Existence of solutions for nonlinear elliptic PDEs with fractional Laplacians on open balls

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June 14, 2023

Abstract

We prove the existence of viscosity solutions for fractional semilinear elliptic PDEs on open balls with bounded exterior condition in dimension $d \geqslant 1$. Our approach relies on a tree-based probabilistic representation based on a $(2s)$-stable branching processes for all $s \in (0,1)$, and our existence results hold for sufficiently small exterior conditions and nonlinearity coefficients. In comparison with existing approaches, we consider a wide class of polynomial nonlinearities without imposing upper bounds on their maximal degree or number of terms. Numerical illustrations are provided in large dimensions.

Keywords: Elliptic PDEs, semilinear PDEs, branching processes, fractional Laplacian, stable processes, subordination, Monte-Carlo method.

Mathematics Subject Classification (2020): 35J15, 35J25, 35J60, 35R11, 35B65, 60J85, 60G51, 60G52, 65C05, 33C05.

1 Introduction

Fully nonlinear Dirichlet problems for nonlocal operators have been studied in Bony et al. (1968) by semi-group methods and in Barles et al. (2008) by the Perron method, for particular types of nonlinearities. For $d \geqslant 1$, let

$$\Delta_s u = -(-\Delta)^s u = \frac{4^s \Gamma(s+d/2)}{\pi^{d/2}|\Gamma(-s)|} \lim_{r \to 0^+} \int_{\mathbb{R}^d \setminus B(x,r)} \frac{u(\,\cdot+z) - u(z)}{|z|^{d+2s}} dz,$$

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denote the fractional Laplacian with parameter $s \in (0,1)$, see, e.g., Kwaśnicki (2017), where $\Gamma(p) := \int_0^\infty e^{-\lambda x} \lambda^{p-1} d\lambda$ is the gamma function and $B(x,r)$ denotes the open ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$. Consider the following nonlinear elliptic PDE on an open set $\mathcal{O}$ in $\mathbb{R}^d$, with fractional Laplacian of the form

$$\begin{cases} \Delta_s u(x) + f(x, u(x)) = u(x), & x = (x_1, \ldots, x_d) \in \mathcal{O}, \\ u(x) = \phi(x), & x \in \mathbb{R}^d \setminus \mathcal{O}, \end{cases} \tag{1.1}$$

with polynomial non-linearities of the form

$$f(x,y) = \sum_{l \in \mathcal{L}} c_l(x) y^l, \qquad x \in \mathcal{O}, \quad y \in \mathbb{R},$$

where

- $\mathcal{L}$ is a subset of $\mathbb{N}$,
- $c_l(x)$, $l \in \mathcal{L}$, are bounded measurable functions on $\mathcal{O}$, and
- $\phi : \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz function bounded on $\mathbb{R}^d \setminus \mathcal{O}$.

Two types of solutions, namely weak solutions, see Definition 2.1 in Ros-Oton and Serra (2014b), and viscosity solutions, see Servadei and Valdinoci (2014), are generally considered for elliptic PDEs. While they coincide in many situations, see Remark 2.11 in Ros-Oton and Serra (2014b), they involve different tools.

Weak solutions can be obtained by the Riesz representation theorem or on the Lax-Milgram theorem, after rewriting the problem in its variational formulation, see Ros-Oton (2016), Felsinger et al. (2015). The study of existence of viscosity solutions by the Perron method, see Barles et al. (2008), does not allow for general polynomial non-linearities as in (1.1), see conditions (A1)-(A6) therein. This restriction can be overcome by semi-group methods Bony et al. (1968) or using branching diffusion processes, see Ikeda et al. (1968-1969), or superprocesses, see Le Gall (1995).

Under strong conditions on the nonlinearity $f(x,y)$, namely $|f(x,y)| \leqslant a_1 + a_2 |y|^{q-1}$ for some $q \in (2, 2n/(n-2s))$ and $\lim_{y\to\infty} f(x,y)/y = 0$, existence of nontrivial weak solutions for problems of the form $\Delta_s u(x) + f(x,u) = 0$ on an open bounded set $\mathcal{O}$ with $u = 0$ on $\mathbb{R}^d \setminus \mathcal{O}$ has been obtained in Servadei and Valdinoci (2012) using the mountain pass theorem. In Servadei and Valdinoci (2014), existence of viscosity solutions has been proved for problems of the form $\Delta_s u(x) + f(x) = 0$ with $u = \phi$ on $\mathbb{R}^d \setminus \mathcal{O}$ under smoothness assumptions on f, ϕ, see also Felsinger et al. (2015) and Mou (2017) for the existence of viscosity solutions, resp. weak solutions, with nonlocal operators.

On the other hand, a large part of the literature on fractional PDEs with nonlinearities is devoted to proving the nonexistence of trivial solutions when the initial datum ϕ vanishes outside $B(0,R)$ and $c_0(x) = 0$ in (1.1), see, e.g., Alves et al. (2020), Stegliński (2021), Correia and Oliveira (2022). This also includes the method of moving spheres, see e.g. de Pablo and Sánchez (2010), Fall and Weth (2012), and the use of the Pohoazev identity for the fractional Laplacian, see Ros-Oton and Serra (2012; 2014a). We note that in this setting, our existence results only yield the null function as a solution, which however does not contradict the nonexistence of nontrivial solutions.

In this paper, we consider existence of (continuous) viscosity solutions for fractional elliptic problems of the form (1.1) using a large class of semilinearities $f(x,y)$ without imposing upper bounds on their maximal degree or number of terms. We denote by $\overline{B}(0,R)$ the closed ball of radius $R > 0$ in $\mathbb{R}^d$, and for $\mathcal{O}$ an open subset of $\mathbb{R}^d$ we will use the fractional Sobolev space

$$H^s(\mathcal{O}) := \left\{ u \in L^2(\mathcal{O}) \ : \ \frac{|u(x)-u(y)|}{|x-y|^{s+d/2}} \in L^2(\mathcal{O}^2) \right\},$$

with $d \geqslant 1$ and $s \in (0,1)$. Our main results can be stated as follows, with $\mathcal{O} = B(0,R)$.

Theorem 1.1 *Assume that* $|\phi|_{L^\infty(B^c(0,R))} < \infty$ *and* $\sum_{l \in \mathcal{L}} |c_l|_{L^\infty(B(0,R))} < \infty$ *are both sufficiently small. Then, the nonlinear PDE* (1.1) *admits a (continuous) viscosity solution on* $\mathcal{O} = B(0,R)$.

We note from the proof of Theorem 1.1 that it suffices in particular to have $|\phi|_{L^\infty(B^c(0,R))} \leqslant 1$ and $\sum_{l \in \mathcal{L}} |c_l|_{L^\infty(B(0,R))} \leqslant 1$ for its conclusion to hold.

Theorem 1.2 *Assume that* $\mathcal{L}$ *is finite and that* $|\phi|_{L^\infty(B^c(0,R))} < \infty$ *and* $|c_l|_{L^\infty(B(0,R))} < \infty$, $l \in \mathcal{L}$. *Then, the nonlinear PDE* (1.1) *admits a (continuous) viscosity solution on* $\mathcal{O} = B(0,R)$ *for sufficiently small* $R > 0$.

We note in Proposition 3.5 that if ϕ belongs to $H^{2s}(\mathbb{R}^d)$, the viscosity solution obtained in Theorems 1.1-1.2 is also a weak solution, and that it is the only weak and viscosity solution of (1.1).

Theorems 1.1-1.2 will be proved through a probabilistic representation of PDE solutions using branching stochastic processes. Stochastic branching processes for the representation of PDE solutions have been introduced by Skorokhod (1964), Ikeda et al. (1968-1969), and have been used to prove blow-up and existence of solutions for parabolic PDEs in Nagasawa and Sirao (1969), López-Mimbela (1996), Penent and Privault (2022).

This branching argument has been recently applied in Henry-Labordère et al. (2019) to the treatment of parabolic PDEs with polynomial gradient nonlinearities, see Agarwal and Claisse (2020) for the elliptic case. In this approach, gradient terms are associated to tree branches to which a Malliavin integrations by parts is applied. In Penent and Privault (2022), this approach has been extended to semilinear parabolic PDEs with pseudo-differential operators of the form $-\eta(-\Delta/2)$ and fractional Laplacians, using a random tree $\mathcal{T}_x$ starting at $x \in \mathbb{R}^d$ and carrying a symmetric $(2s)$-stable process. In the absence of gradient nonlinearities, the tree-based approach has been recently implemented for nonlocal semilinear parabolic PDEs in Belak et al. (2020).

In what follows, we will apply this probabilistic representation approach to the setting of elliptic PDEs with fractional Laplacians. PDE solutions will be constructed as the expectation $u(x) = \mathbb{E}[\mathcal{H}(\mathcal{T}_x)]$ of a random functional $\mathcal{H}(\mathcal{T}_x)$ of the underlying branching process, $x \in \mathbb{R}^d$, which yields a probabilistic representations for the solutions of a wide class of semilinear elliptic PDEs of the form (1.1). Sufficient conditions for the representation of classical solutions of (1.1) as $u(x) = \mathbb{E}[\mathcal{H}(\mathcal{T}_x)]$ are obtained in Proposition 2.2.

As we are dealing with continuous viscosity solutions, we need to ensure that the random variable $\mathcal{H}(\mathcal{T}_x)$ is sufficiently integrable, so that the expected value $\mathbb{E}[\mathcal{H}(\mathcal{T}_x)]$ is a continuous function of $x \in \mathbb{R}^d$, see Lemma 3.3. For this, in Lemma 3.2 we show, using results of Kyprianou et al. (2020), that the exit time of a stable process from the ball $B(0, R)$ is almost surely continuous with respect to its initial condition $x \in B(0, R)$. This allows us to show the uniform integrability of $(\mathcal{H}(\mathcal{T}_x))_{x \in B(0,R)}$ using the fractional Laplacian $\Delta_s = -(-\Delta)^s$ and its associated stable process. We note that the result of Lemma 3.3 may be extended to non spherical domains for which Lemma 3.2 is satisfied.

In Section 4 provide a numerical implementation of our existence results using Monte Carlo simulations for nonlinear fractional PDEs in dimension up to 100. We note that the tree-based Monte Carlo method allows us to solve large dimensional problems, whereas the application of deterministic finite difference methods is generally restricted to one dimension and their extension to higher dimensions still remains a challenge, see e.g. Huang and Oberman (2014) in the linear case.

This paper is organized as follows. The description of the branching mechanism is presented in Section 2. In Section 3 we state and prove Theorem 3.4 which gives the probabilistic representation of the solution and its partial derivatives. Finally, in Section 4 we present numerical simulations to illustrate the method on specific examples.

2 Probabilistic representation of elliptic PDE solutions

This section describes the probabilistic representation for the solution of (1.1), using a branching mechanism giving the solution of (1.1) as the expectation of a multiplicative functional defined on a random tree structure. The probabilistic representations of Theorems 1.1-1.2 use a functional on a random branching process driven by a stable Lévy process $(X_t)_{t\in\mathbb{R}_+}$ such that $\mathbb{E}\big[e^{i\xi X_t}\big] = e^{-t|\xi|^{2s}}$, where $|\xi|$ denotes the Euclidean norm of $\xi \in \mathbb{R}^d$, $t \geqslant 0$, so that the infinitesimal generator of $(X_t)_{t\in\mathbb{R}_+}$ is the fractional Laplacian $\Delta_s = -(-\Delta)^s$, $s \in (0,1]$.

Random tree

Given $\rho : \mathbb{R}^+ \to (0,\infty)$ a probability density function on $\mathbb{R}_+$, consider a probability mass function $(q_l)_{l\in\mathcal{L}}$ on $\mathcal{L}$ with $q_l > 0$, $l \in \mathcal{L}$, and $\sum_{l\in\mathcal{L}} lq_l < \infty$. In addition, we consider, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

- an *i.i.d.* family $(\tau^{i,j})_{i,j\geqslant 1}$ of random variables with distribution $\rho(t)dt$ on $\mathbb{R}_+$,
- an *i.i.d.* family $(I^{i,j})_{i,j\geqslant 1}$ of discrete random variables, with

$$\mathbb{P}\big(I^{i,j} = l\big) = q_l > 0, \qquad l \in \mathcal{L},$$

- an independent family $\big(X^{i,j}\big)_{i,j\geqslant 1}$ of symmetric $(2s)$-stable processes,

where the sequences $(\tau^{i,j})_{i,j\geqslant 1}$, $(I^{i,j})_{i,j\geqslant 1}$ and $\big(X^{i,j}\big)_{i,j\geqslant 1}$ are assumed to be mutually independent.

Branching process

We consider a branching process starting from a particle $x \in B(0,R)$ with label $\overline{1} = (1)$, which evolves according to the process $X^{\overline{1}}_{s,x} = x + X^{1,1}_s$, $s \in [0, T_{\overline{1}}]$ with $T_{\overline{1}} := \min\big(\tau^{1,1}, \tau^B(x)\big)$, where

$$\tau^B(x) := \inf\big\{t \geqslant 0 \ : \ x + X^{1,1}_t \notin B(0,R)\big\}$$

denotes the first exit time of $\big(x + X^{1,1}_s\big)_{s\in\mathbb{R}_+}$ from $B(0,R)$ after starting from $x \in B(0,R)$. Note that by (1.4) in Bogdan et al. (2015) we have $\mathbb{E}\big[\tau^B(x)\big] < \infty$, and therefore $\tau^B(x)$ is almost surely finite for all $x \in B(0,R)$. On the other hand, although $\tau^B(x)$ depends on $(1,1)$, for the sake of clarity we will omit this information in the sequel.

If $\tau^{1,1} < \tau^B(x)$, the process branches at time $\tau^{1,1}$ into new independent copies of $(X_t)_{t\in\mathbb{R}_+}$, each of them started at $X^{\overline{1}}_{x,\tau^{1,1}}$. Based on the values of $I^{1,1} = l \in \mathcal{L}$, a family of l of new branches is created with the probability q_l, where

- the first l_0 branches are indexed by $(1,1), (1,2), \ldots, (1, l_0)$,
- the next l_1 branches are indexed by $(1, l_0 + 1), \ldots, (1, l_0 + l_1)$, and so on.

Each new particle then follows independently the same mechanism as the first one, and every branch stops when it leaves the domain $B(0,R)$. Particles at generation $n \geqslant 1$ are assigned a label of the form $\overline{k} = (1, k_2, \ldots, k_n) \in \mathbb{N}^n$, and their parent is labeled $\overline{k}- := (1, k_2, \ldots, k_{n-1})$. The particle labeled $\overline{k}$ is born at time $T_{\overline{k}-}$ and its lifetime $\tau^{n,\pi_n(\overline{k})}$ is the element of index $\pi_n\big(\overline{k}\big)$ in the *i.i.d.* sequence $(\tau^{n,j})_{j\geqslant 1}$ with $\tau^{1,\pi_1(\overline{1})} = \tau^{1,1}$, defining an injection

$$\pi_n : \mathbb{N}^n \to \mathbb{N}, \qquad n \geqslant 1,$$

such that $\pi_1(1) = 1$. The random evolution of particle $\overline{k}$ is given by

$$X^{\overline{k}}_{s,x} := X^{\overline{k}-}_{T_{\overline{k}-},x} + X^{n,\pi_n(\overline{k})}_{s - T_{\overline{k}-}}, \qquad s \in [T_{\overline{k}-}, T_{\overline{k}}],$$

where $T_{\overline{k}} := T_{\overline{k}-} + \min\big(\tau^{n,\pi_n(\overline{k})}, \tau^B\big(X^{\overline{k}-}_{T_{\overline{k}-},x}\big)\big)$, $\overline{k} \in \mathbb{N}^n$, $n \geqslant 2$, and

$$\tau^B\big(X^{\overline{k}-}_{T_{\overline{k}-},x}\big) := \inf\Big\{t \geqslant 0 \ : \ X^{\overline{k}-}_{T_{\overline{k}-},x} + X^{n,\pi_n(\overline{k})}_t \notin B(0,R)\Big\}.$$

Given $\overline{k} = (1, k_2, \ldots, k_n) \in \mathbb{N}^n$, if $\tau^{n,\pi_n(\overline{k})} < \tau^B\big(X^{\overline{k}-}_{T_{\overline{k}-},x}\big)$, we draw a sample $I_{\overline{k}} := I^{n,\pi_n(\overline{k})} = l$ of $I^{n,\pi_n(\overline{k})}$, and the particle $\overline{k}$ branches into $\big|I^{n,\pi_n(\overline{k})}\big| = l$ offsprings at generation $n+1$, which are indexed by $(1, \ldots, k_n, i)$, $i = 1, \ldots, \big|I^{n,\pi_n(\overline{k})}\big|$.

The set of particles dying inside $B(0,R)$ is denoted by $\mathcal{K}^\circ$, whereas those dying outside form a set denoted by $\mathcal{K}^\partial$. The particles of n-th generation, $n \geqslant 1$, will be denoted by $\mathcal{K}_n^\circ$ (resp. $\mathcal{K}_n^\partial$) if they die inside the domain (resp. outside). We also define the filtration $(\mathcal{F}_n)_{n\geqslant 1}$ as

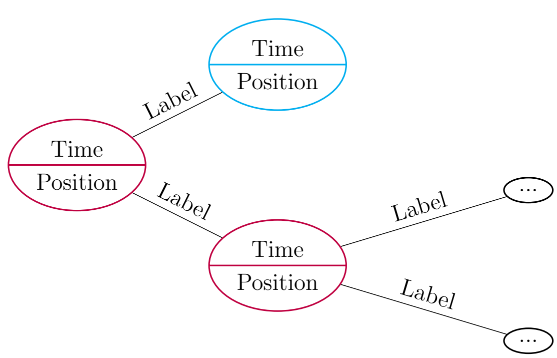
$$\mathcal{F}_n := \sigma\left(T_{\overline{k}}, I_{\overline{k}}, X^{\overline{k}}, \overline{k} \in \bigcup_{i=1}^{n} \mathbb{N}^i\right), \qquad n \geqslant 1.$$

Definition 2.1 *When started from a position $x \in \mathbb{R}^d$, the above construction yields a branching process called a random tree rooted at x, and denoted by $\mathcal{T}_x$.*

The tree $\mathcal{T}_x$ will be used for the stochastic representation of the solution $u(x)$ of the PDE (1.1). The next table summarizes the notation introduced so far.

Object	Notation
Initial position	x
Tree rooted at x	$\mathcal{T}_x$
Particle (or label) of generation $n \geqslant 1$	$\overline{k} = (1, k_2, \ldots, k_n)$
First branching time	$T_{\overline{1}}$
Lifespan of a particle	$T_{\overline{k}} - T_{\overline{k}-}$
Birth time of a particle $\overline{k}$	$T_{\overline{k}-}$
Death time of a particle $\overline{k} \in \mathcal{K}^\circ$	$T_{\overline{k}} = \mathrm{T}_{\overline{k}-} + \tau^{n,\pi_n(\overline{k})}$
Death time of a particle $\overline{k} \in \mathcal{K}^\partial$	$T_{\overline{k}} = T_{\overline{k}-} + \tau^B\big(X^{\overline{k}-}_{T_{\overline{k}-},x}\big)$
Position at birth	$X^{\overline{k}}_{T_{\overline{k}-},x}$
Position at death	$X^{\overline{k}}_{T_{\overline{k}},x}$
Exit time starting from x	$\tau^B(x) := \inf\{t \geqslant 0 \ : \ x + X_t \notin B(0,R)\}$

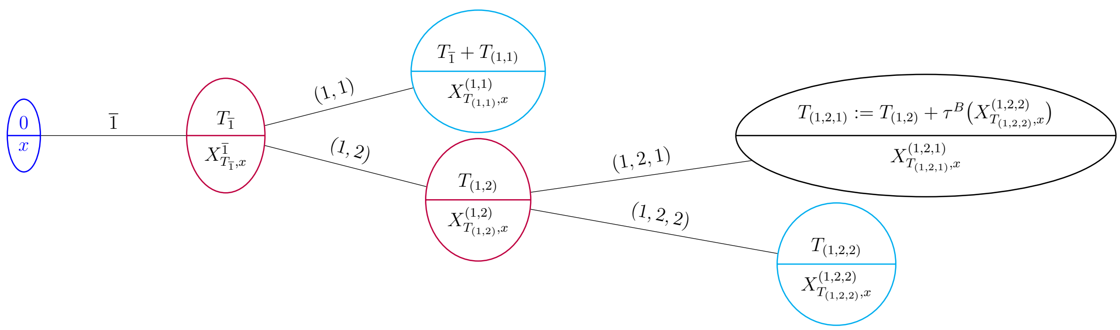
To represent the structure of the tree we use the following conventions, in which different colors represent different ways of branching:



Specifically, let us draw a tree sample for the PDE

$$\Delta_s u(x) + c_0(x) + c_{0,1}(x) u^2(x) = 0$$

in dimension $d = 1$. For this tree, there are two types of branching: we can either branch into no branch at all (which is represented in cyan color), or into two branches. The black color is used for leaves, namely the particles that leave the domain $B(0, R)$.



In the above example we have $\mathcal{K}^\circ = \{\overline{1}, (1,1), (1,2), (1,2,2)\}$ and $\mathcal{K}^\partial = \{(1,2,1)\}$.

Representation of PDE solutions

Given $x \in \mathbb{R}^d$, we consider the functional $\mathcal{H}$ of the random tree $\mathcal{T}_x$ defined as

$$\mathcal{H}(\mathcal{T}_x) := \prod_{\overline{k} \in \mathcal{K}^\circ} \frac{e^{-\Delta T_{\overline{k}}} c_{I_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)}{q_{I_{\overline{k}}} \rho(\Delta T_{\overline{k}})} \prod_{\overline{k} \in \mathcal{K}^\partial} \frac{e^{-\Delta T_{\overline{k}}} \phi\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)}{\overline{F}(\Delta T_{\overline{k}})}, \tag{2.1}$$

where $\Delta T_{\overline{k}} := T_{\overline{k}} - T_{\overline{k}-}$, $\overline{k} \in \mathcal{K}$, and $\overline{F}(t) := 1 - \mathbb{P}(T_{\overline{1}} \leqslant t)$. The next proposition provides a probabilistic representation for the solution of (1.1) as the expected value of the functional $\mathcal{H}(\mathcal{T}_x)$.

Proposition 2.2 *Assume that the PDE (1.1) admits a classical solution $u \in H^{2s}(B(0,R)) \cap \mathcal{C}\big(\overline{B}(0,R)\big)$, such that the sequence*

$$\mathcal{H}_n(\mathcal{T}_x) := \prod_{\overline{k} \in \bigcup_{i=1}^n \mathcal{K}_i^\circ} \frac{e^{-\Delta T_{\overline{k}}} c_{I_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)}{q_{I_{\overline{k}}} \rho(\Delta T_{\overline{k}})} \prod_{\overline{k} \in \bigcup_{i=1}^n \mathcal{K}_i^\partial} \frac{e^{-\Delta T_{\overline{k}}} \phi\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)}{\overline{F}(\Delta T_{\overline{k}})} \prod_{\overline{k} \in \mathcal{K}_{n+1}} u\big(X_{T^{\overline{k}}_{\overline{k}-}}\big)$$

is uniformly integrable in $n \geqslant 1$. Then we have $u(x) = \mathbb{E}[\mathcal{H}(\mathcal{T}_x)]$, $x \in B(0,R)$.

Proof. Applying the Itô-Dynkin formula to the process $\big(e^{-t} u\big(X^{\overline{1}}_{t,x}\big)\big)_{t \in \mathbb{R}_+}$ on the time interval $\big[0, \tau^B(x)\big]$, we find

$$\mathbb{E}\big[e^{-\tau^B(x)} u\big(X^{\overline{1}}_{\tau^B(x),x}\big)\big] = u(x) + \mathbb{E}\bigg[\int_0^{\tau^B(x)} e^{-t} \Delta_s u\big(X^{\overline{1}}_{t,x}\big) dt - \int_0^{\tau^B(x)} e^{-t} u\big(X^{\overline{1}}_{t,x}\big) dt\bigg],$$

which implies that $u(x)$ can be represented as

$$u(x) = \mathbb{E}\Bigg[e^{-\tau^B(x)} u\big(X^{\overline{1}}_{\tau^B(x),x}\big) + \int_0^{\tau^B(x)} e^{-t} f\big(X^{\overline{1}}_{t,x}, u\big(X^{\overline{1}}_{t,x}\big)\big) dt \Bigg],$$

since u solves (1.1). Therefore, since $T_{\overline{1}}$ has the probability density ρ and is independent of $\big(X^{\overline{1}}_{s,x}\big)_{s\in\mathbb{R}_+}$, from the boundary condition $u(x) = \phi(x)$, $x \in \mathbb{R}^d \setminus B(0,R)$, we have

$$\begin{aligned}
u(x) &= \mathbb{E}\left[\mathbb{E}\left[e^{-\tau^B(x)} u\big(X^{\overline{1}}_{\tau^B(x),x}\big) + \int_0^{\tau^B(x)} e^{-t} f\big(X^{\overline{1}}_{t,x}, u\big(X^{\overline{1}}_{t,x}\big)\big) dt \;\Bigg|\; \big(X^{\overline{1}}_{t,x}\big)_{t\in\mathbb{R}_+} \right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\frac{e^{-\tau^B(x)}}{\overline{F}(\tau^B(x))} \phi\big(X^{\overline{1}}_{\tau^B(x),x}\big) \mathbb{1}_{\{T_{\overline{1}}=\tau^B(x)\}} \;\Bigg|\; \big(X^{\overline{1}}_{t,x}\big)_{t\in\mathbb{R}_+} \right]\right] \\
&\quad + \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau^B(x)} \frac{e^{-t}}{\rho(t)} f\big(X^{\overline{1}}_{t,x}, u\big(X^{\overline{1}}_{t,x}\big)\big) \rho(t) dt \;\Bigg|\; \big(X^{\overline{1}}_{t,x}\big)_{t\in\mathbb{R}_+} \right]\right] \\
&= \mathbb{E}\left[\frac{e^{-\tau^B(x)}}{\overline{F}(\tau^B(x))} \phi\big(X^{\overline{1}}_{\tau^B(x),x}\big) \mathbb{1}_{\{T_{\overline{1}}=\tau^B(x)\}} + \frac{e^{-T_{\overline{1}}}}{\rho(T_{\overline{1}})} f\big(X^{\overline{1}}_{t,x}, u\big(X^{\overline{1}}_{t,x}\big)\big) \mathbb{1}_{\{T_{\overline{1}}<\tau^B(x)\}} \right] \\
&= \mathbb{E}\left[\frac{e^{-T_{\overline{1}}}}{\overline{F}(T_{\overline{1}})} \phi\big(X^{\overline{1}}_{\tau^B(x),x}\big) \mathbb{1}_{\{T_{\overline{1}}=\tau^B(x)\}} + \frac{e^{-T_{\overline{1}}}}{\rho(T_{\overline{1}})} \frac{c_{I_{\overline{1}}}\big(X^{\overline{1}}_{T_{\overline{1}},x}\big)}{q_{I_{\overline{1}}}} u^{I_{\overline{1}}}\big(X^{\overline{1}}_{T_{\overline{1}},x}\big) \mathbb{1}_{\{T_{\overline{1}}<\tau^B(x)\}} \right], \quad (2.2)
\end{aligned}$$

showing that $u(x) = \mathbb{E}[\mathcal{H}_1(\mathcal{T}_x)]$, $x \in B(0,R)$, since $\mathcal{K}_1 = \{\overline{1}\}$,

$$\mathcal{H}_1(\mathcal{T}_x) = \prod_{\overline{k}\in\mathcal{K}_1^\circ} \frac{e^{-\Delta T_{\overline{k}}} c_{I_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)}{q_{I_{\overline{k}}}\rho(\Delta T_{\overline{k}})} \prod_{\overline{k}\in\mathcal{K}_1^\partial} \frac{e^{-\Delta T_{\overline{k}}} \phi\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)}{\overline{F}(\Delta T_{\overline{k}})} \prod_{\overline{k}\in\mathcal{K}_2} u\big(X_{T^{\overline{k}}_{\overline{k}-}}\big),$$

and $X^{\overline{1}}_{T_{\overline{1}},x} = X^{\overline{k}}_{T_{\overline{k}-},x}$. Repeating the above argument after starting from $X^{\overline{1}}_{T_{\overline{1}},x} = X^{\overline{k}}_{T_{\overline{k}-},x}$ instead of x, $\overline{k} \in \mathcal{K}_2$, and using the independence of $\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)_{\overline{k}\in\mathcal{K}_2}$ given $\mathcal{F}_1$, we find

$$\begin{aligned}
&u^{I_{\overline{1}}}\big(X^{\overline{1}}_{T_{\overline{1}},x}\big) = \prod_{\overline{k}\in\mathcal{K}_2} u\big(X^{\overline{k}}_{T_{\overline{k}-},x}\big) \\
&= \prod_{\overline{k}\in\mathcal{K}_2} \mathbb{E}\left[\frac{e^{-T_{\overline{k}}}}{\overline{F}(T_{\overline{k}})} \phi\big(X^{\overline{k}}_{T_{\overline{k}},x}\big) \mathbb{1}_{\{X^{\overline{k}}_{T_{\overline{k}},x}\notin B(0,R)\}} + \frac{e^{-T_{\overline{k}}}}{\rho(T_{\overline{k}})} \frac{c_{I_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)}{q_{I_{\overline{k}}}} u^{I_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}},x}\big) \mathbb{1}_{\{X^{\overline{k}}_{T_{\overline{k}},x}\in B(0,R)\}} \;\Bigg|\; \mathcal{F}_1 \right] \\
&= \mathbb{E}\left[\prod_{\overline{k}\in\mathcal{K}_2} \left(\frac{e^{-T_{\overline{k}}}}{\overline{F}(T_{\overline{k}})} \phi\big(X^{\overline{k}}_{T_{\overline{k}},x}\big) \mathbb{1}_{\{X^{\overline{k}}_{T_{\overline{k}},x}\notin B(0,R)\}} + \frac{e^{-T_{\overline{k}}}}{\rho(T_{\overline{k}})} \frac{c_{I_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)}{q_{I_{\overline{k}}}} u^{I_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}},x}\big) \mathbb{1}_{\{X^{\overline{k}}_{T_{\overline{k}},x}\in B(0,R)\}} \right) \;\Bigg|\; \mathcal{F}_1 \right] \\
&= \mathbb{E}\left[\prod_{\overline{k}\in\mathcal{K}_2} \frac{e^{-T_{\overline{k}}}}{\overline{F}(T_{\overline{k}})} \phi\big(X^{\overline{k}}_{T_{\overline{k}},x}\big) \mathbb{1}_{\{X^{\overline{k}}_{T_{\overline{k}},x}\notin B(0,R)\}} + \prod_{\overline{k}\in\mathcal{K}_2} \frac{e^{-T_{\overline{k}}}}{\rho(T_{\overline{k}})} \frac{c_{I_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)}{q_{I_{\overline{k}}}} u^{I_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}},x}\big) \mathbb{1}_{\{X^{\overline{k}}_{T_{\overline{k}},x}\in B(0,R)\}} \;\Bigg|\; \mathcal{F}_1 \right].
\end{aligned}$$

Plugging this expression in (2.2) above and using the tower property of the conditional expectation yields $u(x) = \mathbb{E}[\mathcal{H}_2(\mathcal{T}_x)]$, and repeating this process inductively leads to $u(x) = \mathbb{E}[\mathcal{H}_n(\mathcal{T}_x)]$. From the uniform integrability of $(\mathcal{H}_n(\mathcal{T}_x))_{n\geqslant 1}$ and the fact that the tree goes extinct almost surely, we conclude to $u(x) = \mathbb{E}[\mathcal{H}(\mathcal{T}_x)]$ after letting n tend to infinity. $\square$

3 Proofs of existence results

In Proposition 3.1 we start by showing that existence of solutions holds when $u(x) := \mathbb{E}[\mathcal{H}(\mathcal{T}_x)]$ is continuous in $x \in \overline{B}(0,R)$. We then prove in Lemma 3.3 that continuity of $\mathbb{E}[\mathcal{H}(\mathcal{T}_x)]$ in $x \in \overline{B}(0,R)$ holds when $(\mathcal{H}(\mathcal{T}_x))_{x\in B(0,R)}$ is uniformly integrable. The proofs of Theorems 1.1-1.2 are then completed by showing that $(\mathcal{H}(\mathcal{T}_x))_{x\in B(0,R)}$ is uniformly bounded in $L^p(B(0,R))$ for some $p > 1$, implying the required uniform integrability.

Proposition 3.1 *Suppose that $\mathbb{E}\big[\big|\mathcal{H}(\mathcal{T}_x)\big|\big] < \infty$ for all $x \in B(0,R)$, and that $u(x) := \mathbb{E}[\mathcal{H}(\mathcal{T}_x)]$ is continuous on $\overline{B}(0,R)$. Then, u is a viscosity solution of the PDE (1.1).*

Proof. By conditioning with respect to $X^{\bar{1}}_{T_{\bar{1}},x}$ and $I_{\bar{1}}$ and using the fact that each offspring starts independent identically distributed stable branching processes, we have

$$\mathbb{E}\left[\prod_{i=0}^{I_{\bar{1}}-1} \mathcal{H}\big(\mathcal{T}_{X^{\bar{1}}_{T_{\bar{1}},x}}\big) \;\middle|\; X^{\bar{1}}_{T_{\bar{1}},x}, I_{\bar{1}}\right] \mathbb{1}_{\{T_{\bar{1}}<\tau^B(x)\}} = u^{I_{\bar{1}}}\big(X^{\bar{1}}_{T_{\bar{1}},x}\big)\mathbb{1}_{\{T_{\bar{1}}<\tau^B(x)\}},$$

hence, since $T_{\bar{1}}$ has density ρ, we get

$$\begin{aligned}
u(x) &= \mathbb{E}[\mathcal{H}(\mathcal{T}_x)] \\
&= \mathbb{E}\left[\frac{e^{-T_{\bar{1}}}}{\overline{F}(T_{\bar{1}})}\phi\big(X^{\bar{1}}_{\tau^B(x),x}\big)\mathbb{1}_{\{T_{\bar{1}}=\tau^B(x)\}} + \frac{e^{-T_{\bar{1}}}}{\rho(T_{\bar{1}})}\frac{c_{I_{\bar{1}}}\big(X^{\bar{1}}_{T_{\bar{1}},x}\big)}{q_{I_{\bar{1}}}}\prod_{i=0}^{I_{\bar{1}}-1}\mathcal{H}\big(\mathcal{T}_{X^{\bar{1}}_{T_{\bar{1}},x}}\big)\mathbb{1}_{\{T_{\bar{1}}<\tau^B(x)\}}\right] \\
&= \mathbb{E}\left[e^{-\tau^B_x}\phi\big(X^{\bar{1}}_{\tau^B(x),x}\big) + \int_0^{\tau^B(x)} e^{-t} f\big(X^{\bar{1}}_{t,x}, u\big(X^{\bar{1}}_{t,x}\big)\big)dt\right] \\
&= \mathbb{E}\left[e^{-\delta\wedge\tau^B_x}u\big(X^{\bar{1}}_{\delta\wedge\tau^B(x),x}\big) + \int_0^{\delta\wedge\tau^B(x)} e^{-t} f\big(X^{\bar{1}}_{t,x}, u\big(X^{\bar{1}}_{t,x}\big)\big)dt\right],
\end{aligned}$$

for any $\delta > 0$, by the Markov property. It then follows from a classical argument that u is a viscosity solution of the PDE (1.1). Indeed, let $\xi \in \mathcal{C}^2(B(0,R))$ such that x is a maximum point of $u - \xi$ and $u(x) = \xi(x)$. By the Itô-Dynkin formula, we get

$$\mathbb{E}\big[e^{-\delta\wedge\tau^B(x)}\xi\big(X^{\bar{1}}_{\tau^B(x),x}\big)\big] = \xi(x) + \mathbb{E}\bigg[\int_0^{\delta\wedge\tau^B(x)} e^{-t}\Delta_s\xi\big(X^{\bar{1}}_{t,x}\big)dt - \int_0^{\delta\wedge\tau^B(x)} e^{-t}\xi\big(X^{\bar{1}}_{t,x}\big)dt\bigg].$$

Thus, since $u(x) = \xi(x)$ and $u \leqslant \xi$ we obtain

$$\mathbb{E}\left[\int_0^{\delta\wedge\tau^B(x)} e^{-t}\big(\Delta_s\xi\big(X_{t,x}^{\bar{1}}\big) - \xi\big(X_{t,x}^{\bar{1}}\big) + f\big(X_{t,x}^{\bar{1}}, u\big(X_{t,x}^{\bar{1}}\big)\big)\big)dt\right] \geqslant 0.$$

Since X_t^x converges in distribution to the constant $x \in \mathbb{R}^d$ as t tends to zero, it admits an almost surely convergent subsequence, hence by continuity and boundedness of $f(\cdot, u(\cdot))$ together with the mean-value and dominated convergence theorems, we have

$$\Delta_s\xi(x) + f(x, \xi(x)) - \xi(x) \geqslant 0.$$

We conclude that u is a viscosity subsolution (and similarly a viscosity supersolution) of (1.1). □

Lemma 3.2 *The following statements hold true with probability one.*

a) Let $x \in B(0, R)$. For $\mathbb{P}$-almost every $\omega \in \Omega$ there exists $r_0(\omega) > 0$ such that $\tau^B(y) = \tau^B(x)$, for all $y \in B(x, r_0(\omega))$.

b) Let $x \in \overline{B}(0, R) \setminus B(0, R)$. We have $\lim_{n\to\infty} \tau^B(x_n) = \tau^B(x) = 0$ almost surely for any sequence $(x_n)_{n\geqslant 0}$ in $B(0, R)$ converging fast enough to x.

As a consequence, for any $x \in \overline{B}(0, R)$ we have

$$\mathbb{P}\Big(\lim_{n\to\infty} \tau^B(x_n) = \tau^B(x)\Big) = 1. \tag{3.1}$$

for any sequence $(x_n)_{n\in\mathbb{N}}$ in $B(0, R)$ converging fast enough to $x \in \overline{B}(0, R)$.

Proof. *a*) If $x \in B(0, R)$ we have

$$\mathbb{P}\Bigg(\sup_{s\in[0,\tau^B(x))} |x + X_s| < R\Bigg) = 1,$$

as the distribution of the furthest reach from the origin immediately before exit time admits a density by Theorem 1.3-(*ii*) of Kyprianou et al. (2020). Similarly, letting $\overline{B}^c(0, R) := \mathbb{R}^d \setminus \overline{B}(0, R)$, we have

$$\mathbb{P}\big(x + X_{\tau^B(x)} \in \overline{B}^c(0, R)\big) = 1.$$

as the distribution of $x + X_{\tau^B(x)}$ admits a density on $\mathbb{R}^d \setminus \overline{B}(0, R)$ by Equation (2.2) in Bass and Cranston (1983). Therefore, we have

$$\mathbb{P}\Bigg(\sup_{s\in[0,\tau^B(x))} |x + X_s| < R \quad \text{and} \quad x + X_{\tau^B(x)} \in \overline{B}^c(0, R)\Bigg) = 1$$

and almost surely there exists $r_0(\omega) > 0$ such that

$$(y + X_s(\omega))_{s\in[0,\tau^B(x))} \subset B(0,R) \quad \text{and} \quad y + X_{\tau^B(x)}(\omega) \in \overline{B}^c(0,R),$$

provided that $|y-x| < r_0(\omega)$. Therefore, for any $y \in B(x, r_0(\omega))$ we have $\tau^B(y)(\omega) = \tau^B(x)(\omega)$, which proves (3.1).

b) If $x \in \overline{B}(0,R) \setminus B(0,R)$ we have $\tau^B(x) = 0$, and by (6.3) in Bogdan et al. (2015) there exists $C^* > 0$ such that for sufficiently small ε we have

$$\mathbb{P}\big(\tau^B(x_n) > \varepsilon\big) < \frac{C^*}{\sqrt{\varepsilon}} V(d(x_n, B^c(0,R))), \qquad \varepsilon > 0,$$

where V is the renewal function of the ascending height-process, which satisfies $\lim_{r\to 0} V(r) = 0$, and $B^c(0,R) := \mathbb{R}^d \setminus B(0,R)$. Therefore, if the sequence $(x_n)_{n\geqslant 0}$ in $B(0,R)$ is such that

$$V(d(x_n, B^c(0,R))) < \frac{1}{2^n}, \qquad n \geqslant 0,$$

we have $\sum_{n\geqslant 0} \mathbb{P}\big(\tau^B(x_n) > \varepsilon\big) < \infty$, and we conclude by the Borel-Cantelli Lemma. □

Based on (3.1), we obtain a sufficient condition for the continuity of $x \mapsto \mathbb{E}[\mathcal{H}(\mathcal{T}_x)]$ in $x \in \overline{B}(0,R) \subset \mathbb{R}^d$. We note that this continuity property may be extended to non spherical domains for which (3.1) is satisfied.

Lemma 3.3 *Assume that $(\mathcal{H}(\mathcal{T}_x))_{x\in B(0,R)}$ is uniformly integrable. Then, the function $u(x) := \mathbb{E}[\mathcal{H}(\mathcal{T}_x)]$ is continuous in $x \in \overline{B}(0,R)$.*

Proof. Let $x \in \overline{B}(0,R)$, and let $(x_n)_{n\geqslant 0}$ denote a sequence in $B(0,R)$ converging to x and satisfying (3.1). For any $\overline{k} \in \mathcal{K}$ we let $\tau_{\overline{k},x} := \tau^B\big(X^{\overline{k}-}_{T_{\overline{k}-},x}\big)$, and note that the event

$$A_{\overline{k}} := \Big\{\lim_{n\to\infty} \tau_{\overline{k},x_n} = \tau_{\overline{k},x}\Big\} \bigcap \Big\{\lim_{n\to\infty} X^{\overline{k}}_{\cdot,x_n} = X^{\overline{k}}_{\cdot,x}\Big\},$$

has probability one by Lemma 3.2 and the relation $X^{\overline{k}}_{\cdot,x_n} = X^{\overline{k}}_{\cdot,x} + x_n - x$, $n \geqslant 0$. By Lemma 3.2-*a*), for some $n_0(\omega)$ large enough we have

$$X^{\overline{k}}_{\tau_{\overline{k},x_n}} = X^{\overline{k}}_{\tau_{\overline{k},x}} + x_n - x,$$

and $\tau_{\overline{k},x_n} = \tau_{\overline{k},x}$, $n \geqslant n_0(\omega)$. Therefore, using the continuity of ϕ and $c_l, l \in \mathcal{L}$, we have

$$\lim_{n\to\infty} \phi\big(X^{\overline{k}}_{\tau_{\overline{k},x_n}}\big) \mathbb{1}_{\{T_{\overline{k}} = \tau_{\overline{k},x_n}\}} = \phi\big(X^{\overline{k}}_{\tau_{\overline{k},x}}\big) \mathbb{1}_{\{T_{\overline{k}} = \tau_{\overline{k},x}\}}, \quad \mathbb{P} - \text{a.s.}$$

and

$$\lim_{n\to\infty} \frac{c_{I_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}},x_n}\big)}{q_{I_{\overline{k}}}} \mathbb{1}_{\{T_{\overline{k}}<\tau_{\overline{k},x_n}\}} = \frac{c_{I_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)}{q_{I_{\overline{k}}}} \mathbb{1}_{\{T_{\overline{k}}<\tau_{\overline{k},x}\}}, \quad \mathbb{P}-\text{a.s.}.$$

Hence by (2.1), on the event $A := \bigcap_{\overline{k}\in\mathcal{K}} A_{\overline{k}}$ of probability one, we have

$$\lim_{n\to\infty} \mathcal{H}(\mathcal{T}_{x_n}(\omega)) = \mathcal{H}(\mathcal{T}_x(\omega)).$$

Therefore, for any sequence $(x_n)_{n\geqslant 1}$ converging to $x \in \overline{B}(0,R)$ fast enough, we have

$$\mathbb{P}\big(\lim_{n\to\infty} \mathcal{H}(\mathcal{T}_{x_n}) = \mathcal{H}(\mathcal{T}_x)\big) = 1,$$

which yields $\lim_{n\to\infty} u(x_n) = u(x)$ by uniform integrability of $(\mathcal{H}(\mathcal{T}_x))_{x\in B(0,R)}$. $\square$

Theorem 3.4 *Assume that there exists a sequence $(q_l)_{l\in\mathcal{L}}$ of positive numbers summing to one, such that the partial differential inequality*

$$\Delta_s v(x) + \sum_{l\in\mathcal{L}} \frac{|c_l(x)|^p}{q_l^{p-1}} v^l(x) \leqslant v(x), \qquad x \in B(0,R), \tag{3.2}$$

admits a non-negative solution $v \in H^{2s}(B(0,R)) \bigcap \mathcal{C}\big(\overline{B}(0,R)\big)$ such that $v \geqslant |\phi|^p$ on $\mathbb{R}^d \setminus B(0,R)$ for some $p>1$. Then, the nonlinear PDE (1.1) admits a (continuous) viscosity solution on $\mathcal{O} = B(0,R)$.

Proof. We take $\rho(t) := e^{-t}, t \geqslant 0$, so that $\mathcal{H}(\mathcal{T}_x)$ rewrites as

$$\mathcal{H}(\mathcal{T}_x) := \prod_{\overline{k}\in\mathcal{K}^\circ} \frac{c_{I_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)}{q_{I_{\overline{k}}}} \prod_{\overline{k}\in\mathcal{K}^\partial} \phi\big(X^{\overline{k}}_{T_{\overline{k}},x}\big), \qquad x \in \mathbb{R}^d. \tag{3.3}$$

Applying the Itô-Dynkin formula and (3.2) to the process $\big(e^{-t} v\big(X^{\overline{1}}_{t,x}\big)\big)_{t\in\mathbb{R}_+}$ on the time interval $[0, \tau^B(x)]$ we get

$$v(x) \geqslant \mathbb{E}\left[e^{-\tau^B(x)} \big|\phi\big(X^{\overline{1}}_{\tau^B(x),x}\big)\big|^p + \int_0^{\tau^B(x)} e^{-t} \sum_{l\in\mathcal{L}} \frac{\big|c_l\big(X^{\overline{1}}_{t,x}\big)\big|^p}{q_l^{p-1}} v^l\big(X^{\overline{1}}_{t,x}\big) dt \right].$$

Thus, by the same recursion as in the proof of Proposition 2.2, we obtain

$$v(x) \geqslant \mathbb{E}\left[\prod_{\overline{k}\in\bigcup_{i=1}^n \mathcal{K}_i^\circ} \frac{\big|c_{I_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)\big|^p}{q_{I_{\overline{k}}}^{p-1}} \prod_{\overline{k}\in\bigcup_{i=1}^n \mathcal{K}_i^\partial} \big|\phi\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)\big|^p \prod_{\overline{k}\in\mathcal{K}_{n+1}} v\big(X_{T_{\overline{k}-}}\big) \right], \quad n \geqslant 1.$$

Letting n tend to infinity and applying Fatou's Lemma, since v is non-negative we find

$$v(x) \geqslant \mathbb{E}\left[\prod_{\overline{k}\in\mathcal{K}^\circ} \frac{\left|c_{I_{\overline{k}}}\left(X^{\overline{k}}_{T_{\overline{k}},x}\right)\right|^p}{q_{I_{\overline{k}}}^{p-1}} \prod_{\overline{k}\in\mathcal{K}^\partial} \left|\phi\left(X^{\overline{k}}_{T_{\overline{k}},x}\right)\right|^p\right] = \mathbb{E}[|\mathcal{H}(\mathcal{T}_x)|^p], \quad x \in B(0,R).$$

In particular, $(\mathcal{H}(\mathcal{T}_x))_{x\in B(0,R)}$ is uniformly bounded in $L^p(B(0,R))$ since $v \in \mathcal{C}\big(\overline{B}(0,R)\big)$. Therefore, by Proposition 3.1 and Lemma 3.3 u is a continuous viscosity solution of the PDE (1.1) on $B(0,R)$. □

To prove the integrability required in Theorems 1.1-1.2 we adapt the approach of Agarwal and Claisse (2020) to the fractional setting, by constructing a branching process that stochastically dominates the underlying stable branching process uniformly in $x \in B(0,R)$.

Proof of Theorem 1.1. As in the proof of Theorem 3.4, in order to show that the PDE (1.1) admits a (continuous) viscosity solution it suffices to show that $(\mathcal{H}(\mathcal{T}_x))_{x\in B(0,R)}$ is uniformly bounded in $L^p(B(0,R))$ for some $p > 1$. We take again $\rho(t) := e^{-t}, t \geqslant 0$, in which case $\mathcal{H}(\mathcal{T}_x)$ rewrites as in (3.3). Letting

$$q_k := \frac{|c_k|_{L^\infty(B(0,R))}}{\sum_{l\in\mathcal{L}} |c_l|_{L^\infty(B(0,R))}}, \quad k \in \mathcal{L},$$

we have $|\mathcal{H}(\mathcal{T}_x)| \leqslant 1$ provided that

$$C_0 := \max\left(|\phi|_{L^\infty(B^c(0,R))}, \sup_l \frac{|c_l|_{L^\infty(B(0,R))}}{q_l}\right) = \max\left(|\phi|_{L^\infty(B^c(0,R))}, \sum_{l\in\mathcal{L}} |c_l|_{L^\infty(B(0,R))}\right) \leqslant 1,$$

in which case $(\mathcal{H}(\mathcal{T}_x))_{x\in B(0,R)}$ is uniformly integrable and we conclude by Lemma 3.3 and Proposition 3.1. If $C_0 > 1$, we let

$$\delta := 1 - \inf_{x\in B(0,R)} \mathbb{E}\big[e^{-\tau^B(x)}\big], \tag{3.4}$$

and

$$\widetilde{f}(s) := \sum_{l\in\mathcal{L}} \widetilde{q}_l s^l$$

where

$$\widetilde{q}_0 := 1 - \delta + \frac{\delta|c_0|_{L^\infty(B(0,R))}}{\sum_{l\in\mathcal{L}} |c_l|_{L^\infty(B(0,R))}}, \quad \widetilde{q}_k := \frac{\delta|c_k|_{L^\infty(B(0,R))}}{\sum_{l\in\mathcal{L}} |c_l|_{L^\infty(B(0,R))}}, \qquad k \geqslant 1.$$

By Proposition 3.5 in Agarwal and Claisse (2020), $(\mathcal{H}(\mathcal{T}_x))_{x\in B(0,R)}$ is uniformly bounded in $L^p(B(0,R))$ for some $p > 1$ provided that $C_0 < (\gamma(s^*))^{1/p}$, where

$$\gamma(s^*) := \frac{1}{\widetilde{f}'(s^*)} = \frac{s^*}{\widetilde{f}(s^*)}$$

and s^* is the solution of $s^*\widetilde{f}'(s^*) = \widetilde{f}(s^*)$ if it exists, or $s^* = \zeta$ otherwise, where ζ denotes the radius of convergence of $\widetilde{f}$. As above, we conclude the proof from Lemma 3.3 and Proposition 3.1. $\square$

Proof of Theorem 1.2. When $\mathcal{L}$ is finite we have $\zeta = \infty$ and a solution s^* to $s^*\widetilde{f}'(s^*) = \widetilde{f}(s^*)$ always exists. By part (iv) of the proof of Proposition 3.5 in Agarwal and Claisse (2020), s^* tends to infinity as δ goes to 0 (or equivalently as $\mathrm{Diam}(B(0,R))$ goes to 0 by (3.4)) and $\lim_{\delta\to 0}\gamma(s^*) = \infty$. Hence, if R is sufficiently small we have $C_0 < (\gamma(s^*))^{1/p}$ and $(\mathcal{H}(\mathcal{T}_x))_{x\in B(0,R)}$ is uniformly bounded in $L^p(B(0,R))$ for some $p > 1$ by Proposition 3.5 in Agarwal and Claisse (2020). Therefore, we can conclude from Lemma 3.3 and Proposition 3.1 as in the proof of Theorem 1.1. $\square$

In the next proposition we note that the viscosity solution obtained in Theorems 1.2 and 3.4 is also a weak solution, and that it is the only weak and viscosity solution of (1.1) provided that ϕ belongs to $H^{2s}(\mathbb{R}^d)$.

Proposition 3.5 *Assume that* (1.1) *admits a viscosity solution u, and that ϕ belongs to $H^{2s}(\mathbb{R}^d)$. Then $u \in H^s(\mathbb{R}^d) \cap \mathcal{C}\big(\overline{B}(0,R)\big)$, it is a weak solution and the only weak and viscosity solution of* (1.1).

Proof. As u is a viscosity solution it is continuous, and bounded on $\mathbb{R}^d$. Letting $v := u - \phi$, v solves the equation

$$\begin{cases} \Delta_s v(x) + \Delta_s \phi(x) + \sum_{l\in\mathcal{L}} c_l(x)(v+\phi)^l(x) = v(x) + \phi(x), & x \in B(0,R), \\ v(x) = 0, \quad x \in \mathbb{R}^d \setminus B(0,R), \end{cases}$$

which can be rewritten as:

$$\begin{cases} \Delta_s v(x) + \sum_{l\in\mathcal{L}} \widetilde{c}_l(x) v^l(x) = 0, & x \in B(0,R), \\ v(x) = 0, \quad x \in \mathbb{R}^d \setminus B(0,R), \end{cases} \tag{3.5}$$

where $\widetilde{c}_l$ is a continuous bounded function on $\mathbb{R}^d$ for every $l \in \mathcal{L}$. Letting $g(x) := \sum_{l\in\mathcal{L}} \widetilde{c}_l(x) v^l(x)$, the Dirichlet problem

$$\begin{cases} \Delta_s w = -g(x), & x \in B(0,R), \\ w(x) = 0, \quad x \in \mathbb{R}^d \setminus B(0,R), \end{cases}$$

admits v as viscosity solution. Since g is bounded on $B(0,R)$, it also admits a unique weak solution w which is a viscosity solution by Remark 2.11 in Ros-Oton and Serra (2014b). By

Theorem 5.2 in Caffarelli and Silvestre (2009), the viscosity solutions v, w coincide and v is the unique solution of (3.5) in both in the weak and viscosity senses. Therefore, u is the unique solution of (1.1). In addition, since u is a weak solution it belongs to $H^s(\mathbb{R}^d)$, see Proposition 1.4 in Ros-Oton and Serra (2014b). □

4 Numerical examples

In this section we consider numerical examples involving the fractional Laplacian Δ_s and the s-stable subordinator $(S_t)_{t\in\mathbb{R}_+}$ with Laplace exponent $\eta(\lambda) = (2\lambda)^s$ for $s \in (1/2, 1)$. We represent $(X_t)_{t\in\mathbb{R}_+}$ using the subordination $X_t := B_{S_t}$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard d-dimensional Brownian motion and $(S_t)_{t\in\mathbb{R}_+}$ is a Lévy subordinator with Laplace exponent η, defined by

$$\mathbb{E}\big[e^{-\lambda S_t}\big] = e^{-t(2\lambda)^s}, \qquad \lambda,\, t \geqslant 0,$$

see e.g. Theorem 1.3.23 in Applebaum (2009). For the generation of random samples of S_t, we use the formula

$$\widetilde{S}_t := 2t^{1/s} \frac{\sin(s(U+\pi/2))}{\cos^{1/s}(U)} \left(\frac{\cos\big(U - s(U+\pi/2)\big)}{E} \right)^{-1+1/s}$$

based on the Chambers-Mallows-Stuck (CMS) method, where $U \sim U(-\pi/2, \pi/2)$, and $E \sim \mathrm{Exp}(1)$, see Relation (3.2) in Weron (1996), where $\psi(\lambda)$ denotes the Lévy symbol of $(S_t)_{t\in\mathbb{R}_+}$. For $k \geqslant 0$, we consider the function

$$\Phi_{k,s}(x) := (1 - |x|^2)_+^{k+s}, \qquad x \in \mathbb{R}^d,$$

which is Lipschitz if $k > 1 - s$, and solves the Poisson problem $\Delta_s \Phi_{k,s} = -\Psi_{k,s}$ on $\mathbb{R}^d$, with

$$\Psi_{k,s}(x) := \begin{cases} \dfrac{\Gamma(s+d/2)\Gamma(k+1+s)}{4^{-s}\Gamma(k+1)\Gamma(d/2)}\, {}_2F_1\left(\dfrac{d}{2}+s, -k; \dfrac{d}{2}; |x|^2\right), & |x| \leqslant 1 \\[2ex] \dfrac{4^s\Gamma(s+d/2)\Gamma(k+1+s)}{\Gamma(k+1+s+d/2)\Gamma(-s)|x|^{d+2s}}\, {}_2F_1\left(\dfrac{d}{2}+s, 1+s; k+1+\dfrac{d}{2}+s; \dfrac{1}{|x|^2}\right), & |x| > 1 \end{cases}$$

$x \in \mathbb{R}^d$, where ${}_2F_1(a, b; c; y)$ is Gauss's hypergeometric function, see (5.2) in Getoor (1961) and Lemma 4.1 in Biler et al. (2015), and Relation (36) in Huang and Oberman (2016).

Dirichlet problem

We solve the Dirichlet problem

$$\begin{cases} \Delta_s u(x) + \Psi_{k,s}(x) = 0, \quad x \in B(0,1) \\ u(x) = 0, \quad x \in B^c(0,1). \end{cases} \tag{4.1}$$

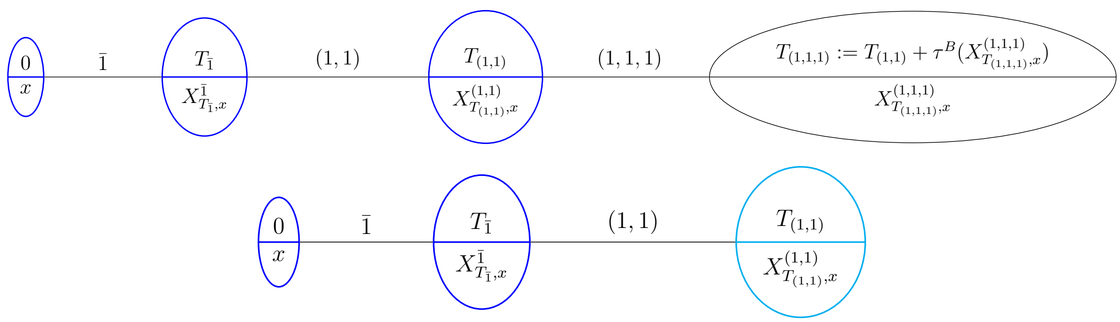
with

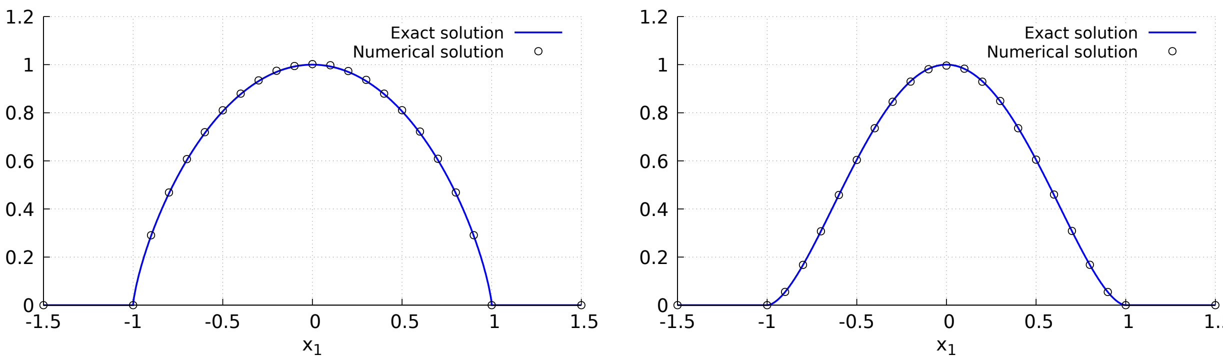
$$c_0(x)u^0(x) := \Psi_{k,s}(x), \qquad c_1(x)u^1(x) := u(x),$$

and explicit solution

$$u(x) = \Phi_{k,s}(x) = (1 - |x|^2)_+^{k+s}, \qquad x \in \mathbb{R}^d. \tag{4.2}$$

The random tree associated to (4.1) starts at a point $x \in B(0,1)$ and branches into zero branch or one branch as in the following random samples:





(a) Numerical solution of (4.1) with $k = 0$. (b) Numerical solution of (4.1) with $k = 1$.

Figure 1: Numerical solution of (4.1) in dimension $d = 1$ with $\alpha = 1.75$.

Figure 1, which uses one million Monte Carlo samples, can be compared to Figure 6.5a in Huang and Oberman (2014).

Linear fractional elliptic PDE

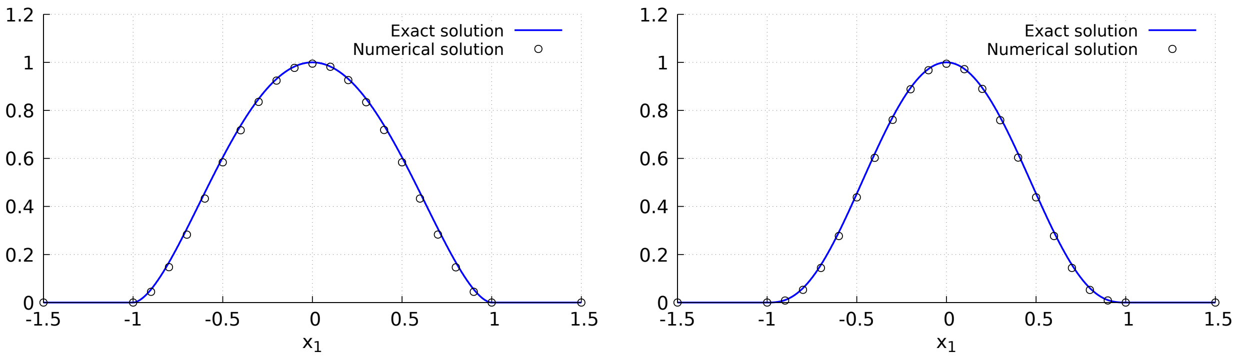
We solve the linear elliptic problem

$$\begin{cases} \Delta_s(x) + \Psi_{k,s}(x) - (1-|x|^2)_+^{k+s} + u(x) = 0, & x \in B(0,1) \\ u(x) = 0, & x \in B^c(0,1), \end{cases} \tag{4.3}$$

with

$$c_0(x)u^0(x) := \Psi_{k,s}(x) - (1-|x|^2)_+^{2k+2s}, \qquad c_1(x)u^1(x) := 2u(x),$$

and explicit solution (4.2). The random tree associated to (4.3) is the same as in the previous example, and the simulations of Figure 2 use five million Monte Carlo samples.



(a) Numerical solution of (4.3) with $k = 1$. (b) Numerical solution of (4.3) with $k = 2$.

Figure 2: Numerical solution of (4.3) in dimension $d = 100$ with $\alpha = 1.75$.

Nonlinear fractional elliptic PDE

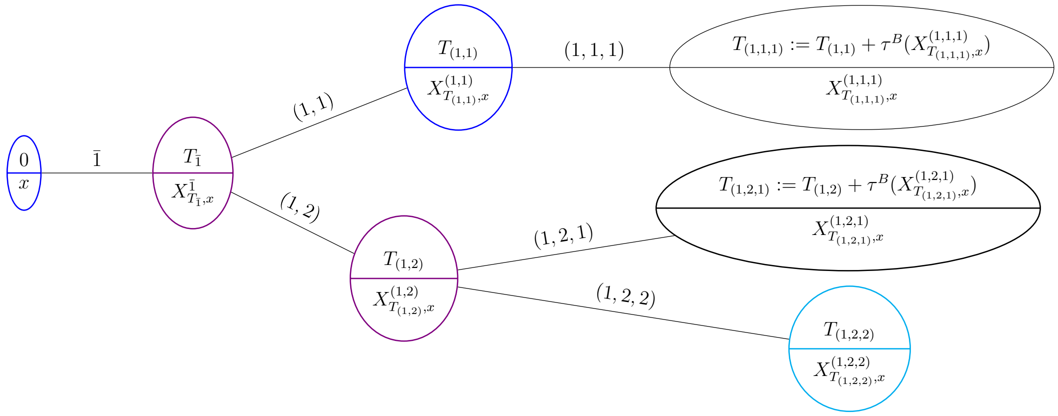
Here we aim at recovering the explicit solution (4.2) of the nonlinear elliptic PDE

$$\begin{cases} \Delta_s u(x) + \Psi_{k,s}(x) - (1-|x|^2)_+^{2k+2s} + u^2(x) = 0, & x \in B(0,1), \\ u(x) = 0, & x \in B^c(0,1), \end{cases} \tag{4.4}$$

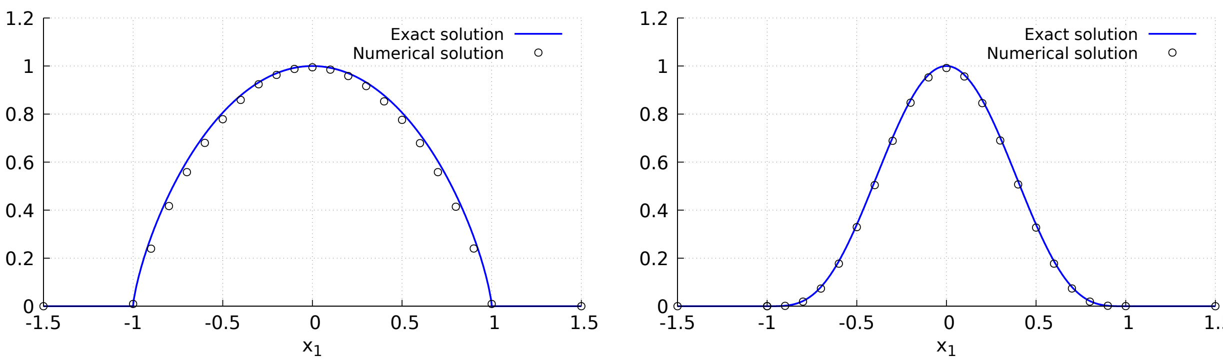
with

$$c_0(x)u^0(x) = \Psi_{k,s}(x) - (1-|x|^2)_+^{2k+2s}, \quad c_1(x)u^1(x) = u(x), \quad c_2(x)u^2(x) = u^2(x).$$

The random tree associated to (4.4) starts at point $x \in B(0,1)$ and branches into zero branch, one branch, or two branches, as in the following random sample tree:



The simulations of Figure 3 use 20 million Monte Carlo samples.



(a) Numerical solution of (4.4) with $k = 0$.

(b) Numerical solution of (4.4) with $k = 3$.

Figure 3: Numerical solution of (4.4) in dimension $d = 10$ with $\alpha = 1.75$.

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