Stochastic ordering by g-expectations

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Abstract

We derive sufficient conditions for the convex and monotonic g-stochastic ordering of diffusion processes under nonlinear g-expectations and g-evaluations. Our approach relies on comparison results for forward-backward stochastic differential equations and on several extensions of convexity, monotonicity and continuous dependence properties for the solutions of associated semilinear parabolic partial differential equations. Applications to contingent claim price comparison under different hedging portfolio constraints are provided.

Keywords: Stochastic ordering, g-expectation, g-evaluation, g-risk measures, forward-backward stochastic differential equations, parabolic PDEs, propagation of convexity.
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1 Introduction

In comparison with standard mean-variance analysis, partial orderings of probability distributions provide additional information which can be used in applications to risk management. In this framework, a random variable $X^{(1)}$ is said to be dominated by another random variable $X^{(2)}$ if

$$\mathbb{E}\big[\phi\big(X^{(1)}\big)\big] \leq \mathbb{E}\big[\phi\big(X^{(2)}\big)\big], \tag{1.1}$$

for all $\phi : \mathbb{R} \to \mathbb{R}$ in a certain class of functions, where $\mathbb{E}[X]$ denotes the usual expectation of the random variable X. For example, if $X^{(1)}$ and $X^{(2)}$ represent the lifetimes of two devices A and B then the stochastic ordering (1.1) for all non-decreasing and bounded functions ϕ, tells that the device B will likely survive longer than the device A. Stochastic ordering has found a wide range of applications in various fields such as reliability theory, economics, actuarial sciences, operation research, risk management, biology, option evaluation, etc., see e.g. Müller and Stoyan (2002), Denuit et al. (2005), Shaked and Shanthikumar (2007), Sriboonchita et al. (2009), Levy (2015), Belzunce et al. (2015), Perrakis (2019). In the von Neumann-Morgenstern expected utility theory, a portfolio with return $X^{(2)}$ dominates a portfolio with return $X^{(1)}$ in the increasing concave order if (1.1) holds for all non-decreasing concave utility functions ϕ, in which case, the second portfolio would be preferred over the first portfolio by risk-averse investors, see Theorem 1.35 in Sriboonchita et al. (2009). Similarly, if (1.1) holds for all non-decreasing convex utility functions ϕ, the second portfolio would be preferred over the first portfolio by risk-seeking investors, see Theorem 1.37 and the notion of stochastic dominance in Theorem 2.4 in Sriboonchita et al. (2009).

Comparison bounds in convex ordering have been established in El Karoui et al. (1998) for option prices with convex payoff functions in the continuous diffusion case, via a martingale approach based on the classical Kolmogorov equation and the propagation of convexity property for Markov semigroups. This approach has been generalized to semimartingales in Gushchin and Mordecki (2002), Bergenthum and Rüschendorf (2006), Bergenthum and Rüschendorf (2007), see also Klein et al. (2006), Arnaudon et al. (2008), Ma and Privault (2013).

On the other hand, empirical experiments have shown that many uncertain phenomena cannot be fully modeled using the linear expectation operator $\mathbb{E}[\cdot]$, as in e.g. the Allais and Ellsberg paradoxes. Choquet's expectation has been proposed as an nonlinear alternative that relies on capacities instead of probability measures, see Grigorova (2014b;a) for the construction of monotonic and increasing convex stochastic orders and application to financial optimization.

The nonlinear g-expectation and g-evaluation $\mathcal{E}_g[\xi]$ of a random variable ξ have been introduced by Peng (1997; 2004) as the initial value Y_0 for a pair $(Y_t, Z_t)_{t\in[0,T]}$ of adapted processes solution of a Backward Stochastic Differential Equation (BSDE) of the form

$$-dY_t = g(t, X_t, Y_t, Z_t)dt - Z_t dB_t, \qquad 0 \le t \le T,$$

with terminal condition $Y_T = \xi$, where the function $g(t, x, y, z)$ is called the BSDE generator, $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion defined on a probability measure space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$, and $(X_t)_{t\in\mathbb{R}_+}$ is a diffusion process driven by $(B_t)_{t\in\mathbb{R}_+}$. The g-expectation $\mathcal{E}_g$ preserves all properties of the classical expectation $\mathbb{E}$, except for linearity, and it generalizes the classical notion of expectation which corresponds to the choice $g(t, x, y, z) := 0$, $t \in [0, T]$, $x, y, z \in \mathbb{R}$.

BSDEs were first introduced by Bismut (1973) in the linear case, and then extended by Pardoux and Peng (1990) to the nonlinear case. BSDEs and the corresponding g-expectations $\mathcal{E}_g$ have been applied to contingent claim pricing, stochastic control theory, utility maximization and dynamic risk measures, see e.g. Pardoux and Peng (1990), Peng (1997), El Karoui et al. (1997), Ma and Yong (1999), Peng (2004; 2010b), Rosazza Gianin (2006), Epstein and Ji (2013; 2014), Jiang et al. (2016).

In this paper, we study stochastic orderings from the point of view of nonlinear g-expectations. Consider two risky assets with positive prices $\big(X_t^{(i)}\big)_{t\in[0,T]}$, $i = 1, 2$, given by

$$dX_t^{(i)} = X_t^{(i)} a_i\big(t, X_t^{(i)}\big)dt + X_t^{(i)} b_i\big(t, X_t^{(i)}\big)dB_t, \qquad i = 1, 2,$$

a risk-free asset $E_t := E_0 e^{rt}$, where r is an interest rate, and two portfolios with prices $Y_t^{(i)} = p_t^{(i)} E_t + q_t^{(i)} X_t^{(i)}$, under the self-financing conditions

$$\begin{aligned} dY_t^{(i)} &= q_t^{(i)} dE_t + p_t^{(i)} dX_t^{(i)} \\ &= \big(rY_t^{(i)} + \big(a_i\big(t, X_t^{(i)}\big) - r\big)p_t^{(i)} X_t^{(i)}\big)dt + p_t^{(i)} X_t^{(i)} b_i\big(t, X_t^{(i)}\big)dB_t, \quad i = 1, 2, \end{aligned} \tag{1.2}$$

which lead to the BSDEs

$$Y_t^{(i)} = Y_T^{(i)} + \int_t^T g_i\big(s, X_s^{(i)}, Y_s^{(i)}, Z_s^{(i)}\big)ds - \int_t^T Z_s^{(i)} dB_s,$$

where $Z_t^{(i)} := p_t^{(i)} X_t^{(i)} b_i\big(t, X_t^{(i)}\big)$ and

$$g_i(t, x, y, z) := -ry - z\theta_i(t, x), \quad \text{with} \quad \theta_i(t, x) = \frac{a_i(t, x) - r}{b_i(t, x)}, \quad i = 1, 2.$$

We say that $X_T^{(2)}$ dominates $X_T^{(1)}$ in the convex g_1, g_2-stochastic ordering, i.e. $X_T^{(1)} \leq_{g_1,g_2}^{conv} X_T^{(2)}$ if

$$\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big]$$

for all convex functions ϕ, where $Y_0^{(i)} := \mathcal{E}_{g_i}[\phi\big(X_T^{(i)}\big)]$ represent the fair prices at time $t = 0$ of the options with convex payoffs $Y_T^{(i)} := \phi\big(X_T^{(i)}\big)$, $i = 1, 2$.

Here, the use of distinct generators $g_i(t, x, y, z)$ is motivated by the comparison of different contingent claims under different hedging strategies, for example in the case of misspecified volatility coefficients or for hedging under constraints, see the examples presented in Section 6. In case $g(t, x, y, z) = \beta(t)z$, the increasing convex g-stochastic ordering $\leq_g^{iconv}$ is equivalent to the (classical) increasing convex ordering with respect to the capacities μ_g, see Grigorova (2014b), where $\mu_g[A] := \mathcal{E}_g[\mathbf{1}_A]$, $A \in \mathcal{F}$, see Chen et al. (2005).

Main results

In Theorem 3.1, we derive sufficient conditions on two BSDE generators $g_1(t,x,y,z)$, $g_2(t,x,y,z)$ for the convex ordering

$$\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big] \le \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big],$$

in nonlinear expectations $\mathcal{E}_{g_1}$, $\mathcal{E}_{g_2}$, for all convex functions $\phi(x)$ with polynomial growth, where $X_T^{(1)}$ and $X_T^{(2)}$ are the terminal values of the solutions of two forward Stochastic Differential Equations (SDEs)

$$\begin{cases} dX_t^{(1)} = \mu_1\big(t, X_t^{(1)}\big)dt + \sigma_1\big(t, X_t^{(1)}\big)dB_t, \\ \\ dX_t^{(2)} = \mu_2\big(t, X_t^{(2)}\big)dt + \sigma_2\big(t, X_t^{(2)}\big)dB_t, \end{cases}$$

with $X_0^{(1)} = X_0^{(2)}$, under the bound

$$0 < \sigma_1(t,x) \le \sigma_2(t,x), \qquad t \in [0,T], \quad x \in \mathbb{R}.$$

The proof of Theorem 3.1 is based on the comparison Theorem 2.4 in Appendix C of Peng (2010a), provided that

$$z\mu_1(t,x) + g_1(t,x,y,z\sigma_1(t,x)) \le z\mu_2(t,x) + g_2(t,x,y,z\sigma_2(t,x)), \quad x,y,z \in \mathbb{R}, \; t \in [0,T],$$

and both functions $(x,y,z) \mapsto f_i(t,x,y,z) := z\mu_i(t,x) + g_i(t,x,y,z\sigma_i(t,x))$ are convex in (x,y) and in (y,z) on $\mathbb{R}^2$ for $i = 1,2$ and $t \in [0,T]$. Several extensions are considered on increasing convex and monotonic orderings in Theorem 3.2 and Corollaries 3.3-3.4, with the particular cases of equal drifts and equal volatilities treated in Corollaries 3.5 and 3.6.

This approach requires convexity of the function $(x,y,z) \mapsto f_i(t,x,y,z)$ for both $i = 1$ and $i = 2$. In Section 4 we relax those conditions using a stochastic calculus approach, by only requiring the convexity of the function $f_i(t,x,y,z)$ for $i = 1$ or $i = 2$ in Theorems 4.1 and 4.2, which respectively deal with the convex and increasing convex orders.

Related comparison results for g-risk measures are presented in Corollaries 5.2-5.3, using the quantity $\mathcal{E}_{g_i}\big[-\phi\big(X_T^{(i)}\big) \;\big|\mathcal{F}_t\big] = -\mathcal{E}_{g_i^{(-1)}}\big[\phi\big(X_T^{(i)}\big) \;\big|\mathcal{F}_t\big]$ which makes sense as a dynamic g-risk measure, where $g_i^{(-1)}(t,x,y,z) := -g_i(t,x,-y,-z)$. Here, the choice of generator function g_i determines the investor's portfolio strategy and the corresponding risk measures, see Section 6 for examples.

The proofs of Theorems 3.1-5.3 rely on an extension of convexity properties of the solutions of nonlinear parabolic Partial Differential Equations (PDEs) which is proved in Theorem 7.2. The convexity properties of solutions of nonlinear PDEs have been studied by several authors, see e.g. Theorem 3.1 in Lions and Musiela (2006), Theorem 2.1 in Giga et al. (1991), and Theorem 1.1 in Bian and Guan (2008), see also Theorem 1 in Alvarez et al. (1997) in the elliptic case. Those works typically require global convexity of the nonlinear drifts $f(t,x,y,z)$ in all state variables (x,y,z), a condition which is too strong for our applications to finance in Examples 6.1-6.5 below. For this reason, in Theorem 7.2 we extend Theorem 1.1 of Bian and Guan (2008) in dimension one, by replacing the global convexity of the nonlinear drift $f_i(t,x,y,z)$ in (x,y,z) with its convexity in (x,y) and (y,z), $i=1,2$.

Finally, Section 8 deals with monotonicity properties and continuous dependence results for the solutions Forward-Backward Stochastic Differential Equations (FBSDEs) and PDEs, which are used in the proofs of Theorems 3.1-4.2 and Corollaries 3.3-3.6.

2 Preliminaries

In this section, we recall some notation and background on FBSDEs, g-expectations, g-evaluations and g-stochastic orderings. Given $T>0$, let $(B_t)_{t\in[0,T]}$ be a standard Brownian motion on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Denote by $(\mathcal{F}_t)_{t\in[0,T]}$ the augmented filtration such that $\mathcal{F}_t=\sigma(B_s, 0\le s\le t)\vee\mathcal{N}$, $t\in[0,T]$, where $\mathcal{N}$ is the collection of all $\mathbb{P}$-null sets. We also let $L^2(\Omega,\mathcal{F}_t):=L^2(\Omega,\mathcal{F}_t,\mathbb{P})$, $t\in[0,T]$.

Forward-Backward SDEs

Consider a forward SDE of the form

$$dX_s^{t,x}=\mu\big(s,X_s^{t,x}\big)dt+\sigma\big(s,X_s^{t,x}\big)dB_s, \qquad 0\le t\le s\le T.$$

with initial condition $X_t^{t,x}=x$, and whose coefficients are assumed throughout this paper to satisfy the following condition:

(A_1) For every $t\in[0,T]$, the functions $x\mapsto\mu(t,x)$ and $x\mapsto\sigma(t,x)$ are globally Lipschitz, i.e.

$$|\mu(t,x)-\mu(t,y)|\le C|x-y| \quad\text{and}\quad |\sigma(t,x)-\sigma(t,y)|\le C|x-y|, \quad x,y\in\mathbb{R},$$

In particular, $x \mapsto \mu(t,x)$ and $x \mapsto \sigma(t,x)$ satisfy the linear growth conditions

$$|\mu(t,x)| \leq C(1+|x|) \quad \text{and} \quad |\sigma(t,x)| \leq C(1+|x|), \quad x \in \mathbb{R},$$

for some positive constant $C > 0$. The associated backward SDE is defined by

$$Y_s^{t,x} = \phi\big(X_T^{t,x}\big) + \int_s^T g\big(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x}\big) d\tau - \int_s^T Z_\tau^{t,x} dB_\tau, \quad 0 \leq t \leq s \leq T, \tag{2.1}$$

with terminal condition $Y_T^{t,x} = \phi\big(X_T^{t,x}\big) \in L^2(\Omega, \mathcal{F}_T)$, where the generator $g(\cdot, x, y, z)$ of (2.1) is an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process in $L^2\left(\Omega \times [0,T]\right)$ for all $x, y, z \in \mathbb{R}$, which satisfies the following Conditions (A_2)-(A_3).

(A_2) The function $g(t,x,y,z)$ is uniformly Lipschitz in (x,y,z), i.e., there exists $C > 0$ such that

$$|g(t,x_2,y_2,z_2) - g(t,x_1,y_1,z_1)| \leq C\left(|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1|\right)$$

a.s., $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$, $t \in [0,T]$,

(A_3) We have $g(\cdot, x, 0, 0) = 0$ a.s. for all $x \in \mathbb{R}$.

By Theorem 2.1 of El Karoui et al. (1997), see Proposition 2.2 of Pardoux and Peng (1990), under (A_1)-(A_2) there exists a unique pair $\big(Y_s^{t,x}, Z_s^{t,x}\big)_{s\in[t,T]}$ of adapted processes in $L^2\left(\Omega \times [0,T]\right)$ that solves the BSDE (2.1).

In the sequel we will also consider the following condition:

(A_4) The function ϕ is continuous on $\mathbb{R}$ and has the polynomial growth

$$|\phi(x)| \leq C(1+|x|^p), \qquad x \in \mathbb{R}, \text{ for some } p \geq 1 \text{ and } C > 0. \tag{2.2}$$

g-evaluation and g-expectation

Next, we state the definition of the g-evaluation.

Definition 2.1 *Given $\xi \in L^2(\Omega, \mathcal{F}_T)$ and the backward SDE*

$$\begin{cases} dY_t^{0,x} = -g\big(t, X_t^{0,x}, Y_t^{0,x}, Z_t^{0,x}\big)dt + Z_t^{0,x} dB_t, \quad 0 \leq t \leq T, \\ Y_T^{0,x} = \xi, \end{cases} \tag{2.3}$$

we respectively call

$$\mathcal{E}_g[\xi] := Y_0^{0,x} \quad \textit{and} \quad \mathcal{E}_g[\xi \mid \mathcal{F}_t] := Y_t^{0,x}$$

the g-evaluation and the $\mathcal{F}_t$-conditional g-evaluation of ξ, $t \in [0,T]$.

Under (A_3), one can show in addition that the map $\xi \mapsto \mathcal{E}_g[\xi]$ preserves all properties of the classical expectation $\mathbb{E}$, except for linearity and the property $\mathcal{E}_g[c] = c$ for constant $c \in \mathbb{R}$, see Relation (34) and Theorem 3.4 in Peng (2004).

In the sequel we make the (stronger than (A_3)) assumption

(A_3') $g(\cdot, x, y, 0) = 0$ a.s. for all $x, y \in \mathbb{R}$,

under which the g-evaluation $\mathcal{E}_g$ becomes the g-expectation, which satisfies the property $\mathcal{E}_g[c] = c$ for constant $c \in \mathbb{R}$, see Relation (36.2) and Lemma 36.3 in Peng (1997). We note that the results of Sections 3, 4 and 5 remain valid for g-expectations if we assume (A_3') instead of (A_3).

Remark 2.2
- *When $g(t, x, y, z)$ is convex in $(y, z) \in \mathbb{R}^2$ for all $(t, x) \in [0, T] \times \mathbb{R}$, which is the case in Theorems 3.1-3.2, Corollaries 3.4-3.5, and Theorems 4.1-4.2, we have the representation*

$$\mathcal{E}_g[\xi] = \sup_{\mathbb{Q} \in \mathcal{P}_g} \left(\mathbb{E}_{\mathbb{Q}}[\xi] - F_g \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right), \tag{2.4}$$

where $F_g : L^2(\mathcal{F}_T) \mapsto \mathbb{R} \cup \{+\infty\}$ is the convex functional defined by

$$F_g(X) := \sup_{\xi \in L^2(\Omega, \mathcal{F}_T)} (\mathbb{E}[\xi X] - \mathcal{E}_g[\xi]), \quad X \in L^2(\Omega, \mathcal{F}_T),$$

and $\mathcal{P}_g$ is the non-empty convex set of prior probability measures representing model uncertainty and defined by

$$\mathcal{P}_g := \left\{ \mathbb{Q} \in \mathcal{M} \ : \ \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^2(\mathcal{F}_T) \ \text{and} \ F_g \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) < \infty \right\}, \tag{2.5}$$

where $\mathcal{M}$ is the set of probability measures on $(\Omega, \mathcal{F}_T)$ which are absolutely continuous with respect to $\mathbb{P}$, see Corollary 12 of Rosazza Gianin (2006).

- *If $g(t, x, y, z)$ is both convex and sublinear in $(y, z) \in \mathbb{R}^2$ for all $(t, x) \in [0, T] \times \mathbb{R}$, which is the case in Examples 6.3-6.5, then (2.4) becomes*

$$\mathcal{E}_g[\xi] = \sup_{\mathbb{Q} \in \mathcal{P}_g} \mathbb{E}_{\mathbb{Q}}[\xi], \tag{2.6}$$

see Corollary 12 in Rosazza Gianin (2006) and also Chen and Peng (2000), Chen et al. (2003).

- *If $g(t,x,y,z) = \alpha(t,x)y + \beta(t,x)z$ is linear in $(y,z) \in \mathbb{R}^2$ for all $(t,x) \in [0,T] \times \mathbb{R}$, where $\alpha(t,x)$ and $\beta(t,x)$ are bounded functions, see Example 6.1, then the g-expectation $\mathcal{E}_g[\,\cdot\,]$ satisfies*

$$\mathcal{E}_g\big[\xi \;\big|\mathcal{F}_t\big] = \mathbb{E}_{\mathbb{Q}}\left[\xi \exp\left(\int_t^T \alpha(s,X_s)ds\right) \;\Big|\; \mathcal{F}_t\right],$$

where $\mathbb{Q}$ is the probability measure defined as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \exp\left(\int_0^T \beta(s,X_s)dB_s - \frac{1}{2}\int_0^T \beta^2(s,X_s)ds\right), \tag{2.7}$$

as follows by applying the Itô formula to $Y_t \exp\big(\int_0^t \alpha(s,X_s)ds\big)$ using the Brownian motion $\widetilde{B}_t := B_t + \int_0^t \beta(s,X_s)ds$ under $\mathbb{Q}$.

- *If $g(t,x,y,z) = \alpha_t|z| + \beta_t z$, $t \in [0,T]$, $z \in \mathbb{R}$, where $(\alpha_t)_{t\in[0,T]}$ and $(\beta_t)_{t\in[0,T]}$ are time-continuous processes then $\mathcal{E}_g[\xi]$ coincides with Choquet's expectation*

$$\mathcal{E}_g[\xi] = \mathbb{E}_{\mu_g}[\xi] := \int_{-\infty}^0 (\mu_g(\xi > x) - 1)dx + \int_0^\infty \mu_g(\xi > x)dx$$

for ξ of the form $\xi = y + zB_T$, where $\mu_g[A] := \mathcal{E}_g[\mathbf{1}_A]$, $A \in \mathcal{F}$, is the corresponding capacity, see Chen et al. (2005).

Moreover, $\mathcal{E}_g[\,\cdot\,]$ coincides with the linear expectation $\mathcal{E}_{\mu_g}[\,\cdot\,]$ if and only if $\alpha_t = 0$ a.s., $t \in [0,T]$, see Theorem 1 in Chen et al. (2005), and in case $\beta_t = 0$, $t \in [0,T]$, we have

$$\mathcal{E}_g[\xi] = \begin{cases} \sup\limits_{\mathbb{Q}\in\mathcal{P}_g} \mathbb{E}_{\mathbb{Q}}[\xi], \text{ if } \alpha_t > 0, \text{ a.s., } t \in [0,T], & \text{(2.8a)} \\ \inf\limits_{\mathbb{Q}\in\mathcal{P}_g} \mathbb{E}_{\mathbb{Q}}[\xi], \text{ if } \alpha_t < 0, \text{ a.s., } t \in [0,T], & \text{(2.8b)} \end{cases}$$

where $\mathcal{P}_g$ is the set of probability measures $\mathbb{Q} \in \mathcal{M}$ such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \exp\left(-\int_0^T v_t dB_t - \frac{1}{2}\int_0^T v_t^2 dt\right),$$

where $(v_t)_{t\in[0,T]}$ is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted and $|v_t| \leq |\alpha_t|$, $t \in [0,T]$, see Theorem 2.2 in Chen and Epstein (2002) and also Example 1 in Chen et al. (2005).

In the sequel, we state ordering results for the g-evaluation $\mathcal{E}_g[\,\cdot\,]$, and more generally for the conditional g-evaluation and the conditional g-expectation $\mathcal{E}_g[\,\cdot \mid \mathcal{F}_t]$, $t \in [0,T]$, under (A_3') instead of (A_3).

g-stochastic orderings

Stochastic orderings with respect to capacity have been studied in Grigorova (2014b;a) by using Choquet's expectation and uncertainty orders have been constructed in Tian and Jiang (2016) on the sublinear G-expectation space. Here, we extend their approaches to the comparison of random variables $X^{(1)}$, $X^{(2)}$ in the settings of Peng's g-expectations and g-evaluations, which are not sublinear in general, via the condition

$$\mathcal{E}_{g_1}\big[\phi\big(X^{(1)}\big)\big] \le \mathcal{E}_{g_2}\big[\phi\big(X^{(2)}\big)\big], \tag{2.9}$$

in nonlinear expectations $\mathcal{E}_{g_1}$, $\mathcal{E}_{g_2}$, for all $\phi(x)$ in a certain class of functions having polynomial growth. In general, different portfolios or hedging strategies may corresponding to different generators g_1, g_2 as can be seen in Examples 6.3 and 6.5.

Definition 2.3 *Let g_1, g_2 satisfy (A_2)-(A_3). For any $X^{(1)}, X^{(2)} \in L^2(\Omega, \mathcal{F}_T)$, we say that*

1) $X^{(1)}$ is dominated by $X^{(2)}$ in the monotonic g_1, g_2-ordering, i.e. $X^{(1)} \le^{\mathrm{mon}}_{g_1,g_2} X^{(2)}$, if (2.9) holds for all non-decreasing functions $\phi(x)$ satisfying (2.2).

2) $X^{(1)}$ is dominated by $X^{(2)}$ in the convex g_1, g_2-ordering, i.e. $X^{(1)} \le^{\mathrm{conv}}_{g_1,g_2} X^{(2)}$, if (2.9) holds for all convex functions $\phi(x)$ satisfying (2.2).

3) $X^{(1)}$ is dominated by $X^{(2)}$ in the increasing convex g_1, g_2-ordering, i.e. $X^{(1)} \le^{\mathrm{iconv}}_{g_1,g_2} X^{(2)}$, if (2.9) holds for all non-decreasing convex functions $\phi(x)$ satisfying (2.2).

We note that

(i) $X^{(1)} \le^{\mathrm{mon}}_{g_1,g_2} X^{(2)} \Longrightarrow X^{(1)} \le^{\mathrm{iconv}}_{g_1,g_2} X^{(2)}$, and

(ii) $X^{(1)} \le^{\mathrm{conv}}_{g_1,g_2} X^{(2)} \Longrightarrow X^{(1)} \le^{\mathrm{iconv}}_{g_1,g_2} X^{(2)}$,

and we simply write $\le^{\mathrm{mon}}_{g}$, $\le^{\mathrm{conv}}_{g}$, $\le^{\mathrm{iconv}}_{g}$ if $g_1 = g_2 := g$. Replacing convex functions with concave functions yields the corresponding notions of concave and increasing concave g_1, g_2-orderings denoted by $X^{(1)} \le^{\mathrm{conc}}_{g_1,g_2} X^{(2)}$ and $X^{(1)} \le^{\mathrm{iconc}}_{g_1,g_2} X^{(2)}$, respectively, which are characterized in the next proposition.

Proposition 2.4 *Given two random variables $X^{(i)} \in L^2(\Omega, \mathcal{F}_T)$, $i = 1, 2$ and the generator $g^{(-1)}(t, x, y, z) := -g(t, x, -y, -z)$, for $(t, x, y, z) \in [0, T] \times \mathbb{R}^3$, we have that*

(i) $X^{(1)} \le^{\mathrm{conc}}_{g} X^{(2)}$ if and only if $-X^{(2)} \le^{\mathrm{conv}}_{g^{(-1)}} -X^{(1)}$.

(ii) $X^{(1)} \leq_g^{\mathrm{iconc}} X^{(2)}$ *if and only if* $-X^{(2)} \leq_{g^{(-1)}}^{\mathrm{iconv}} -X^{(1)}$.

Proposition 2.4 is a direct consequence of the following lemma.

Lemma 2.5 *Letting* $g^{(a)}(t,x,y,z) := ag(t,x,y/a,z/a)$, $x,y,z \in \mathbb{R}$, $t \in [0,T]$, *we have*

$$\mathcal{E}_{g^{(a)}}[a\xi] = a\mathcal{E}_{g^{(1)}}[\xi], \qquad a \neq 0,$$

Proof. Given $(X_t)_{t\in[0,T]}$ an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process, let $(Y_t, Z_t)_{t\in[0,T]}$ denote the solution of (2.3) with terminal condition $Y_T = \xi \in L^2(\Omega, \mathcal{F}_T)$, and let $(\overline{Y}_t, \overline{Z}_t)_{t\in[0,T]}$ denote the solution of the backward SDE

$$\overline{Y}_t = a\xi + \int_t^T g^{(a)}\left(s, X_s, \overline{Y}_s, \overline{Z}_s\right) ds - \int_t^T \overline{Z}_s dB_s, \qquad t \in [0,T],$$

with generator $g^{(a)}(t,x,y,z)$ and terminal condition $\overline{Y}_T = a\xi$, i.e.

$$\frac{\overline{Y}_t}{a} = \xi + \int_t^T g\left(s, X_s, \frac{\overline{Y}_s}{a}, \frac{\overline{Z}_s}{a}\right) ds - \frac{1}{a}\int_t^T \overline{Z}_s dB_s, \qquad t \in [0,T].$$

The uniqueness of the solution (Y_t, Z_t) of the backward SDEs

$$Y_t = \xi + \int_t^T g\left(s, X_s, Y_s, Z_s\right) ds - \int_t^T Z_s dB_s, \qquad t \in [0,T],$$

yields $\overline{Z}_t/a = Z_t$ and $\overline{Y}_t/a = Y_t$, $t \in [0,T]$, and we conclude by taking $t = 0$. $\square$

We also note that the monotonic g-ordering admits the following characterization in the case of sublinear generator functions.

Proposition 2.6 *Assume that* $g_i(t,x,y,z)$ *is sublinear in* $(y,z) \in \mathbb{R}^2$ *for all* $(t,x) \in [0,T] \times \mathbb{R}$, $i = 1,2$, *and that* $g_1(t,x,y,z) \leq g_2(t,x,y,z)$. *Then* $X^{(1)} \leq_{g_1,g_2}^{mon} X^{(2)}$ *is equivalent to*

$$\inf_{Q\in\mathcal{P}_{g_1}} Q(X^{(1)} \leq c) \geq \inf_{Q\in\mathcal{P}_{g_2}} Q(X^{(2)} \leq c), \qquad c \in \mathbb{R}, \tag{2.10}$$

where $\mathcal{P}_{g_1}$ *and* $\mathcal{P}_{g_2}$ *are defined in* (2.5).

Proof. $(i) \Rightarrow (ii)$: We apply (2.6) to the non-decreasing function $\phi(x) := \mathbf{1}_{\{x>c\}}$ for $c \in \mathbb{R}$, after noting that $\mathcal{P}_{g_1} \subseteq \mathcal{P}_{g_2}$ since $g_1 \leq g_2$ by Remark 13 in Rosazza Gianin (2006).
$(ii) \Rightarrow (i)$: By relation (2.10), for any $Q \in \mathcal{P}_{g_1}$ and non-decreasing functions ϕ we have

$$\mathbb{E}_Q\big[\phi\big(X^{(1)}\big)\big] \leq \mathbb{E}_Q\big[\phi\big(X^{(2)}\big)\big] \leq \sup_{Q\in\mathcal{P}_{g_2}} \mathbb{E}_Q\big[\phi\big(X^{(2)}\big)\big],$$

hence by (2.6) we find

$$\mathcal{E}_{g_1}\big[\phi\big(X^{(1)}\big)\big] = \sup_{Q\in\mathcal{P}_{g_1}} \mathbb{E}_Q\big[\phi\big(X^{(1)}\big)\big] \leq \sup_{Q\in\mathcal{P}_{g_2}} \mathbb{E}_Q\big[\phi\big(X^{(2)}\big)\big] = \mathcal{E}_{g_2}\big[\phi\big(X^{(2)}\big)\big].$$

$\square$

Associated PDE

Throughout the remaining of this paper we assume that $g(t,x,y,z)$ is a deterministic function, in addition to (A_1)-(A_3). The function $u(t,x) := Y_t^{t,x}$ can be shown to be a viscosity solution of the backward PDE

$$\frac{\partial u}{\partial t}(t,x) + \mu(t,x)\frac{\partial u}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 u}{\partial x^2}(t,x) + g\Big(t,x,u(t,x),\sigma(t,x)\frac{\partial u}{\partial x}(t,x)\Big) = 0, \quad (2.11)$$

$x \in \mathbb{R}$, $t \in [0,T]$, with a terminal condition $u(T,x) = \phi(x)$ satisfying (A_4), see Theorem 2.2 in Pardoux (1998), Theorem 4.3 of Pardoux and Peng (1992) and Theorem 4.2 of El Karoui et al. (1997).

In the sequel, we let $\mathcal{C}^{p,q}([0,T] \times \mathbb{R})$ denote the space of functions $f(t,x)$ which are p times continuously differentiable in $t \in [0,T]$, $p \geq 1$, and q times differentiable in $x \in \mathbb{R}$, $q \geq 1$. We also let $\mathcal{C}_b^k(\mathbb{R}^n)$ denote the space of continuously differentiable functions whose partial derivatives of orders one to k are uniformly bounded on $\mathbb{R}^n$. In Theorem 2.7 below we state an existence result for classical solutions under stronger smoothness assumptions on BSDE coefficients, see Theorem 3.2 of Pardoux and Peng (1992), Theorem 8.1 in § V.8 page 495, and Theorem 7.1 in § VII.7 page 596 of Ladyženskaja et al. (1968).

Theorem 2.7 *Assume (A_3) and in addition that $\mu(t,\cdot)$, $\sigma(t,\cdot)$, $\phi \in \mathcal{C}_b^3(\mathbb{R})$, and that $g(t,\cdot,\cdot,\cdot) \in \mathcal{C}_b^3(\mathbb{R}^3)$ for any $t \in [0,T]$. Then the function $u(t,x) := Y_t^{t,x}$ is a classical solution in $\mathcal{C}^{1,2}([0,T] \times \mathbb{R})$ of the backward PDE (2.11) with terminal condition $u(T,\cdot) = \phi$.*

Under the conditions of Theorem 2.7, by Proposition 4.3 of El Karoui et al. (1997) the solution $\big(Y_s^{t,x}, Z_s^{t,x}\big)_{s\in[t,T]}$ of (3.2) satisfies $Y_s^{t,x} = u(s,X_s^{t,x})$ and $Z_s^{t,x} = \sigma(s,X_s^{t,x})\dfrac{\partial u}{\partial x}(s,X_s^{t,x})$, $0 \leq t \leq s \leq T$. In addition, by Theorem 2.2 in Ma and Yong (1999) or Proposition 3.3 in Ma et al. (1994) we have the following result.

Theorem 2.8 *Under the assumptions of Theorem 2.7, suppose additionally that $\sigma(t,x)$ is bounded above and below by strictly positive constants. Then the first derivative in $t \in [0,T]$ and the first and second derivatives in $x \in \mathbb{R}$ of $u(t,x)$ are bounded in $(t,x) \in [0,T] \times \mathbb{R}$.*

As in Douglas et al. (1996), we denote by $\mathcal{C}^{1+\eta/2,2+\eta}([0,T] \times \mathbb{R})$, $\eta \in (0,1)$, the space of functions $f(t,x)$ which are differentiable in $t \in [0,T]$ and twice differentiable in $x \in \mathbb{R}$ with $\frac{\partial f}{\partial t}(t,x)$ and $\frac{\partial^2 f}{\partial x^2}(t,x)$ being respectively $\eta/2$-Hölder continuous and η-Hölder continuous in $(t,x) \in [0,T] \times \mathbb{R}$, and define the space $\mathcal{C}^{k+\eta}(\mathbb{R})$ analogously for $k \geq 1$. By Theorem 2.3 in Douglas et al. (1996), see also page 236 of Ma and Yong (1999), we have the following result.

Theorem 2.9 *In addition to the assumptions of Theorem 2.8, suppose that for some $\eta \in (0,1)$ the functions $\mu(\cdot,\cdot)$, $\sigma(\cdot,\cdot)$ and $g(\cdot,\cdot,y,z)$ are in $\mathcal{C}^{1+\eta/2,2+\eta}([0,T]\times\mathbb{R})$ for all $y,z \in \mathbb{R}$, and that $\phi \in \mathcal{C}^{4+\eta}(\mathbb{R})$. Then the function $u(t,x)$ is a classical solution in $\mathcal{C}^{2+\eta/2,4+\eta}([0,T]\times\mathbb{R})$ of the backward PDE (2.11) with terminal condition $u(T,\cdot) = \phi$.*

3 Ordering with convex drifts

Consider the forward SDEs

$$\begin{cases} dX_t^{(1)} = \mu_1\big(t, X_t^{(1)}\big)dt + \sigma_1\big(t, X_t^{(1)}\big)dB_t, & \text{(3.1a)} \\ \\ dX_t^{(2)} = \mu_2\big(t, X_t^{(2)}\big)dt + \sigma_2\big(t, X_t^{(2)}\big)dB_t, & \text{(3.1b)} \end{cases}$$

and the associated BSDEs

$$\begin{cases} dY_t^{(1)} = -g_1\big(t, X_t^{(1)}, Y_t^{(1)}, Z_t^{(1)}\big)dt + Z_t^{(1)}dB_t, \quad Y_T^{(1)} = \phi\big(X_T^{(1)}\big), \\ \\ dY_t^{(2)} = -g_2\big(t, X_t^{(2)}, Y_t^{(2)}, Z_t^{(2)}\big)dt + Z_t^{(2)}dB_t, \quad Y_T^{(2)} = \phi\big(X_T^{(2)}\big), \end{cases}$$

and let

$$f_i(t,x,y,z) := z\mu_i(t,x) + g_i(t,x,y,z\sigma_i), \quad t \in [0,T], \ x,y,z \in \mathbb{R}, \ i = 1,2. \tag{3.2}$$

In all following propositions, the convexity of

$$(x,y) \mapsto f_i(t,x,y,z), \quad \text{resp.} \quad (y,z) \mapsto f_i(t,x,y,z)$$

on $\mathbb{R}^2$ is understood to hold for all $(t,z) \in [0,T]\times\mathbb{R}$, resp. for all $(t,x) \in [0,T]\times\mathbb{R}$. The next result is a consequence of the comparison Theorem 2.4 in Appendix C of Peng (2010a). We note that Condition (B_1) can be shown to be necessary for convex ordering by taking $\phi(x) = x$ as in Theorem 3.2 of Briand et al. (2000).

Theorem 3.1 (Convex order). *Assume that $X_0^{(1)} = X_0^{(2)}$, and*

$$0 < \sigma_1(t,x) \le \sigma_2(t,x), \qquad t \in [0,T], \quad x \in \mathbb{R},$$

together with the conditions

(B_1) $f_1(t,x,y,z) \le f_2(t,x,y,z)$, $t \in [0,T]$, $x,y,z \in \mathbb{R}$,

(B_2) $(x,y) \mapsto f_i(t,x,y,z)$ and $(y,z) \mapsto f_i(t,x,y,z)$ are convex on $\mathbb{R}^2$ for $i = 1,2$.

Then we have $X_T^{(1)} \leq_{g_1,g_2}^{\mathrm{conv}} X_T^{(2)}$, *i.e.,*

$$\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big], \tag{3.3}$$

for all convex functions $\phi(x)$ *satisfying* (2.2).

Proof. We start by assuming that the function ϕ and the coefficients $\mu_i(t,\cdot)$, $\sigma_i(t,\cdot)$ and $g_i(t,\cdot,\cdot,\cdot)$ are $\mathcal{C}_b^3$ functions for all $t \in [0,T]$. By Theorem 2.7, the functions $u_1(t,x) := Y_t^{(1),t,x}$ and $u_2(t,x) := Y_t^{(2),t,x}$ are solutions of the backward PDEs (3.5) which are continuous in t and x. Letting

$$h_i(t,x,y,z,w) := f_i(t,x,y,z) + \frac{w}{2}\sigma_i^2(t,x), \qquad i = 1,2, \tag{3.4}$$

we rewrite (2.11) as

$$\frac{\partial u_i}{\partial \tau}(\tau,x) = h_i\left(\tau, x, u_i(\tau,x), \frac{\partial u_i}{\partial x}(\tau,x), \frac{\partial^2 u_i}{\partial x^2}(\tau,x)\right) \text{ with } u_i(0,x) = \phi(x), \quad i = 1,2, \tag{3.5}$$

by setting $\tau := T - t$. We also assume that there exists constants $c, C' > 0$ such that

$$0 < c \leq \sigma_1(t,x) \leq \sigma_2(t,x) \leq C', \qquad t \in [0,T], \quad x \in \mathbb{R}. \tag{3.6}$$

In this case, by Theorem 2.8 the second derivative $\left|\frac{\partial^2 u_i}{\partial x^2}(t,x)\right|$ is bounded by $C'' > 0$. In addition, under (B_2), both solutions $u_1(t,x)$ and $u_2(t,x)$ of (3.5) are convex functions of x by Theorem 7.2 below, hence we have $\frac{\partial^2 u_i}{\partial x^2}(\tau,x) \geq 0$, $\tau \in [0,T]$, $x \in \mathbb{R}$. Therefore, in (3.5) we can replace $h_i(t,x,y,z,w)$ in (3.4) with

$$\tilde{h}_i(t,x,y,z,w) := f_i(t,x,y,z) + \frac{(\min(w,C''))^+}{2}\sigma_i^2(t,x), \qquad i = 1,2, \tag{3.7}$$

where $w^+ = \max(w,0)$, and rewrite the backward PDEs (3.5) as

$$\frac{\partial u_i}{\partial \tau}(\tau,x) = \tilde{h}_i\left(\tau, x, u_i(\tau,x), \frac{\partial u_i}{\partial x}(\tau,x), \frac{\partial^2 u_i}{\partial x^2}(\tau,x)\right) \text{ with } u_i(0,x) = \phi(x), \quad i = 1,2.$$

Next, for all $\tau \in [0,T]$ and $x_1, x_2, y, z \in \mathbb{R}$ we have

$$\begin{aligned}
&|f_i(\tau,x_2,y,z) - f_i(\tau,x_1,y,z)| \\
&\qquad \leq \ |z||\mu_i(\tau,x_2) - \mu_i(\tau,x_1)| + |g_i(\tau,x_2,y,z\sigma_i(\tau,x_2)) - g_i(\tau,x_1,y,z\sigma_i(\tau,x_1))| \\
&\qquad \leq \ C|z||x_2 - x_1| + C(|x_2 - x_1| + |z|\,|\sigma_i(\tau,x_2) - \sigma_i(\tau,x_1)|) \\
&\qquad \leq \ C|z||x_2 - x_1| + C(1 + |z|)|x_2 - x_1|, \qquad i = 1,2,
\end{aligned}$$

hence

$$
\begin{aligned}
&|\tilde{h}_i(\tau,x_2,y,z,w)-\tilde{h}_i(\tau,x_1,y,z,w)| \\
&\quad \leq \ |f_i(\tau,x_2,y,z)-f_i(\tau,x_1,y,z)|+\frac{(\min(w,C''))^+}{2}|\sigma_i^2(\tau,x_2)-\sigma_i^2(\tau,x_1)| \\
&\quad \leq \ C|z||x_2-x_1|+C(1+|z|)|x_2-x_1|+\frac{C''}{2}\left|\sigma_i(\tau,x_2)-\sigma_i(\tau,x_1)\right|(\sigma_i(\tau,x_1)+\sigma_i(\tau,x_2)) \\
&\quad \leq \ C|z||x_2-x_1|+C(1+|z|)|x_2-x_1|+CC'C''|x_2-x_1| \\
&\quad \leq \ (C+CC'C'')\,(1+|x_1|+|x_2|+|y|)\,(1+|z|)|x_2-x_1|, \quad i=1,2,
\end{aligned}
$$

which shows that Condition (G) of Theorem 2.4 in Appendix C of Peng (2010a) is satisfied with $\omega(x)=\bar{\omega}(x):=Cx$. In addition, by the conditions (3.6) and (B_1) we have

$$
\begin{aligned}
&\tilde{h}_2(\tau,x,y,z,w)-\tilde{h}_1(\tau,x,y,z,w) \\
&\qquad = \ f_2(\tau,x,y,z)-f_1(\tau,x,y,z)+\frac{(\min(w,C''))^+}{2}\big(\sigma_2^2(\tau,x)-\sigma_1^2(\tau,x)\big) \\
&\qquad \geq \ 0, \qquad x,y,z,w\in\mathbb{R}, \quad \tau\in[0,T].
\end{aligned}
$$

Besides, we have $\tilde{h}_2(\tau,x,y,z,w_1)\leq \tilde{h}_2(\tau,x,y,z,w_2)$ when $w_1\leq w_2$, and

$$
\begin{aligned}
|\tilde{h}_2(\tau,x,y_1,z,w_1)-\tilde{h}_2(\tau,x,y_2,z,w_2)| &\leq \frac{1}{2}\sigma_2^2(\tau,x)\,|w_1-w_2| \\
&\quad +|f_2(\tau,x,y_1,z)-f_2(\tau,x,y_2,z)| \\
&\leq \ C\left(|y_1-y_2|+|w_1-w_2|\right),
\end{aligned}
$$

$(\tau,x)\in[0,T]\times\mathbb{R}$, (y_1,z,w_1), $(y_2,z,w_2)\in\mathbb{R}^3$, hence $\tilde{h}_2(t,x,y,z,w)$ is Lipschitz in y and w. Therefore, by the comparison Theorem 2.4 in Appendix C of Peng (2010a) it follows that $u_1(t,x)\leq u_2(t,x)$ for all $t\in[0,T]$, from which we conclude to

$$
Y_0^{(1)}=u_1\big(0,X_0^{(1)}\big)\leq Y_0^{(2)}=u_2\big(0,X_0^{(1)}\big)=u_2\big(0,X_0^{(2)}\big),
$$

hence

$$
\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big]\leq \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big],
$$

for all convex functions ϕ in $\mathcal{C}_b^3(\mathbb{R})$. In order to extend (3.3) to coefficients satisfying (A_1)-(A_4) without assuming the bound (3.6), we apply the above argument to sequences $(\mu_{n,i})_{n\geq 1}$, $(\sigma_{n,i})_{n\geq 1}$, $(g_{n,i})_{n\geq 1}$, $(\phi_n)_{n\geq 1}$ of $\mathcal{C}_b^3$ functions as in Theorem 2.7, with

$$
0<c_n\leq \sigma_{n,1}(t,x)\leq \sigma_{n,2}(t,x)\leq C_n, \qquad t\in[0,T], \quad x\in\mathbb{R}, \quad n\geq 1, \tag{3.8}
$$

for some constants $c_n, C_n>0$ satisfying (A_1)-(A_4) with a same constant $C>0$ for all $n\geq 1$, and converging respectively pointwise to μ_i, σ_i, g_i and strongly to ϕ, i.e. $\phi_n(x_n)\to\phi(x)$

whenever $x_n \to x \in \mathbb{R}$, while preserving the convexity of the approximations $(\phi_n)_{n\geq 1}$ and $(f_{n,i})_{n\geq 1}$ defined by (3.2), see Azagra (2013), Lemma 1 of Lepeltier and San Martin (1997), and Problem 1.4.14 in Zhang (2017). The continuous dependence Proposition 8.4 then yields the convergence of the corresponding sequences $(Y^{(i)}_{n,0})_{n\geq 1}$ of BSDE solutions, concluding the proof. $\square$

By similar arguments, we derive the following Theorem 3.2 for the increasing convex ordering. The proof of Theorem 3.2 is first stated for $\mathcal{C}^3_b$ coefficients ϕ, $\mu_i(t,x)$, $\sigma_i(t,x)$ and $g_i(t,x,y,z)$ under (3.8), and then extended to coefficients satisfying (A_1)-(A_4) by applying the continuous dependence Proposition 8.4 as in the proof of Theorem 3.1.

Theorem 3.2 (Increasing convex order). *Assume that* $X^{(1)}_0 \leq X^{(2)}_0$ *and*

$$0 < \sigma_1(t,x) \leq \sigma_2(t,x), \qquad t \in [0,T], \quad x \in \mathbb{R},$$

together with the conditions

(B'_1) $f_1(t,x,y,z) \leq f_2(t,x,y,z)$, $t \in [0,T]$, $x, y \in \mathbb{R}$, $z \in \mathbb{R}_+$,

(B'_2) $(x,y) \mapsto f_i(t,x,y,z)$ *and* $(y,z) \mapsto f_i(t,x,y,z)$ *are both convex respectively on* $\mathbb{R}^2$ *and* $\mathbb{R}\times\mathbb{R}_+$, *for* $i = 1,2$, $x, y \in \mathbb{R}$, $z \in \mathbb{R}_+$, $t \in [0,T]$,

(B'_3) $x \mapsto g_i(t,x,y,z)$ *is non-decreasing on* $\mathbb{R}$ *for* $i = 1,2$, $y \in \mathbb{R}$, $z \in \mathbb{R}_+$, $t \in [0,T]$.

Then we have $X^{(1)}_T \leq^{\text{iconv}}_{g_1,g_2} X^{(2)}_T$, *i.e.,*

$$\mathcal{E}_{g_1}\big[\phi\big(X^{(1)}_T\big)\big] \leq \mathcal{E}_{g_2}\big[\phi\big(X^{(2)}_T\big)\big],$$

for all non-decreasing convex functions $\phi(x)$ *satisfying* (2.2).

Proof. Under (B'_3), when $\phi(x)$ and $g_i(t,x,y,z)$, $i = 1,2$, are non-decreasing in x, Proposition 8.2 tells us that the PDE solutions $u_1(t,x)$ and $u_2(t,x)$ satisfy

$$\frac{\partial u_1}{\partial x}(t,x) \geq 0 \quad \text{and} \quad \frac{\partial u_2}{\partial x}(t,x) \geq 0, \qquad t \in [0,T],$$

hence Conditions (B_1)-(B_2) only need to hold for $z \geq 0$, and the conclusion follows by repeating the arguments in the proof of Theorem 3.1. $\square$

We note that in case $\sigma_1(t,x) = \sigma_2(t,x)$ the convexity of $u_i(t,x)$, $i = 1,2$, is no longer required in the proofs of Theorems 3.1-3.2, and one can then remove Condition (B'_2) to obtain a result for the monotonic order.

Corollary 3.3 (Monotonic order with equal volatilities). *Assume that* $X_0^{(1)} \leq X_0^{(2)}$ *and*

$$0 < \sigma(t,x) := \sigma_1(t,x) = \sigma_2(t,x), \quad t \in [0,T], \quad x \in \mathbb{R},$$

together with the conditions

(B_1'') $f_1(t,x,y,z) \leq f_2(t,x,y,z)$, $t \in [0,T]$, $x, y \in \mathbb{R}$, $z \in \mathbb{R}_+$,

(B_2'') $x \mapsto g_i(t,x,y,z)$ *is non-decreasing on* $\mathbb{R}$ *for* $i = 1,2$, $y \in \mathbb{R}$, $z \in \mathbb{R}_+$, $t \in [0,T]$.

Then we have $X_T^{(1)} \leq_{g_1,g_2}^{\mathrm{mon}} X_T^{(2)}$, *i.e.,*

$$\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big],$$

for all non-decreasing functions $\phi(x)$ *satisfying* (2.2).

Proof. When $\sigma_1(t,x) = \sigma_2(t,x)$ we can repeat the proof of Theorem 3.1 by using h_i in (3.4), without defining $\tilde{h}_i$ in (3.7) and without assuming (B_2), and then follow the proof argument of Theorem 3.2 without requiring the convexity of $u_i(t,x)$, $i = 1,2$. □

Ordered drifts

Theorem 3.2 also admits the following version in the case of ordered drifts.

Corollary 3.4 (Increasing convex order). *Assume that* $X_0^{(1)} \leq X_0^{(2)}$ *and*

$$\mu_1(t,x) \leq \mu_2(t,x) \quad \text{and} \quad 0 < \sigma_1(t,x) \leq \sigma_2(t,x), \qquad t \in [0,T], \quad x \in \mathbb{R},$$

together with the following conditions:

(C_1) $g_1(t,x,y,z) \leq g_2(t,x,y,z)$, $t \in [0,T]$, $x, y \in \mathbb{R}$, $z \in \mathbb{R}_+$,

(C_2) $g_i(t,x,y,z)$ *is non-decreasing in* z *for* $i = 1$ *or* $i = 2$, $x, y \in \mathbb{R}$, $z \in \mathbb{R}_+$, $t \in [0,T]$,

(C_3) $g_i(t,x,y,z)$ *is non-decreasing in* x *for* $i = 1,2$, $x, y \in \mathbb{R}$, $z \in \mathbb{R}_+$, $t \in [0,T]$,

(C_4) $(x,y) \mapsto f_i(t,x,y,z)$ *and* $(y,z) \mapsto f_i(t,x,y,z)$ *are both convex respectively on* $\mathbb{R}^2$ *and* $\mathbb{R} \times \mathbb{R}_+$ *for* $i = 1,2$, $x, y \in \mathbb{R}$, $z \in \mathbb{R}_+$, $t \in [0,T]$.

Then we have $X_T^{(1)} \leq_{g_1,g_2}^{\mathrm{iconv}} X_T^{(2)}$, *i.e.,*

$$\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big],$$

for all non-decreasing convex functions $\phi(x)$ *satisfying* (2.2).

Proof. Under (C_3), since $\phi(x)$ and $g_i(t,x,y,z), i = 1,2$, are non-decreasing in x, by Proposition 8.2 the solutions $u_1(t,x)$ and $u_2(t,x)$ of (3.5) are non-decreasing in x and, as in the proof of Theorem 3.2, one can take $z \geq 0$ since $\frac{\partial u_i}{\partial x}(t,x) \geq 0$. Assuming that e.g. $g_1(t,x,y,z)$ is non-decreasing in z under (C_2), then by $z\sigma_1(t,x) \leq z\sigma_2(t,x)$, $(t,x) \in [0,T]\times\mathbb{R}$, $z \in \mathbb{R}_+$, and (C_1), we have

$$g_1(t,x,y,z\sigma_1(t,x)) \leq g_1(t,x,y,z\sigma_2(t,x)) \leq g_2(t,x,y,z\sigma_2(t,x)).$$

Combining the above with the inequality $z\mu_1(t,x) \leq z\mu_2(t,x), (t,x) \in [0,T] \times \mathbb{R}$, $z \in \mathbb{R}_+$, one finds $f_1(t,x,y,z) \leq f_2(t,x,y,z)$, and by Theorem 3.2 we conclude that $\mathcal{E}_{g_1}\big[\phi(X_T^{(1))}\big] \leq \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big]$ for all convex non-decreasing functions $\phi(x)$ satisfying (2.2). □

When the drift coefficients $\mu(t,x) = \mu_1(t,x) = \mu_2(t,x)$ are equal and $g_i(t,x,y,z)$ is independent of z, $i = 1,2$, the following proposition can be proved for the convex g-ordering similarly to Corollary 3.4, by applying Theorem 3.1 which deals with convex ordering, instead of Theorem 3.2.

Corollary 3.5 (Convex order with equal drifts). *Assume that $X_0^{(1)} = X_0^{(2)}$ and*

$$\mu_1(t,x) = \mu_2(t,x), \quad \textit{and} \quad 0 < \sigma_1(t,x) \leq \sigma_2(t,x), \qquad t \in [0,T], \quad x \in \mathbb{R},$$

together with the conditions

(C_1') *$g_i(t,x,y,z) = g_i(t,x,y)$ is independent of $z \in \mathbb{R}$ for $i = 1,2$, $t \in [0,T]$, $x,y \in \mathbb{R}$,*

(C_2') *$g_1(t,x,y) \leq g_2(t,x,y)$, $t \in [0,T]$, $x,y \in \mathbb{R}$,*

(C_3') *$(x,y) \mapsto f_i(t,x,y,z)$ and $(y,z) \mapsto f_i(t,x,y,z)$ are convex on $\mathbb{R}^2$ for $i = 1,2$.*

Then we have $X_T^{(1)} \leq_{g_1,g_2}^{\text{conv}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big],$$

for all convex functions $\phi(x)$ satisfying (2.2).

We note that the convexity of $u_1(t,x)$ and $u_2(t,x)$ is not needed in the proof of Theorem 3.1 when $\sigma_1(t,x) = \sigma_2(t,x)$, and in this case we can remove Condition (B_2') in Theorem 3.2 as in the next corollary.

Corollary 3.6 (Monotonic order with equal volatilities). *Assume that* $X_0^{(1)} \leq X_0^{(2)}$ *and*

$$0 < \sigma(t,x) := \sigma_1(t,x) = \sigma_2(t,x), \quad t \in [0,T], \quad x \in \mathbb{R},$$

together with the following conditions:

(D_1) $\mu_1(t,x) \leq \mu_2(t,x)$, $x \in \mathbb{R}$, $t \in [0,T]$,

(D_2) $g_1(t,x,y,z) \leq g_2(t,x,y,z)$ *for all* $(x,y,z) \in \mathbb{R}^2 \times \mathbb{R}_+$, $t \in [0,T]$,

(D_3) $g_i(t,x,y,z)$ *is non-decreasing in* x *for* $i = 1,2$ *and* $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}_+$.

Then we have $X_T^{(1)} \leq_{g_1,g_2}^{\mathrm{mon}} X_T^{(2)}$, *i.e.,*

$$\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big],$$

for all non-decreasing functions $\phi(x)$ *satisfying* (2.2).

Proof. Similarly to the proof of Corollary 3.3, under the condition $\sigma_1(t,x) = \sigma_2(t,x)$ the convexity of $u_i(t,x)$ and the non-decreasing property of $g_i(t,x,y,z)$ with respect to z, $i=1$ or $i=2$, are no longer required. In addition, the condition

$$f_1(t,x,y,z) \leq f_2(t,x,y,z), \quad x,y \in \mathbb{R}, \ z \in \mathbb{R}_+, \ t \in [0,T],$$

clearly holds from (D_1)-(D_2), and we can conclude as in the proof of Corollary 3.4. $\square$

4 Ordering with partially convex drifts

Theorems 3.1 and 3.2 require the convexity assumptions (B_2) and (B_2') on

$$(x,y,z) \mapsto f_i(t,x,y,z) := z\mu_i(t,x) + g_i(t,x,y,z\sigma_i(t,x))$$

in (x,y) and (y,z) to hold for both $i = 1,2$. In this section, we develop different convex g-ordering results under weaker convexity conditions, based on a measurable function $\zeta(t,x)$ such that

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \left(\frac{\mu_i\big(t,X_t^{(i)}\big) - \zeta\big(t,X_t^{(i)}\big)}{\sigma_i\big(t,X_t^{(i)}\big)}\right)^2 dt\right)\right] < \infty, \quad i = 1,2.$$

As in Section 3, the proofs of Theorems 4.1-4.2 are first stated for $\mathcal{C}_b^3$ BSDE coefficients as in Theorem 2.7, and then extended under (A_1)-(A_4) using Proposition 8.4.

Theorem 4.1 (Convex order). *Assume that $X_0^{(1)} = X_0^{(2)}$ and*

$$0 < \sigma_1(t,x) \le \sigma_2(t,x), \qquad t \in [0,T], \quad x \in \mathbb{R}, \tag{4.1}$$

together with the conditions

(E_1) $f_1(t,x,y,z) \le z\zeta(t,x) \le f_2(t,x,y,z)$, $t \in [0,T]$, $x,y,z \in \mathbb{R}$,

(E_2) $(x,y) \mapsto f_i(t,x,y,z)$ *and* $(y,z) \mapsto f_i(t,x,y,z)$ *are convex on* $\mathbb{R}^2$ *for* $i=1$ *or* $i=2$.

Then we have $X_T^{(1)} \le_{g_1,g_2}^{\mathrm{conv}} X_T^{(2)}$, i.e.,

$$\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big] \le \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big], \tag{4.2}$$

for all convex functions $\phi(x)$ satisfying (2.2).

Proof. (*i*) We start by assuming that the function ϕ and the coefficients $\mu_i(t,\cdot)$, $\sigma_i(t,\cdot)$ and $g_i(t,\cdot,\cdot,\cdot)$ are $\mathcal{C}_b^3$ functions for all $t \in [0,T]$ as in Theorem 2.7, and that (E_2) holds with $i=1$. Let

$$\theta_2(t,x) := \frac{\mu_2(t,x) - \zeta(t,x)}{\sigma_2(t,x)}, \qquad x \in \mathbb{R}, \quad t \in [0,T].$$

By the Girsanov theorem, the process

$$\widetilde{B}_t := B_t + \int_0^t \theta_2\big(s, X_s^{(2)}\big) ds, \qquad t \in [0,T],$$

is a standard Brownian motion under the probability measure $\mathbb{Q}_2$ defined by

$$\frac{d\mathbb{Q}_2}{d\mathbb{P}} := \exp\left(- \int_0^T \theta_2\big(s, X_s^{(2)}\big) dB_s - \frac{1}{2} \int_0^T \big(\theta_2\big(s, X_s^{(2)}\big)\big)^2 ds \right),$$

and the forward SDEs (3.1a)-(3.1b) can be rewritten as

$$\begin{cases} dX_t^{(1)} &= \big(\mu_1\big(t, X_t^{(1)}\big) - \theta_2\big(t, X_t^{(2)}\big)\sigma_1\big(t, X_t^{(1)}\big)\big)dt + \sigma_1\big(t, X_t^{(1)}\big)d\widetilde{B}_t, \quad X_0^{(1)} = x_0^{(1)}, \\ dX_t^{(2)} &= \zeta\big(t, X_t^{(2)}\big)dt + \sigma_2\big(t, X_t^{(2)}\big)d\widetilde{B}_t, \qquad X_0^{(2)} = x_0^{(2)}, \end{cases}$$

with the associated BSDEs

$$\begin{cases} dY_t^{(1)} = -\big(g_1\big(t, X_t^{(1)}, Y_t^{(1)}, Z_t^{(1)}\big) + Z_t^{(1)}\theta_2\big(t, X_t^{(2)}\big)\big)dt + Z_t^{(1)}d\widetilde{B}_t, \quad Y_T^{(1)} = \phi\big(X_T^{(1)}\big), \\ dY_t^{(2)} = -\big(g_2\big(t, X_t^{(2)}, Y_t^{(2)}, Z_t^{(2)}\big) + Z_t^{(2)}\theta_2\big(t, X_t^{(2)}\big)\big)dt + Z_t^{(2)}d\widetilde{B}_t, \quad Y_T^{(2)} = \phi\big(X_T^{(2)}\big). \end{cases}$$

By Theorem 2.7 we have $Y_t^{(1)} = u_1\big(t, X_t^{(1)}\big)$ and $Y_t^{(2)} = u_2\big(t, X_t^{(2)}\big)$, where the functions $u_1(t,x)$ and $u_2(t,x)$ are in $\mathcal{C}^{1,2}([0,T] \times \mathbb{R})$ and solve the PDEs

$$\frac{\partial u_i}{\partial t}(t,x) + \frac{1}{2}\sigma_i^2(t,x)\frac{\partial^2 u_i}{\partial x^2}(t,x) + f_i\Big(t, x, u_i(t,x), \frac{\partial u_i}{\partial x}(t,x)\Big) = 0, \tag{4.3}$$

with $u_i(T,x)=\phi(x)$, $i=1,2$. Applying Itô's formula to $u_1\big(t,X_t^{(2)}\big)$ and using (4.3), we have

$$
\begin{aligned}
u_1\big(t,X_t^{(2)}\big) &= u_1\big(0,X_0^{(2)}\big)+\int_0^t \frac{\partial u_1}{\partial s}\big(s,X_s^{(2)}\big)ds+\int_0^t \zeta\big(s,X_s^{(2)}\big)\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)ds \\
&\quad +\frac{1}{2}\int_0^t \sigma_2^2\big(s,X_s^{(2)}\big)\frac{\partial^2 u_1}{\partial x^2}\big(s,X_s^{(2)}\big)ds+\int_0^t \sigma_2\big(s,X_s^{(2)}\big)\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)d\widetilde{B}_s \\
&= u_1\big(0,X_0^{(2)}\big)+\int_0^t \zeta\big(s,X_s^{(2)}\big)\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)ds \\
&\quad -\int_0^t f_1\Big(s,X_s^{(2)},u_1\big(s,X_s^{(2)}\big),\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)\Big)ds \\
&\quad +\frac{1}{2}\int_0^t \big(\sigma_2^2\big(s,X_s^{(2)}\big)-\sigma_1^2\big(s,X_s^{(2)}\big)\big)\frac{\partial^2 u_1}{\partial x^2}\big(s,X_s^{(2)}\big)ds \\
&\quad +\int_0^t \sigma_2\big(s,X_s^{(2)}\big)\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)d\widetilde{B}_s.
\end{aligned}
$$

Taking expectation at time $t=T$ under $\mathbb{Q}_2$, we find

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}_2}\big[\phi\big(X_T^{(2)}\big)\big] &= u_1\big(0,X_0^{(2)}\big)+\mathbb{E}_{\mathbb{Q}_2}\left[\int_0^T \zeta\big(s,X_s^{(2)}\big)\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)ds\right] \\
&\quad -\mathbb{E}_{\mathbb{Q}_2}\left[\int_0^T f_1\Big(s,X_s^{(2)},u_1\big(s,X_s^{(2)}\big),\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)\Big)ds\right] \\
&\quad +\frac{1}{2}\mathbb{E}_{\mathbb{Q}_2}\left[\int_0^T \big(\sigma_2^2\big(s,X_s^{(2)}\big)-\sigma_1^2\big(s,X_s^{(2)}\big)\big)\frac{\partial^2 u_1}{\partial x^2}\big(s,X_s^{(2)}\big)ds\right].
\end{aligned}
$$

Next, applying similarly Itô's formula to $u_2\big(t,X_t^{(2)}\big)$ and then taking expectation at $t=T$ under $\mathbb{Q}_2$ we obtain, from (4.3),

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}_2}\big[\phi\big(X_T^{(2)}\big)\big] &= u_2\big(0,X_0^{(2)}\big)-\mathbb{E}_{\mathbb{Q}_2}\left[\int_0^T f_2\Big(s,X_s^{(2)},u_2\big(s,X_s^{(2)}\big),\frac{\partial u_2}{\partial x}\big(s,X_s^{(2)}\big)\Big)ds\right] \\
&\quad +\mathbb{E}_{\mathbb{Q}_2}\left[\int_0^T \zeta\big(s,X_s^{(2)}\big)\frac{\partial u_2}{\partial x}\big(s,X_s^{(2)}\big)ds\right].
\end{aligned}
$$

From Assumption (E_1) and Condition (4.1) we get

$$
\begin{aligned}
u_2\big(0,X_0^{(2)}\big)-u_1\big(0,X_0^{(2)}\big) &= \mathbb{E}_{\mathbb{Q}_2}\left[\int_0^T f_2\Big(s,X_s^{(2)},u_2\big(s,X_s^{(2)}\big),\frac{\partial u_2}{\partial x}\big(s,X_s^{(2)}\big)\Big)ds\right] \\
&\quad -\mathbb{E}_{\mathbb{Q}_2}\left[\int_0^T f_1\Big(s,X_s^{(2)},u_1\big(s,X_s^{(2)}\big),\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)\Big)ds\right] \\
&\quad -\mathbb{E}_{\mathbb{Q}_2}\left[\int_0^T \zeta\big(s,X_s^{(2)}\big)\frac{\partial u_2}{\partial x}\big(s,X_s^{(2)}\big)ds\right]+\mathbb{E}_{\mathbb{Q}_2}\left[\int_0^T \zeta\big(s,X_s^{(2)}\big)\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)ds\right] \\
&\quad +\frac{1}{2}\mathbb{E}_{\mathbb{Q}_2}\left[\int_0^T \big(\sigma_2^2\big(s,X_s^{(2)}\big)-\sigma_1^2\big(s,X_s^{(2)}\big)\big)\frac{\partial^2 u_1}{\partial x^2}\big(s,X_s^{(2)}\big)ds\right]
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
&\geq \ \mathbb{E}_{\mathbb{Q}_2}\left[\int_0^T \left(f_2\big(s,X_s^{(2)},u_2\big(s,X_s^{(2)}\big),\frac{\partial u_2}{\partial x}\big(s,X_s^{(2)}\big)\big) - \zeta\big(s,X_s^{(2)}\big)\frac{\partial u_2}{\partial x}\big(s,X_s^{(2)}\big)\right) ds\right] \\
&\quad -\mathbb{E}_{\mathbb{Q}_2}\left[\int_0^T \left(f_1\big(s,X_s^{(2)},u_1\big(s,X_s^{(2)}\big),\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)\big) - \zeta\big(s,X_s^{(2)}\big)\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)\right) ds\right] \\
&\geq \ 0,
\end{aligned}
$$

where we have used (E_1) and the fact that $\dfrac{\partial^2 u_1}{\partial x^2}(t,x) \geq 0$, as follows from Theorem 7.2. (ii) The case $i = 2$ in Assumption (E_2) is dealt with similarly by applying Itô's formula to $u_2\big(t,X_t^{(1)}\big)$ and then to $u_1\big(t,X_t^{(1)}\big)$, and by taking expectation at $t = T$ under the probability measure $\mathbb{Q}_1$ defined by

$$
\frac{d\mathbb{Q}_1}{d\mathbb{P}} := \exp\left(-\int_0^T \theta_1\big(s,X_s^{(1)}\big)dB_s - \frac{1}{2}\int_0^T \big(\theta_1\big(s,X_s^{(1)}\big)\big)^2 ds\right),
$$

where

$$
\theta_1(t,x) := \frac{\mu_1(t,x) - \zeta(t,x)}{\sigma_1(t,x)}, \qquad x \in \mathbb{R}, \quad t \in [0,T].
$$

In this case, from (4.1) and (E_1) we get

$$
\begin{aligned}
& u_2\big(0,X_0^{(2)}\big) - u_1\big(0,X_0^{(2)}\big) = \mathbb{E}_{\mathbb{Q}_1}\left[\int_0^T f_2\Big(s,X_s^{(1)},u_2\big(s,X_s^{(1)}\big),\frac{\partial u_2}{\partial x}\big(s,X_s^{(1)}\big)\Big) ds\right] \\
&\qquad -\mathbb{E}_{\mathbb{Q}_1}\left[\int_0^T f_1\Big(s,X_s^{(1)},u_1\big(s,X_s^{(1)}\big),\frac{\partial u_1}{\partial x}\big(s,X_s^{(1)}\big)\Big) ds\right] \\
&\qquad -\mathbb{E}_{\mathbb{Q}_1}\left[\int_0^T \zeta\big(s,X_s^{(1)}\big)\frac{\partial u_2}{\partial x}\big(s,X_s^{(1)}\big) ds\right] + \mathbb{E}_{\mathbb{Q}_1}\left[\int_0^T \zeta\big(s,X_s^{(1)}\big)\frac{\partial u_1}{\partial x}\big(s,X_s^{(1)}\big) ds\right] \\
&\qquad +\frac{1}{2}\mathbb{E}_{\mathbb{Q}_1}\left[\int_0^T \big(\sigma_2^2\big(s,X_s^{(1)}\big) - \sigma_1^2\big(s,X_s^{(1)}\big)\big)\frac{\partial^2 u_2}{\partial x^2}\big(s,X_s^{(1)}\big) ds\right] \\
&\quad \geq \ 0,
\end{aligned}
$$

since $\dfrac{\partial^2 u_2}{\partial x^2}(t,x) \geq 0$ by Theorem 7.2. By the relations $Y_0^{(1)} = u_1\big(0,X_0^{(1)}\big)$, $Y_0^{(2)} = u_2\big(0,X_0^{(2)}\big)$ and $X_0^{(1)} = X_0^{(2)}$ we conclude to $Y_0^{(2)} - Y_0^{(1)} \geq 0$, which shows (4.2). The extension of (4.2) to coefficients satisfying (A_1)-(A_4) follows as in the proof of Theorem 3.1. $\square$

The next proposition deals with the increasing convex order, for which only the Conditions (E_1')-(E_2') and $X_0^{(1)} \leq X_0^{(2)}$ are required in addition to Condition (4.5) and (E_3') below.

Theorem 4.2 (Increasing convex order). *Assume that $X_0^{(1)} \leq X_0^{(2)}$ and*

$$
0 < \sigma_1(t,x) \leq \sigma_2(t,x), \quad t \in [0,T], \quad x \in \mathbb{R}, \tag{4.5}
$$

together with the conditions

(E_1') $f_1(t,x,y,z) \leq z\zeta(t,x) \leq f_2(t,x,y,z)$, $t \in [0,T]$, $x,y \in \mathbb{R}$, $z \in \mathbb{R}_+$,

(E_2') $(x,y) \mapsto f_i(t,x,y,z)$ *and* $(y,z) \mapsto f_i(t,x,y,z)$ *are respectively convex on* $\mathbb{R}^2$ *and* $\mathbb{R}\times\mathbb{R}_+$ *for* $i=1$ *or* $i=2$,

(E_3') $x \mapsto g_i(t,x,y,z)$ *is non-decreasing on* $\mathbb{R}$ *for* $i=1,2$, $y \in \mathbb{R}$, $z \in \mathbb{R}_+$, $t \in [0,T]$.

Then we have $X_T^{(1)} \leq_{g_1,g_2}^{\mathrm{iconv}} X_T^{(2)}$, *i.e.*,

$$\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big],$$

for all non-decreasing convex functions $\phi(x)$ *satisfying* (2.2).

Proof. As in the proof of Theorem 4.1 we start with $\mathcal{C}_b^3$ coefficients, and then extend the conclusion to coefficients satisfying (A_1)-(A_4) using Proposition 8.4. If $\phi(x)$ and $g_i(t,x,y,z)$ are non-decreasing in x by (E_3'), $i=1,2$, then by Proposition 8.2 the solutions $u_1(t,x)$ and $u_2(t,x)$ of the PDE (4.3) are nondecreasing in x and satisfy

$$\frac{\partial u_1}{\partial x}(t,X_t^{(2)}) \geq 0 \quad \text{and} \quad \frac{\partial u_2}{\partial x}(t,X_t^{(2)}) \geq 0,$$

a.s., $t \in [0,T]$, hence Conditions (E_1)-(E_2) only need to hold for $z \geq 0$, showing the sufficiency of (E_i'), $i=1,2,3$. In addition, we have $u_1\big(0,X_0^{(1)}\big) = Y_0^{(1)} \leq u_1\big(0,X_0^{(2)}\big)$ by the assumption $X_0^{(1)} \leq X_0^{(2)}$, hence by repeating arguments in the proof of Theorem 4.1 for $i=1$ we find by (4.4) that

$$\begin{aligned}
Y_0^{(2)} - Y_0^{(1)} &\geq u_2\big(0,X_0^{(2)}\big) - u_1\big(0,X_0^{(2)}\big) \\
&= \mathbb{E}_{Q_2}\left[\int_0^T f_2\Big(s,X_s^{(2)},u_2\big(s,X_s^{(2)}\big),\frac{\partial u_2}{\partial x}\big(s,X_s^{(2)}\big)\Big)ds\right] \\
&\quad -\mathbb{E}_{Q_2}\left[\int_0^T f_1\Big(s,X_s^{(2)},u_1\big(s,X_s^{(2)}\big),\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)\Big)ds\right] \\
&\quad -\mathbb{E}_{Q_2}\left[\int_0^T \zeta\big(s,X_s^{(2)}\big)\frac{\partial u_2}{\partial x}\big(s,X_s^{(2)}\big)ds\right] + \mathbb{E}_{Q_2}\left[\int_0^T \zeta\big(s,X_s^{(2)}\big)\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)ds\right] \\
&\quad +\frac{1}{2}\mathbb{E}_{Q_2}\left[\int_0^T \big(\sigma_2^2\big(s,X_s^{(2)}\big) - \sigma_1^2\big(s,X_s^{(2)}\big)\big)\frac{\partial^2 u_1}{\partial x^2}\big(s,X_s^{(2)}\big)ds\right] \\
&\geq 0,
\end{aligned}$$

under Assumption (E_2') for $i=1$. The case $i=2$ is treated similarly according to the proof of Theorem 4.1. $\square$

5 Comparison in g-risk measures

A g-risk measure is a mapping $\rho : L^2(\Omega, \mathcal{F}_T) \to \mathbb{R}$ satisfying the following conditions.

Definition 5.1 *Let g satisfy Conditions (A_2)-(A_3) and $\xi \in L^2(\Omega, \mathcal{F}_T)$.*

1) The static g-risk measure is defined in terms of g-evaluation as

$$\rho^g(X) := \mathcal{E}_g[-\xi].$$

2) The dynamic g-risk measure is defined in terms of conditional g-evaluation as

$$\rho_t^g(\xi) := \mathcal{E}_g[-\xi \mid \mathcal{F}_t], \qquad t \in [0, T].$$

We refer to Rosazza Gianin (2006) for the relations between coherent and convex risk measures, and the g-expectation.

We note that, taking $\phi(x) := -x$ in (2.9), $X^{(1)} \leq_{g_1,g_2}^{conv} X^{(2)}$ implies $\rho^{g_1}\big(X^{(1)}\big) \leq \rho^{g_2}\big(X^{(2)}\big)$. In addition, we have $\rho^{g_1}\big(\phi\big(X_T^{(1)}\big)\big) \leq \rho^{g_2}\big(\phi\big(X_T^{(2)}\big)\big)$ for all convex function $\phi(x)$ if and only if

$$X_T^{(2)} \leq_{g_1^{(-1)}, g_2^{(-1)}}^{\text{conv}} X_T^{(1)},$$

where

$$g_1^{(-1)}(t, x, y, z) := -g_1(t, x, -y, -z) \quad \textit{and} \quad g_2^{(-1)}(t, x, y, z) = -g_2(t, x, -y, -z),$$

as from Lemma 2.5 with $a = -1$ we have $\mathcal{E}_{g^{(-1)}}[\phi(X_T)] = -\mathcal{E}_{g^{(1)}}[-\phi(X_T)]$. A stochastic ordering via G-expectations has also been defined in Tian and Jiang (2016) by combining (2.9) with the inequality

$$-\rho^{g_1}\big(\phi\big(X^{(1)}\big)\big) = -\mathcal{E}_{g_1}\big[-\phi\big(X^{(1)}\big)\big] \leq -\mathcal{E}_{g_2}\big[-\phi\big(X^{(2)}\big)\big] = -\rho^{g_2}\big(\phi\big(X^{(2)}\big)\big),$$

where $-\mathcal{E}_{g_i}\big[-\phi\big(X_T^{(i)}\big)\big]$ and $\mathcal{E}_{g_i}\big[\phi\big(X_T^{(i)}\big)\big]$ respectively represent bid and ask prices of the contingent claim in financial markets, $i = 1, 2$.

Theorems 3.1 and 4.1 admit the following versions for the comparison of risks. First, we have the next consequence of Theorem 3.1 and Lemma 2.5 below, where we let

$$f_i^{(-1)}(t, x, y, z) := z\mu_i(t, x) + g_i^{(-1)}(t, x, y, z\sigma_1(t, x)), \quad t \in \mathbb{R}_+, \ x, y, z \in \mathbb{R}, \ i = 1, 2.$$

Corollary 5.2 *Assume that* $X_0^{(1)} = X_0^{(2)}$ *and*

$$0 < \sigma_1(t,x) \leq \sigma_2(t,x), \qquad t \in [0,T], \quad x \in \mathbb{R},$$

together with the conditions

(F_1) $f_1^{(-1)}(t,x,y,z) \leq f_2^{(-1)}(t,x,y,z)$, $t \in [0,T]$, $x,y,z \in \mathbb{R}$,

(F_2) $(x,y) \mapsto f_i^{(-1)}(t,x,y,z)$ *and* $(y,z) \mapsto f_i^{(-1)}(t,x,y,z)$ *are convex on* $\mathbb{R}^2$ *for* $i = 1,2$.

Then we have

$$-\mathcal{E}_{g_1}\big[-\phi\big(X_T^{(1)}\big)\big] \leq -\mathcal{E}_{g_2}\big[-\phi\big(X_T^{(2)}\big)\big],$$

for all convex functions $\phi(x)$ *satisfying* (2.2).

Similarly, we have the next consequence of Theorem 4.1 and Lemma 2.5.

Corollary 5.3 *Assume that* $X_0^{(1)} = X_0^{(2)}$ *and*

$$0 < \sigma_1(t,x) \leq \sigma_2(t,x), \qquad t \in [0,T], \quad x \in \mathbb{R},$$

together with the conditions

(G_1) $f_1^{(-1)}(t,x,y,z) \leq z\zeta(t,x) \leq f_2^{(-1)}(t,x,y,z)$, $t \in [0,T]$, $x,y,z \in \mathbb{R}$,

(G_2) $(x,y) \mapsto f_i^{(-1)}(t,x,y,z)$ *and* $(y,z) \mapsto f_i^{(-1)}(t,x,y,z)$ *are convex on* $\mathbb{R}^2$ *for* $i = 1$ *or* $i = 2$.

Then we have

$$-\mathcal{E}_{g_1}\big[-\phi\big(X_T^{(1)}\big)\big] \leq -\mathcal{E}_{g_2}\big[-\phi\big(X_T^{(2)}\big)\big],$$

for all convex functions $\phi(x)$ *satisfying* (2.2).

We note that Corollaries 5.2 and 5.3 can be applied to Example 6.1 below for the comparison of g_i-risk measures with $g_i(t,x,y,z)$ linear in y and z, in which case $g_i^{(-1)}(t,x,y,z) = g_i(t,x,y,z)$ and the bid and ask prices $-\mathcal{E}_{g_i}\big[-\phi\big(X_T^{(i)}\big)\big] = \mathcal{E}_{g_i}\big[\phi\big(X_T^{(i)}\big)\big]$ are equal, $i = 1,2$.

Furthermore, we can also derive results for the increasing convex and monotonic orderings of g-risk measures under Conditions (F_1)-(F_2) and (G_1)-(G_2). For example, if (F_1)-(F_2) or (G_1)-(G_2) only holds for $z \geq 0$ and g_i is non-decreasing in x, $i = 1,2$, we then get versions of Corollaries 5.2-5.3 for the increasing convex g-risk comparisons as in Theorems 3.2 and 4.2. Similarly, under additional the assumption $\sigma_1(t,x) = \sigma_2(t,x)$ and by removing (F_2) and (G_2), we can obtain versions of Corollaries 5.2-5.3 for the monotonic g-risk ordering as in Corollary 3.3.

6 Application examples

In the following examples we aim at comparing option prices of the form $Y_0^{(i)} := \mathcal{E}_{g_i}\big[\phi\big(X_T^{(i)}\big)\big]$ for two risky assets with positive prices $\big(X_t^{(i)}\big)_{t\in[0,T]}$, $i = 1,2$, given by

$$dX_t^{(i)} = X_t^{(i)} a_i\big(t, X_t^{(i)}\big)dt + X_t^{(i)} b_i\big(t, X_t^{(i)}\big)dB_t, \qquad i = 1,2,$$

where the coefficients $\mu_i(t,x) = xa_i(t,x)$ and $\sigma_i(t,x) = xb_i(t,x)$ satisfy (A_1), $i = 1,2$, for example $a_i(t,x)$ and $b_i(t,x)$ can be bounded functions. Example 6.1 compares option prices in classical expectation for standard self-financing portfolios with $X_t^{(1)} = X_t^{(2)} := X_t$, $a_1(t,x) = a_2(t,x) := r$, $\sigma_t = b_1(t, X_t)$, with a misspecified volatility coefficient $b_2\big(t,x\big)$ such that $\sigma_t \le b_2(t, X_t)$, *a.s.*, and is consistent with Theorem 6.2 in El Karoui et al. (1998).

Example 6.1 *Taking*

$$g_i(t,x,y,z) := -ry - z\frac{a_i(t,x) - r}{b_i(t,x)}, \qquad i = 1,2, \tag{6.1}$$

under the conditions

$$X_0^{(1)} = X_0^{(2)} \qquad \text{and} \qquad 0 < b_1(t,x) \le b_2(t,x), \quad t \in [0,T], \; x > 0, \tag{6.2}$$

we have $X_T^{(1)} \le^{\text{conv}}_{g_1,g_2} X_T^{(2)}$, i.e. the values of the self-financing portfolios hedging the claim payoffs $\phi\big(X_T^{(1)}\big)$ and $\phi\big(X_T^{(2)}\big)$ satisfy

$$\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big] \le \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big],$$

for all convex functions $\phi(x)$ satisfying (2.2).

Proof. Consider the risk-free asset $E_t := E_0 e^{rt}$ and the portfolio valued

$$V_t^{(i)} := p_t^{(i)} X_t^{(i)} + q_t^{(i)} E_t, \qquad t \in \mathbb{R}_+,$$

where $p_t^{(i)}$ is the quantity of risky assets and $q_t^{(i)}$ is the quantity of risk-free assets. When the strategy $(p_t^{(i)}, q_t^{(i)})_{t\in\mathbb{R}_+}$ is self-financing, we have

$$\begin{aligned} dV_t^{(i)} &= q_t^{(i)} dE_t + p_t^{(i)} dX_t^{(i)} \\ &= \big(rV_t^{(i)} + \theta_i\big(t, X_t^{(i)}\big)p_t^{(i)} X_t^{(i)} b_i\big(t, X_t^{(i)}\big)\big)dt + p_t^{(i)} X_t^{(i)} b_i\big(t, X_t^{(i)}\big)dB_t, \end{aligned} \tag{6.3}$$

where

$$\theta_i(t,x) := \frac{a_i(t,x) - r}{b_i(t,x)}, \qquad i = 1,2.$$

Hence, letting

$$Z_t^{(i)} := p_t^{(i)} X_t^{(i)} b_i\big(t, X_t^{(i)}\big),$$

and discounting as

$$\widetilde{V}_t^{(i)} := e^{-rt} V_t^{(i)}, \quad \widetilde{X}_t^{(i)} := e^{-rt} X_t^{(i)}, \quad \text{and} \quad \widetilde{Z}_t^{(i)} := e^{-rt} Z_t^{(i)},$$

with $V_T^{(i)} = \phi\big(X_T^{(i)}\big)$, we find the linear BSDE

$$\widetilde{V}_t^{(i)} = \widetilde{V}_T^{(i)} + \int_t^T \tilde{g}_i\big(s, \widetilde{X}_s^{(i)}, \widetilde{V}_s^{(i)}, \widetilde{Z}_s^{(i)}\big) ds - \int_t^T \widetilde{Z}_s^{(i)} dB_s,$$

with $\tilde{g}_i(t,x,y,z) = -z\theta_i(t, xe^{rt})$, $i = 1, 2$. Since

$$xz(a_i(t, xe^{rt}) - r) + \tilde{g}_i(t, x, y, zxb_i(t, xe^{rt})) = 0, \quad x, y, z \in (0, \infty) \times \mathbb{R}^2, \ t \in [0, T],$$

$i = 1, 2$, we check that Conditions (B_1) and (E_1) are satisfied (with $\zeta(t,x) = 0$) together with (B_2) and (E_2), hence, under (6.2), Theorems 3.1 and 4.1 show that

$$\widetilde{V}_0^{(1)} = \mathcal{E}_{\widetilde{g}_1}\big[e^{-rT}\phi\big(e^{rT}\widetilde{X}_T^{(1)}\big)\big] \le \widetilde{V}_0^{(2)} = \mathcal{E}_{\widetilde{g}_2}\big[e^{-rT}\phi\big(e^{rT}\widetilde{X}_T^{(2)}\big)\big],$$

that is $\widetilde{X}_T^{(1)} \le_{\widetilde{g}_1, \widetilde{g}_2}^{\text{conv}} \widetilde{X}_T^{(2)}$. Therefore the price of the first claim is upper bounded by the price of the second claim, i.e.

$$\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big] \le \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big],$$

for all convex functions $\phi(x)$ satisfying (2.2), or $X_T^{(1)} \le_{g_1, g_2}^{\text{conv}} X_T^{(2)}$, with $g_i(t,x,y,z)$ as in (6.1), $i = 1, 2$. □

The next example considers the kernel $g(t,x,y,z) := \alpha(t)|z|$ with $\alpha(t) > 0$, $t \in [0,T]$, and shows that the second risky asset would be preferred over the first one by risk-seeking investors whose preferences are modeled by g since in this case $\mathcal{E}_g[\,\cdot\,]$ is represented as in (2.8a), see also Theorem 1.37 in Sriboonchita et al. (2009).

Example 6.2 *Consider the kernel*

$$g_i(t,x,y,z) = g(t,x,y,z) := \alpha(t)|z|, \quad z \in \mathbb{R}, \quad i = 1, 2,$$

where $\alpha(t)$ is a positive bounded function on $[0,T]$. Then, assuming that

$$X_0^{(1)} \le X_0^{(2)}, \quad a_1(t,x) \le a_2(t,x), \quad \text{and} \quad b_1(t,x) \le b_2(t,x), \ t \in [0,T], \ x > 0, \tag{6.4}$$

and that the functions $x \mapsto xa_i(t,x)$ and $x \mapsto xb_i(t,x)$ are convex in $x \in \mathbb{R}_+$ for $i = 1, 2$ and $t \in [0,T]$, we have

$$\mathcal{E}_g\big[\phi\big(X_T^{(1)}\big)\big] \le \mathcal{E}_g\big[\phi\big(X_T^{(2)}\big)\big], \tag{6.5}$$

for all non-decreasing convex functions $\phi(x)$ satisfying (2.2).

Proof. Under (6.4), we check that when $\alpha(t) > 0$, $t \in [0,T]$, we have

$$xza_1(t,x) + z\alpha(t)b_1(t,x) \leq xza_2(t,x) + z\alpha(t)b_2(t,x), \qquad x, z \in \mathbb{R}_+,$$

and we conclude by Theorem 3.2. □

In the case $\alpha(t) < 0$, $t \in [0,T]$, the function $z \mapsto g(t,x,y,z) = \alpha(t)|z|$ is concave on $\mathbb{R}$, and a similar result can be obtained for risk-averse investors by taking $b_1(t,x) \geq b_2(t,x)$ in (6.4), provided that the functions and $x \mapsto xa_i(t,x)$, $x \mapsto xb_i(t,x)$ are concave in $x \in \mathbb{R}_+$ for $= 1,2$ and $t \in [0,T]$, and that the function ϕ in (6.5) is non-decreasing concave.

In Examples 6.3-6.5 we consider portfolios under constraints, in which case the BSDE generators are sublinear functions. In Example 6.3 we consider the comparison of option prices for a standard self-financing portfolio and a second self-financing hedging portfolio in which borrowing occurs at the rate $R \geq r$, as in Example 1.1 in El Karoui et al. (1997).

Example 6.3 *Taking*

$$g_1(t,x,y,z) := -ry - z\frac{a_1(t,x) - r}{b_1(t,x)}$$

and

$$g_2(t,x,y,z) := -ry - z\frac{a_2(t,x) - r}{b_2(t,x)} + (R-r)\left(y - \frac{z}{b_2(t,x)}\right)^-,$$

where $w^- = -\min(w,0)$, *and under the conditions*

$$X_0^{(1)} = X_0^{(2)} \qquad \textit{and} \qquad 0 < b_1(t,x) \leq b_2(t,x), \quad t \in [0,T], \ x > 0, \tag{6.6}$$

we have $X_T^{(1)} \leq_{g_1,g_2}^{\text{conv}} X_T^{(2)}$, *i.e.*

$$\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big],$$

for all convex functions $\phi(x)$ *satisfying* (2.2), *i.e., the fair price of the unconstrained portfolio is less than that of the one with constraints.*

Proof. The first portfolio value is the discounted wealth process of Example 6.1, which satisfies the BSDE

$$\widetilde{V}_t^{(1)} = e^{-rT}\phi\big(e^{rT}\widetilde{X}_T^{(1)}\big) + \int_t^T \tilde{g}_1(s,\widetilde{X}_s^{(1)},\widetilde{V}_s^{(1)},\widetilde{Z}_s^{(1)})ds - \int_t^T \widetilde{Z}_s^{(1)}dB_s,$$

where $\theta_1(t,x) := (a_1(t,x) - r)/b_1(t,x)$, with the generator $\tilde{g}_1(t,x,y,z) := -z\theta_1(t,xe^{rt})$. In the second portfolio the investor is only allowed to borrow at the rate $R \geq r$, which yields the discounted wealth process

$$\widetilde{V}_t^{(2)} = e^{-rT}\phi\big(e^{rT}\widetilde{X}_T^{(2)}\big) + \int_t^T \tilde{g}_2\big(s,\widetilde{X}_s^{(2)},\widetilde{V}_s^{(2)},\widetilde{Z}_s^{(2)}\big)ds - \int_t^T \widetilde{Z}_s^{(2)}dB_s,$$

which is a BSDE with the generator

$$\tilde{g}_2(t,x,y,z) := -z\theta_2(t,xe^{rt}) + (R-r)\left(y - \frac{z}{b_2(t,xe^{rt})}\right)^{-}.$$

We check that

$$\begin{aligned} xz(a_1(t,xe^{rt})-r) + \tilde{g}_1(t,x,y,zxb_1(t,xe^{rt})) &= 0 \qquad (6.7)\\ &\leq (R-r)(y-xz)^{-} \\ &= zx(a_2(t,xe^{rt})-r) + \tilde{g}_2(t,x,y,zxb_2(t,xe^{rt})), \end{aligned}$$

$x,y,z \in (0,\infty)\times\mathbb{R}^2$, $t\in[0,T]$, where both functions $(x,y)\mapsto(R-r)(y-xz)^{-}$ and $(y,z)\mapsto(R-r)(y-xz)^{-}$ are convex, hence (B_2) and (E_2) are satisfied. In addition, (B_1) and (E_1) hold (with $\zeta(t,x)=0$) hence by (6.6), Theorems 3.1 and 4.1 both show that

$$\widetilde{V}_0^{(1)} = \mathcal{E}_{\tilde{g}_1}\big[e^{-rT}\phi\big(e^{rT}\widetilde{X}_T^{(1)}\big)\big] \leq \widetilde{V}_0^{(2)} = \mathcal{E}_{\tilde{g}_2}\big[e^{-rT}\phi\big(e^{rT}\widetilde{X}_T^{(2)}\big)\big],$$

or

$$\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big],$$

with $g_i(t,x,y,z) = -ry + \tilde{g}_i(t,xe^{-rt},y,z)$, $i=1,2$, for all convex functions $\phi(x)$ satisfying (2.2), which shows the part (a). □

In particular, when $X_t^{(1)} = X_t^{(2)} := X_t$, $a_1(t,x) = a_2(t,x) := a(t,x)$ and $b_1(t,x) = b_2\big(t,x\big) := b(t,x)$, Example 6.3 shows that

$$\mathcal{E}_{g_1}\big[\phi\big(X_T\big)\big] \leq \mathcal{E}_{g_2}\big[\phi\big(X_T\big)\big],$$

for all convex functions $\phi(x)$ satisfying (2.2). Here, using the same underlying risky asset $(X_t)_{t\in[0,T]}$, the first agent hedges the contingent claim $\phi(X_T)$ by a self-financing strategy without borrowing money, while the second agent hedges the same claim under constraints on the difference between the borrowing and lending rates. In this case, the initial investment of the second agent is higher as we have $V_0^{(2)} \geq V_0^{(1)}$.

In the next example we assume that both self-financing hedging portfolios require borrowing at the rate $R \geq r$.

Example 6.4 *Taking*

$$g_i(t,x,y,z) := -ry - z\frac{a_i(t,x)-r}{b_i(t,x)} + (R-r)\left(y - \frac{z}{b_i(t,x)}\right)^{-}, \qquad i=1,2,$$

under the conditions

$$X_0^{(1)} = X_0^{(2)} \qquad \textit{and} \qquad 0 < b_1(t,x) \le b_2(t,x), \quad t \in [0,T], \ x > 0, \tag{6.8}$$

we have $X_T^{(1)} \le_{g_1,g_2}^{\mathrm{conv}} X_T^{(2)}$, *i.e.*

$$\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big] \le \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big],$$

for all convex functions $\phi(x)$ *satisfying* (2.2).

Proof. We consider two portfolios constructed as in Example 6.3, with discounted wealth processes given by the BSDEs

$$\widetilde{V}_t^{(i)} = e^{-rT}\phi\big(e^{rT}\widetilde{X}_T^{(i)}\big) + \int_t^T \tilde{g}_i\big(s, \widetilde{X}_s^{(i)}, \widetilde{V}_s^{(i)}, \widetilde{Z}_s^{(i)}\big)ds - \int_t^T \widetilde{Z}_s^{(i)} dB_s, \quad i = 1, 2,$$

with the generators

$$\tilde{g}_i(t,x,y,z) := -z\theta_i(t, xe^{rt}) + (R-r)\left(y - \frac{z}{b_i(t, xe^{rt})}\right)^-, \quad i = 1, 2.$$

We check that

$$\begin{aligned} xz(a_1(t, xe^{rt}) - r) + \tilde{g}_1(t,x,y,zxb_1(t,xe^{rt})) &= (R-r)(y-xz)^- \\ &= zx(a_2(t, e^{rt}x) - r) + \tilde{g}_2(t,x,y,zxb_2(t,xe^{rt})), \end{aligned}$$

$x, y, z \in (0,\infty) \times \mathbb{R}^2$, $t \in [0,T]$, hence (B_1) is satisfied. Here, Theorem 4.1 cannot be applied because (E_1) is not satisfied. However, (B_2) is satisfied since both functions $(x,y) \mapsto (R-r)(y-xz)^-$ and $(y,z) \mapsto (R-r)(y-xz)^-$ are convex, hence by (6.8), Theorem 3.1 shows that

$$\widetilde{V}_0^{(1)} = \mathcal{E}_{\tilde{g}_1}\big[e^{-rT}\phi\big(e^{rT}\widetilde{X}_T^{(1)}\big)\big] \le \widetilde{V}_0^{(2)} = \mathcal{E}_{\tilde{g}_2}\big[e^{-rT}\phi\big(e^{rT}\widetilde{X}_T^{(2)}\big)\big],$$

or

$$\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big] \le \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big],$$

with $g_i(t,x,y,z) = -ry + \tilde{g}_i(t, xe^{-rt}, y, z)$, $i = 1, 2$, for all convex functions $\phi(x)$ satisfying (2.2). $\square$

The next Example 6.5 is based on three risky asset prices $\big(X_t^{(1)}\big)_{t\in[0,T]}$, $\big(X_t^{(2)}\big)_{t\in[0,T]}$ and $\big(X_t^{(3)}\big)_{t\in[0,T]}$, see §3.2 of Jouini and Kallal (1995). The portfolio of the first investor is based on the risky asset $X_t^{(1)}$ and on the risk-free asset $E_t := E_0e^{rt}$ as in Example 6.1. On the other hand, the second investor is longing $X_t^{(2)}$ and E_t while shorting $X_t^{(3)}$ and the risk-free asset $\overline{E}_t := \overline{E}_0e^{Rt}$, with $R \ge r$.

Example 6.5 *In addition to* $\big(X_t^{(1)}\big)_{t\in[0,T]}$ *and* $\big(X_t^{(2)}\big)_{t\in[0,T]}$*, consider a third asset with positive price* $\big(X_t^{(3)}\big)_{t\in[0,T]}$ *given by*

$$dX_t^{(3)} = X_t^{(3)} a_3\big(t, X_t^{(2)}\big) dt + X_t^{(3)} b_3\big(t, X_t^{(2)}\big) dB_t, \quad t \in [0,T].$$

Let

$$g_1(t,x,y,z) := -ry - z\frac{a_1(t,x)-r}{b_1(t,x)},$$

and

$$g_2(t,x,y,z) := -ry - z^+\frac{a_2(t,x)-r}{b_2(t,x)} + z^-\frac{a_3(t,x)-r}{b_3(t,x)} + (R-r)\left(y - \frac{z^+}{b_2(t,x)} + \frac{z^-}{b_3(t,x)}\right)^-,$$

and assume that

$$\theta_2(t,x) := \frac{a_2(t,x)-r}{b_2(t,x)} \le \theta_3(t,x) := \frac{a_3(t,x)-r}{b_3(t,x)}, \qquad t\in[0,T],\ x>0. \tag{6.9}$$

Then, under the conditions

$$X_0^{(1)} = X_0^{(2)} \qquad \text{and} \qquad 0 < b_1(t,x) \le b_2(t,x), \quad t\in[0,T],\ x>0, \tag{6.10}$$

we have $X_T^{(1)} \le_{g_1,g_2}^{\mathrm{conv}} X_T^{(2)}$*, i.e.*

$$\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big] \le \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big],$$

for all convex functions $\phi(x)$ *satisfying (2.2), i.e., the fair price of the short-selling constrained portfolio is greater than that of the one without constraints.*

Proof. Under Condition (6.9) the model is without arbitrage by Theorems 3.1 and 3.2 in Jouini and Kallal (1995). In the optimal solution of Theorem 3.2 therein, the optimal hedging strategy at time t for the second investor is to long

$$\Delta_t^{(2)} := \left(\frac{\partial v}{\partial x}\big(t, X_t^{(2)}\big)\right)^+$$

units of $X_t^{(2)}$, and to short

$$\Delta_t^{(3)} := -\frac{b_2\big(t,X_t^{(2)}\big)X_t^{(2)}}{b_3\big(t,X_t^{(2)}\big)X_t^{(3)}}\left(\frac{\partial v}{\partial x}\big(t, X_t^{(2)}\big)\right)^-,$$

units of $X_t^{(3)}$, while longing $\big(\Delta_t^{(0)}\big)^+$ units of E_t, and shorting $-\big(\Delta_t^{(0)}\big)^-$ units of $\overline{E}_t$, where

$$\Delta_t^{(0)} := v\big(t,X_t^{(2)}\big) - X_t^{(2)}\Delta_t^{(2)} - X_t^{(3)}\Delta_t^{(3)}, \qquad t\in[0,T].$$

In other words, the discounted portfolio asset price processes $\widetilde{V}_t^{(i)} := e^{-rt}V_t^{(i)}$ and $\widetilde{X}_t^{(i)} := e^{-rt}X_t^{(i)}$ satisfy the BSDEs

$$\widetilde{V}_t^{(i)} = e^{-rT}\phi\big(e^{rT}\widetilde{X}_T^{(i)}\big) + \int_t^T \tilde{g}_i\big(s, \widetilde{X}_s^{(i)}, \widetilde{V}_s^{(i)}, \widetilde{Z}_s^{(i)}\big)ds - \int_t^T \widetilde{Z}_s^{(i)}dB_s, \quad i = 1, 2,$$

with

$$\widetilde{Z}_t^{(2)} := e^{-rt}X_t^{(2)}b_2\big(t, X_t^{(2)}\big)\frac{\partial v}{\partial x}\big(t, X_t^{(2)}\big), \qquad t \in [0, T],$$

and the generators $\tilde{g}_1(t, x, y, z) := -z\theta_1(t, xe^{rt})$ and

$$\tilde{g}_2(t, x, y, z) := -z\theta_2(t, xe^{rt}) + z^-\big(\theta_3(t, xe^{rt}) - \theta_2(t, xe^{rt})\big) + (R-r)\left(y - \frac{z^+}{b_2(t, xe^{rt})} + \frac{z^-}{b_3(t, xe^{rt})}\right)^-.$$

Hence the second portfolio price $V_t^{(2)} = v\big(t, X_t^{(2)}\big)$ satisfies the PDE

$$\begin{aligned}\frac{\partial v}{\partial t}(t, x) + rx\frac{\partial v}{\partial x}(t, x) + \frac{1}{2}x^2 b_2(t, x)\frac{\partial^2 v}{\partial x^2}(t, x) + x\left(\theta_3(t, x) - \theta_2(t, x)\right) b_2(t, x)\left(\frac{\partial v}{\partial x}(t, x)\right)^- \\ - rv(t, x) + (R - r)\left(v(t, x) - x\left(\frac{\partial v}{\partial x}(t, x)\right)^+ + x\frac{b_2(t, x)}{b_3(t, x)}\left(\frac{\partial v}{\partial x}(t, x)\right)^-\right)^- = 0.\end{aligned}$$

Then, by Conditions (6.9)-(6.10) we have

$$\begin{aligned} & xz(a_1(t, xe^{rt}) - r) + \tilde{g}_1(t, x, y, zxb_1(t, xe^{rt})) = 0 \\ \leq \; & xz^- b_2(t, x)\big(\theta_3(t, xe^{rt}) - \theta_2(t, xe^{rt})\big) + (R - r)\left(y - xz^+ + \frac{b_2(t, xe^{rt})}{b_3(t, xe^{rt})}xz^-\right)^- \\ = \; & xz(a_2(t, xe^{rt}) - r) + \tilde{g}_2(t, x, y, zxb_2(t, xe^{rt})), \qquad (x, y, z) \in \mathbb{R}_+ \times \mathbb{R}^2, \; t \in [0, T],\end{aligned}$$

hence (E_2) is satisfied. Condition (E_1) is satisfied with $\zeta(t, x) = 0$ hence, under Condition (6.10), Theorem 4.1 shows that

$$\widetilde{V}_0^{(1)} = \mathcal{E}_{\tilde{g}_1}\big[e^{-rT}\phi\big(e^{rT}\widetilde{X}_T^{(1)}\big)\big] \leq \widetilde{V}_0^{(2)} = \mathcal{E}_{\tilde{g}_2}\big[e^{-rT}\phi\big(e^{rT}\widetilde{X}_T^{(2)}\big)\big],$$

or

$$\mathcal{E}_{g_1}\big[\phi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{g_2}\big[\phi\big(X_T^{(2)}\big)\big],$$

with $g_i(t, x, y, z) = -ry + \tilde{g}_i(t, xe^{-rt}, y, z)$, $i = 1, 2$, for all convex functions $\phi(x)$ satisfying (2.2), that is $X_T^{(1)} \leq_{g_1,g_2}^{\mathrm{conv}} X_T^{(2)}$. Note that here, Theorem 3.1 cannot be applied since the function

$$(x, y) \longmapsto xz(a_2(t, xe^{rt}) - r) + \tilde{g}_2(t, x, y, zxb_2(t, xe^{rt}))$$

may not be convex, hence (B_2) is not satisfied. □

We note that the conclusions of the above examples also imply the comparison of terminal portfolios values

$$\mathcal{E}_g\big[u\big(V_T^{(1)}\big)\big] \le \mathcal{E}_g\big[u\big(V_T^{(2)}\big)\big],$$

for all non-decreasing convex utility and payoff functions u and ϕ, since $u \circ \phi(x)$ is also non-decreasing and convex, hence in this case the second portfolio would be preferred over the first portfolio by risk-seeking investors whose preferences are modeled by g.

7 Convexity of nonlinear PDE solutions

In this section, we extend the convexity result Theorem 1.1 in Bian and Guan (2008) for nonlinear PDEs under a weaker convexity condition on the nonlinear drift $(x, y, z) \mapsto f(t, x, y, z)$ in the one-dimensional case, as required by applications in finance, see the nonlinear Examples 6.3-6.5. For this we remark that, in our one-dimensional setting, the constant rank Theorem 2.3 in Bian and Guan (2008), see also Theorem 1.2 in Bian and Guan (2009), only requires convexity of the nonlinear drift $f(t, x, y, z)$ in $(x, y) \in \mathbb{R}^2$ for every $(t, z) \in [0, T] \times \mathbb{R}$, instead of global convexity in (x, y, z). Precisely, we note that Condition (2.6) in Theorem 2.3 of Bian and Guan (2008) reduces to (7.2) below.

Theorem 7.1 *(Theorem 2.3, Bian and Guan (2008)). Assume that $u(t, x)$ is a $\mathcal{C}^{2,4}\big([0, T) \times \mathbb{R}\big)$ convex solution of the PDE*

$$\frac{\partial u}{\partial t}(t, x) + F\Big(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), \frac{\partial^2 u}{\partial x^2}(t, x)\Big) = 0, \tag{7.1}$$

and that $F(t, x, y, z, w)$ is a $\mathcal{C}^{1,2}\big([0, T) \times \mathbb{R}^4\big)$ function that satisfies the elliptic condition

$$\frac{\partial F}{\partial w}\Big(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), \frac{\partial^2 u}{\partial x^2}(t, x)\Big) > 0, \quad t \in [0, T], \ x \in \mathbb{R},$$

and

$$\begin{aligned}\frac{\partial^2 F}{\partial x^2}\Big(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), 0\Big) &+ 2b\frac{\partial^2 F}{\partial x \partial y}\Big(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), 0\Big) \\ &+ b^2\frac{\partial^2 F}{\partial y^2}\Big(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), 0\Big) \ge 0,\end{aligned} \tag{7.2}$$

for all $t \in [0, T]$, $x \in \mathbb{R}$, and $b \in \mathbb{R}$. Then the sign $\operatorname{sgn}\left(\frac{\partial^2 u}{\partial x^2}(t, x)\right)$ of $\frac{\partial^2 u}{\partial x^2}(t, x)$ is constant in $x \in \mathbb{R}$ for any $t \in (0, T)$, and we have

$$\operatorname{sgn}\left(\frac{\partial^2 u}{\partial x^2}(s, x)\right) \ge \operatorname{sgn}\left(\frac{\partial^2 u}{\partial x^2}(t, x)\right), \qquad 0 \le s \le t < T.$$

We note that in our one-dimensional setting, the rank of $\frac{\partial^2 u}{\partial x^2}(t,x)$ is $\{0,1\}$-valued, with the relation $\operatorname{sgn}\Big(\frac{\partial^2 u}{\partial x^2}(t,x)\Big) \equiv \operatorname{rank}\Big(\frac{\partial^2 u}{\partial x^2}(t,x)\Big)$ provided that $x \mapsto u(t,x)$ convex in $x \in \mathbb{R}$, $t \in [0,T]$. By adapting arguments of Bian and Guan (2008), using the constant rank Theorem 7.1 and a new Lemma 7.3, we will prove the following Theorem 7.2 which has been used in the proofs of Theorems 3.1-3.2, Corollaries 3.4-3.5, and Theorems 4.1-4.2 and Corollaries 5.2-5.3.

We note that by Theorem 2.3 and Condition (2.6) in Bian and Guan (2008), convexity of the solution $u(t,x)$ of the PDE (2.11) is ensured by the joint convexity of $(x,y,z) \mapsto f(t,x,y,z) := F(t,x,y,z,0)$, where $F(t,x,y,z,w)$ is the semilinear function

$$F(t,x,y,z,w) := z\mu(t,x) + g\left(t,x,y,z\sigma(t,x)\right) + \frac{w}{2}\sigma^2(t,x).$$

However, this joint convexity condition is too strong for our applications in mathematical finance, and in Theorem 7.2 we show that it can be relaxed into the partial convexity of $F(t,x,y,z,0)$ in (x,y) and in (y,z). Namely, in Theorem 7.2 we extend Theorem 1.1 of Bian and Guan (2008) on the convexity of the solution $u(t,x)$ by only assuming convexity in (x,y) and in (y,z) of the function $f(t,x,y,z)$ in (7.3) below, instead of joint convexity in $(x,y,z) \mapsto f(t,x,y,z)$. For this, we start by assuming additional regularity conditions on coefficients, and then use approximation arguments with help of the continuous dependence (stability) Theorem 8.4, see Lemma 7.3.

Theorem 7.2 *Assume that the coefficients μ, σ, g and ϕ satisfy (A_1)-(A_4). Suppose that $u(t,x)$ is a $\mathcal{C}^{1,2}([0,T)\times\mathbb{R})$ solution of (2.11) with terminal condition $u(T,x)=\phi(x)$, and that the function*

$$f(t,x,y,z) := z\mu(t,x) + g\left(t,x,y,z\sigma(t,x)\right) \tag{7.3}$$

satisfies the following conditions:

(H_1) $(x,y) \mapsto f(t,x,y,z)$ is convex on $\mathbb{R}^2$ for every $(t,z) \in [0,T]\times\mathbb{R}$,

(H_2) $(y,z) \mapsto f(t,x,y,z)$ is convex on $\mathbb{R}^2$ for every $(t,x) \in [0,T]\times\mathbb{R}$.

Then the function $x \mapsto u(t,x)$ is convex on $\mathbb{R}$ for all $t\in[0,T]$, provided that $u(T,x)=\phi(x)$ convex in $x\in\mathbb{R}$.

Proof. We proceed by extending the proof argument of Theorem 1.1 in Bian and Guan (2008) by using Theorem 7.1 and an approximation argument. We start by assuming that the following conditions, which are stronger than (A_1)-(A_4), hold for some $\eta \in (0,1)$.

(H_3) $\mu(\cdot,\cdot)$, $\sigma(\cdot,\cdot)$, $g(\cdot,\cdot,y,z) \in \mathcal{C}_b^{1+\eta/2,2+\eta}\big([0,T]\times\mathbb{R}\big)$ for all $y,z\in\mathbb{R}$, and $\phi(\cdot)\in C_b^{4+\eta}(\mathbb{R})$,

(H_4) $\sigma(\cdot,\cdot)$ satisfies the bound

$$0 < c \le \sigma(t,x) \le C, \qquad t\in[0,T], \quad x\in\mathbb{R}, \tag{7.4}$$

for some constants $c, C > 0$.

(H_5) For some $C > 0$ and $\alpha > 0$ we have

$$\left|\frac{\partial^2 g}{\partial z^2}(t,x,y,z)\right| \le \frac{C}{(1+x^2)^{\alpha+1}}, \qquad (t,x,y,z)\in[0,T]\times\mathbb{R}^3.$$

Under Conditions (H_3)-(H_4) the function $F(t,y,z,w)$ defined as

$$F(t,x,y,z,w) := f(t,x,y,z) + \frac{w}{2}\sigma^2(t,x), \quad t\in[0,T),\ x,\ y,\ z,\ w\in\mathbb{R},$$

is in $\mathcal{C}^{1,2}\big([0,T)\times\mathbb{R}^4\big)$ and the solution $u(t,x)$ of (7.1) is in $\mathcal{C}^{2,4}([0,T)\times\mathbb{R})$ by Theorem 2.7. Besides, we note that by Theorem 2.8, the first and the second partial derivatives of $u(t,x)$ with respect to x are bounded uniformly in $(t,x)\in[0,T]\times\mathbb{R}$. Setting $h(x) := (1+x^2)^{\alpha+1}$, for any $K\in\mathbb{R}$ and $\varepsilon>0$, we define

$$v_K(t,x) := e^{-Kt}h(x) \quad \text{and} \quad u_\varepsilon(t,x) := u(t,x) + \varepsilon v_K(t,x), \qquad x\in\mathbb{R}, \quad t\in[0,T].$$

Next, we let

$$E_\varepsilon := \left\{(t,x)\in[0,T]\times\mathbb{R} \ : \ \frac{\partial^2 u_\varepsilon}{\partial x^2}(t,x) \le 0\right\}$$

and suppose that $E_\varepsilon \neq \emptyset$. From the relation $h''(x) \ge (1+x^2)^\alpha$ and the bound $\left|\frac{\partial^2 u}{\partial x^2}(t,x)\right| \le C$ we get

$$\frac{\partial^2 u_\varepsilon}{\partial x^2}(t,x) \ge \varepsilon e^{-Kt}(1+x^2)^\alpha - C,$$

therefore there exists $R_\varepsilon > 0$ such that $\frac{\partial^2 u_\varepsilon}{\partial x^2}(t,x) > 0$ for all $|x| \ge R_\varepsilon$, and we have $E_\varepsilon \subseteq [0,T]\times B(0,R_\varepsilon)$, where $B(0,R_\varepsilon)$ is the centered open ball with radius R_ε, so that E_ε is compact. Consequently, the supremum

$$\tau_0 := \sup\{t\in[0,T] \ : \ (t,x)\in E_\varepsilon \text{ for some } x\in\mathbb{R}\}$$

is attained at some $(\tau_0,x_0)\in E_\varepsilon$ with $x_0\in B(0,R_\varepsilon)$, such that $\frac{\partial^2 u_\varepsilon}{\partial x^2}(\tau_0,x_0) \le 0$. In addition, by the convexity assumption on $x\mapsto u(T,x)$ we have

$$\frac{\partial^2 u_\varepsilon}{\partial x^2}(T,x) = \frac{\partial^2 u}{\partial x^2}(T,x) + \varepsilon\frac{\partial^2 v_K}{\partial x^2}(T,x) \ge \varepsilon e^{-KT}h''(x) > 0, \qquad x\in\mathbb{R},$$

hence $\tau_0 < T$ and by the continuity of u_ε we have $\dfrac{\partial^2 u_\varepsilon}{\partial x^2}(\tau_0, x) \geq 0$, $x \in \mathbb{R}$, since $\dfrac{\partial^2 u_\varepsilon}{\partial x^2}(t, x) > 0$ for all $t \in (\tau_0, T)$ and $x \in \mathbb{R}$. Consequently, the function $u_\varepsilon(t, x)$ is convex in x on $[\tau_0, T] \times B(0, R_\varepsilon)$.

On the other hand, we note that $\dfrac{\partial^2 u_\varepsilon}{\partial x^2}(\tau_0, x_0) = 0$ for $x_0 \in B(0, R_\varepsilon)$, and that $u_\varepsilon(t, x)$ satisfies the equation

$$\frac{\partial u_\varepsilon}{\partial t}(t, x) + F_{K,\varepsilon}\left(t, x, u_\varepsilon(t, x), \frac{\partial u_\varepsilon}{\partial x}(t, x), \frac{\partial^2 u_\varepsilon}{\partial x^2}(t, x)\right) = 0,$$

where

$$\begin{aligned} F_{K,\varepsilon}(t, x, y, z, w) \ &:= \ -\varepsilon \frac{\partial v_K}{\partial t}(t, x) + \frac{1}{2}\sigma^2(t, x)\left(w - \varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x)\right) \\ &\quad + f\left(t, x, y - \varepsilon v_K(t, x), z - \varepsilon \frac{\partial v_K}{\partial x}(t, x)\right). \end{aligned} \tag{7.5}$$

By the constant rank Theorem 7.1 and Lemma 7.3 below, we deduce that for small enough $T = T(\varepsilon, \alpha) > 0$ the second derivative $\dfrac{\partial^2 u_\varepsilon}{\partial x^2}(t, x)$ vanishes on $[\tau_0, T) \times B(0, R_\varepsilon)$ hence $\tau_0 = T$, which is a contradiction showing that $E_\varepsilon = \emptyset$. Therefore we have

$$\frac{\partial^2 u}{\partial x^2}(t, x) + \varepsilon \frac{\partial^2 v_K}{\partial x^2}(t, x) = \frac{\partial^2 u_\varepsilon}{\partial x^2}(t, x) > 0, \qquad (t, x) \in [0, T] \times \mathbb{R},$$

and after letting ε tend to 0, we conclude that

$$\frac{\partial^2 u}{\partial x^2}(t, x) \geq 0, \qquad (t, x) \in [0, T] \times \mathbb{R},$$

for small enough $T > 0$. This conclusion extends to all $T > 0$ by decomposing $[0, T]$ into subintervals of lengths at most $T(\varepsilon, \alpha) > 0$. Finally, we relax the above Conditions (H_3)-(H_5) under the hypotheses (A_1)-(A_4) by applying the above argument to sequences $(\mu_n)_{n\geq 1}$, $(\sigma_n)_{n\geq 1}$, $(g_n)_{n\geq 1}$ of $\mathcal{C}_b^{2,3}$ functions and $(\phi)_{n\geq 1}$ of $\mathcal{C}_b^5$ functions satisfying (H_3) and (A_1)-(A_4), and converging pointwise respectively to μ, σ, g, and ϕ while preserving the convexity of the approximations $(\phi_n)_{n\geq 1}$ and $(f_n)_{n\geq 1}$ defined in (7.3), as well as Condition (7.4). In order to satisfy (H_5), we replace g_n with $\tilde{g}_n$ obtained by smoothing out the piecewise $\mathcal{C}^1$ function

$$\begin{aligned} x \ \mapsto \ &\left(g_n(t, -n, y, z) + (x + n)\frac{\partial g_n}{\partial x}(t, -n, y, z)\right) \mathbf{1}_{(-\infty, -n)}(x) \\ &+ g_n(t, x, y, z)\mathbf{1}_{[-n, n]}(x) + \left(g_n(t, n, y, z) + (x - n)\frac{\partial g_n}{\partial x}(t, n, y, z)\right) \mathbf{1}_{(n, \infty)}(x), \end{aligned}$$

by convolution in x with the Gaussian kernel $e^{-nx^2/2}/\sqrt{2\pi/n}$, and we conclude by the continuous dependence Proposition 8.4. $\square$

The proof of Theorem 7.2 relies on the following lemma.

Lemma 7.3 *Under Conditions (H_1)-(H_5) above, for $T = T(\varepsilon, \alpha) > 0$ small enough we can choose $K \in \mathbb{R}$ such that the function $(x, y) \mapsto F_{K,\varepsilon}(t, x, y, z, 0)$ in (7.5) satisfies Condition (7.2).*

Proof. We need to show that

$$\begin{aligned} S(b) \; := \; & \frac{\partial^2 F_{K,\varepsilon}}{\partial x^2}\Big(t, x, u_\varepsilon(t,x), \frac{\partial u_\varepsilon}{\partial x}(t,x), 0\Big) + 2b \frac{\partial^2 F_{K,\varepsilon}}{\partial x \partial y}\Big(t, x, u_\varepsilon(t,x), \frac{\partial u_\varepsilon}{\partial x}(t,x), 0\Big) \\ & + b^2 \frac{\partial^2 F_{K,\varepsilon}}{\partial y^2}\Big(t, x, u_\varepsilon(t,x), \frac{\partial u_\varepsilon}{\partial x}(t,x), 0\Big) \\ \geq \; & 0, \end{aligned}$$

for all $b \in \mathbb{R}$. By (7.5), we have

$$\begin{aligned} S(b) = & \, 2\frac{\partial^2 f}{\partial x^2} - 2\Big(\varepsilon \frac{\partial v_K}{\partial x}(t,x) - b\Big)\frac{\partial^2 f}{\partial x \partial y} + \frac{1}{2}\Big(\varepsilon \frac{\partial v_K}{\partial x}(t,x) - b\Big)^2 \frac{\partial^2 f}{\partial y^2} \\ & + \frac{1}{2}\Big(\varepsilon \frac{\partial v_K}{\partial x}(t,x) - b\Big)^2 \frac{\partial^2 f}{\partial y^2} + 2\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t,x)\Big(\varepsilon \frac{\partial v_K}{\partial x}(t,x) - b\Big)\frac{\partial^2 f}{\partial y \partial z} + 2\Big(\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t,x)\Big)^2 \frac{\partial^2 f}{\partial z^2} \\ & - \frac{\partial^2 f}{\partial x^2} - \Big(\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t,x)\Big)^2 \frac{\partial^2 f}{\partial z^2} - \varepsilon\left(\frac{1}{2}\frac{\partial^2 \sigma^2}{\partial x^2}(t,x) + \frac{\partial f}{\partial y} + 2\frac{\partial^2 f}{\partial x \partial z}\right)\frac{\partial^2 v_K}{\partial x^2}(t,x) \\ & - \varepsilon \frac{\partial^3 v_K}{\partial x^2 \partial t}(t,x) - \varepsilon\left(\frac{\partial f}{\partial z} + 2\sigma(t,x)\frac{\partial \sigma}{\partial x}(t,x)\right)\frac{\partial^3 v_K}{\partial x^3}(t,x) - \frac{\varepsilon}{2}\sigma^2(t,x)\frac{\partial^4 v_K}{\partial x^4}(t,x), \end{aligned}$$

for all $b \in \mathbb{R}$, where the derivatives of f are evaluated at the point $\big(t, x, u_\varepsilon(t,x), \frac{\partial u_\varepsilon}{\partial x}(t,x)\big)$. Since $f(t,x,y,z)$ is convex in (x,y), and in (y,z), we have

$$2\frac{\partial^2 f}{\partial x^2} - 2\Big(\varepsilon \frac{\partial v_K}{\partial x}(t,x) - b\Big)\frac{\partial^2 f}{\partial x \partial y} + \frac{1}{2}\Big(\varepsilon \frac{\partial v_K}{\partial x}(t,x) - b\Big)^2 \frac{\partial^2 f}{\partial y^2} \geq 0,$$

and

$$\frac{1}{2}\Big(\varepsilon \frac{\partial v_K}{\partial x}(t,x) - b\Big)^2 \frac{\partial^2 f}{\partial y^2} + 2\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t,x)\Big(\varepsilon \frac{\partial v_K}{\partial x}(t,x) - b\Big)\frac{\partial^2 f}{\partial y \partial z} + 2\Big(\varepsilon \frac{\partial^2 v_K}{\partial x^2}(t,x)\Big)^2 \frac{\partial^2 f}{\partial z^2} \geq 0,$$

for all $b \in \mathbb{R}$. To conclude it suffices to show the inequality

$$\begin{aligned} S' \; := \; & -\frac{\partial^2 f}{\partial x^2} - \varepsilon\Big(\frac{\partial^2 v_K}{\partial x^2}(t,x)\Big)^2 \frac{\partial^2 f}{\partial z^2} - \left(\frac{1}{2}\frac{\partial^2 \sigma^2}{\partial x^2}(t,x) + \frac{\partial f}{\partial y} + 2\frac{\partial^2 f}{\partial x \partial z}\right)\frac{\partial^2 v_K}{\partial x^2}(t,x) \\ & - \frac{\partial^3 v_K}{\partial x^2 \partial t}(t,x) - \left(\frac{\partial f}{\partial z} + 2\sigma(t,x)\frac{\partial \sigma}{\partial x}(t,x)\right)\frac{\partial^3 v_K}{\partial x^3}(t,x) - \frac{1}{2}\sigma^2(t,x)\frac{\partial^4 v_K}{\partial x^4}(t,x) \\ \geq \; & 0, \end{aligned}$$

Thanks to Conditions (H_3)-(H_4), we find that there exists $C > 0$ such that

$$\Big|\frac{\partial^2 f}{\partial x^2}(t,x,y,z)\Big| \leq C\left(1 + z^2\Big|\frac{\partial^2 g}{\partial x^2}(t,x,y,z\sigma(t,x))\Big|\right), \qquad \Big|\frac{\partial f}{\partial y}(t,x,y,z)\Big| \leq C,$$

and

$$\left|\frac{\partial^2 f}{\partial x \partial z}(t,x,y,z)\right| \leq C\left(1+|z|\left|\frac{\partial^2 g}{\partial z^2}(t,x,y,z)\right|\right), \qquad \left|\frac{\partial f}{\partial z}(t,x,y,z)\right| \leq C(1+|x|),$$

for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^3$, hence by Condition ($H_5$) for some $C' > 0$ and $C'(\varepsilon,\alpha) > 0$ we have

$$\begin{aligned} \left|\frac{\partial^2 f}{\partial x^2}\left(t,x,u_\varepsilon(t,x),\frac{\partial u_\varepsilon}{\partial x}(t,x)\right)\right| &\leq C + \frac{C}{(1+x^2)^{\alpha+1}}\Big(\frac{\partial u_\varepsilon}{\partial x}(t,x)\Big)^2 \\ &\leq C' + C'(\varepsilon,\alpha)e^{-2Kt}(1+x^2)^\alpha \end{aligned} \tag{7.6}$$

and

$$\begin{aligned} \left|\frac{\partial^2 f}{\partial x \partial z}\left(t,x,u_\varepsilon(t,x),\frac{\partial u_\varepsilon}{\partial x}(t,x)\right)\right| &\leq C + \frac{C}{(1+x^2)^{\alpha+1}}\Big|\frac{\partial u_\varepsilon}{\partial x}(t,x)\Big| \\ &\leq C' + C'(\varepsilon,\alpha)e^{-Kt}, \end{aligned} \tag{7.7}$$

$(t,x) \in [0,T] \times \mathbb{R}$. By the relation $v_K(t,x) := e^{-Kt}(1+x^2)^{\alpha+1}$ and Condition (H_5), we find

$$\left|\left(\frac{\partial^2 v_K}{\partial x^2}(t,x)\right)^2 \frac{\partial^2 f}{\partial z^2}\Big(t,x,u_\varepsilon(t,x),\frac{\partial u_\varepsilon}{\partial x}(t,x)\Big)\right| \leq C(\alpha)e^{-2Kt}(1+x^2)^\alpha, \tag{7.8}$$

for some constant $C(\alpha) > 0$. Next, we note that for some $C'(\alpha) > 0$ we have

$$\left|\frac{\partial^2 v_K}{\partial x^2}(t,x)\right| \leq C'(\alpha)e^{-Kt}(1+x^2)^\alpha, \qquad \left|\frac{\partial^3 v_K}{\partial x^3}(t,x)\right| \leq C'(\alpha)e^{-Kt}(1+|x|)(1+x^2)^{\alpha-1},$$

and

$$\left|\frac{\partial^4 v_K}{\partial x^4}(t,x)\right| \leq C'(\alpha)e^{-Kt}(1+x^2)^{\alpha-1},$$

hence from (H_3), (7.4) and (7.7) we check that

$$\left|\frac{1}{2}\frac{\partial^2 \sigma^2}{\partial x^2} + \frac{\partial f}{\partial y} + 2\frac{\partial^2 f}{\partial x \partial z}\right|\left|\frac{\partial^2 v_K}{\partial x^2}(t,x)\right| \leq C'(\varepsilon,\alpha)e^{-Kt}(1+x^2)^\alpha.$$

Similarly, thanks to the conditions (H_3) and (7.4)

$$\left|\frac{\partial f}{\partial z} + 2\sigma(t,x)\frac{\partial \sigma}{\partial x}(t,x)\right|\left|\frac{\partial^3 v_K}{\partial x^3}(t,x)\right| \leq C''(\alpha)e^{-Kt}(1+x^2)^\alpha,$$

and

$$\frac{1}{2}\sigma^2(t,x)\left|\frac{\partial^4 v_K}{\partial x^4}(t,x)\right| \leq C''(\alpha)e^{-Kt}(1+x^2)^\alpha. \tag{7.9}$$

Combining (7.8)-(7.9), the inequality $\dfrac{\partial^3 v_K}{\partial x^2 \partial t}(t,x) \leq -KC'''(\alpha)e^{-Kt}(1+x^2)^\alpha$ for some constant $C'''(\alpha) > 0$, and (7.6), and letting $K := 1/T$ we obtain

$$S' \geq (e^{-KT}KC'''(\alpha) - 2C''(\alpha) - 2C'(\varepsilon,\alpha) - C(\alpha))(1+x^2)^\alpha - C'$$

$$\geq \left(\frac{C'''(\alpha)}{eT} - 2C''(\alpha) - 2C'(\varepsilon, \alpha) - C(\alpha) \right) (1 + x^2)^\alpha - C'$$
$$\geq 0, \qquad (t,x) \in [0,T] \times \mathbb{R},$$

for small enough $T = T(\alpha, \varepsilon)$. $\square$

8 Monotonicity and continuous dependence results

Monotonicity of FBSDEs

In Proposition 8.1 and Proposition 8.2 we prove the monotonicity results needed in the proofs of Theorem 3.2, Corollaries 3.3-3.4, 3.6 and Theorem 4.2. We apply Lemma 8.3 below to derive monotonicity results for FBSDE flows of the form

$$\begin{cases} dX_s^{t,x} = \mu\big(t, X_s^{t,x}\big)ds + \sigma\big(s, X_t^{t,x}\big)dB_s, \quad X_t^{t,x} = x, & \text{(8.1a)} \\ \\ dY_s^{t,x} = -g\big(s, X_s^{s,x}, Y_s^{t,x}, Z_s^{t,x}\big)ds + Z_s^{t,x}dB_s, \quad Y_T^{t,x} = \phi(X_T^{t,x}), & \text{(8.1b)} \end{cases}$$

$0 \leq t \leq s \leq T$. We first prove a monotonicity result for the solution $(X_s^{t,x})_{s\in[t,T]}$ of the SDE (8.1a), which will be used to prove non-decreasing property of $Y_s^{t,x}$ and $u(t,x)$ in Proposition 8.2.

Proposition 8.1 *Under the assumption (A_1) the solution $\big(X_s^{t,x}\big)_{s\in[t,T]}$ of (8.1a) is a.s. non-decreasing in x for all $t \in [0,T]$ and $s \in [t,T]$.*

Proof. Let $\widehat{X}_s^{t,x,y} := X_s^{t,y} - X_s^{t,x}$ for $x \leq y$, $s \in [t,T]$, and consider the processes

$$\widehat{\mu}_u := \frac{\mu(u, X_u^{t,y}) - \mu(u, X_u^{t,x})}{X_u^{t,y} - X_u^{t,x}} \mathbf{1}_{\{X_u^{t,y} \neq X_u^{t,x}\}} \quad \text{and} \quad \widehat{\sigma}_u := \frac{\sigma(u, X_u^{t,y}) - \sigma(u, X_u^{t,x})}{X_u^{t,y} - X_u^{t,x}} \mathbf{1}_{\{X_u^{t,y} \neq X_u^{t,x}\}},$$

$u \in [t,T]$. We note that the processes $(\widehat{\mu}_u)_{u\in[t,T]}$ and $(\widehat{\sigma}_u)_{u\in[t,T]}$ are bounded since $\mu(t,x)$ and $\sigma(t,x)$ are Lipschitz in x, and that $(\widehat{X}_s^{t,x,y})_{s\in[t,T]}$ satisfies the equation

$$\widehat{X}_s^{t,x,y} = y - x + \int_t^s \widehat{\mu}_u \widehat{X}_u^{t,x,y} du + \int_t^s \widehat{\sigma}_u \widehat{X}_u^{t,x,y} dB_u, \qquad s \in [t,T],$$

which yields

$$\widehat{X}_s^{t,x,y} = (y - x) \exp\left(\int_t^s \widehat{\mu}_u du + \int_t^s \widehat{\sigma}_u dB_u - \frac{1}{2} \int_t^s \widehat{\sigma}_u^2 du \right) \geq 0, \quad 0 \leq t \leq s \leq T.$$

$\square$

Monotonicity of nonlinear PDE solutions

The next monotonicity result is used for the proofs of Theorem 3.2, Corollaries 3.3-3.4, 3.6 and Theorem 4.2.

Proposition 8.2 *Assume that the coefficients μ, σ, g and ϕ satisfy (A_1)-(A_4). If $\phi(x)$ and $g(t,x,y,z)$ are non-decreasing in $x \in \mathbb{R}$ for all $t \in [0,T]$ and $y,z \in \mathbb{R}$, then the solution $\big(Y_s^{t,x}\big)_{s\in[t,T]}$ of (8.1b) is a.s. non-decreasing in x for all $s \in [t,T]$. As a consequence, if $u(t,x)$ is solution of the backward PDEs (2.11), then $u(t,x)$ is also a non-decreasing function of $x \in \mathbb{R}$ for all $t \in [0,T]$.*

Proof. Letting $\widehat{X}_s = X_s^{t,y} - X_s^{t,x}$, $\widehat{Y}_s = Y_s^{t,y} - Y_s^{t,x}$, $\widehat{Z}_s = Z_s^{t,y} - Z_s^{t,x}$ and $\widehat{Y}_T = \phi_2\big(X_T^{t,y}\big) - \phi_1\big(X_T^{t,x}\big)$, we have

$$\widehat{Y}_s = \widehat{Y}_T + \int_s^T \big(g\big(u,X_u^{t,y},Y_u^{t,y},Z_u^{t,y}\big) - g\big(u,X_u^{t,x},Y_u^{t,x},Z_u^{t,x}\big)\big)du - \int_s^T \widehat{Z}_u dB_u.$$

Defining the processes a_u, b_u, and c_u as

$$\begin{aligned}
a_u &:= \frac{g\big(u,X_u^{t,y},Y_u^{t,y},Z_u^{t,y}\big) - g\big(u,X_u^{t,x},Y_u^{t,y},Z_u^{t,y}\big)}{X_u^{t,y} - X_u^{t,x}} \mathbf{1}_{\{X_u^{t,y} \neq X_u^{t,x}\}}, \\
b_u &:= \frac{g\big(u,X_u^{t,x},Y_u^{t,y},Z_u^{t,y}\big) - g\big(u,X_u^{t,x},Y_u^{t,x},Z_u^{t,y}\big)}{Y_u^{t,y} - Y_u^{t,x}} \mathbf{1}_{\{Y_u^{t,y} \neq Y_u^{t,x}\}}, \\
c_u &:= \frac{g\big(u,X_u^{t,x},Y_u^{t,x},Z_u^{t,y}\big) - g\big(u,X_u^{t,x},Y_u^{t,x},Z_u^{t,x}\big)}{Z_u^{t,y} - Z_u^{t,x}} \mathbf{1}_{\{Z_u^{t,x} \neq Z_u^{t,x}\}}, \qquad u \in [0,T],
\end{aligned}$$

which are $(\mathcal{F}_t)_{t\in[0,T]}$-adapted and bounded since $g(u,x,y,z)$ is Lipschitz, and using the decomposition

$$\begin{aligned}
&g\left(u,X_u^{t,y},Y_u^{t,y},Z_u^{t,y}\right) - g\big(u,X_u^{t,x},Y_u^{t,x},Z_u^{t,x}\big) \\
&= g\left(u,X_u^{t,y},Y_u^{t,y},Z_u^{t,y}\right) - g\left(u,X_u^{t,x},Y_u^{t,y},Z_u^{t,y}\right) + g\left(u,X_u^{t,x},Y_u^{t,y},Z_u^{t,y}\right) - g\left(u,X_u^{t,x},Y_u^{t,x},Z_u^{t,y}\right) \\
&\quad + g\big(u,X_u^{t,x},Y_u^{t,x},Z_u^{t,y}\big) - g\big(u,X_u^{t,x},Y_u^{t,x},Z_u^{t,x}\big),
\end{aligned}$$

we have

$$\widehat{Y}_s = \widehat{Y}_T + \int_s^T \big(a_u\widehat{X}_u + b_u\widehat{Y}_u + c_u\widehat{Z}_u\big)du - \int_s^T \widehat{Z}_u dB_u, \qquad s \in [0,T].$$

Hence, by Lemma 8.3 below we get

$$\widehat{Y}_s = \frac{1}{\Gamma_s}\mathbb{E}\left[\Gamma_T\widehat{Y}_T + \int_s^T a_u\widehat{X}_u\Gamma_u du \;\middle|\; \mathcal{F}_s\right], \qquad s \in [t,T], \tag{8.2}$$

where

$$\Gamma_s := \exp\left(\int_t^s c_u dB_u - \frac{1}{2}\int_t^s c_u^2 du + \int_0^s b_u du\right), \qquad s \in [t,T].$$

By Proposition 8.1 the solution $(X_s^{t,x})_{s\in[t,T]}$ of the forward SDE (8.1a) satisfies $\widehat{X}_s = X_s^{t,y} - X_s^{t,x} \geq 0$ for all $s \in [t,T]$ if $x \leq y$, and since $g(s,x,y,z)$ is non-decreasing in x we have $a_s \geq 0$ a.s., $s \in [t,T]$. Since $\phi(x)$ is non-decreasing we have $\widehat{Y}_T = \phi\left(X_T^{t,x}\right) - \phi\left(X_T^{t,y}\right) \geq 0$ a.s., hence by (8.2) we have $\widehat{Y}_s = Y_s^{t,x} - Y_s^{t,y} \geq 0$, $s \in [t,T]$, if $x \leq y$, which implies the monotonicity of $\left(Y_s^{t,x}\right)_{s\in[t,T]}$, therefore we also get $u(t,x) \leq u(t,y)$, $x \leq y$, $t \in [0,T]$, since $u(t,x) = Y_t^{t,x}$.

□

Linear FBSDEs

The following Lemma 8.3, which has been used in the proof of Proposition 8.2, extends a classical result from linear BSDEs to linear FBSDEs. Let $(X_t)_{t\in[0,T]}$ satisfy the forward diffusion equation

$$dX_t = \mu\,(t, X_t)\,dt + \sigma\,(t, X_t)\,dB_t, \tag{8.3}$$

where μ, σ satisfy (A_1), with associated linear backward SDE

$$dY_t = -\left(a_t X_t + b_t Y_t + c_t Z_t + k_t\right)dt + Z_t dB_t, \tag{8.4}$$

with terminal condition $Y_T = \phi(X_T)$, where $(a_t)_{t\in[0,T]}$, $(b_t)_{t\in[0,T]}$ and $(c_t)_{t\in[0,T]}$ are real-valued, $(\mathcal{F}_t)_{t\in[0,T]}$-adapted bounded processes, and $(k_t)_{t\in[0,T]}$ is a real-valued $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process such that

$$\mathbb{E}\left[\int_0^T k_t^2 dt\right] < \infty.$$

Lemma 8.3 *Let $(X_t)_{t\in[0,T]}$ be the solution of (8.3), and let $(Y_t, Z_t)_{t\in[0,T]}$ be the solution of (8.4). Then the process $(Y_t)_{t\in[0,T]}$ is given in explicit form as*

$$Y_t = \frac{1}{\Gamma_t}\mathbb{E}\left[\Gamma_T \phi\left(X_T\right) + \int_t^T \left(a_s X_s + k_s\right)\Gamma_s ds \,\middle|\, \mathcal{F}_t\right],$$

where $(\Gamma_t)_{t\in[0,T]}$ is the geometric Brownian motion

$$\Gamma_t := \exp\left(\int_0^t b_s ds + \int_0^t c_s dB_s - \frac{1}{2}\int_0^t c_s^2 ds\right), \qquad t \in [0,T].$$

Proof. We have

$$d(\Gamma_s Y_s) \;=\; \Gamma_s dY_s + Y_s d\Gamma_s + d\langle \Gamma_s, Y_s\rangle$$

$$\begin{aligned}
&= \Gamma_s \left(-\left(a_s X_s + b_s Y_s + c_s Z_s + k_s\right) ds + Z_s dB_s\right) + Y_s \Gamma_s \left(b_s ds + c_s dB_s\right) + c_s Z_s \Gamma_s ds \\
&= -\left(a_s X_s \Gamma_s + k_s \Gamma_s\right) ds + \left(c_s Y_s \Gamma_s + Z_s \Gamma_s\right) dB_s,
\end{aligned}$$

hence

$$\Gamma_T Y_T - \Gamma_t Y_t = -\int_t^T \left(a_s X_s \Gamma_s + k_s \Gamma_s\right) ds + \int_t^T \left(c_s Y_s \Gamma_s + Z_s \Gamma_s\right) dB_s, \tag{8.5}$$

and by taking conditional expectation on both sides of (8.5) we find

$$\begin{aligned}
Y_t &= \mathbb{E}\left[Y_t \mid \mathcal{F}_t\right] \\
&= \frac{1}{\Gamma_t}\mathbb{E}\left[\Gamma_T Y_T \mid \mathcal{F}_t\right] + \frac{1}{\Gamma_t}\mathbb{E}\left[\int_t^T \left(a_s X_s \Gamma_s + k_s \Gamma_s\right) ds \,\middle|\, \mathcal{F}_t\right] \\
&= \frac{1}{\Gamma_t}\mathbb{E}\left[\Gamma_T \phi\left(X_T\right) + \int_t^T \left(a_s X_s + k_s\right) \Gamma_s ds \,\middle|\, \mathcal{F}_t\right], \qquad t \in [0, T].
\end{aligned}$$

□

Continuous dependence of FBSDE solutions

The next Proposition 8.4 result extends the argument of Theorem 9.7 in Mishura and Shevchenko (2017) to the setting of FBSDEs. Other continuous dependence results are available in the literature such as Theorem 3.3 of Jakobsen and Karlsen (2002), which however requires uniform estimates on coefficients.

Proposition 8.4 *Consider the family of forward-backward stochastic differential equations*

$$\begin{cases}
X_{n,t} = X_{n,0} + \displaystyle\int_0^t \mu_n(s, X_{n,s}) ds + \int_0^t \sigma_n(s, X_{n,s}) dB_s, \\[2ex]
Y_{n,t} = \phi_n(X_{n,T}) + \displaystyle\int_t^T g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) ds - \int_t^T Z_{n,s} dB_s,
\end{cases}$$

where, for every $n \geq 1$, the coefficients μ_n, σ_n, g_n and ϕ_n satisfy (A_1)-(A_4) for a same $C > 0$. Assume the pointwise convergences $X_{n,0} \to X_0$ and

$$\mu_n(t, x) \to \mu(t, x), \quad \sigma_n(t, x) \to \sigma(t, x), \quad g_n(t, x, y, z) \to g(t, x, y, z),$$

for all $t \in [0, T]$, and $x, y, z \in \mathbb{R}$ as $n \to \infty$, and the strong convergence $\phi_n(x_n) \to \phi(x)$ whenever $x_n \to x \in \mathbb{R}$, where μ, σ, g and ϕ satisfy (A_1)-(A_4) for a same constant $C > 0$. Then for all $t \in [0, T]$ we have

$$\lim_{n \to \infty} \mathbb{E}\left[\left|Y_{n,t} - Y_t\right|^2\right] = 0,$$

where $(Y_t)_{t\in\mathbb{R}_+}$ *is solution of the FBSDE*

$$\begin{cases} X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \\ Y_t = \phi(X_T) + \int_t^T g(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dB_s. \end{cases}$$

Proof. For $n \geq 1$, let

$$\widehat{Y}_{n,t} := Y_{n,t} - Y_t, \quad \widehat{X}_{n,t} := X_{n,t} - X_t, \quad \widehat{Z}_{n,t} := Z_{n,t} - Z_t, \quad \text{and} \quad \widehat{\phi}_n(x) := \phi_n(x) - \phi(x),$$

$t \in [0, T]$, $x \in \mathbb{R}$. Applying the Itô formula to $|\widehat{Y}_{n,t}|^2$ and taking expectation on both sides yields

$$\begin{aligned} \mathbb{E}\big[|\widehat{Y}_{n,t}|^2\big] &= \mathbb{E}\big[\big(\phi_n(X_{n,T}) - \phi(X_T)\big)^2\big] \\ &\quad +2\mathbb{E}\left[\int_t^T \widehat{Y}_{n,s}(g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - g(s, X_s, Y_s, Z_s))ds\right] - \mathbb{E}\left[\int_t^T |\widehat{Z}_{n,s}|^2 ds\right]. \end{aligned}$$

By the inequality $2ab \leq (6C^2)a^2 + b^2/(6C^2)$, we have

$$\begin{aligned} 2\widehat{Y}_{n,s}(g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - g(s, X_s, Y_s, Z_s)) &\leq 6C^2|\widehat{Y}_{n,s}|^2 \\ &\quad + \frac{1}{6C^2}(g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - g(s, X_s, Y_s, Z_s))^2. \end{aligned}$$

Next, letting

$$\widehat{g}_n(t, x, y, z) := g_n(t, x, y, z) - g(t, x, y, z), \qquad t \in [0, T], \quad x, y, z \in \mathbb{R},$$

we have

$$\begin{aligned} &(g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - g(s, X_s, Y_s, Z_s))^2 \\ &\leq 2(g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - g_n(s, X_s, Y_s, Z_s))^2 + 2(\widehat{g}_n(s, X_s, Y_s, Z_s))^2 \\ &\leq 2C^2(|X_{n,s} - X_s| + |Y_{n,s} - Y_s| + |Z_{n,s} - Z_s|)^2 + 2(\widehat{g}_n(s, X_s, Y_s, Z_s))^2 \\ &\leq 6C^2(|\widehat{X}_{n,s}|^2 + |\widehat{Y}_{n,s}|^2 + |\widehat{Z}_{n,s}|^2) + 2(\widehat{g}_n(s, X_s, Y_s, Z_s))^2, \end{aligned}$$

by the inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$. Combining the above estimates, we find

$$\begin{aligned} \mathbb{E}\big[|\widehat{Y}_{n,t}|^2\big] &\leq \mathbb{E}\big[\big(\phi_n(X_{n,T}) - \phi(X_T)\big)^2\big] \\ &\quad +(6C^2 + 1)\mathbb{E}\left[\int_t^T |\widehat{Y}_{n,s}|^2 ds\right] + \mathbb{E}\left[\int_t^T |\widehat{X}_{n,s}|^2 ds\right] \\ &\quad + \frac{1}{3C^2}\mathbb{E}\left[\int_t^T \big(\widehat{g}_n(s, X_s, Y_s, Z_s)\big)^2 ds\right], \end{aligned}$$

which yields

$$\mathbb{E}\big[|\widehat{Y}_{n,t}|^2\big] \leq C'' \left(\mathbb{E}\big[\big(\phi_n(X_{n,T}) - \phi(X_T)\big)^2\big] + \int_t^T \mathbb{E}\big[|\widehat{X}_{n,s}|^2\big] ds + \mathbb{E}\left[\int_t^T \big(\widehat{g}_n(s, X_s, Y_s, Z_s)\big)^2 ds\right]\right)$$

by Gronwall's inequality, and therefore

$$\begin{aligned}\mathbb{E}\big[|\widehat{Y}_{n,t}|^2\big] &\leq C''\mathbb{E}\big[\big(\phi_n(X_{n,T}) - \phi(X_T)\big)^2\big] \\ &\quad + C''(T-t) \sup_{s\in[t,T]} \mathbb{E}\big[|\widehat{X}_{n,s}|^2\big] + C''\mathbb{E}\left[\int_t^T \big(\widehat{g}_n(s, X_s, Y_s, Z_s)\big)^2\big] ds\right].\end{aligned}$$

We note that since $X_{n,0} \to X_0$, $\mu_n(t,x) \to \mu$ and $\sigma_n(t,x) \to \sigma(t,x)$ pointwise when $n \to \infty$, by Theorem 9.7 of Mishura and Shevchenko (2017) we have $\lim_{n\to\infty} \mathbb{E}\big[\sup_{t\in[0,T]} |\widehat{X}_{n,t}|^2\big] = 0$. Hence, by the condition $|\widehat{g}_n(t,x,y,z)| \leq 2C(|x| + |y| + |z|)$ and the pointwise limit $\lim_{n\to\infty} \widehat{g}_n(t,x,y,z) = 0$, by Lebesgue dominated convergence we find

$$\lim_{n\to\infty} \mathbb{E}\left[\int_t^T \big(\widehat{g}_n(s, X_s, Y_s, Z_s)\big)^2\big] ds\right] = 0, \quad t \in [0,T].$$

Finally, by the strong convergence of $(\phi_n)_{n\geq 1}$ to ϕ and the uniform integrability

$$\sup_{n\geq 1} \mathbb{E}\left[\sup_{t\in[0,T]} |X_{n,t}|^{2p}\right] < \infty, \qquad p \geq 1,$$

see Theorem 9.2 in Mishura and Shevchenko (2017), we obtain

$$\lim_{n\to\infty} \mathbb{E}\big[\big(\phi_n(X_{n,T}) - \phi(X_T)\big)^2\big] = 0,$$

and we conclude to

$$\lim_{n\to\infty} \mathbb{E}\left[|Y_{n,t} - Y_t|^2\right] = 0, \quad t \in [0,T].$$

□

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