BLOW-UP AND STABILITY OF SEMILINEAR PDE'S WITH GAMMA GENERATORS

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Abstract

We investigate finite-time blow-up and stability of semilinear partial differential equations of the form $\partial w_t/\partial t = \Gamma w_t + \nu t^\sigma w_t^{1+\beta}$, $w_0(x) = \varphi(x) \geq 0$, $x \in \mathbb{R}_+$, where Γ is the generator of the standard gamma process and $\nu > 0$, $\sigma \in \mathbb{R}$, $\beta > 0$ are constants. We show that any initial value satisfying $c_1 x^{-a_1} \leq \varphi(x)$, $x > x_0$ for some positive constants x_0, c_1, a_1, yields a non-global solution if $a_1\beta < 1 + \sigma$. If $\varphi(x) \leq c_2 x^{-a_2}$, $x > x_0$, where $x_0, c_2, a_2 > 0$, and $a_2\beta > 1 + \sigma$, then the solution w_t is global and satisfies $0 \leq w_t(x) \leq C t^{-a_2}$, $x \geq 0$, for some constant $C > 0$. This complements the results previously obtained in [3, 10, 22] for symmetric α-stable generators. Systems of semilinear PDE's with gamma generators are also considered.

Key words: Semilinear partial differential equations, Feynman-Kac representation, blow-up of semilinear systems, gamma processes.
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1 Introduction

Critical exponents for blow-up of semilinear Cauchy problems of the prototype

$$\frac{\partial w_t(x)}{\partial t} = L w_t(x) + w_t^{1+\beta}(x), \qquad w_0(x) = \varphi(x), \quad x \in \mathbb{R}^d, \tag{1.1}$$

where L is a Lévy generator, $\beta > 0$ is constant and $\varphi \geq 0$, have been investigated by many authors, specially in the case of the d-dimensional Laplacian $L = \Delta$ (see [13] and [6] for surveys). When L is the fractional power $\Delta_\alpha = -(-\Delta)^{\alpha/2}$ of the Laplacian, $0 < \alpha \leq 2$, it was shown in a series of papers [3, 15, 17, 20, 22] that the critical parameter for blow-up of (1.1) is $d_c := \alpha/\beta$, meaning that if $d \leq d_c$ then (1.1) possesses no global nontrivial solutions, and if $d > d_c$, then (1.1) admits a nontrivial global solution for all sufficiently small initial values. Critical parameters for semilinear equations with time-dependent non-linearities of the form $t^\sigma w_t^{1+\beta}(x)$ were studied in [3] and [10] for $L = \Delta_\alpha$. The case of an elliptic operator L on an exterior domain was investigated in [1] for general time-dependent reaction terms.

The approaches developed in the works quoted above use subtle comparison arguments [1, 22] or probabilistic representations of solutions [3, 15, 17, 20]. In [3] the Feynman-Kac formula is used to construct subsolutions $0 \le f_t \le g_t \le h_t$ of (1.1), where $f_t = e^{tL}\varphi$, and g_t, h_t are the mild solutions, respectively, of

$$\frac{\partial g_t}{\partial t} = Lg_t + f_t^\beta g_t, \quad g_0 = \varphi, \qquad \text{and} \qquad \frac{\partial h_t}{\partial t} = Lh_t + g_t^\beta h_t, \quad h_0 = \varphi.$$

It is shown that g_t (resp. h_t) grows locally to ∞ if $d < d_c$ (resp. if $d = d_c$, and in this case a second application of the Feynman-Kac formula is required). In proving this a crucial step consists in bounding from below, for suitable classes of Borel sets $B \subset \mathbb{R}^d$ and end points $x, y \in \mathbb{R}^d$, the bridges

$$P_x(X_s \in B \,|\, X_t = y), \quad 0 \le s \le t, \tag{1.2}$$

of a process $\{X_t\}$ with generator L. Finite-time blow-up of any non-trivial solution of (1.1), in dimensions $d \le d_c$, then follow from a classical argument that goes back to [12].

Motivated by the method developed in [3], in this paper we investigate finite-time blow-up and existence of non-trivial global solutions of semilinear equations of the form

$$\frac{\partial w_t}{\partial t} = \mathit{\Gamma} w_t + \nu t^\sigma w_t^{1+\beta}, \qquad w_0(x) = \varphi(x), \qquad x \in \mathbb{R}_+, \tag{1.3}$$

where φ is a nonnegative function, ν, σ and β are positive constants, and $\mathit{\Gamma}$ is the pseudo-differential operator

$$\mathit{\Gamma} f(x) = \int_0^\infty (f(x+y) - f(x)) \frac{e^{-y}}{y}\, dy,$$

i.e. $\mathit{\Gamma}$ is the generator of the (conservative) transition semigroup of the standard gamma process. In the linear case $\beta = 0$, equations of the type (1.3) are of interest in reliability models based on the gamma process [25], where the solutions represent failure probabilities of a system undergoing random impacts according to a space-time gamma noise.

Notice that the symmetrized generator

$$\tilde{\Gamma} f(x) = \int_{-\infty}^{\infty} (f(x+y) - f(x)) \frac{e^{-|y|}}{|y|} \, dy$$

has symbol

$$\log(1 + |\xi|) = \lim_{\alpha \to 0} \alpha^{-1}((1 + |\xi|)^\alpha - 1), \qquad \xi \in \mathbb{R},$$

(see e.g. [11]) and can be viewed as the weak limit of $\alpha^{-1}((I - \Delta_{1/2})^\alpha - I)$ as α goes to 0. In other terms, α-stable processes can be suitably normalized to converge in distribution to a gamma process, see [5] and [23]. Thus, the study of the behavior of (1.3) (finite-time blow up *vs* existence of global solutions) constitutes a natural follow-up to previous investigations, as it can be considered in a sense as a "limiting case" $\alpha \to 0$.

Let us recall that the gamma process belongs to a special class of Lévy processes called subordinators (see e.g. [2] or [21]), a subordinator being a purely non-Gaussian Lévy process $\{X_t\}$ in $\mathbb{R}$, whose Lévy measure ν satisfies $\nu((-\infty, 0)) = 0$ and $\int_{(0,1]} x \, \nu(dx) < \infty$. In particular, the trajectories $t \mapsto X_t(\omega)$ are increasing functions a.s. and the transition kernels $P_{t-s}(x, dy) := P(X_t \in dy | X_s = x)$ are supported in $[x, \infty)$.

In contrast with the Brownian and α-stable Lévy motions, subordinators enjoy in general no scaling or symmetry property, nor dimensional-dependent behavior. This circumstance makes it difficult to carry out the methods in the papers quoted above to investigate (1.1), since most of those methods rely significantly on the symmetry and scaling properties of Gaussian and stable distributions. However, in the case we are considering here the transition densities of the motion process are explicitly given, and, moreover, it is known that the bridges (1.2) are beta distributed. Together with the estimates of [18] for the median of beta distributions, this allows us to obtain lower bounds for the bridge distribution of the gamma subordinator, making it possible to exploit the Feynman-Kac approach of [3] to derive criteria for finite-time blow of (1.3).

We emphasize that the class of Lévy processes for which both, the Lévy measure and probability density function are explicitly known, is restricted and essentially limited to Brownian motion and the Poisson, Gamma and Meixner processes (cf. [16], [9]). On the other hand, the d-dimensional gamma process (i.e. an $\mathbb{R}^d$-valued stochastic process having as coordinates d independent copies of a Gamma subordinator) seems to require the use of arguments different from the ones used in this paper, starting with the fundamental Lemma 2.1 below.

Our solutions will be understood in the mild sense (see e.g. [19]), and therefore we can consider bounded, measurable initial values $\varphi \geq 0$. We will show as a consequence of Corollary 4.3 and Theorem 5.1 that any initial value satisfying

$$c_1 x^{-a_1} \leq \varphi(x), \qquad x > x_0,$$

for some positive constants x_0, c_1, a_1, yields a non-global solution of (1.3) provided $a_1\beta < 1 + \sigma$. Similarly, if the initial value of (1.3) satisfies

$$\varphi(x) \leq c_2 x^{-a_2}, \qquad x > x_0,$$

where x_0, c_2, a_2 are positive numbers and $a_2\beta > 1 + \sigma$, then the solution u_t is global and satisfies $0 \leq u_t(x) \leq Ct^{-a_2}$, $x \geq 0$, for some constant $C > 0$. For the particular case $\sigma = 0$, if $\varphi(x) \sim_{x\to\infty} cx^{-a}$ for some $c > 0$ and $a > 0$, then blow-up of (1.3) occurs if $a\beta < 1$, and a global solution exists if $a\beta > 1$. Hence, if $\sigma = 0$ and for some $\varepsilon > 0$

$$\liminf_{x\to\infty} x^{-\varepsilon+1/\beta}\varphi(x) > 0,$$

then the solution of (1.3) blows up, whereas if

$$\limsup_{x\to\infty} x^{\varepsilon+1/\beta}\varphi(x) = 0,$$

then the solution of (1.3) exists globally.

Note that without additional difficulty we may replace the operator Γ in (1.3) with the generator Γ_λ given by

$$\Gamma_\lambda f(x) = \int_0^\infty (f(x+y) - f(x)) \frac{e^{-\lambda y}}{y}\, dy, \qquad x \in \mathbb{R}_+,$$

where λ is a strictly positive parameter. Indeed, for $f \in \mathrm{Dom}(\varGamma_\lambda)$ we have the relation $\varGamma_\lambda f(x) = \varGamma f_\lambda(\lambda x)$, where $f_\lambda(x) = f(x/\lambda)$. This means that f_λ is solution of (1.3) if and only if f is solution of (1.3) with $\varGamma_\lambda$ in place of $\varGamma$.

We remark that, for the parameter constellation $a\beta = \sigma + 1$ in Corollary 4.3 and in Theorem 5.1, our semigroup bounds (see Lemma 2.1) seem to be not sharp enough to yield, using our present methods, a subsolution h_t of (1.3) growing uniformly on a ball. Therefore, the blow-up behavior of (1.3) for such constellation remains open.

In the case of systems of equations of the form

$$\begin{cases} \dfrac{\partial u_t}{\partial t} = \varGamma_\lambda u_t + \nu u_t^{1+\beta_1} v_t^{\beta_2}, & u_0 = \varphi_1, \\ \\ \dfrac{\partial v_t}{\partial t} = \varGamma_\mu v_t + F_t(u_t, v_t), & v_0 = \varphi_2, \end{cases}$$

where $\lambda \neq \mu$, $\nu > 0$, and F_t is a positive and measurable function, the solution cannot be constructed directly from the case $\lambda = \mu = 1$, nevertheless the existence and blow-up criteria for solutions are independent of the values of $\lambda, \mu > 0$. In this case we show that if $\varphi_1(x) \geq cx^{-a_1}$ and $\varphi_2(x) \geq cx^{-a_2}$, for x large enough, then blow-up occurs provided $a_1\beta_1 + a_2\beta_2 < 1$. We also study the semilinear system

$$\begin{cases} \dfrac{\partial u_t}{\partial t} = \varGamma_{\lambda_1} u_t + \nu_1 u_t^{\beta_{11}} v_t^{\beta_{12}}, & u_0 = \varphi_1, \\ \\ \dfrac{\partial v_t}{\partial t} = \varGamma_{\lambda_2} v_t + \nu_2 u_t^{\beta_{21}} v_t^{\beta_{22}}, & v_0 = \varphi_2, \end{cases}$$

$\nu_1, \nu_2 > 0$, with integer exponents $\beta_{ij} \geq 1$ and initial values satisfying $\varphi_1(x) \leq c_1 x^{-a_1}$ and $\varphi_2(x) \leq c_2 x^{-a_2}$ for x large enough, where $a_1, a_2 \in (1, \infty)$. We show that this system admits a global solution provided the constants $c_1, c_2 > 0$ are sufficiently small. In particular, the solution of the system

$$\begin{cases} \dfrac{\partial u_t}{\partial t} = \varGamma u_t + u_t v_t \\ \\ \dfrac{\partial v_t}{\partial t} = \varGamma v_t + u_t v_t, \end{cases} \tag{1.4}$$

with $u_0(x) \sim cx^{-a_1}$ and $v_0(x) \sim cx^{-a_2}$ for x large enough, is global if $\min(a_2, a_1) > 1$ and c is sufficiently small. We also show that blow-up occurs if $\min(a_2, a_1) < 1$, and deal under additional assumptions with critical cases with time-dependent non-linearities.

We point out that critical blow-up and global existence parameters for semilinear systems of the form (1.4), with the Laplacian Δ instead of the operator Γ, were investigated in [7] for more general non-linear terms.

Our methods of proof are motivated by the approaches developed in [3] and [17]. As mentioned before, to prove explosion of semilinear equations we use the Feynman-Kac representation as well as estimates of probability transition densities and bridge distributions. Existence of global solutions is proved using a general criterion originally obtained in [17]. Global existence for systems of equations is proved along the lines of [14] and [17].

The paper is organized as follows. In Section 2 we recall some basic facts about the gamma process and its infinitesimal generator, and obtain bounds for the gamma semigroup that will be useful in the sequel. In Section 3 we recall the Feynman-Kac representation of (1.3), and derive from this representation a criterion for blow-up of semilinear PDE's. Using a general argument deduced from [24], we show existence of global solutions in Section 4. Blow-up of solutions of (1.3) is dealt with in Section 5, and systems of semilinear PDE's with gamma generators are considered in Section 6.

2 Estimates of the gamma semigroup

Let G denote the gamma function, i.e.

$$G(t) = \int_0^\infty x^{t-1} e^{-x}\, dx, \qquad t > 0,$$

and let $(X_t^\Gamma)_{t\in\mathbb{R}_+}$ denote the standard gamma process starting from 0, having transition densities

$$\gamma_t(x) = \frac{x^{t-1}}{G(t)} e^{-x} 1_{[0,\infty)}(x), \qquad x \in \mathbb{R}, \quad t > 0,$$

and generator

$$\Gamma f(x) = \int_0^\infty (f(x+y) - f(x)) \frac{e^{-y}}{y}\, dy.$$

Let $\{T_t^\Gamma,\, t \ge 0\}$ denote the operator semigroup generated by Γ, which is given by

$$T_t^\Gamma \varphi(y) = E[\varphi(X_t^\Gamma + y)] = \int_0^\infty \varphi(x+y)\gamma_t(x)dx = \int_y^\infty \varphi(x)\gamma_t(x-y)dx, \qquad (2.1)$$

$y \in \mathbb{R}_+$. In the next lemma we prove asymptotic estimates for the semigroup $\{T_t^\Gamma,\, t \ge 0\}$, using results of [4] on the median of the gamma density. Recall that for $t > 1$, γ_t is increasing on $[0, t-1]$ and decreasing on $[t-1, \infty)$.

Lemma 2.1 *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be bounded and measurable. Assume that there exist $c_1 \in [0, \infty)$, $c_2 \in (0, \infty]$, and $a_1 \ge a_2 > 0$ such that for all x large enough,*

$$c_1 x^{-a_1} \le \varphi(x) \le c_2 x^{-a_2}. \qquad (2.2)$$

Then, for all $\eta \ge 0$ and $0 < \varepsilon \le 1$ there exists $t_0 = t_0(\varepsilon, \eta) > 0$ such that

1. *For all $t > t_0$ and all $y \ge 0$,*

$$\left(\frac{1-\varepsilon}{2+\varepsilon}\right)^{a_1} \frac{c_1}{2} t^{-a_1} 1_{[0,t+\eta]}(y) \le T_t^\Gamma \varphi(y) \le c_2(1+\varepsilon)t^{-a_2}. \qquad (2.3)$$

2. *For all $t > t_0$ and any $0 \le y \le \eta + t/2$,*

$$(1-\varepsilon)\frac{c_1}{2^{1+a_1}} t^{-a_1} 1_{[0,\eta+t/2]}(y) \le T_t^\Gamma (1_{[t-1/3,2t]}\varphi)(y) \le c_2(1+\varepsilon)t^{-a_2}. \qquad (2.4)$$

3. *For all $t > t_0$ and any $0 \le y \le \eta \le 1$,*

$$(1-\varepsilon)\frac{\eta c_1}{\sqrt{2\pi}} t^{-a_1-1/2} 1_{[0,\eta]}(y) \le T_t^\Gamma (1_{[t-\eta,t]}\varphi)(y) \le (1+\varepsilon)\frac{\eta c_2}{\sqrt{2\pi}} t^{-a_2-1/2}. \qquad (2.5)$$

Proof. There exists $x_0 > 0$ such that for all $0 < y < t + \eta$,

$$\begin{aligned}
T_t^\Gamma \varphi(y) &= \int_0^\infty \varphi(x+y)\gamma_t(x)dx \\
&\ge c_1 \int_{x_0}^\infty (x+y)^{-a_1}\gamma_t(x)dx \\
&\ge c_1 \int_{x_0}^\infty (x+t+\eta)^{-a_1}\gamma_t(x)dx \\
&= c_1 \frac{G(t-a_1)}{G(t)} \int_{x_0}^\infty (1+(t+\eta)/x)^{-a_1}\gamma_{t-a_1}(x)dx \\
&\ge c_1 \frac{G(t-a_1)}{G(t)} \int_{t-a_1-1/3}^\infty (1+(t+\eta)/x)^{-a_1}\gamma_{t-a_1}(x)dx
\end{aligned}$$

$$
\begin{aligned}
&\geq c_1 \frac{G(t-a_1)}{G(t)} \int_{t-a_1-1/3}^{\infty} \left(1+\frac{t+\eta}{t-a_1-1/3}\right)^{-a_1} \gamma_{t-a_1}(x)dx \\
&\geq c_1 \frac{G(t-a_1)}{G(t)} \frac{(1-\varepsilon)^{a_1}}{2^{a_1}} \int_{t-a_1-1/3}^{\infty} \gamma_{t-a_1}(x)dx \\
&\geq \frac{c_1}{2} \frac{(1-\varepsilon)^{a_1}}{(2+\varepsilon)^{a_1}} t^{-a_1},
\end{aligned}
$$

for all sufficiently large t, provided $(a_1+1/3)/t < \varepsilon$ and $\eta/t < \varepsilon$. Here we used the equivalence $G(t-a_1)/G(t) \sim t^{-a_1}$ as $t \to \infty$, which follows from Stirling's formula $G(t) \sim \sqrt{2\pi} t^{t-1/2} e^{-t}$, and the fact that the median of the gamma distribution with parameter $t-a_1$ is greater than $t-a_1-1/3$ (and smaller than $t-a_1$), see Theorem 2 of [4]. Analogously we have, for all $y>0$ and all t, x_0 big enough,

$$
\begin{aligned}
T_t^{\Gamma}\varphi(y) &= \int_0^{\infty} \varphi(x+y)\gamma_t(x)dx \\
&\leq c_2 \int_{x_0}^{\infty} (x+y)^{-a_2}\gamma_t(x)dx + \|\varphi\|_\infty \int_0^{x_0} \gamma_t(x)dx \\
&\leq c_2 \int_{x_0}^{\infty} x^{-a_2}\gamma_t(x)dx + x_0\|\varphi\|_\infty \gamma_t(x_0) \\
&\leq c_2 \frac{G(t-a_2)}{G(t)} \int_0^{\infty} \gamma_{t-a_2}(x)dx + x_0\|\varphi\|_\infty \gamma_t(x_0) \\
&\leq \left(c_2(1+\varepsilon/2) + x_0 e^{-x_0}\|\varphi\|_\infty \frac{x_0^{t-1}t^{a_2}}{G(t)}\right) t^{-a_2} \\
&\leq c_2(1+\varepsilon)t^{-a_2},
\end{aligned}
$$

which proves (2.3). Concerning (2.4) we have for $0 < y \leq \eta + t/2$ and t sufficiently large:

$$
\begin{aligned}
\int_{t-1/3}^{2t} \varphi(x)\gamma_t(x-y)dx &\geq c_1 \int_{t-1/3}^{2t} x^{-a_1}\gamma_t(x-y)dx \\
&\geq c_1(2t)^{-a_1} \int_{t-1/3}^{2t} \gamma_t(x-y)dx \\
&\geq c_1(2t)^{-a_1} \int_{t-1/3}^{-\eta+3t/2} \gamma_t(x)dx \\
&\geq c_1(2t)^{-a} \left(\frac{1}{2} - \int_{-\eta+3t/2}^{\infty} \gamma_t(x)dx\right) \\
&\geq (1-\varepsilon)\frac{c_1}{2}(2t)^{-a},
\end{aligned}
$$

since $\int_{t-1/3}^{\infty} \gamma_t(x)dx \geq 1/2$ and $\int_{-\eta+3t/2}^{\infty} \gamma_t(x)dx = P(X_t^\Gamma \geq -\eta + \frac{3t}{2}) \to 0$ as $t \to \infty$ by the law of large numbers. Similarly, for t large enough,

$$\begin{aligned}\int_{t-1/3}^{2t} \varphi(x)\gamma_t(x-y)\,dx &\leq c_2 \int_{t-1/3}^{2t} x^{-a_2}\gamma_t(x-y)\,dx \\ &\leq c_2(t-1/3)^{-a_2} \int_{t-1/3}^{2t} \gamma_t(x-y)\,dx \\ &\leq c_2(t-1/3)^{-a_2} \\ &\leq (1+\varepsilon)c_2 t^{-a_2}.\end{aligned}$$

Concerning (2.5) we have, for $0 < y \leq \eta \leq 1$ and $t > 2$,

$$\gamma_t(t-1)\int_{t-\eta}^{t} \varphi(x)\,dx \geq \int_{t-\eta}^{t} \varphi(x)\gamma_t(x-y)\,dx \geq (\gamma_t(t) \wedge \gamma_t(t-2)) \int_{t-\eta}^{t} \varphi(x)\,dx.$$

Since for any $l \geq 0$,

$$\gamma_t(t-l) = \frac{(t-l)^{t-1}}{G(t)} e^{-t+l} \sim \frac{(t-l)^{t-1}e^l}{\sqrt{2\pi}t^{t-1/2}} \sim \frac{t^{-1/2}}{\sqrt{2\pi}}, \quad t \to \infty,$$

it follows that for any $0 < \varepsilon < 1$ and for all sufficiently large t,

$$(1+\varepsilon)\frac{t^{-1/2}}{\sqrt{2\pi}} \int_{t-\eta}^{t} \varphi(x)\,dx \geq \int_{t-\eta}^{t} \varphi(x)\gamma_t(x-y)\,dx \geq (1-\varepsilon)\frac{t^{-1/2}}{\sqrt{2\pi}} \int_{t-\eta}^{t} \varphi(x)\,dx.$$

It remains to note that

$$\int_{t-\eta}^{t} x^{-a}dx = \frac{t^{-a}}{1-a}(1-(1-\eta/t)^{1-a}) \sim \eta t^{-a}$$

for all $a \geq 0$ as t goes to infinity, and to use (2.2). $\square$

Remark 2.2 *Let $\{T_t^\lambda,\ t \geq 0\}$ be the operator semigroup with generator Γ_λ. From the relation $T_t^\lambda\varphi(x) = [T_t^\Gamma\varphi_\lambda](\lambda x)$ we get, for $t > t_0$ and $y \geq 0$,*

$$\frac{c_1}{2}\left(\frac{1-\varepsilon}{3}\right)^{a_1}\left(\frac{t}{\lambda}\right)^{-a_1} 1_{[0,t+\eta]}(y) \leq T_t^\lambda\varphi(y) \leq c_2(1+\varepsilon)\left(\frac{t}{\lambda}\right)^{-a_2},$$

$$(1-\varepsilon)\frac{c_1}{2^{1+a_1}}\left(\frac{t}{\lambda}\right)^{-a_1} 1_{[0,\eta+t/2]}(y) \leq T_t^\Gamma(1_{[t-1/3,2t]}\varphi)(y) \leq c_2(1+\varepsilon)\left(\frac{t}{\lambda}\right)^{-a_2},$$

$$(1-\varepsilon)\frac{\eta c_1}{\sqrt{2\pi}}\left(\frac{t}{\lambda}\right)^{-a_1} 1_{[0,\eta]}(y) \leq T_t^\Gamma(1_{[t-\eta,t]}\varphi)(y) \leq (1+\varepsilon)\frac{\eta c_2}{\sqrt{2\pi}}\left(\frac{t}{\lambda}\right)^{-a_2}.$$

Recall that for $0 \le s < t$ and $x > 0$, the conditional law of X_s^Γ, given that $X_t^\Gamma = x$, is the beta distribution with density

$$\beta_{s,t}(z,x) := \frac{\gamma_s(z)\gamma_{t-s}(x-z)}{\gamma_t(x)} = \frac{1}{x}\frac{G(t)}{G(s)G(t-s)}\left(\frac{z}{x}\right)^{s-1}\left(1-\frac{z}{x}\right)^{t-s-1}, \qquad 0 \le z \le x. \tag{2.6}$$

Using the result of [18] on the median of the beta distribution we obtain the following estimates.

Lemma 2.3 *Let $\eta > 0$. We have*

$$P_y(0 < X_s^\Gamma < s+\eta | X_t^\Gamma = x) \ge 1/2 \tag{2.7}$$

for all $0 < s < t/2$, $0 < y < \eta$, $0 < t - 2\eta < t - \eta < x < t$, and

$$P_y(0 < X_s^\Gamma < 2s + t/2 | X_t^\Gamma = x) \ge 1/2 \tag{2.8}$$

for all $0 < s < t/2$, $0 < y < t/2$ and $0 < t/2 < x < 2t$.

Proof. We have

$$\begin{aligned}
P_y(0 < X_s^\Gamma < s+\eta | X_t^\Gamma = x) &= P(0 < y + X_s^\Gamma < s+\eta | X_t^\Gamma = x - y) \\
&\ge P(0 < X_s^\Gamma < s | X_t^\Gamma = x - y) \\
&= \int_0^s \beta_{s,t}(z, x-y)dz \\
&= \frac{G(t)}{G(s)G(t-s)}\int_0^{s/(x-y)} z^{s-1}(1-z)^{t-s-1}dz \\
&\ge \frac{G(t)}{G(s)G(t-s)}\int_0^{s/t} z^{s-1}(1-z)^{t-s-1}dz \\
&= \int_0^{s/t} \beta_{s,t}(z,1)dz \\
&\ge 1/2,
\end{aligned}$$

because, from Theorem 1 of [18], the median $m_{s,t}$ of the standard beta density $\beta_{s,t}(\cdot, 1)$ with mean s/t satisfies

$$0 < m_{s,t} < \frac{s}{t} < m_{s,t} + \frac{t-2s}{(t-2)t},$$

provided $s < t/2$. Similarly we have

$$\begin{aligned} P_y(0 < X_s^\Gamma < 2s + t/2 | X_t^\Gamma = x) &= P(0 < y + X_s^\Gamma < 2s + t/2 | X_t^\Gamma = x - y) \\ &\geq P(0 < X_s^\Gamma < 2s | X_t^\Gamma = x - y) \\ &= \int_0^{2s} \beta_{s,t}(z, x-y) dz \\ &= \frac{G(t)}{G(s)G(t-s)} \int_0^{2s/(x-y)} z^{s-1}(1-z)^{t-s-1} dz \\ &\geq \frac{G(t)}{G(s)G(t-s)} \int_0^{s/t} z^{s-1}(1-z)^{t-s-1} dz \\ &\geq 1/2. \end{aligned}$$

□

3 Feynman-Kac representation and subsolutions

Let $(X_t)_{t\in\mathbb{R}_+}$ be a Lévy process in $\mathbb{R}_+$ with generator L and transition semigroup $\{T_t^L,\ t \geq 0\}$. Recall (see e.g. [8]) that the mild solution of

$$\frac{\partial w_t}{\partial t}(y) = Lw_t(y) + \zeta_t(y)w_t(y), \qquad w_0 = \varphi, \tag{3.1}$$

admits the Feynman-Kac representation

$$w_t(y) = E\left[\varphi(y + X_t) \exp \int_0^t \zeta_{t-s}(y + X_s) ds\right], \qquad t \geq 0, \quad y \geq 0. \tag{3.2}$$

If ζ_t is positive (3.2) implies

$$w_t(y) \geq E\left[\varphi(y + X_t)\right] = T_t^L \varphi(y), \qquad y \in \mathbb{R}_+, \quad t \geq 0.$$

Thus, the solution of

$$\frac{\partial w_t}{\partial t} = Lw_t, \qquad w_0 = \varphi \geq 0,$$

is also a subsolution of (3.1) provided $\zeta_t \geq 0$. By linearity this implies the following lemma.

Lemma 3.1 *Let $\varphi \geq 0$ be bounded and measurable. If u_t, v_t respectively solve*

$$\frac{\partial u_t}{\partial t}(y) = Lu_t(y) + \zeta_t(y)u_t(y), \qquad \frac{\partial v_t}{\partial t}(y) = Lv_t(y) + \xi_t(y)v_t(y),$$

with $u_0 \geq v_0$ and $\zeta_t \geq \xi_t$, then $u_t \geq v_t$.

We will use the fact that if u_t is a subsolution of

$$\frac{\partial w_t}{\partial t}(y) = Lw_t(y) + \nu w_t^{1+\beta}(y), \qquad w_0 = \varphi, \tag{3.3}$$

where $\nu, \beta > 0$, then any solution of

$$\frac{\partial v_t}{\partial t}(y) = Lv_t(y) + \nu u_t^{\beta}(y) v_t(y), \qquad v_0 = \varphi,$$

remains a subsolution of (3.3). This follows from Lemma 3.1.

Going back to (3.1), notice that from the Feynman-Kac representation,

$$\begin{aligned} w_t(y) &= \int_{-\infty}^{\infty} \varphi(y+x) E\left[\exp \int_0^t \zeta_{t-s}(y+X_s)\, ds \Big| X_t = x\right] p_t(x) dx \\ &= \int_{-\infty}^{\infty} \varphi(x) p_t(x-y) E\left[\exp \int_0^t \zeta_{t-s}(y+X_s) ds \Big| y + X_t = x\right] dx \\ &= \int_{-\infty}^{\infty} \varphi(x) p_t(x-y) E_y\left[\exp \int_0^t \zeta_{t-s}(X_s) ds \Big| X_t = x\right] dx \\ &\geq \int_{-\infty}^{\infty} \varphi(x) p_t(x-y) \exp\left(E_y\left[\int_0^t \zeta_{t-s}(X_s) ds \Big| X_t = x\right]\right) dx, \end{aligned} \tag{3.4}$$

where on the last line we used Jensen's inequality. Hence, when $L = \varGamma$, (3.4) reads

$$w_t(y) \;\geq\; \int_y^{\infty} \varphi(x) \gamma_t(x-y) \exp\left(\int_0^t \int_y^x \beta_{s,t}(z-y, x-y) \zeta_{t-s}(z) dz ds\right) dx,$$

where $\beta_{s,t}(z-y, x-y)$ is given by (2.6).

We close this section with a lemma that will be helpful in the proof of explosion, see §4 in [12] for the case $L = \Delta$.

Lemma 3.2 *Let $\sigma \in \mathbb{R}$ and $\nu > 0$. Assume that the solution u_t of*

$$\frac{\partial w_t}{\partial t}(y) = \varGamma w_t(y) + \nu t^{\sigma} v_t(y) w_t(y), \qquad w_0 = \varphi, \tag{3.5}$$

satisfies

$$\lim_{t\to\infty} \inf_{0 \leq x \leq 1} u_t(x) = \infty,$$

where $v : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a measurable function such that $u_t^{\beta} \leq v_t$ for all $t \geq 0$. Then u_t blows up in finite time, in the sense that there exists $t > 0$ such that

$$\int_0^1 u_t(x)\, dx = \infty.$$

In particular, explosion in $L^p(\mathbb{R}_+)$-norm occurs for all $p \in [1, \infty]$.

Proof. Given $t_0 > 0$, let $u_t = w_{t_0+t}$ and $K(t_0) = \inf_{0\le y\le 1} w_{t_0}(y)$. The mild solution of (3.5) is given by

$$u_t(x) = \int_0^\infty \gamma_t(y-x)u_0(y)\,dy + \nu\int_0^t s^\sigma \int_0^\infty \gamma_{t-s}(y-x)u_s(y)v_{s+t_0}(y)\,dy\,ds.$$

Thus, for any $\varepsilon \in (0,1)$ and $t < (1-\varepsilon)\beta \wedge 1$,

$$\begin{aligned}
&\int_0^1 u_t(x)\,dx \\
&\ge \int_0^1\int_0^\infty \gamma_t(y-x)u_0(y)dydx + \nu\int_0^t s^\sigma\int_0^1\int_0^\infty \gamma_{t-s}(y-x)u_s^{1+\beta}(y)dydxds \\
&\ge \int_0^1\int_x^1 \gamma_t(y-x)u_0(y)dydx + \nu\int_0^t s^\sigma\int_0^1\int_x^1 \gamma_{t-s}(y-x)u_s^{1+\beta}(y)dydxds \\
&\ge K(t_0)\int_0^1\int_0^y \gamma_t(x-y)dxdy + \nu\int_0^t s^\sigma\int_0^1 u_s^{1+\beta}(y)\int_0^y \gamma_{t-s}(x-y)dxdyds \\
&\ge K(t_0)\int_0^1\int_0^y \gamma_t(x)dxdy + \nu\int_0^t s^\sigma\int_0^1 u_s^{1+\beta}(y)\int_0^y \gamma_{t-s}(x)dxdyds \\
&\ge \frac{1}{4}K(t_0)\int_0^1\int_0^y \frac{x^{t-1}}{G(t)}dxdy + \frac{\nu}{4}\int_0^t s^\sigma\int_0^1 u_s^{1+\beta}(y)\int_0^y \frac{x^{t-s-1}}{G(t-s)}dxdyds \\
&\ge \frac{1}{4}\frac{K(t_0)}{tG(t)}\int_0^1 y^t dy + \frac{\nu}{4}\int_0^t s^\sigma\int_0^1 u_s^{1+\beta}(y)\frac{y^{t-s}}{(t-s)\gamma(t-s)}dyds \\
&\ge \frac{1}{4}K(t_0)\int_0^1 y^\beta dy + \frac{\nu}{4}\int_0^t s^\sigma\int_0^1 u_s^{1+\beta}(y)y^{(1-\varepsilon)\beta}dyds \\
&\ge \frac{K(t_0)}{4(1+\beta)} + \frac{\nu}{4}\int_0^t s^\sigma\int_0^1 u_s^{1+\beta}(y)y^{(1-\varepsilon)\beta}dyds,
\end{aligned}$$

where we used the inequalities $0 \le t-s \le t < (1-\varepsilon)\beta$ and $0 \le tG(t) \le 1$, $0 \le t \le 1$. Hölder's inequality yields

$$\begin{aligned}
\left(\int_0^1 u_s(y)dy\right)^{1+\beta} &\le \left(\int_0^1 u_s^{1+\beta}(y)y^{(1-\varepsilon)\beta}dy\right)\left(\int_0^1 y^{-(1-\varepsilon)}dy\right)^\beta \\
&= \varepsilon^{-\beta}\int_0^1 u_s^{1+\beta}(y)y^{(1-\varepsilon)\beta}dy,
\end{aligned}$$

hence letting $\tilde{u}(t) = \int_0^1 u_t(x)\,dx$ we get

$$\tilde{u}(t) \ge \frac{K(t_0)}{4(1+\beta)} + \frac{\nu\varepsilon^\beta}{4}\int_0^t s^\sigma \tilde{u}^{1+\beta}(s)\,ds, \qquad t < (1-\varepsilon)\beta \wedge 1.$$

It remains to choose t_0 such that the blow-up time of the equation

$$\tilde{u}(t) = \frac{K(t_0)}{4(1+\beta)} + \frac{\nu\varepsilon^\beta}{4}\int_0^t s^\sigma \tilde{u}^{1+\beta}(s)ds, \qquad t < (1-\varepsilon)\beta \wedge 1,$$

is smaller than $(1-\varepsilon)\beta \wedge 1$. □

Choosing $v_t = u_t^\beta$ in Lemma 3.2 yields immediately:

Corollary 3.3 *Let $\sigma \in \mathbb{R}$ and $\nu > 0$. If the solution u_t of*

$$\frac{\partial w_t}{\partial t}(y) = \Gamma w_t(y) + \nu t^\sigma w_t^{1+\beta}(y), \qquad w_0 = \varphi,$$

satisfies

$$\lim_{t\to\infty} \inf_{0\le x\le 1} u_t(x) = \infty,$$

then u_t blows up in finite time, in the sense that there exists $t > 0$ such that

$$\int_0^1 u_t(x)\,dx = \infty.$$

□

4 Existence of global solutions

We have the following non-explosion result, obtained originally by Nagasawa and Sirao [17] for integer $\beta \ge 1$. In the next theorem, as in the previous section, $\{T_t^L,\ t \ge 0\}$ denotes the transition semigroup of a Lévy process with generator L.

Theorem 4.1 *Let $\sigma \in \mathbb{R}$ and $\beta, \nu > 0$. Assume that*

$$\int_0^\infty r^\sigma \|T_r^L\varphi\|_\infty^\beta\,dr < \frac{b}{\nu\beta}$$

for some $b \in (0,1)$. Then the equation

$$\frac{\partial w_t}{\partial t} = Lw_t + \nu t^\sigma w_t^{1+\beta}, \qquad w_0 = \varphi, \tag{4.1}$$

admits a global solution $u_t(x)$ which satisfies

$$0 \le u_t(x) \le \frac{b^{1/\beta} T_t^L \varphi(x)}{\left(b - \nu\beta\int_0^t r^\sigma \|T_r^L\varphi\|_\infty^\beta dr\right)^{1/\beta}}, \qquad x \in \mathbb{R}_+, \quad t \ge 0.$$

Proof. This is an adaptation of the proof of Theorem 3 in [24] to our context of time-dependent non-linearities. Recall that the mild solution of (4.1) is given by

$$u_t(x) = T_t^L\varphi(x) + \nu \int_0^t r^\sigma T_{t-r}^L u_r^{1+\beta}(x)\, dr. \tag{4.2}$$

Defining

$$B(t) = \left(b - \beta\nu \int_0^t r^\sigma \|T_r^L\varphi\|_\infty^\beta dr\right)^{-1/\beta}, \qquad t \geq 0,$$

we have $B(0) = b^{-1/\beta}$ and

$$\frac{d}{dt}B(t) = \nu t^\sigma \|T_t^L\varphi\|_\infty^\beta \left(b - \beta\nu \int_0^t r^\sigma \|T_r^L\varphi\|_\infty^\beta dr\right)^{-1-1/\beta} = \nu t^\sigma \|T_t^L\varphi\|_\infty^\beta B^{1+\beta}(t),$$

hence

$$B(t) = b^{-1/\beta} + \nu \int_0^t r^\sigma \|T_r^L\varphi\|_\infty^\beta B^{1+\beta}(r)\, dr.$$

Let $(t,x) \mapsto v_t(x)$ be a continuous function such that $v_t(\cdot) \in C_0(\mathbb{R}_+)$, $t \geq 0$, and

$$T_t^L\varphi(x) \leq v_t(x) \leq b^{1/\beta}B(t)T_t^L\varphi(x), \qquad t \geq 0,\ x \in \mathbb{R}_+.$$

Let now

$$R(v)(t,x) = T_t^L\varphi(x) + \nu \int_0^t r^\sigma T_{t-r}^L v_r^{1+\beta}(x) dr.$$

We have

$$\begin{aligned} R(v)(t,x) &\leq T_t^L\varphi(x) + \nu b^{1+1/\beta} \int_0^t r^\sigma B^{1+\beta}(r) T_{t-r}^L (T_r^L\varphi(x))^{1+\beta} dr \\ &\leq T_t^L\varphi(x) + \nu b^{1+1/\beta} \int_0^t r^\sigma B^{1+\beta}(r) T_{t-r}^L T_r^L\varphi(x) \|T_r^L\varphi\|_\infty^\beta dr \\ &= b^{1/\beta}T_t^L\varphi(x) \left(b^{-1/\beta} + \nu b \int_0^t r^\sigma B^{1+\beta}(r) \|T_r^L\varphi\|_\infty^\beta dr\right), \end{aligned}$$

hence

$$T_t^L\varphi(x) \leq R(v)(t,x) \leq b^{1/\beta}B(t)T_t^L\varphi(x), \qquad t \geq 0,\ x \in \mathbb{R}_+.$$

Let

$$u_t^0(x) = T_t^L\varphi(x), \quad \text{and} \quad u_t^{n+1}(x) = R(u^n)(t,x), \quad n \in \mathbb{N}.$$

Then $u_t^0(x) \leq u_t^1(x)$, $t \geq 0$, $x \in \mathbb{R}_+$. Since T_t^L is non-negative, using induction we obtain

$$0 \leq u_t^n(x) \leq u_t^{n+1}(x), \qquad n \geq 0.$$

Letting $n \to \infty$ yields, for $t \geq 0$ and $x \in \mathbb{R}_+$,

$$0 \leq u_t(x) = \lim_{n\to\infty} u_t^n(x) \leq b^{1/\beta} B(t) T_t^L \varphi(x) \leq \frac{b^{1/\beta} T_t^L \varphi(x)}{\left(b - \nu\beta \int_0^t r^\sigma \|T_r^L \varphi\|_\infty^\beta dr\right)^{1/\beta}} < \infty.$$

Consequently, u_t is a global solution of (4.2) due to the monotone convergence theorem. $\square$

As an application of Theorem 4.1 to the case $L = \Gamma$, a global existence result can be obtained under an integrability condition on φ.

Corollary 4.2 *Let $1 \leq q < \infty$, $\sigma > -1$, $\nu > 0$, and $\beta > 2q(1+\sigma)$. If $\varphi \in L^q(\mathbb{R}_+)$ is non-negative and $\|\varphi\|_q$ is sufficiently small, then the solution u_t of*

$$\frac{\partial w_t}{\partial t} = \Gamma w_t + \nu t^\sigma w_t^{1+\beta}, \qquad w_0 = \varphi,$$

is global and satisfies, for some $c > 0$,

$$0 \leq u_t(x) \leq c t^{-1/(2q)}, \qquad x \in \mathbb{R}_+,$$

for all t large enough.

Proof. From Hölder's inequality and (2.1) we have

$$|T_t^\Gamma \varphi(y)| \leq \|\varphi\|_q \|\gamma_t\|_p, \qquad 1/p = 1 - 1/q,$$

where

$$\begin{aligned}
\|\gamma_t\|_p &= \left(\int_0^\infty \frac{x^{p(t-1)}}{G(t)^p} e^{-px} dx\right)^{1/p} \\
&= \frac{G(p(t-1)+1)^{1/p}}{p^{t-1} G(t)} \left(\int_0^\infty \frac{(px)^{p(t-1)}}{G(p(t-1)+1)} e^{-px} dx\right)^{1/p} \\
&= \frac{G(p(t-1)+1)^{1/p}}{p^{t-1+1/p} G(t)} \\
&\sim \frac{(p(t-1)+1)^{(t-1)+1/(2p)} e^{1-1/p}}{p^{t-1+1/p} t^{t-1/2}} (2\pi)^{-1/2+1/(2p)} \\
&\sim t^{1/2} \frac{(1 - 1/t + 1/(pt))^t (p(t-1)+1)^{1/(2p)} e^{1-1/p}}{(t-1+1/p) p^{1/p}} (2\pi)^{-1/(2q)}
\end{aligned}$$

$$
\begin{aligned}
&\sim \quad t^{1/2}\frac{(p(t-1)+1)^{1/(2p)}}{(t-1+1/p)p^{1/p}}(2\pi)^{-1/(2q)} \\
&\sim \quad t^{-1/2}t^{1/(2p)}p^{-1/(2p)}(2\pi)^{-1/(2q)} \\
&\sim \quad (2\pi t)^{-1/(2q)}p^{-1/(2p)},
\end{aligned}
$$

as $t \to \infty$. Hence for some $t_0 > 0$ and $c > 0$,

$$
\int_0^\infty t^\sigma \|T_t^\Gamma \varphi\|_\infty^\beta dt \le \|\varphi\|_\infty^\beta \int_0^{t_0} t^\sigma dt + c\|\varphi\|_q^\beta \int_{t_0}^\infty t^\sigma \|\gamma_t\|_p^\beta dt < \infty
$$

provided $\beta > 2q(1+\sigma)$, and the conclusion follows from Theorem 4.1. □

Under a polynomial growth assumption on φ we get the following more specialized result as another corollary of Theorem 4.1.

Corollary 4.3 *Let $\sigma \in \mathbb{R}$ and assume that there exist $c \ge 0$, $a \ge 0$ and $x_0 \ge 0$ such that*

$$
\varphi(x) \le cx^{-a}, \qquad x > x_0.
$$

If $a\beta > 1+\sigma$, then the solution u_t of

$$
\frac{\partial w_t}{\partial t} = \mathit{\Gamma} w_t + \nu t^\sigma w_t^{1+\beta}, \qquad w_0 = \varphi
$$

is global, and there exists $C > 0$ such that

$$
0 \le u_t(x) \le Ct^{-a}, \qquad x \in \mathbb{R}_+,
$$

for all t large enough.

Proof. Let $\tilde{a}$ be such that $a < \tilde{a} < (1+\sigma)/\beta$. For any $\tilde{c} > 0$ there exists $\tilde{x}_0$ such that $\varphi(x) \le \tilde{c}x^{-\tilde{a}}$, $x > \tilde{x}_0$. It remains to apply (2.3) of Lemma 2.1 and Theorem 4.1. □

5 Blow-up of solutions

In this section we obtain a partial converse to Corollary 4.3.

Theorem 5.1 *Assume that $\varphi \geq 0$ satisfies $\varphi(x) \geq cx^{-a}$ for all x large enough, where $a, c \geq 0$. Let $\nu > 0$, $\beta > 0$ and $a\beta < 1+\sigma$. Then the equation*

$$\frac{\partial w_t}{\partial t} = \Gamma w_t + \nu t^\sigma w_t^{1+\beta}, \qquad w_0 = \varphi,$$

blows up in finite time.

This result is a consequence of the lemmas 3.1 and 3.2 above, and of the following lemma.

Lemma 5.2 *Assume that $\varphi \geq 0$ is such that $\varphi(x) \geq cx^{-a}$ for all x large enough, where $a, c \geq 0$. Let $\nu > 0$, $\beta > 0$, and let g_t be the solution of*

$$\frac{\partial w_t}{\partial t}(y) = \Gamma w_t(y) + \nu t^\sigma (T_t^\Gamma \varphi)^\beta(y) w_t(y), \qquad w_0 = \varphi. \tag{5.1}$$

If $a\beta < 1+\sigma$, then

$$\lim_{t\to\infty} \inf_{0\leq x\leq 1} g_t(x) = \infty.$$

Proof. Let $0 < \eta < 1$. The Feynman-Kac representation and (2.3) yield, for $0 < y < \eta + t/2$, $t > 6t_0$ (where t_0 is defined in Lemma 2.1), and some $c_0 > 0$:

$$\begin{aligned}
g_t(y) &= \int_y^\infty \varphi(x)\gamma_t(x-y) E_y\left[\exp\left(\nu \int_0^t (t-s)^\sigma (T_{t-s}^\Gamma \varphi(X_s^\Gamma))^\beta ds\right) \Big| X_t^\Gamma = x\right] dx \\
&\geq \int_y^\infty \varphi(x)\gamma_t(x-y) E_y\left[\exp\left(c_0\nu \int_{t_0}^{t-t_0} 1_{[0,\eta+t-s]}(X_s^\Gamma)(t-s)^{\sigma-a\beta} ds\right) \Big| X_t^\Gamma = x\right] dx \\
&\geq \int_{t-1/3}^{2t} \varphi(x)\gamma_t(x-y) \exp\left(c_0\nu \int_{t_0}^{t-t_0} (t-s)^{\sigma-a\beta} P_y(0 < X_s^\Gamma < \eta + t - s | X_t^\Gamma = x) ds\right) dx \\
&\geq \int_{t-1/3}^{2t} \varphi(x)\gamma_t(x-y) \exp\left(c_0\nu \int_{t_0}^{t/6} (t-s)^{\sigma-a\beta} P_y(0 < X_s^\Gamma < 2s + t/2 | X_t^\Gamma = x) ds\right) dx \\
&\geq c_1 1_{[0,\eta+t/2]}(y) t^{-a} \exp\left(\frac{c_0\nu}{2} \int_{t_0}^{t/6} (t-s)^{\sigma-a\beta} ds\right),
\end{aligned}$$

where we used (2.4) and (2.8) to obtain the last inequality. Hence

$$\begin{aligned}
g_t(y) &\geq 1_{[0,\eta+t/2]}(y) c_1 t^{-a} \exp\left(\frac{c_0\nu}{2} \int_{t_0}^{t/6} (t-s)^{\sigma-a\beta} ds\right) \qquad (5.2) \\
&= 1_{[0,\eta+t/2]}(y) c_1 t^{-a} \exp\left(\frac{c_0\nu}{2(1+\sigma-a\beta)} \left((t-t_0)^{1+\sigma-a\beta} - \left(\frac{5t}{6}\right)^{1+\sigma-a\beta}\right)\right),
\end{aligned}$$

and it suffices that $a\beta < 1+\sigma$ in order to get $\inf_{0<y<1} g_t(y) \to \infty$ as $t \to \infty$. $\square$

Notice that the criteria for blow-up of Lemma 5.2 can easily be adapted to other time-dependent non-linearities. More precisely, a time-dependent non-linearity of the form $\alpha(t)$ will lead to finite-time blow-up provided

$$\liminf_{t\to\infty} t^{-\varepsilon}\int_{t_0}^{t/6}(t-s)^{-a\beta}\alpha(t-s)\,ds>0,$$

for some $\varepsilon>0$.

6 Systems of semilinear equations

First we consider the following system of semilinear equations

$$\begin{cases} \dfrac{\partial u_t}{\partial t} = \Gamma_{\lambda_1}u_t+\nu_1 u_t^{\beta_{11}}v_t^{\beta_{12}} \\ \\ \dfrac{\partial v_t}{\partial t} = \Gamma_{\lambda_2}v_t+\nu_2 u_t^{\beta_{21}}v_t^{\beta_{22}}, \end{cases} \tag{6.1}$$

where $u_0=\varphi_1$ and $v_0=\varphi_2$ are nonnegative bounded measurable functions, $\nu_1,\nu_2>0$, and $\beta_{ij}\in\{1,2,\ldots\}$, $i,j=1,2$. The solution of this system can be expressed in terms of a continuous-time, two-type branching process evolving in the following way. The particles of type $i=1,2$ live independent exponential lifetimes of mean $1/\nu_i$. During its lifetime a type-i particle develops an independent Markov motion of generator Γ_{λ_i} and, at the end of its life, it branches, leaving behind β_{i1} individuals of type 1 and β_{i2} individuals of type 2 that appear where the parent particle died, and evolve independently under the same rules. The state space of such branching process is the space $\mathcal{N}_f(S)$ of finite counting measures on $S:=\mathbb{R}_+\times\{1,2\}$, where a measure

$$\mu=\sum_{i=1}^{n}\delta_{(x_i,1)}+\sum_{j=1}^{m}\delta_{(y_j,2)}$$

represents a population consisting of n individuals of type 1 at positions $x_1,\ldots,x_n$, and m individuals of type 2 at positions $y_1,\ldots,y_m$. Let X_t^μ be the random element of $\mathcal{N}_f(S)$ representing the population configuration at time $t\geq 0$, starting from a given $\mu\in\mathcal{N}_f(S)$. For any bounded measurable $f:S\to[0,\infty)$ we define

$$w_t(\mu)=E_\mu\left[e^{S_t}\prod_{z\in\operatorname{supp}(X_t^\mu)}f(z)\right],\qquad \mu\in\mathcal{N}_f(S),\quad t\geq 0, \tag{6.2}$$

where E_μ denotes expectation with respect to $P(\,\cdot\,|X_0 = \mu)$, and $S_t = \nu_1 \int_0^t N_{s,1}\,ds + \nu_2 \int_0^t N_{s,2}\,ds$, where $N_{s,i}$ is the number of particles of type i in the population at time s. Choosing f so that $f(\cdot, i) = \varphi_i$ for $i = 1, 2$, one can show, similarly as in [14], that the solution of (6.1) is given by $u_t = w_t(\cdot, 1)$ and $v_t = w_t(\cdot, 2)$, where for shortness of notation we write $w_t(x, i)$ when $\mu = \delta_{(x,i)}$. We now prove the following theorem.

Theorem 6.1 *Let the initial values φ_1, φ_2 of* (6.1) *be bounded measurable functions such that, for some constants $c_1, c_2 > 0$ and $a_1, a_2 \in (1, \infty)$,*

$$0 \le \varphi_1(x) \le c_1 x^{-a_1} \quad \text{and} \quad 0 \le \varphi_2(x) \le c_2 x^{-a_2}$$

for all x large enough. If c_1, c_2 are sufficiently small, then the solution of (6.1) *is global.*

The proof of the above theorem uses substantially the probabilistic representation of (u_t, v_t) derived from (6.2); see [14] for a short argument in the case $\beta_{ij} = 1$, $i, j = 1, 2$, and for a detailed description of the probabilistic framework.

Proof. Without loss of generality we assume that $f(x, i) := \varphi_i(x) \le c_i(x^{-a_i} \wedge 1)$ for all $x > 0$ and $i = 1, 2$. Let $\kappa = \kappa(t)$ denote the number of branchings occurring in the interval $[0, t]$, and let $w_t^{(k)}(\mu) = E_\mu\left[e^{S_t} \prod_{z \in \mathrm{supp}(X_t^\mu)} f(z);\ \kappa = k\right]$, $\mu \in \mathcal{N}_f(S)$, $k \in \mathbb{N}$. Therefore,

$$w_t(\mu) = \sum_{k=0}^{\infty} w_t^{(k)}(\mu), \qquad \mu \in \mathcal{N}_f(S), \quad t \ge 0.$$

Writing $\gamma_t^{\lambda_i}$ for the transition densities of the gamma process of parameter λ_i, $i = 1, 2$, and defining

$$\pi_t f(x, i) := \int_{\mathbb{R}} f(y, i) \gamma_t^{\lambda_i}(y - x)\,dy, \qquad (x, i) \in S, \quad t \ge 0,$$

we see that, for $\mu = \sum_{i=1}^n \delta_{(x_i,1)} + \sum_{j=1}^m \delta_{(y_j,2)}$,

$$w_t^{(0)}(\mu) = \left(\prod_{i=1}^n \pi_t f(x_i, 1)\right)\left(\prod_{j=1}^m \pi_t f(y_j, 2)\right).$$

Recall that the first branching time, τ, is exponentially distributed with parameter $n\nu_1 + m\nu_2$. Given that $\tau \ge s$, the evolution of the population up to time s follows a stochastic translation originated by the particle motion processes. Hence,

$$S_s = \int_0^s (n\nu_1 + m\nu_2)\, dr.$$

Notice that a given particle of type i performs the first branching with probability $\nu_i/(n\nu_1 + m\nu_2)$. Consequently, conditioning on the first branching time, the above equality yields

$$\begin{aligned} w_t^{(1)}(\mu) \quad &= \quad 1_{\{n\neq 0\}} \frac{\nu_1}{n\nu_1 + m\nu_2} \sum_{i=1}^{n} \int_0^t (n\nu_1 + m\nu_2) e^{-(n\nu_1+m\nu_2)s} e^{\int_0^s (n\nu_1+m\nu_2)\, dr} \\ &\quad \cdot \int_{\mathbb{R}} \gamma_s^{\lambda_1}(z - x_i) \left(\pi_{t-s} f(z,1)\right)^{\beta_{11}} \left(\pi_{t-s} f(z,2)\right)^{\beta_{12}}\, dz \\ &\quad \times \prod_{\substack{l=1 \\ l\neq i}}^{n} \pi_s w_{t-s}^{(0)}(x_l, 1) \prod_{h=1}^{m} \pi_s w_{t-s}^{(0)}(y_h, 2)\, ds \\ &+ \quad 1_{\{m\neq 0\}} \frac{\nu_2}{n\nu_1 + m\nu_2} \sum_{j=1}^{m} \int_0^t (n\nu_1 + m\nu_2) e^{-(n\nu_1+m\nu_2)s} e^{\int_0^s (n\nu_1+m\nu_2)\, dr} \\ &\quad \cdot \int_{\mathbb{R}} \gamma_s^{\lambda_2}(z - y_j) \left(\pi_{t-s} f(z,1)\right)^{\beta_{21}} \left(\pi_{t-s} f(z,2)\right)^{\beta_{22}}\, dz \\ &\quad \times \prod_{l=1}^{n} \pi_s w_{t-s}^{(0)}(x_l, 1) \prod_{\substack{h=1 \\ h\neq j}}^{m} \pi_s w_{t-s}^{(0)}(y_h, 2)\, ds, \end{aligned}$$

and therefore,

$$\begin{aligned} w_t^{(1)}(\mu) \quad &\leq \quad \nu_1 n \prod_{l=1}^{n} \pi_t f(x_l, 1) \prod_{h=1}^{m} \pi_t f(y_h, 2) \int_0^t \left(\sup_{z\in S} \pi_s f(z)\right)^{\beta_{11}+\beta_{12}-1} ds \\ &+ \quad \nu_2 m \prod_{l=1}^{n} \pi_t f(x_l, 1) \prod_{h=1}^{m} \pi_t f(y_h, 2) \int_0^t \left(\sup_{z\in S} \pi_s f(z)\right)^{\beta_{21}+\beta_{22}-1} ds, \end{aligned}$$

where we used that $\|f\|_\infty \leq 1$ and $\pi_s w_{t-s}^{(0)}(z,i) = \pi_t f(z,i)$, $(z,i) \in S$, $t \geq 0$. Hence,

$$w_t^{(1)}(\mu) \leq (\nu_1 \vee \nu_2)(n+m) w_t^{(0)}(\mu) \int_0^t \left(\sup_{z\in S} \pi_s f(z)\right)^{[(\beta_{11}+\beta_{12})\wedge(\beta_{21}+\beta_{22})]-1} ds. \qquad (6.3)$$

More generally, for $k = 0, 1, \ldots,$

$$w_t^{(k+1)}(\mu) \quad = \quad 1_{\{n\neq 0\}} \nu_1 \sum_{i=1}^{n} \int_0^t \pi_s \left(\int_{\mathcal{N}_f(S)} w_{t-s}^{(k)}(\chi) K^{(\cdot)}(d\chi) \right)(x_i, 1) \qquad (6.4)$$

$$\times \prod_{\substack{l=1\\ l\neq i}}^{n} \pi_s w_{t-s}^{(k)}(x_l,1) \prod_{h=1}^{m} \pi_s w_{t-s}^{(k)}(y_h,2)\, ds$$

$$+ \quad 1_{\{m\neq 0\}}\nu_2 \sum_{j=1}^{m} \int_0^t \pi_s \left(\int_{\mathcal{N}_f(S)} w_{t-s}^{(k)}(\chi) K^{(\cdot)}(d\chi) \right) (y_h,2)$$

$$\times \prod_{l=1}^{n} \pi_s w_{t-s}^{(k)}(x_l,1) \prod_{\substack{h=1\\ h\neq j}}^{m} \pi_s w_{t-s}^{(k)}(y_h,2)\, ds,$$

where the measure $K^{(z,i)}(d\chi)$ is supported by a population at site z, consisting of β_{i1} type-1, and β_{i1} type-2 individuals. By induction on k we will prove that for $t \geq 0$, $\mu = \sum_{i=1}^n \delta_{(x_i,1)} + \sum_{j=1}^m \delta_{(y_j,2)}$ and $k \geq 1$,

$$w_t^{(k)}(\mu) \leq \frac{\nu^k}{k!} \prod_{i=0}^{k-1} (n+m+i(\beta^*-1)) \left(\int_0^t \left(\sup_{z\in S} \pi_s f(z) \right)^{\beta_*-1} ds \right)^k w_t^{(0)}(\mu), \quad (6.5)$$

where $\nu = \nu_1 \vee \nu_2$, $\beta_* = (\beta_{11}+\beta_{12}) \wedge (\beta_{21}+\beta_{22})$ and $\beta^* = (\beta_{11}+\beta_{12}) \vee (\beta_{21}+\beta_{22})$. In fact, we have seen above that (6.5) is valid for $k=1$. If (6.5) holds for some $k \geq 1$, then, using that a branching of type i contributes to the current population with $\beta_{i1} - \delta_{i1}$ individuals of type 1, and $\beta_{i2} - \delta_{i2}$ individuals of type 2, we obtain from (6.4) that

$$\begin{aligned}
& w_t^{(k+1)}(\mu) \\
\leq \quad & \frac{\nu^k}{k!} \prod_{\iota=0}^{k-1} (n+m+\beta_{11}+\beta_{12}-1+\iota(\beta^*-1)) \\
& \cdot \int_0^t \left[\int_0^{t-s} \left(\sup_{z\in S} \pi_r f(z) \right)^{\beta_*-1} dr \right]^k ds \\
& \cdot \nu_1 \sum_{i=1}^{n} \pi_s \pi_{t-s} f(x_i,1) \prod_{\substack{l=1\\ l\neq i}}^{n} \pi_s \pi_{t-s} f(x_l,1) \prod_{h=1}^{m} \pi_s \pi_{t-s} f(y_h,2) \left(\sup_{z\in S} \pi_s f(z) \right)^{\beta_*-1} \\
+ \quad & \frac{\nu^k}{k!} \prod_{\iota=0}^{k-1} (n+m+\beta_{21}+\beta_{22}-1+\iota(\beta^*-1)) \\
& \cdot \int_0^t \left[\int_0^{t-s} \left(\sup_{z\in S} \pi_r f(z) \right)^{\beta_*-1} dr \right]^k ds \\
& \cdot \nu_2 \sum_{j=1}^{m} \pi_s \pi_{t-s} f(y_j,2) \prod_{l=1}^{n} \pi_s \pi_{t-s} f(x_l,1) \prod_{\substack{h=1\\ h\neq j}}^{m} \pi_s \pi_{t-s} f(y_h,2) \left(\sup_{z\in S} \pi_s f(z) \right)^{\beta_*-1}
\end{aligned}$$

$$\leq \frac{\nu^{k+1}}{k!}\prod_{\iota=0}^{k-1}(n+m+(\iota+1)(\beta^*-1))w_t^{(0)}(\mu)\frac{n+m}{k+1}\left[\int_0^t\left(\sup_{z\in S}\pi_r f(z)\right)^{\beta_*-1}dr\right]^{k+1}$$

$$\leq \frac{\nu^{k+1}}{(k+1)!}\prod_{\iota=0}^{k}(n+m+\iota(\beta^*-1))w_t^{(0)}(\mu)\left[\int_0^t\left(\sup_{z\in S}\pi_r f(z)\right)^{\beta_*-1}dr\right]^{k+1},$$

which proves the desired estimate.

Setting $X_0=\mu=\delta_{(z,i)}$ in (6.5) yields

$$w_t(z,i)\leq \pi_t f(z,i)\left(1+\sum_{k=1}^{\infty}v_k(t)\right), \qquad t\geq 0, \tag{6.6}$$

where

$$v_k(t)=\frac{1}{k!}\prod_{i=0}^{k-1}(1+i(\beta^*-1))\left(\nu\int_0^t\left(\sup_{z\in S}\pi_s f(z)\right)^{\beta_*-1}ds\right)^k.$$

Taking $M>0$ large enough we obtain from Remark 2.2 that

$$v_k(t)\leq\left(\beta^*\nu\left(\int_0^M\left(\sup_{z\in S}\pi_s f(z)\right)^{\beta_*-1}ds+\text{Const.}\int_M^\infty\left((c_1\vee c_2)s^{-a_1\wedge a_2}\right)^{\beta_*-1}ds\right)\right)^k.$$

Thus, if c_1, c_2 are so small that $v_k(t)<1$ uniformly in t for all k, then, due to $(a_1\wedge a_2)(\beta_*-1)>1$, the solution of (6.1) is global. □

Next, consider the nonlinear system of equations:

$$\begin{cases}\dfrac{\partial u_t}{\partial t}=\mathit{\Gamma}_\lambda u_t+\nu t^\sigma u_t^{1+\beta_1}v_t^{\beta_2}\\[2ex] \dfrac{\partial v_t}{\partial t}=\mathit{\Gamma}_\mu v_t+F_t(u_t,v_t),\end{cases} \tag{6.7}$$

$u_0=\varphi_1$, $v_0=\varphi_2$, $\lambda,\mu,\nu>0$, where F_t is a positive and measurable function.

Proposition 6.2 *Assume that $\varphi_1(x)\geq cx^{-a_1}$ and $\varphi_2(x)\geq cx^{-a_2}$ for x large enough, with $a_1,a_2\geq 0$. Then* (6.7) *blows up if $a_1\beta_1+a_2\beta_2<1+\sigma$.*

Proof. From Lemma 3.1 and Lemma 2.1 we have $T_t^\Gamma\varphi_2(y)\geq c_2\mu^{a_2}t^{-a_2}1_{[0,t]}(y)$, and

$$v_t^{\beta_2}(y)\geq(T_t^\Gamma\varphi_2(y))^{\beta_2}\geq c_2^{\beta_2}\mu^{a_2\beta_2}t^{-a_2\beta_2}1_{[0,t]}(y).$$

We conclude by an application of Theorem 5.1 and Lemma 3.1. □

In the remaining part of this section we obtain conditions for explosion in finite time of the system

$$\begin{cases} \dfrac{\partial u_t}{\partial t} = \mathit{\Gamma} u_t + t^{\sigma_1} u_t v_t \\ \\ \dfrac{\partial v_t}{\partial t} = \mathit{\Gamma} v_t + (1 \vee t)^{\sigma_2} u_t v_t, \end{cases} \tag{6.8}$$

with $u_0 = \varphi_1$, $v_0 = \varphi_2$, and $\sigma_1, \sigma_2 \in \mathbb{R}$.

Lemma 6.3 *Assume that $\sigma_2 \geq \sigma_1$ and that for some initial conditions $\varphi_1 \leq \varphi_2$, the solution u_t of* (6.8) *satisfies*

$$\inf_{0 \leq x \leq 1} u_t(x) \to \infty$$

as $t \to \infty$. Then u_t blows up in finite time, in the sense that there exists $t > 0$ such that

$$\int_0^1 u_t(x) dx = \infty.$$

Proof. By linearity, $u_t - v_t$ is solution of

$$\frac{\partial}{\partial t}(u_t - v_t) = \mathit{\Gamma}(u_t - v_t) + u_t v_t (t^{\sigma_1} - (1 \vee t)^{\sigma_2}), \tag{6.9}$$

with $u_0 - v_0 = \varphi_1 - \varphi_2 \leq 0$, hence from the integral form of (6.9):

$$(u_t - v_t)(x) = T_t^\Gamma (u_0 - v_0) + \int_0^t (s^{\sigma_1} - (1 \vee s)^{\sigma_2}) T_{t-s}^\Gamma (u_s v_s)(x) ds,$$

we have $u_t - v_t \leq 0$, $t \geq 0$. It remains to apply Lemma 3.2 to the equation

$$\frac{\partial u_t}{\partial t}(y) = \mathit{\Gamma} u_t(y) + t^{\sigma_1} v_t(y) u_t(y),$$

with $\beta = 1$, $\nu = 1$, and to use the inequality $v_t \geq u_t$. $\square$

The above explosion criterion also implies blow-up in all L^p norms, $p \in [1, \infty]$, and is used in the next proposition.

Proposition 6.4 *Assume that $\sigma_2 \geq \sigma_1$ and $\varphi_1(x) \geq c x^{-a_1}$, $\varphi_2(x) \geq c x^{-a_2}$, for x large enough. Then* (6.8) *blows up if* $\min(a_1, a_2) < 1 + \sigma_1$. *In the critical case* $\min(a_1, a_2) = 1 + \sigma_1$, *blow-up occurs if* $\max(a_1, a_2) < 1 + \sigma_2$.

Proof. It suffices to prove blow-up for any pair of functions φ_1, φ_2 such that $\varphi_1(x) = cx^{-a_1}$ and $\varphi_2(x) = cx^{-a_2}$ for x large enough. Moreover, without loss of generality we may assume that $a_1 \geq a_2$ and $\varphi_1 \leq \varphi_2$. From (2.3) of Lemma 2.1, there exists $t_0 > 0$ such that for all $t \geq t_0$ and $y \in \mathbb{R}_+$,

$$u_t(y) \geq T_t^\Gamma \varphi_1(y) \geq ct^{-a_1} 1_{[0,t+\eta]}(y)$$

and

$$v_t(y) \geq T_t^\Gamma \varphi_2(y) \geq ct^{-a_2} 1_{[0,t+\eta]}(y).$$

The Feynman-Kac formula, (2.4) and (2.8) yield, for $0 \leq y \leq \eta + t/2$ and $t > 2 \vee t_0$,

$$\begin{aligned}
u_t(y) &= \int_{-\infty}^{\infty} \varphi_1(x)\gamma_t(x-y)E_y\left[\exp\int_0^t v_{t-s}(X_s^\Gamma)ds \Big| X_t^\Gamma = x\right]dx \\
&\geq \int_y^{\infty} \varphi_1(x)\gamma_t(x-y)E_y\left[\exp\left(c\int_{t_0}^{t/6}(t-s)^{-a_2+\sigma_1}1_{[0,\eta+t-s]}(X_s^\Gamma)ds\right)\Big| X_t^\Gamma = x\right]dx \\
&\geq \int_{t-1/3}^{2t} \varphi_1(x)\gamma_t(x-y) \\
&\quad \times \exp\left(c\int_{t_0}^{t/6}(t-s)^{-a_2+\sigma_1}P_y(0 < X_{t-s}^\Gamma < \eta + t - s | X_t^\Gamma = x)ds\right)dx \\
&\geq \int_{t-1/3}^{2t} \varphi_1(x)\gamma_t(x-y) \\
&\quad \times \exp\left(c\int_{t_0}^{t/6}(t-s)^{-a_2+\sigma_1}P_y(0 < X_{t-s}^\Gamma < 2s + t/2 | X_t^\Gamma = x)ds\right)dx \\
&\geq \int_{t-1/3}^{2t} \varphi_1(x)\gamma_t(x-y)\exp\left(\frac{c}{2}\int_{t_0}^{t/6}(t-s)^{-a_2+\sigma_1}ds\right)dx \\
&\geq c_2 t^{-a_1}\exp\left(\frac{1}{2}\int_{t_0}^{t/6}(t-s)^{-a_2+\sigma_1}ds\right) \\
&\geq c_2 t^{-a_1}\exp\left(\frac{c}{2(1+\sigma_1-a_2)}\left((t-t_0)^{\sigma_1-a_2+1} - \left(\frac{5t}{6}\right)^{\sigma_1-a_2+1}\right)\right).
\end{aligned}$$

Hence, with $\eta = 1$, we infer blow-up from Lemma 6.3 if $a_2 < 1 + \sigma_1$. Turning to the critical case, if $a_2 = 1 + \sigma_1$ the above estimate yields $u_t(y) \geq c_2 1_{[0,\eta+t/2]}(y)t^{-a_1}$, and from (2.7) and (2.5) we have, for all $0 \leq y \leq \eta$,

$$v_t(y) \;=\; \int_{-\infty}^{\infty} \varphi_2(x)\gamma_t(x-y)E_y\left[\exp\int_0^t u_{t-s}(X_s^\Gamma)ds \Big| X_t^\Gamma = x\right]dx$$

$$
\begin{aligned}
&\geq \int_{t-\eta}^{t} \varphi_2(x)\gamma_t(x-y) \\
&\quad \times \exp\left(c_2 \int_{t_0}^{t}(t-s)^{-a_1+\sigma_2} P_y(0 < X_s^\Gamma < \eta + (t-s)/2 | X_t^\Gamma = x) ds\right) dx \\
&\geq \int_{t-\eta}^{t} \varphi_2(x)\gamma_t(x-y) \\
&\quad \times \exp\left(c_2 \int_{t_0}^{t/3}(t-s)^{-a_1+\sigma_2} P_y(0 < X_s^\Gamma < \eta + s | X_t^\Gamma = x) ds\right) dx \\
&\geq c_2 \int_{t-\eta}^{t} \varphi_2(x)\, dx\, t^{-1/2} \exp\left(\frac{c_2}{2} \int_{t_0}^{t/3}(t-s)^{-a_1+\sigma_2} ds\right) \\
&\geq c_2 t^{-a_2-1/2} \exp\left(\frac{c_2}{2} \int_{t_0}^{t/3}(t-s)^{-a_1+\sigma_2} ds\right).
\end{aligned}
$$

Hence, Lemma 6.3 implies blow-up provided $a_1 < 1 + \sigma_2$. □

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References

[1] C. Bandle and H.A. Levine. Fujita type results for convective-like reaction diffusion equations in exterior domains. *Z. Angew. Math. Phys.*, 40(5):665–676, 1989.

[2] J. Bertoin. Sur une intégrale pour les processus à α-variation bornée. *Ann. Probab.*, 17(4):1521–1535, 1989.

[3] M. Birkner, J.A. López-Mimbela, and A. Wakolbinger. Blow-up of semilinear PDE's at the critical dimension. A probabilistic approach. *Proc. Amer. Math. Soc.*, 130(8):2431–2442 (electronic), 2002.

[4] J. Chen and H. Rubin. Bounds for the difference between median and mean of gamma and Poisson distributions. *Statist. Probab. Lett.*, 4(6):281–283, 1986.

[5] N. Cressie. A note on the behaviour of the stable distributions for small index α. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 33(1):61–64, 1975/76.

[6] K. Deng and H.A. Levine. The role of critical exponents in blow-up theorems: the sequel. *J. Math. Anal. Appl.*, 243(1):85–126, 2000.

[7] M. Escobedo and H.A. Levine. Critical blowup and global existence numbers for a weakly coupled system of reaction-diffusion equations. *Arch. Rational Mech. Anal.*, 129(1):47–100, 1995.

[8] M. Freidlin. *Functional integration and partial differential equations*, volume 109 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.

[9] B. Grigelionis. Processes of Meixner type. *Liet. Mat. Rink.*, 39(1):40–51, 1999.

[10] M. Guedda and M. Kirane. A note on nonexistence of global solutions to a nonlinear integral equation. *Bull. Belg. Math. Soc. Simon Stevin*, 6(4):491–497, 1999.

[11] N. Jacob and R.L. Schilling. Lévy-type processes and pseudodifferential operators. In S. Resnick, O. Barndorff-Nielsen, and Th. Mikosch, editors, *Lévy processes*, pages 139–168. Birkhäuser Boston, Boston, MA, 2001.

[12] K. Kobayashi, T. Sirao, and H. Tanaka. On the growing up problem for semilinear heat equations. *J. Math. Soc. Japan*, 29(3):407–424, 1977.

[13] H.A. Levine. The role of critical exponents in blowup theorems. *SIAM Rev.*, 32(2):262–288, 1990.

[14] J. A. López-Mimbela and A. Wakolbinger. A probabilistic proof of non-explosion of a non-linear PDE system. *J. Appl. Probab.*, 37(3):635–641, 2000.

[15] J.A. López-Mimbela and A. Wakolbinger. Length of Galton-Watson trees and blow-up of semilinear systems. *J. Appl. Probab.*, 35(4):802–811, 1998.

[16] J. Meixner. Orthogonale polynomsysteme mit einer besonderen gestalt der erzeugenden funktion. *J. London Math. Soc.*, 9:6–13, 1934.

[17] M. Nagasawa and T. Sirao. Probabilistic treatment of the blowing up of solutions for a nonlinear integral equation. *Trans. Amer. Math. Soc.*, 139:301–310, 1969.

[18] M. E. Payton, L.J. Young, and J.H. Young. Bounds for the difference between median and mean of beta and negative binomial distributions. *Metrika*, 36(6):347–354, 1989.

[19] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[20] S. Portnoy. Transience and solvability of a non-linear diffusion equation. *Ann. Probability*, 3(3):465–477, 1975.

[21] K. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.

[22] S. Sugitani. On nonexistence of global solutions for some nonlinear integral equations. *Osaka J. Math.*, 12:45–51, 1975.

[23] A. Vershik and M. Yor. Multiplicativité du processus gamma et étude asymptotique des lois stables d'indice α, lorsque α tend vers 0. Prépublication No 289, Laboratoire de Probabilités de l'Université Paris VI, 1995.

[24] F. B. Weissler. Existence and nonexistence of global solutions for a semilinear heat equation. *Israel J. Math.*, 38(1-2):29–40, 1981.

[25] M. L. Wenocur. A reliability model based on the gamma process and its analytic theory. *Adv. in Appl. Probab.*, 21(4):899–918, 1989.

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