \prod_{p=0}^{k} g(\tau(\omega, t_p)) = 0, \qquad t_0, \ldots t_k \in X,$$

for $k = 1, \ldots, N$.

Proof. We use the relation

$$D_{s_0} \cdots D_{s_j} \prod_{p=0}^{n} g(\tau(\omega, s_p)) = \sum_{\Theta_0 \cup \cdots \cup \Theta_n = \{0,1,\ldots,j\}} D_{\Theta_0} g(\tau(\omega, s_0)) \cdots D_{\Theta_n} g(\tau(\omega, s_n)),$$

$$(4.9)$$

$s_0, \ldots, s_n \in X$, where $D_\Theta := \prod_{j \in \Theta} D_{s_j}$ when $\Theta \subset \{0, 1, \ldots, j\}$, $0 \leq j \leq n$, which follows from the product rule

$$D_t(FG) = F D_t G + G D_t F + D_t F D_t G, \qquad t \in X, \qquad (4.10)$$

which is satisfied by D_t as a finite difference operator. Without loss of generality we may assume that $\Theta_0 \neq \emptyset, \ldots, \Theta_j \neq \emptyset$ and $\Theta_k \cap \Theta_l = \emptyset$, $0 \leq k \neq l \leq j$. In this case we can construct a sequence $(k_1, \ldots, k_i)$ by choosing

$$0 \neq k_1 \in \Theta_0, \ \ k_2 \in \Theta_{k_1}, \ \ \ldots, k_{i-1} \in \Theta_{k_{i-2}},$$

until $k_i = 0 \in \Theta_{k_{i-1}}$ for some $i \in \{2, \ldots, j\}$ since $\Theta_0 \cap \cdots \cap \Theta_j = \emptyset$ and $\Theta_0 \cup \cdots \cup \Theta_j = \{0, 1, \ldots, j\}$. Hence by (4.8) we have

$$D_{s_{k_1}} g(\tau(\omega, s_{s_0})) D_{s_{k_2}} g(\tau(\omega, s_{s_{k_1}})) \cdots D_{s_{k_{i-1}}} g(\tau(\omega, s_{s_{k_{i-2}}})) D_{s_0} g(\tau(\omega, s_{s_{k_{i-1}}})) = 0,$$

by (4.8), which implies

$$D_{\Theta_0} g(\tau(\omega, s_0)) D_{\Theta_{k_1}} g(\tau(\omega, s_{k_1})) \cdots D_{\Theta_{k_{i-2}}} g(\tau(\omega, s_{k_{i-2}})) D_{\Theta_{k_{i-1}}} g(\tau(\omega, s_{k_{i-1}})) = 0,$$

since

$$(k_1, \ldots, k_{i-1}, 0) \in \Theta_0 \times \Theta_{k_1} \times \cdots \times \Theta_{k_{i-1}}.$$

$\square$

5 Moment identities for Poisson integrals

In this section we state some results obtained in [22] on the moments of Poisson stochastic integrals, and we reformulate them in view of our applications to Girsanov identities and to random Charlier polynomial functionals.

The Poisson-Skorohod integral operator δ is defined on any measurable process $u : \Omega^X \times X \to \mathbb{R}$ by the expression

$$\delta(u) = \int_X u(\omega \setminus \{t\}, t)(\omega(dt) - \sigma(dt)), \tag{5.1}$$

provided $E_\sigma \left[\int_X |u(\omega, t)| \sigma(dt) \right] < \infty$, cf., for example, [14], [21].

Note that if $D_t u_t = 0$, $t \in X$, and in particular when applying (5.1) to $u \in L^1_\sigma(X)$ a deterministic function, we have

$$\delta(u) = \int_X u(t)(\omega(dt) - \sigma(dt)), \tag{5.2}$$

that is, $\delta(u)$ with the compensated Poisson-Stieltjes integral of u. In addition, if $X = \mathbb{R}_+$ and $\sigma(dt) = \lambda_t dt$, we have

$$\delta(u) = \int_0^\infty u_t (dN_t - \lambda_t dt) \tag{5.3}$$

for all square-integrable predictable processes $(u_t)_{t \in \mathbb{R}_+}$, where $N_t = \omega([0, t])$, $t \in \mathbb{R}_+$, is a Poisson process with intensity $\lambda_t > 0$; cf., for instance, the Example page 518 of [15].

From Corollaries 1 and 5 in [15] or Proposition 6.4.3 in [21] the operators D and δ are closable and satisfy the duality relation

$$E_\sigma[\langle DF, u \rangle_{L^2_\sigma(X)}] = E_\sigma[F\delta(u)], \tag{5.4}$$

which can be seen as a formulation of the Mecke [11] identity for Poisson random measures, on their L^2 domains $\mathrm{Dom}\,(\delta) \subset L^2(\Omega^X \times X, \pi_\sigma \otimes \sigma)$ and $\mathrm{Dom}\,(D) = \mathbb{D}_{2,1} \subset L^2(\Omega^X, \pi_\sigma)$ under the Poisson measure π_σ with intensity σ.

The operator δ is continuous on the space $\mathbb{L}_{2,1} \subset \operatorname{Dom}(\delta)$ defined by the norm

$$\|u\|_{2,1}^2 = E_\sigma\left[\int_X |u_t|^2 \sigma(dt)\right] + E_\sigma\left[\int_X |D_s u_t|^2 \sigma(ds)\sigma(dt)\right],$$

and the operator δ satisfies the Skorohod isometry

$$E_\sigma\left[\delta(u)^2\right] = E_\sigma\left[\int_X |u_t|^2 \sigma(dt)\right] + E_\sigma\left[\int_X \int_X D_s u_t D_t u_s \sigma(ds)\sigma(dt)\right], \tag{5.5}$$

for any $u \in \mathbb{L}_{2,1}$, cf. Corollary 4 and pages 517-518 of [15].

In addition, from (5.1), for any $u \in \operatorname{Dom}(\delta)$ we have the commutation relation

$$D_t \delta(u) = \delta(D_t u) + u_t, \qquad t \in X, \tag{5.6}$$

or

$$(I + D_t)\delta(u) = \delta((I + D_t)u) + u_t, \qquad t \in X, \tag{5.7}$$

provided $D_t u \in \mathbb{L}_{2,1}$, $t \in X$.

The following lemma relies on the application of Relations (5.4) and (5.6), and extends (5.5) to powers of order greater than two; cf. Lemma 2.4 in [22].

Lemma 5.1 *[22] Let $u \in \mathbb{L}_{2,1}$ be such that $D_t u \in \mathbb{L}_{2,1}$, $t \in X$, $\delta(u)^n \in \mathbb{D}_{2,1}$, and*

$$E_\sigma\left[\int_X |u_t|^{n-k+1}|\delta((I + D_t)u)|^k \sigma(dt)\right] < \infty, \ E_\sigma\left[|\delta(u)|^k \int_X |u_t|^{n-k+1}\sigma(dt)\right] < \infty,$$

$0 \leq k \leq n$. Then we have

$$
\begin{aligned}
E_\sigma[\delta(u)^{n+1}] &= \sum_{k=0}^{n-1} \binom{n}{k} E_\sigma\left[\delta(u)^k \int_X u_t^{n-k+1}\sigma(dt)\right] \\
&\quad + \sum_{k=1}^{n} \binom{n}{k} E_\sigma\left[\int_X u_t^{n-k+1}(\delta((I + D_t)u)^k - \delta(u)^k)\sigma(dt)\right],
\end{aligned}
$$

for all $n \geq 1$.

When h is a deterministic function, Lemma 5.1 yields the recursive covariance identity

$$E_\sigma[\delta(h)^{n+1}] = \sum_{k=1}^{n} \binom{n}{k} \int_X h^{k+1}(t)\sigma(dt)E_\sigma\left[\delta(h)^{n-k}\right], \qquad n \geq 0, \tag{5.8}$$

for the Poisson stochastic integral

$$\delta(h) = \int_X h(x)(\omega(dx) - \sigma(dx)).$$

By induction, (5.8) shows that the moments of the above Poisson stochastic integral can be computed as

$$E_\sigma[\delta(h)^n] = \sum_{a=1}^{n-1} \sum_{0=k_1 \ll \cdots \ll k_{a+1}=n} \prod_{l=1}^{a} \binom{k_{l+1}-1}{k_l} \prod_{l=1}^{a} \int_X h^{k_{l+1}-k_l} d\sigma, \qquad (5.9)$$

for all $n \geq 1$ and deterministic $h \in \bigcap_{p=2 \wedge n}^{n} L_\sigma^p(X)$, where $a \ll b$ means $a < b - 1$, $a, b \in \mathbb{N}$. This result can also be recovered from the relation

$$E_\sigma[\delta(h)^n] = \sum_{d=1}^{n} \sum_{B_1,\ldots,B_d} \kappa_{|B_1|} \cdots \kappa_{|B_d|} \qquad (5.10)$$

where the sum runs over all partitions of $\{1, \ldots, n\}$, $|B_i|$ denotes the cardinality of B_i, and $\kappa_1 = 0$, $\kappa_n = \int_X h^n(t)\sigma(dt)$, $n \geq 2$, denote the cumulants of $\delta(h)$.

In particular, Relations (5.9) and (5.10) yield the identity

$$E_\lambda[(Z-\lambda)^n] = \sum_{a=0}^{n} \lambda^a S_2(n,a) \qquad (5.11)$$

for the central moments of a Poisson random variable Z with intensity λ, where

$$S_2(n,a) := \sum_{0=k_1 \ll \cdots \ll k_{a+1}=n} \prod_{l=1}^{a} \binom{k_{l+1}-1}{k_l},$$

represents the number of partitions of a set of size m into a subsets of size at least 2.

In the sequel we let

$$C(l_1,\ldots,l_a,b) = \sum_{0=r_{b+1}<\cdots<r_0=a+b+1} \prod_{q=0}^{b} \prod_{p=r_{q+1}-(b-q-1)}^{r_q-1-(b-q)} \binom{l_1+\cdots+l_p+q-1}{l_1+\cdots+l_{p-1}+q}, \qquad (5.12)$$

which represents the number of partitions of a set of $l_1 + \cdots + l_a + b$ elements into a subsets of lengths $l_1, \ldots, l_a$ and b singletons. We will need the following result; cf. Lemma 5.2 of [22].

19

Proposition 5.2 *[22] Let $F : \Omega^X \to \mathbb{R}$ be a bounded random variable, and let $u :$ $\Omega^X \times X \to \mathbb{R}$ be a bounded process with compact support in X. For all $n \geq 0$ we have*

$$E_\sigma \left[F \delta_\sigma(u)^n \right] = \sum_{a=0}^{n} \sum_{b=0}^{n-a} (-1)^b \sum_{\substack{l_1+\cdots+l_a=n-b \\ l_1,\ldots,l_a \geq 1}} C(l_1,\ldots,l_a,b) E_\sigma \left[\int_{X^{a+b}} \left(\prod_{i=1}^{a} (I + D_{s_i}) F \right) \right.$$

$$\left. \times \left(\prod_{q=a+1}^{a+b} \prod_{i=1}^{a} (I + D_{s_i}) u_{s_q} \right) \prod_{p=1}^{a} \left(\prod_{\substack{i=1 \\ i \neq p}}^{a} (I + D_{s_i}) u_{s_p} \right)^{l_p} \sigma(ds_1) \cdots \sigma(ds_{a+b}) \right].$$

In the above proposition, by saying that $u : \Omega^X \times X \to \mathbb{R}$ has a compact support in X we mean that there exists a compact $K \in \mathcal{B}(X)$ such that $u(\omega, x) = 0$ for all $\omega \in \Omega^X$ and $x \in X \setminus K$.

In particular when $u = \mathbf{1}_A$ is a (random) indicator function we get the following proposition, which will be used to prove Proposition 7.1 below. We let

$$S(n, c) = \frac{1}{c!} \sum_{l=0}^{c} (-1)^{c-l} \binom{c}{l} l^n \tag{5.13}$$

denote the Stirling number of the second kind, that is, the number of ways to partition a set of n objects into c non-empty subsets. In the next proposition, which is an application of Proposition 5.2, the random indicator function $(x, \omega) \mapsto \mathbf{1}_{A(\omega)}(x)$ on $\Omega^X \times X$ denotes a measurable process $u : \Omega^X \times X \to \mathbb{R}$ such that $u^2(\omega, t) = u(\omega, t)$, $\omega \in \Omega^X$, $t \in X$.

Proposition 5.3 *Let $F : \Omega^X \to \mathbb{R}$ be a bounded random variable, and consider a measurable random indicator function $(x, \omega) \mapsto \mathbf{1}_{A(\omega)}(x)$ on $\Omega^X \times X$, with compact support in X. Then for all $n \geq 0$ we have*

$$E_\sigma \left[F \delta(\mathbf{1}_A)^n \right] = \sum_{c=0}^{n} \sum_{a=0}^{c} (-1)^a \binom{n}{a} S(n-a, c-a)$$

$$E_\sigma \left[\int_{X^a} \left(\prod_{i=1}^{a} (I + D_{s_i})(F \sigma(A)^{c-a}) \right) \prod_{p=1}^{a} \prod_{\substack{i=1 \\ i \neq p}}^{a} (I + D_{s_i}) \mathbf{1}_A(s_p) \sigma(ds_1) \cdots \sigma(ds_a) \right].$$

Proof. Taking $u = \mathbf{1}_A$ in Proposition 5.2 yields

$$E_\sigma\left[F(\delta(u))^n\right] = \sum_{a=0}^{n}\sum_{b=0}^{n-a}(-1)^b \sum_{\substack{l_1+\cdots+l_a=n-b \\ l_1,\ldots,l_a \geq 1}} C(l_1,\ldots,l_a,b)E_\sigma\left[\int_{X^{a+b}}\left(\prod_{i=1}^{a}(I+D_{s_i})F\right)\right.$$

$$\left.\prod_{p=1}^{a+b}\prod_{\substack{i=1 \\ i\neq p}}^{a}(I+D_{s_i})\mathbf{1}_A(s_p)\sigma(ds_1)\cdots\sigma(ds_{a+b})\right]$$

$$= \sum_{c=0}^{n}\sum_{a=0}^{c}(-1)^a\binom{n}{a}S(n-a,c-a)$$

$$E_\sigma\left[\int_{X^c}\left(\prod_{i=1}^{a}(I+D_{s_i})F\right)\prod_{p=1}^{c}\prod_{\substack{i=1 \\ i\neq p}}^{a}(I+D_{s_i})\mathbf{1}_A(s_p)\sigma(ds_1)\cdots\sigma(ds_c)\right]$$

$$= \sum_{c=0}^{n}\sum_{a=0}^{c}(-1)^a\binom{n}{a}S(n-a,c-a)$$

$$E_\sigma\left[\int_{X^a}\left(\prod_{i=1}^{a}(I+D_{s_i})(F\sigma(A)^{c-a})\right)\prod_{p=1}^{a}\prod_{\substack{i=1 \\ i\neq p}}^{a}(I+D_{s_i})\mathbf{1}_A(s_p)\sigma(ds_1)\cdots\sigma(ds_a)\right],$$

after checking that we have

$$\binom{n}{b}S(n-b,a) = \sum_{\substack{l_1+\cdots+l_a=n-b \\ l_1,\ldots,l_a \geq 1}} C(l_1,\ldots,l_a,b),$$

which is the number of partitions of a set of n elements into a non-empty subsets and one subset of size b. $\qquad\square$

When the set A is deterministic, Proposition 5.3 yields

$$E_\lambda\left[(Z-\lambda)^n\right] = \sum_{c=0}^{n}\lambda^c\sum_{a=0}^{c}(-1)^a\binom{n}{a}S(n-a,c-a),$$

for the central moments of a Poisson random variable $Z = \omega(A)$ with intensity $\lambda = \sigma(A)$, which, from (5.11), shows the combinatorial identity

$$S_2(n,c) = \sum_{a=0}^{c}(-1)^a\binom{n}{a}S(n-a,c-a). \tag{5.14}$$

6   Poisson moments and polynomials

As mentioned in the introduction we need to introduce another family of polynomials whose generating function and associated combinatorics will be better adapted to our approach, making it possible to apply the moment identities of Proposition 5.3 and the integration by parts formula (5.4).

In terms of polynomials the identity (4.3) is easy to check for $n = 1$ and $n = 2$, in which case we have

$$C_1(\omega(A), \sigma(A)) = \omega(A) - \sigma(A) = \delta(\mathbf{1}_A),$$

and

$$
\begin{aligned}
C_2(\omega(A), \sigma(A)) &= (\omega(A) - \sigma(A))^2 - (\omega(A) - \sigma(A)) - \sigma(A) \\
&= \delta(\mathbf{1}_A)^2 - \delta(\mathbf{1}_A) - \sigma(A),
\end{aligned}
\tag{6.1}
$$

hence

$$
\begin{aligned}
E_\sigma \left[C_2(\omega(A), \sigma(A)) \right] &= E_\sigma[\delta(\mathbf{1}_A)^2] - \sigma(A) \\
&= E_\sigma \left[\int_X \int_X D_s \mathbf{1}_A(t) D_t \mathbf{1}_A(s) \sigma(ds)\sigma(dt) \right],
\end{aligned}
$$

from the Skorohod isometry (5.5).

In the sequel we will need to extend the above calculations and the proof of (4.3) to Charlier polynomials $C_n(x, \lambda)$ of all orders. For this, in Section 7 we will use the moment identities for the Skorohod integral $\delta(\mathbf{1}_A)$ of Proposition 5.3, and for this reason we will need to rewrite $C_n(\omega(A), \sigma(A))$, a linear combination of polynomials of the form $B_n(\delta(\mathbf{1}_A), \sigma(A))$, where $B_n(x, \lambda)$ is another polynomial of degree n. This construction is done using Stirling numbers and combinatorial arguments; cf. Proposition 6.1 below.

In other words, instead of using the identity (1.2) we need its Laplace form (2.1), that is,

$$E_\sigma \left[\exp \left(\delta(f) - \int_X (e^{f(x)} - f(x) - 1)\sigma(dx) \right) \right] = 1,
\tag{6.2}$$

obtained from (1.2) by taking

$$f(x) = \log(1 + g(x)), \qquad x \in X.$$

In particular when $f = \mathbf{1}_A$ with $A \in \mathcal{B}(X)$ a fixed compact subset of X, Relation (6.2) reads

$$E_\sigma\left[e^{t\delta(\mathbf{1}_A) - \sigma(A)(e^t - t - 1)}\right] = 1, \qquad t \in \mathbb{R}, \tag{6.3}$$

where $\delta(\mathbf{1}_A) = \omega(A) - \sigma(A)$ is a compensated Poisson random variable with intensity $\sigma(A) > 0$.

We let $(B_n(x, \lambda))_{n \in \mathbb{N}}$ denote the family of polynomials defined by the generating function

$$e^{ty - \lambda(e^t - t - 1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(y, \lambda), \qquad t \in \mathbb{R}, \tag{6.4}$$

for all $y, \lambda \in \mathbb{R}$. This definition implies in particular that

$$B_n(y, -\lambda) = E_\lambda[(Z + y - \lambda)^n],$$

where Z is a Poisson random variable with intensity $\lambda > 0$, and

$$B_n(y, \lambda) = \sum_{k=0}^{n} \binom{n}{k} y^k B_{n-k}(0, \lambda), \qquad \lambda \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{6.5}$$

For example one has that $B_1(y, \lambda) = y$ and $B_2(y, \lambda) = y^2 - \lambda$; hence (6.1) reads

$$C_2(x, \lambda) = B_2(x - \lambda, \lambda) - B_1(x - \lambda, \lambda),$$

and these relations will extended to all polynomial degrees in Proposition 6.1 below.

In addition, the definition of $B_n(x, \lambda)$ generalizes that of the Bell (or Touchard) polynomials $B_n(\lambda)$ defined by the generating function

$$e^{\lambda(e^t - 1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(\lambda),$$

which satisfy

$$B_n(\lambda) = B_n(\lambda, -\lambda) = E_\lambda[Z^n] = \sum_{c=0}^{n} \lambda^c S(n, c), \tag{6.6}$$

23

where Z is a Poisson random variable with intensity $\lambda > 0$, cf., for example, Proposition 2 of [5] or Section 3.1 of [13].

Next we show that the Charlier polynomials $C_n(x, \lambda)$ with exponential generating function

$$e^{-\lambda t}(1+t)^x = \sum_{n=0}^{\infty} \frac{t^n}{n!} C_n(x, \lambda), \qquad x, t, \lambda \in \mathbb{R},$$

are dual to the generalized Bell polynomials $B_n(x - \lambda, \lambda)$ under the Stirling transform.

Proposition 6.1 *We have the relations*

$$C_n(y, \lambda) = \sum_{k=0}^{n} s(n, k) B_k(y - \lambda, \lambda) \quad and \quad B_n(y, \lambda) = \sum_{k=0}^{n} S(n, k) C_k(y + \lambda, \lambda),$$

$y, \lambda \in \mathbb{R}$, $n \in \mathbb{N}$.

Proof. For the first relation, for all fixed $y, \lambda \in \mathbb{R}$ we let

$$A(t) = e^{-\lambda t}(1+t)^{y+\lambda} = \sum_{n=0}^{\infty} \frac{t^n}{n!} C_n(y + \lambda, \lambda), \qquad t \in \mathbb{R},$$

and note that

$$A(e^t - 1) = e^{t(y+\lambda) - \lambda(e^t - 1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(y, \lambda), \qquad t \in \mathbb{R},$$

which implies

$$B_n(y, \lambda) = \sum_{k=0}^{n} S(n, k) C_k(y + \lambda, \lambda), \qquad n \in \mathbb{N},$$

(see e.g. [3] page 2). The second part can be proved by inversion using Stirling numbers of the first kind, as

$$
\begin{aligned}
\sum_{k=0}^{n} S(n, k) C_k(y + \lambda, \lambda) &= \sum_{k=0}^{n} \sum_{l=0}^{k} S(n, k) s(k, l) B_l(y, \lambda) \\
&= \sum_{l=0}^{n} B_l(y, \lambda) \sum_{k=l}^{n} S(n, k) s(k, l) \\
&= B_n(y, \lambda),
\end{aligned}
$$

from the inversion formula

$$\sum_{k=l}^{n} S(n, k) s(k, l) = \mathbf{1}_{\{n=l\}}, \qquad n, l \in \mathbb{N}, \tag{6.7}$$

for Stirling numbers; cf., for example, page 825 of [1]. $\square$

24

The combinatorial identity proved in the next lemma will be used in Section 7 for the proof of Proposition 7.1. For $b = 0$ it yields the identity

$$S(n, a) = \sum_{c=0}^{a} \binom{n}{c} S_2(n - c, a - c), \tag{6.8}$$

which is the inversion formula of (5.14), and has a natural interpretation by stating that $S_2(m, b)$ is the number of partitions of a set of m elements made of b sets of cardinal greater or equal to 2.

Lemma 6.2 *For all $a, b \in \mathbb{N}$ we have*

$$\binom{a+b}{a} S(n, a+b) = \sum_{l=0}^{b} \sum_{k=l}^{n} \binom{n}{k} \binom{k}{l} S(k - l, a) S_2(n - k, b - l).$$

Proof. This identity can be proved by a combinatorial argument. For each value of $k = 0, \ldots, n$ one chooses a subset of $\{1, \ldots, n\}$ of size k which is partitioned into a non-empty subsets, the remaining set of size $n - k$ being partitioned into l singletons and $b - l$ subsets of size at least 2. In this process the b subsets mentioned above are counted including their combinations within $a + b$ sets, which explains the binomial coefficient $\binom{a+b}{a}$ on the right-hand side. $\qquad\square$

7 Random Charlier polynomials

In order to simplify the presentation of our results it will sometimes be convenient to use the symbolic notation

$$\Delta_{s_0} \cdots \Delta_{s_j} \prod_{p=0}^{n} u_{s_p} = \sum_{\substack{\Theta_0 \cup \cdots \cup \Theta_n = \{0,1,\ldots,j\} \\ 0 \notin \Theta_0, \ldots, j \notin \Theta_j}} D_{\Theta_0} u_{s_0} \cdots D_{\Theta_n} u_{s_n}, \tag{7.1}$$

$s_0, \ldots, s_n \in X$, $0 \leq j \leq n$, for any measurable process $u : \Omega^X \times X \to \mathbb{R}$.

The above formula implies in particular $\Delta_{s_0} u_{s_0} = 0$, and it can be used to rewrite the Skorohod isometry (5.5) as

$$E_\sigma \left[\delta(u)^2 \right] = E_\sigma \left[\|u\|_{L^2_\sigma(X)}^2 \right] + E_\sigma \left[\int_X \int_X \Delta_s \Delta_t (u_t u_s) \sigma(ds) \sigma(dt) \right],$$

since by definition we have

$$\Delta_s \Delta_t (u_s u_t) = D_s u_t D_t u_s, \qquad s, t \in X.$$

In this section we show the following proposition.

Proposition 7.1 *Let $n \geq 1$ and let $A_1(\omega), \ldots, A_n(\omega)$ be a.e. disjoint random Borel sets, all of them being a.s. contained in a fixed compact set K of X. Then we have*

$$E_\sigma \left[\prod_{i=1}^n C_{k_i} (\delta(\mathbf{1}_{A_i}) + \sigma(A_i), \sigma(A_i)) \right]$$
$$= E_\sigma \left[\int_{K^N} \Delta_{s_1} \cdots \Delta_{s_N} (\mathbf{1}_{A_1^{k_1}} \otimes \cdots \otimes \mathbf{1}_{A_n^{k_n}})(s_1, \ldots, s_N) \sigma(ds_1) \cdots \sigma(ds_N) \right],$$

$k_1, \ldots, k_n \in \mathbb{N}$, *with* $N = k_1 + \cdots + k_n$.

For $n = 1$, Proposition 7.1 yields, in particular,

$$E_\sigma \left[C_n(\omega(A), \sigma(A)) \right] = E_\sigma \left[\int_{K^n} \Delta_{s_1} \cdots \Delta_{s_n} \prod_{p=1}^n \mathbf{1}_A(s_p) \sigma(ds_1) \cdots \sigma(ds_n) \right],$$

for A a.s. contained in a fixed compact set K of X, which leads to (4.3) by Lemma 7.3 under Condition (4.2), as in the following corollary which is used for the proof of Proposition 4.1.

Corollary 7.2 *Assume that* $\tau : \Omega^X \times X \to X$ *satisfies*

$$D_t \tau(\omega, t) = 0, \qquad \omega \in \Omega^X, \quad t \in X. \tag{7.2}$$

Then for all deterministic disjoint $B_1, \ldots, B_n \in \mathcal{B}(X)$ *we have*

$$E_\sigma \left[\prod_{i=1}^n C_{k_i} (\omega(A \cap \tau^{-1}(B_i)), \sigma(A \cap \tau^{-1}(B_i))) \right]$$
$$= E_\sigma \left[\int_{A^N} D_{s_1} \cdots D_{s_N} ((\mathbf{1}_{B_1^{k_1}} \otimes \cdots \otimes \mathbf{1}_{B_n^{k_n}})(\tau(\omega, s_1), \ldots, \tau(\omega, s_N))) \sigma(ds_1) \cdots \sigma(ds_N) \right],$$

$k_1, \ldots, k_n \in \mathbb{N}$, *with* $N = k_1 + \cdots + k_n$, *for all compact* $A \in \mathcal{B}(X)$.

Proof. We apply Proposition 7.1 by letting $A_i(\omega) = A \cap \tau^{-1}(\omega, B_i)$, and we note that we have

$$
\begin{aligned}
\sigma(A_i(\omega)) &= \int_A \mathbf{1}_{B_i}(\tau(\omega, t))\sigma(dt) \\
&= \sigma(A \cap \tau^{-1}(\omega, B_i)).
\end{aligned}
$$

On the other hand, by (7.2) we have $D_t \mathbf{1}_{A_i}(t) = D_t \mathbf{1}_{B_i}(\tau(\omega, t)) = 0$; hence from Lemma 7.4 below we have

$$
\begin{aligned}
\delta(\mathbf{1}_{A_i}) + \sigma(A_i) &= \delta(\mathbf{1}_A \mathbf{1}_{B_i} \circ \tau) + \sigma(A \cap \tau^{-1}(B_i)) \\
&= \omega(A \cap \tau^{-1}(B_i)).
\end{aligned}
$$

Finally we note that from (7.1) and (7.2) we have

$$
D_{s_1} \cdots D_{s_N} = \Delta_{s_1} \cdots \Delta_{s_N},
$$

and we apply Proposition 7.1. $\qquad\square$

The proof of Proposition 7.1 relies on the following lemma.

Lemma 7.3 *Let $F : \Omega^X \to \mathbb{R}$ be a bounded random variable, and consider a random set A, a.s. contained in a fixed compact set K of X. For all $k \geq 1$ we have*

$E_\sigma \left[F C_k(\delta(\mathbf{1}_A) + \sigma(A), \sigma(A)) \right]$

$$
= \sum_{z=0}^{k} (-1)^{k-z} \binom{k}{z} E_\sigma \left[\int_{X^k} \prod_{j=1}^{z} (I + D_{s_j}) F \prod_{p=1}^{k} \prod_{\substack{j=1 \\ j \neq p}}^{z} (I + D_{s_j}) \mathbf{1}_A(s_p) \sigma(ds_1) \cdots \sigma(ds_k) \right].
$$

Proof. Using Proposition 5.3 and Lemma 6.2 we have

$E_\sigma \left[F B_n(\delta(\mathbf{1}_A), \sigma(A)) \right]$

$$
\begin{aligned}
&= \sum_{i=0}^{n} \binom{n}{i} E_\sigma \left[F(\delta(\mathbf{1}_A))^i B_{n-i}(0, \sigma(A)) \right] \\
&= \sum_{i=0}^{n} \binom{n}{i} \sum_{c=0}^{n-i} (-1)^c S_2(n-i, c) E_\sigma \left[F(\delta(\mathbf{1}_A))^i \sigma(A)^c \right] \\
&= \sum_{i=0}^{n} \binom{n}{i} \sum_{c=0}^{n-i} (-1)^c S_2(n-i, c) \sum_{e=0}^{i} \sum_{z=0}^{e} (-1)^{e-z} \binom{i}{z} S(i-z, e-z)
\end{aligned}
$$

27

$$
E_\sigma \left[\int_{X^z} \left(\prod_{j=1}^z (I + D_{s_j})(F\sigma(A)^{c+e-z}) \right) \prod_{p=1}^z \prod_{\substack{j=1 \\ j \neq p}}^z (I + D_{s_j}) \mathbf{1}_A(s_p)\sigma(ds_1)\cdots\sigma(ds_z) \right]
$$

$$
= \sum_{k=0}^{n-1} \sum_{i=0}^n \binom{n}{i} \sum_{c=0}^{n-i} S_2(n-i,c) \sum_{z=0}^{k-c} (-1)^{k-z} \binom{i}{z} S(i-z, k-c-z)
$$

$$
E_\sigma \left[\int_{X^z} \left(\prod_{j=1}^z (I + D_{s_j})(F\sigma(A)^{k-z}) \right) \prod_{p=1}^z \prod_{\substack{j=1 \\ j \neq p}}^z (I + D_{s_j}) \mathbf{1}_A(s_p)\sigma(ds_1)\cdots\sigma(ds_z) \right]
$$

$$
= \sum_{k=0}^{n-1} \sum_{z=0}^k (-1)^{k-z} \sum_{i=0}^n \binom{n}{i} \sum_{c=0}^{n-i} \binom{i}{z} S_2(n-i,c) S(i-z, k-c-z)
$$

$$
E_\sigma \left[\int_{X^z} \left(\prod_{j=1}^z (I + D_{s_j})(F\sigma(A)^{k-z}) \right) \prod_{p=1}^z \prod_{\substack{j=1 \\ j \neq p}}^z (I + D_{s_j}) \mathbf{1}_A(s_p)\sigma(ds_1)\cdots\sigma(ds_z) \right]
$$

$$
= \sum_{k=0}^n S(n,k) \sum_{z=0}^k (-1)^{k-z} \binom{k}{z}
$$

$$
E_\sigma \left[\int_{X^z} \left(\prod_{j=1}^z (I + D_{s_j})(F\sigma(A)^{k-z}) \right) \prod_{p=1}^z \prod_{\substack{j=1 \\ j \neq p}}^z (I + D_{s_j}) \mathbf{1}_A(s_p)\sigma(ds_1)\cdots\sigma(ds_z) \right].
$$

Hence from Proposition 6.1 or the inversion formula (6.7) we get

$$
E_\sigma \left[FC_k(\delta(\mathbf{1}_A) + \sigma(A), \sigma(A)) \right]
$$

$$
= \sum_{z=0}^k (-1)^{k-z} \binom{k}{z} E_\sigma \left[\int_{X^k} \prod_{j=1}^z (I + D_{s_j}) F \prod_{p=1}^k \prod_{\substack{j=1 \\ j \neq p}}^z (I + D_{s_j}) \mathbf{1}_A(s_p)\sigma(ds_1)\cdots\sigma(ds_k) \right].
$$

$\square$

In particular, Lemma 7.3 applied to $F = \mathbf{1}$ shows that

$$
E_\sigma \left[C_k(\delta(\mathbf{1}_A) + \sigma(A), \sigma(A)) \right]
$$

$$
= \sum_{z=0}^k (-1)^{k-z} \binom{k}{z} E_\sigma \left[\int_{X^k} \prod_{p=1}^k \prod_{\substack{j=1 \\ j \neq p}}^z (I + D_{s_j}) \mathbf{1}_A(s_p)\sigma(ds_1)\cdots\sigma(ds_k) \right]
$$

$$
= \sum_{z=0}^k (-1)^{k-z} \binom{k}{z} E_\sigma \left[\int_{X^k} \prod_{p=1}^k \prod_{j=1}^z (I + \Delta_{s_j}) \mathbf{1}_A(s_p)\sigma(ds_1)\cdots\sigma(ds_k) \right]
$$

$$= E_\sigma \left[\int_{X^k} \left(\prod_{j=1}^{k} (I + \Delta_{s_j} - I) \right) \prod_{p=1}^{k} \mathbf{1}_A(s_p) \sigma(ds_1) \cdots \sigma(ds_k) \right]$$

$$= E_\sigma \left[\int_{X^k} \Delta_{s_1} \cdots \Delta_{s_k} \prod_{p=1}^{k} \mathbf{1}_A(s_p) \sigma(ds_1) \cdots \sigma(ds_k) \right],$$

which is Proposition 7.1 for $n = 1$. Next we will apply this argument to prove Proposition 7.1 from Lemma 7.3 by induction.

Proof of Proposition 7.1. From Lemma 7.3 we have

$$E_\sigma \left[FC_{k_1}(\delta(\mathbf{1}_{A_1}) + \sigma(A_1), \sigma(A_1)) \right] \tag{7.3}$$

$$= \sum_{z_1=0}^{k_1} (-1)^{k_1 - z_1} \binom{k_1}{z_1} E_\sigma \left[\int_{K^{k_1}} \prod_{j=1}^{z_1} (I + D_{s_{1,j}}) F \prod_{p=1}^{k_1} \prod_{\substack{j=1 \\ j \neq p}}^{z_1} (I + D_{s_{1,j}}) \mathbf{1}_{A_1}(s_{1,p}) \sigma(ds_{1,1}) \cdots \sigma(ds_{1,k_1}) \right].$$

The first induction step is to apply the above equality to the random variable

$$F = \prod_{i=2}^{n} C_{k_i}(\delta(\mathbf{1}_{A_i}) + \sigma(A_i), \sigma(A_i)).$$

Here F is not bounded, however since $A_i(\omega) \subset K$, a.s., $i = 1, \ldots, n$, for a fixed compact $K \in \mathcal{B}(X)$, we check that $|F|$ is bounded by a polynomial in $\omega(K)$, and

$$\left| \prod_{j=1}^{z_1} (I + D_{s_{1,j}}) F \right|$$

is bounded by another a polynomial in $\omega(K)$, uniformly in $s_1, \ldots, s_{k_1} \in X$. Hence by dominated convergence we can extend (7.3) from the bounded random variable $\max(\min(F, -C), C)$, $C > 0$, to F by letting C go to infinity. From Relation (5.7) we have

$$\prod_{j=1}^{z_1} (I + D_{s_{1,j}}) \delta(\mathbf{1}_{A_i}) = \delta \left(\prod_{j=1}^{z_1} (I + D_{s_{1,j}}) \mathbf{1}_{A_i} \right) + \sum_{k=1}^{z_1} \prod_{\substack{j=1 \\ j \neq k}}^{z_1} (I + D_{s_{1,j}}) \mathbf{1}_{A_i}(s_{1,k})$$

$$= \delta \left(\prod_{j=1}^{z_1} (I + D_{s_{1,j}}) \mathbf{1}_{A_i} \right),$$

$0 \le z_1 \le k_1$, $i \ge 2$, when $s_{1,k} \in \prod\limits_{\substack{j=1 \\ j \ne k}}^{z_1}(I + D_{s_{1,j}})A_1$, $1 \le k \le k_1$, since

$$\prod_{\substack{j=1 \\ j \ne k}}^{z_1}(I + D_{s_{1,j}})A_1(\omega), \ldots, \prod_{\substack{j=1 \\ j \ne k}}^{z_1}(I + D_{s_{1,j}})A_n(\omega)$$

are disjoint, $1 \le k \le k_1$, $\omega \in \Omega^X$, hence

$$\prod_{j=1}^{z_1}(I + D_{s_{1,j}})F = \prod_{j=1}^{z_1}(I + D_{s_{1,j}}) \prod_{i=2}^{n} C_{k_i}(\delta(\mathbf{1}_{A_i}) + \sigma(A_i), \sigma(A_i))$$

$$= \prod_{i=2}^{n} C_{k_i}\left(\prod_{j=1}^{z_1}(I + D_{s_{1,j}})\delta(\mathbf{1}_{A_i}) + \prod_{j=1}^{z_1}(I + D_{s_{1,j}})\sigma(A_i), \prod_{j=1}^{z_1}(I + D_{s_{1,j}})\sigma(A_i)\right)$$

$$= \prod_{i=2}^{n} C_{k_i}\left(\delta\left(\prod_{j=1}^{z_1}(I + D_{s_{1,j}})\mathbf{1}_{A_i}\right) + \prod_{j=1}^{z_1}(I + D_{s_{1,j}})\sigma(A_i), \prod_{j=1}^{z_1}(I + D_{s_{1,j}})\sigma(A_i)\right),$$

which yields, from (7.3),

$$E_\sigma\left[\prod_{i=1}^{n} C_{k_i}(\delta(\mathbf{1}_{A_i}) + \sigma(A_i), \sigma(A_i))\right]$$

$$= \sum_{z_1=0}^{k_1}(-1)^{k_1-z_1}\binom{k_1}{z_1}$$

$$E_\sigma\left[\int_{X^{k_1}} \prod_{i=2}^{n} C_{k_i}\left(\delta\left(\prod_{j=1}^{z_1}(I + D_{s_{1,j}})\mathbf{1}_{A_i}\right) + \prod_{j=1}^{z_1}(I + D_{s_{1,j}})\sigma(A_i), \prod_{j=1}^{z_1}(I + D_{s_{1,j}})\sigma(A_i)\right)\right.$$

$$\left. \prod_{p=1}^{k_1}\prod_{\substack{j=1 \\ j \ne p}}^{z_1}(I + D_{s_{1,j}})\mathbf{1}_{A_1}(s_{1,p})\sigma(ds_{1,1})\cdots\sigma(ds_{1,k_1})\right].$$

Next, we apply Lemma 7.3 again to

$$C_{k_2}\left(\delta\left(\prod_{j=1}^{z_1}(I + D_{s_{1,j}})\mathbf{1}_{A_2}\right) + \prod_{j=1}^{z_1}(I + D_{s_{1,j}})\sigma(A_2), \prod_{j=1}^{z_1}(I + D_{s_{1,j}})\sigma(A_2)\right)$$

and to

$$F = \prod_{p=1}^{k_1}\prod_{\substack{j=1 \\ j \ne p}}^{z_1}(I + D_{s_{1,j}})\mathbf{1}_{A_1}(s_{1,p})$$

$$\times \prod_{i=3}^{n} C_{k_i} \left(\delta \left(\prod_{j=1}^{z_1}(I + D_{s_{1,j}})\mathbf{1}_{A_i} \right) + \prod_{j=1}^{z_1}(I + D_{s_{1,j}})\sigma(A_i), \prod_{j=1}^{z_1}(I + D_{s_{1,j}})\sigma(A_i) \right),$$

and by iteration of this argument we obtain

$$E_\sigma \left[\prod_{i=1}^{n} C_{k_i}(\delta(\mathbf{1}_{A_i}) + \sigma(A_i), \sigma(A_i)) \right]$$

$$= \sum_{z_n=0}^{k_n} \cdots \sum_{z_1=0}^{k_1} \prod_{l=1}^{n}(-1)^{k_l-z_l} \prod_{l=1}^{n} \binom{k_l}{z_l}$$

$$E_\sigma \left[\int_{X^{k_1}} \prod_{\substack{i=1 \\ }}^{n} \prod_{\substack{j=1 \\ j\neq p}}^{z_i}(I + D_{s_{i,j}}) \prod_{i=1}^{n}\prod_{j=1}^{k_i} \mathbf{1}_{A_i}(s_{i,j})\sigma(ds_{1,1}) \cdots \sigma(ds_{n,k_n}) \right]$$

$$= \sum_{z_n=0}^{k_n} \cdots \sum_{z_1=0}^{k_1} \prod_{l=1}^{n}(-1)^{k_l-z_l} \prod_{l=1}^{n} \binom{k_l}{z_l}$$

$$E_\sigma \left[\int_{X^{k_1}} \left(\prod_{i=1}^{n}\prod_{j=1}^{z_i}(I + \Delta_{s_{i,j}}) \right) \prod_{i=1}^{n}\prod_{j=1}^{k_i} \mathbf{1}_{A_i}(s_{i,j})\sigma(ds_{1,1}) \cdots \sigma(ds_{n,k_n}) \right]$$

$$= E_\sigma \left[\int_{X^N} \left(\prod_{i=1}^{n}\prod_{j=1}^{k_i}(I + \Delta_{s_{i,j}} - I) \right) \prod_{i=1}^{n}\prod_{j=1}^{k_i} \mathbf{1}_{A_i}(s_{i,j})\sigma(ds_{1,1}) \cdots \sigma(ds_{n,k_n}) \right]$$

$$= E_\sigma \left[\int_{X^N} \left(\prod_{i=1}^{n}\prod_{j=1}^{k_i}\Delta_{s_{i,j}} \right) \prod_{i=1}^{n}\prod_{j=1}^{k_i} \mathbf{1}_{A_i}(s_{i,j})\sigma(ds_{1,1}) \cdots \sigma(ds_{n,k_n}) \right].$$

□

Next we prove Proposition 4.2.

Proof of Proposition 4.2. Taking $g : Y \to \mathbb{R}$ to be the step function

$$g = \sum_{i=1}^{m} c_i \mathbf{1}_{B_i},$$

where $c_1, \ldots, c_m \in \mathbb{R}$ and $B_1, \ldots, B_m \in \mathcal{B}(Y)$ are disjoint Borel subsets of Y, Corollary 7.2 shows that for compact $A \in \mathcal{B}(X)$ we have

$$E_\sigma \left[e^{-\int_A g(\tau(\omega,t))\sigma(dt)} \prod_{x \in A \cap \omega} (1 + g(\tau(\omega,x))) \right] = E_\sigma \left[\prod_{l=1}^{m} e^{-c_l\sigma(A \cap \tau^{-1}(B_l))} \prod_{l=1}^{m}(1 + c_l)^{\omega(A \cap \tau^{-1}(B_l))} \right]$$

$$= \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \left(\prod_{i=1}^{m} \frac{c_i^{k_i}}{k_i!} \right) E_\sigma \left[\prod_{i=1}^{m} C_{k_i}(\omega(A \cap \tau^{-1}(B_i)), \sigma(A \cap \tau^{-1}(B_i))) \right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} E_{\sigma} \left[I_n(\mathbf{1}_{A^n}(\cdot) g^{\otimes n}(\tau^{\otimes n}(\omega, \cdot))) \right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} E_{\sigma} \left[\int_{A^n} \Delta_{s_1} \cdots \Delta_{s_n} \prod_{p=1}^{n} g(\tau(\omega, s_p)) \sigma(ds_1) \cdots \sigma(ds_n) \right].$$

In the general case with $g : Y \to \mathbb{R}$ bounded measurable the conclusion follows by approximation of g by step functions and dominated convergence under (2.4), followed by extension to $A = X$ using the bound (4.7). $\qquad\square$

Finally we state the following lemma which has been used in the proof of Corollary 7.2.

Lemma 7.4 *Assume that*

$$D_t \tau(\omega, t) = 0, \qquad \omega \in \Omega^X, \quad t \in X. \tag{7.4}$$

Then we have

$$\int_X \mathbf{1}_A(t) h(\tau(\omega, t)) \omega(dt) = \delta(\mathbf{1}_A h \circ \tau)(\omega) + \int_A h \circ \tau(\omega, t) \sigma(dt), \qquad \omega \in \Omega^X,$$

for all compact $A \in \mathcal{B}(X)$ and all bounded measurable functions $h : X \to \mathbb{R}$.

Proof. We note that Condition (7.4) above means that $\tau(\omega, t)$ does not depend on the presence or absence of a point in ω at t, and in particular,

$$\tau(\omega, t) = \tau(\omega \cup \{t\}, t), \qquad t \notin \omega,$$

and

$$\tau(\omega, t) = \tau(\omega \setminus \{t\}, t), \qquad t \in \omega.$$

Hence we have

$$\delta(\mathbf{1}_A h \circ \tau) + \int_A h \circ \tau(\omega, t) \sigma(dt)$$

$$= \int_X \mathbf{1}_A(t) h(\tau(\omega \setminus \{t\}, t))(\omega(dt) - \sigma(dt)) + \int_X \mathbf{1}_A(t) h(\tau(\omega, t)) \sigma(dt)$$

$$= \int_X \mathbf{1}_A(t) h(\tau(\omega \setminus \{t\}, t)) \omega(dt)$$

$$= \int_X \mathbf{1}_A(t) h(\tau(\omega, t)) \omega(dt).$$

$\qquad\square$

8 Link with the Carleman-Fredholm determinant

In this section we make some remarks on differences between the Poisson and Wiener cases, in relation to the quasi-nilpotence of random transformations. We consider a Poisson random measure on $\mathbb{R}_+ \times [-1, 1]^d$ on the real line with flat intensity measure, in which case it is known [16], [17], [18], that, building the Poisson measure as a product of exponential and uniform densities on the sequence space $\mathbb{R}^\mathbb{N}$, we have the Girsanov identity,

$$E\left[F(I+u)|\det_2(I+\nabla u)| \exp(-\nabla^*(u))\right] = E\left[F\right],$$

where $u : \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ is a random shift satisfying certain conditions, $\det_2(I + \nabla u)$ is the Carleman-Fredholm determinant of $I + \nabla u$ and $\nabla^*(u)$ is a Skorohod-type integral of the discrete-time process u.

When it is invertible, $(I + u)_* \pi_\sigma$ is absolutely continuous with respect to π_σ with

$$\frac{d(I+u)_*^{-1}\pi_\sigma}{d\pi_\sigma} = |\det_2(I + \nabla u)| \exp(-\nabla^*(u)).$$

It can be checked (cf. [16], [17], [18]) that in the adapted case this yields the usual Girsanov theorem for the change of intensity of Poisson random measures when the configuration points are shifted by an adapted smooth diffeomorphism $\phi : \Omega^X \times \mathbb{R}_+ \times [0,1]^d \longrightarrow \mathbb{R}_+ \times [0,1]^d$, in which case $I + Du$ becomes a block diagonal matrix, each $d \times d$ block having the Jacobian determinant $|\partial_{t,x}\phi(\omega, T_k, x_k^1, \ldots, x_k^d)|$, and we have

$$\det_2(I+\nabla u) \exp(-\nabla^*(u)) = e^{-\int_{\mathbb{R}_+ \times [0,1]^d} (|\partial_{s,x}\phi(\omega,s,x)|-1)dsdx} \prod_{k=1}^{\infty} |\partial_{t,x}\phi(\omega, T_k, x_k^1, \ldots, x_k^d)|.$$

The main difference with the Wiener case is that here ∇u is not quasi-nilpotent on $\ell^2(\mathbb{N})$ and we do not have $\det_2(I + \nabla u) = 1$. Nevertheless it should be possible to recover Proposition 4.1 in a weaker form by checking the relation

$$\det(I + \nabla u) = \prod_{k=1}^{\infty} |\partial_{t,x}\phi(\omega, T_k, x_k^1, \ldots, x_k^d)|$$

for anticipating shifts $\phi : \Omega^X \times \mathbb{R}_+ \times [0,1]^d \longrightarrow \mathbb{R}_+ \times [0,1]^d$, under smoothness and quasi-nilpotence assumptions stronger than those assumed in this paper.

References

[1] M. Abramowitz and I.A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55. Dover Publications, New York, 1972. 9th Edition.

[2] S. Albeverio and N. V. Smorodina. A distributional approach to multiple stochastic integrals and transformations of the Poisson measure. *Acta Appl. Math.*, 94(1):1–19, 2006.

[3] M. Bernstein and N. J. A. Sloane. Some canonical sequences of integers. *Linear Algebra Appl.*, 226/228:57–72, 1995.

[4] K. Bichteler. *Stochastic integration with jumps*, volume 89 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2002.

[5] K.N. Boyadzhiev. Exponential polynomials, Stirling numbers, and evaluation of some gamma integrals. *Abstr. Appl. Anal.*, pages Art. ID 168672, 18, 2009.

[6] Y. Davydov and S. Nagaev. On the convex hulls of point processes. Manuscript, 2000.

[7] A. Dermoune, P. Krée, and L. Wu. Calcul stochastique non adapté par rapport à la mesure aléatoire de Poisson. In *Séminaire de Probabilités, XXII*, volume 1321 of *Lecture Notes in Math.*, pages 477–484, Berlin, 1988. Springer.

[8] Y. Ito. Generalized Poisson functionals. *Probab. Theory Related Fields*, 77:1–28, 1988.

[9] S. Kusuoka. The nonlinear transformation of Wiener measure on Banach space and its absolute continuity. *Journal of the Faculty of Science of Tokyo University, Section IA, Mathematics*, 29:567–598, 1982.

[10] D. Lépingle and J. Mémin. Sur l'intégrabilité uniforme des martingales exponentielles. *Z. Wahrsch. Verw. Gebiete*, 42(3):175–203, 1978.

[11] J. Mecke. Stationäre zufällige Masse auf lokalkompakten Abelschen Gruppen. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, 9:36–58, 1967.

[12] P.A. Meyer. Un cours sur les intégrales stochastiques. In *Séminaire de Probabilités, X (Seconde partie: Théorie des intégrales stochastiques, Univ. Strasbourg, Strasbourg, année universitaire 1974/1975)*, pages 245–400. Lecture Notes in Math., Vol. 511, Berlin, 1976. Springer. Available at http://www.numdam.org/numdam-bin/item?id=SPS_1976__10__245_0.

[13] E. Di Nardo and D. Senato. Umbral nature of the Poisson random variables. In *Algebraic combinatorics and computer science*, pages 245–266. Springer Italia, Milan, 2001.

[14] D. Nualart and J. Vives. A duality formula on the Poisson space and some applications. In R. Dalang, M. Dozzi, and F. Russo, editors, *Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1993)*, volume 36 of *Progress in Probability*, pages 205–213. Birkhäuser, Basel, 1995.

[15] J. Picard. Formules de dualité sur l'espace de Poisson. *Ann. Inst. H. Poincaré Probab. Statist.*, 32(4):509–548, 1996.

[16] N. Privault. Girsanov theorem for anticipative shifts on Poisson space. *Probab. Theory Related Fields*, 104:61–76, 1996.

[17] N. Privault. Absolute continuity in infinite dimensions and anticipating stochastic calculus. *Potential Analysis*, 8(4):325–343, 1998.

[18] N. Privault. Quasi-invariance for Lévy processes under anticipating shifts. In U. Çapar and A.S. Üstünel, editors, *Stochastic analysis and related topics VIII*, volume 53 of *Progress in Probability*, pages 181–202. Birkäuser, 2003.

[19] N. Privault. Moment identities for Poisson-Skorohod integrals and application to measure invariance. *C. R. Math. Acad. Sci. Paris*, 347:1071–1074, 2009.

[20] N. Privault. Moment identities for Skorohod integrals on the Wiener space and applications. *Electron. Commun. Probab.*, 14:116–121 (electronic), 2009.

[21] N. Privault. *Stochastic Analysis in Discrete and Continuous Settings*, volume 1982 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.

[22] N. Privault. Invariance of Poisson measures under random transformations. Preprint, to appear in Ann. Inst. H. Poincaré Probab. Statist., 2011.

[23] R. Ramer. On nonlinear transformations of Gaussian measures. *J. Funct. Anal.*, 15:166–187, 1974.

[24] A. Segall and T. Kailath. Orthogonal functionals of independent-increment processes. *IEEE Trans. Information Theory*, IT-22(3):287–298, 1976.

[25] A.S. Üstünel and M. Zakai. Random rotations of the Wiener path. *Probab. Theory Relat. Fields*, 103(3):409–429, 1995.

[26] A.S. Üstünel and M. Zakai. *Transformation of measure on Wiener space*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000.

[27] M. Zakai and O. Zeitouni. When does the Ramer formula look like the Girsanov formula ? *Ann. Probab.*, 20(3):1436–1440, 1992.