A direct solution to the Fokker-Planck equation for exponential Brownian functionals*

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Abstract

The solution of the Fokker-Planck equation for exponential Brownian functionals usually involves spectral expansions that are difficult to compute explicitly. In this paper we propose a direct solution based on heat kernels and a new integral representation for the square modulus of the Gamma function. A financial application to bond pricing in the Dothan model is also presented.

Key words: Brownian exponential functionals, Fokker-Planck equation, spectral expansions, Bessel functions.

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1 Introduction

Brownian exponential functionals of the form

$$\mathcal{A}_T = \int_0^T e^{\sigma W_s - \sigma^2 s/2 + \lambda s} ds, \qquad T \geq 0,$$

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where $(W_s)_{s\in\mathbb{R}_+}$ is a standard Brownian motion, play an important role in the statistical physics of disordered systems where $\mathcal{A}_T$ is the partition function and $\log \mathcal{A}_T$ represents the free energy of the system. It is also used in financial mathematics for the pricing of Asian options and of bonds, cf. e.g. [3], [6], [11], [17], and references therein.

The Laplace transform

$$F(t,x) = E\left[\exp\left(-x\int_0^t e^{\sigma W_s-\sigma^2 s/2+\lambda s}ds\right)\right], \qquad x\in\mathbb{R}_+,$$

of $\mathcal{A}_t$ is known to solve the PDE

$$\begin{cases} \dfrac{\partial F}{\partial t}(t,x) = \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 F}{\partial x^2}(t,x) + \lambda x \dfrac{\partial F}{\partial x}(t,x) - xF(t,x) \\ \\ F(0,x) = 1, \qquad x\in\mathbb{R}_+, \end{cases} \tag{1.1}$$

and the probability density $\Psi(t,y)$ of $\mathcal{A}_t$, defined by

$$F(t,x) = \int_{-\infty}^{\infty} e^{-xy}\Psi(t,y)dy, \qquad x,t\in\mathbb{R}_+,$$

satisfies the Fokker-Planck equation

$$\begin{cases} \dfrac{\partial \Psi}{\partial t}(t,y) = \dfrac{1}{2}\sigma^2 \dfrac{\partial^2}{\partial y^2}\left(y^2\Psi(t,y)\right) - \lambda\dfrac{\partial}{\partial y}\left(y\Psi(t,y)\right) - \dfrac{\partial \Psi}{\partial y}(t,y), \\ \\ \Psi(0,y) = \delta_0(y). \end{cases} \tag{1.2}$$

The solution of (1.2) has been given in [18] using spectral expansions, and this solution is commonly used in the mathematical physics literature, cf. e.g. [7], [8], [9] and references therein, as well as in finance [15].

However, the full derivation of the normalization constants in the continuous spectrum was not presented in [18], cf. page 1641 therein, due to severe analytical difficulties in the computation of the normalization constants in the spectral expansion via the use of Meijer functions. Recently a computation of the spectral expansion has been given in [15] using a limiting procedure, nevertheless this method relies on a complex and delicate limiting argument.

In this paper we propose a direct solution to (1.1) and (1.2) via a simple argument based on heat kernels and a new integral representation (1.9) for the square modulus of the Gamma function. This approach relies on asymptotic expansions, cf. Lemma 3.1, and on combinatorial identities for modified Bessel functions such as

$$\sum_{k=0}^{p} \frac{p-2k}{k!(p-k)!} I_{-(p-2k)}(y) = 0, \qquad y \in \mathbb{R}, \quad p \in \mathbb{N},$$

cf. Relation (3.10) below, which might be of independent interest.

Under the change of variable

$$F_p(t,x) := \left(\frac{\sqrt{8x}}{\sigma} \right)^p U_p \left(\frac{\sigma^2 t}{8}, \frac{\sqrt{8x}}{\sigma} \right), \qquad x,t > 0, \tag{1.3}$$

and $p = 1 - 2\lambda/\sigma^2$, we can rewrite (1.1) as the modified Bessel heat equation

$$\begin{cases} \dfrac{\partial U_p}{\partial s}(s,y) = \left(H - p^2 \right) U_p(s,y), \qquad s,y > 0, \\ \\ U_p(0,y) = y^{-p}, \end{cases} \tag{1.4}$$

where H is the operator

$$H = y^2 \frac{\partial^2}{\partial y^2} + y \frac{\partial}{\partial y} - y^2,$$

which is known to admit the modifed Bessel function

$$K_w(t) = \int_0^\infty e^{-t\cosh x} \cosh(wx) dx, \qquad t \in \mathbb{R}, \tag{1.5}$$

of the second kind with parameter $w \in \mathbb{C}$ as eigenvector with corresponding eigenvalue w^2.

Using the integral representation (4.2) of the kernel $q_t(x,y)$ of $(e^{tH})_{t \in \mathbb{R}_+}$ proved in Proposition 4.2 below, the solution

$$U_p(s,y) = e^{-p^2 s} \int_{-\infty}^\infty U_p(0,x) q_s(x,y) dx \tag{1.6}$$

of (1.4) is given by

$$U_p(s,y) = \frac{2}{\pi^2} \int_0^\infty U_p(0,x) \int_0^\infty u \sinh(\pi u) e^{-(p^2+u^2)s} K_{iu}(x) K_{iu}(y) du \frac{dx}{x}, \tag{1.7}$$

$z \in \mathbb{R}$, $s > 0$. As noted in Relation (3.6) page 640 of [7] and in Corollary 3.2 of [17], when $U_p(0, y) = y^{-p}$ and $p < 0$, the Fubini theorem can be used to derive the expression

$$U_p(s, y) = \frac{2^{-p-1}}{\pi^2} \int_0^\infty u e^{-(p^2+u^2)s} \sinh(\pi u) \left| \Gamma\left(-\frac{p}{2} + i\frac{u}{2} \right) \right|^2 K_{iu}(y) du,$$

based on the integral representation

$$\left| \Gamma\left(-\frac{p}{2} + i\frac{s}{2} \right) \right|^2 = 4 \int_0^\infty \left(\frac{2}{y} \right)^{-p} K_{is}(y) dy, \qquad s \in \mathbb{R}, \tag{1.8}$$

which is only true for $p < 0$ due to integrability restrictions, cf. Lemma 4.1 in the Appendix. In case $p > 0$, the Fubini theorem does not apply and one generally relies on the spectral expansion of the density $\Psi(s, y)$, cf. [7], page 640. Here the problem amounts to decompose the initial condition y^{-p} into an integral series of Bessel eigenfunctions K_ω of H. However, although

$$H = y\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial y} \right) - y^2$$

is formally self-adjoint with respect to dy/y, the computation of this expansion involves several divergent terms, cf. Relation (2.3) below.

In this paper we give a simple solution to (1.1) by noting that the Fubini theorem can still be applied when $p > 0$, provided we use the extension

$$\left| \Gamma\left(-\frac{p}{2} + i\frac{s}{2} \right) \right|^2 = 4 \int_0^\infty \left(\frac{2}{x} \right)^{p} K_{is}(x) \left(1 - \sum_{0 \le m < p/2} \frac{2^{1-p}(p-2m)}{m!(p-m)!} K_{p-2m}(x) \right) \frac{dx}{x} \tag{1.9}$$

of (1.8) to all $p + is \in \mathbb{C}$, $p \notin \mathbb{Z}$, proved in Corollary 3.4 below, where $x! = \Gamma(x+1) \in (0, \infty]$, $x \in \mathbb{R}$, denotes the generalized factorial. In particular this allows us to recover the complete spectral expansion of the Hamiltonian H in a simple way, as shown in Proposition 2.1 below.

The remaining of this paper is organised as follows. In Section 2 we present the main result and its proof, with a financial application to bond pricing in the Dothan model, extending to $p > 0$ the expansions given in [17]. In Section 3 we prove the integral representation (1.9) and the associated integrability property using combinatorial arguments and analytic continuation. The results of Section 3 may be already known, but, since we were not able to find them in the literature, we provide their complete proofs. In the Appendix Section 4 we derive some identities for special functions that are used in the paper and might also be difficult to find in the literature.

2 Main result and its consequences

In the next proposition we provide a simple derivation of the spectral expansion of H using Relation (1.9) which is proved in the next sections.

Proposition 2.1 *The solution $U_p(s, y)$ of (1.4) is given for all $p \in \mathbb{R}$ by*

$$\begin{aligned} U_p(s,y) &= \frac{2^{-p}}{2\pi^2}\int_0^\infty u e^{-(p^2+u^2)s}\sinh(\pi u)\left|\Gamma\left(-\frac{p}{2}+i\frac{u}{2}\right)\right|^2 K_{iu}(y)du \\ &\quad +2^{-p}\sum_{k=0}^\infty \frac{2(p-2k)^+}{k!(p-k)!}e^{4k(k-p)s}K_{p-2k}(y), \qquad y\in\mathbb{R}, \quad s\in\mathbb{R}_+. \end{aligned} \tag{2.1}$$

Proof. From the integrability property established in Lemma 3.2 below we may apply the Fubini theorem, hence from (1.6), (1.7) and the integral representation (1.9) of Corollary 3.4 we get, for $p \in \mathbb{R}\setminus\mathbb{Z}$ and $s \in \mathbb{R}_+$,

$$\begin{aligned} U_p(s,y) &= e^{-p^2 s}\int_0^\infty x^{-p} q_s(x,y)dx \\ &= e^{-p^2 s}\int_0^\infty x^{-p} q_s(x,y)\left(1-\sum_{0\le m<p/2}\frac{2^{1-p}(p-2m)}{m!(p-m)!}K_{p-2m}(x)\right)dx \\ &\quad +e^{-p^2 s}\sum_{0\le m<p/2}\frac{2^{1-p}(p-2m)}{m!(p-m)!}\int_0^\infty q_s(x,y)K_{p-2m}(x)dx \\ &= \frac{2}{\pi^2}\int_0^\infty u\sinh(\pi u)K_{iu}(y)e^{-(p^2+u^2)s} \\ &\quad \times\int_0^\infty x^{-p}K_{iu}(x)\left(1-\sum_{0\le m<p/2}\frac{2^{1-p}(p-2m)}{m!(p-m)!}K_{p-2m}(x)\right)\frac{dx}{x}du \\ &\quad +e^{-p^2 s}\sum_{0\le m<p/2}\frac{2^{1-p}(p-2m)}{m!(p-m)!}e^{sH}K_{p-2m}(y) \end{aligned}$$

$$= \frac{2^{-p}}{2\pi^2}\int_0^\infty u\sinh(\pi u)e^{-(p^2+u^2)s}\left|\Gamma\left(-\frac{p}{2}+i\frac{u}{2}\right)\right|^2 K_{iu}(y)du$$
$$+\sum_{0\le m<p/2} e^{4((p-2m)^2-p^2)s}\frac{2^{1-p}(p-2m)}{m!(p-m)!}K_{p-2m}(y). \qquad (2.2)$$

The formula extends to $p\in\mathbb{Z}$ by continuity and dominated convergence. □

Taking $s=0$ in (2.1) yields the spectral expansion

$$y^{-p} = \frac{2^{-p}}{2\pi^2}\int_0^\infty u\sinh(\pi u)\left|\Gamma\left(-\frac{p}{2}+i\frac{u}{2}\right)\right|^2 K_{iu}(y)du + 2^{-p}\sum_{k=0}^\infty \frac{2(p-2k)^+}{k!(p-k)!}K_{p-2k}(y), \qquad (2.3)$$

$y>0$, of the initial condition y^{-p} in terms of the modified Bessel eigenfunctions of H. The knowledge of this expansion is sufficient to recover Proposition 2.1, however (2.3) can not be obtained by a simple orthogonality argument because the functions K_{iu} and K_{p-2k} are not in $L^2(\mathbb{R}_+, dy/y)$, cf. e.g. Relation (6.576.4) page 684 of [12].

From the change of variable (1.3) we recover the expression of the Laplace transform $F_p(t,x)$ in the next corollary.

Corollary 2.2 *For all $p\in\mathbb{R}$ we have*

$$F_p(t,x) = \frac{(2x)^{p/2}}{2\pi^2\sigma^p}\int_0^\infty ue^{-\sigma^2(p^2+u^2)t/8}\sinh(\pi u)\left|\Gamma\left(-\frac{p}{2}+i\frac{u}{2}\right)\right|^2 K_{iu}\left(\frac{\sqrt{8x}}{\sigma}\right)du$$
$$+\frac{(2x)^{p/2}}{\sigma^p}\sum_{k=0}^\infty \frac{2(p-2k)^+}{k!(p-k)!}e^{\sigma^2 k(k-p)t/2}K_{p-2k}\left(\frac{\sqrt{8x}}{\sigma}\right), \qquad (2.4)$$

$x>0$, $t>0$.

By inversion of the Laplace transform, in the next proposition we obtain the probability density of $\mathcal{A}_t$ solution of the Fokker-Planck equation (1.2). This provides another expression for the density of the exponential functional $\mathcal{A}_t$, to be compared with [21], Proposition 2, and [16], as noted in [7], [8], [9].

Proposition 2.3 *For all $t,z>0$ we have*

$$\Psi_p(t,z) = \frac{e^{-1/(\sigma^2 z)}}{4z\pi^2}\left(\frac{2}{\sigma^2 z}\right)^{(p-1)/2}\int_0^\infty s\sinh(\pi s)e^{-\sigma^2(p^2+s^2)t/8}\left|\Gamma\left(-\frac{p}{2}+i\frac{z}{2}\right)\right|^2 W_{\frac{p+1}{2},\frac{is}{2}}\left(\frac{2}{\sigma^2 z}\right)ds$$

$$
+e^{-2/(\sigma^2 z)} \sum_{0\le m<p/2} \frac{(-1)^m}{z} \left(\frac{2}{\sigma^2 z}\right)^{p-m} \frac{p-2m}{(p-m)!} e^{-\sigma^2 m(p-m)t/2} L_m^{p-2m}\left(\frac{2}{\sigma^2 z}\right), \tag{2.5}
$$

where $\mathtt{W}_{(p+1)/2,is/2}$ *and* L_m^{p-2m} *are respectively the Whittaker function and generalized Laguerre polynomials.*

Proof. From Relation (9.237.3), page 1028, Relation (13.6.27), page 510, and Relation (7.629.2) page 828 of [12] we get, for $x > 0$,

$$
\begin{aligned}
&(-1)^m(\sigma^2/2)^{m-p}\int_0^\infty z^{m-p-1}\exp\left(-xz-\frac{2}{\sigma^2 z}\right)L_m^{p-2m}\left(\frac{2}{\sigma^2 z}\right)dz\\
&\quad = (-1)^m\int_0^\infty y^{p-m-1}e^{-y-2x/(\sigma^2 y)}L_m^{p-2m}(y)dy\\
&\quad = \frac{1}{m!}\int_0^\infty y^{\frac{p}{2}-\frac{3}{2}}e^{-y/2-2x/(\sigma^2 y)}\mathtt{W}_{\frac{p+1}{2},p/2-m}(y)dy\\
&\quad = \frac{x^{(p-1)/2}}{m!}\left(\frac{2}{\sigma^2}\right)^{(p-1)/2}\int_0^\infty y^{-\frac{p+1}{2}}e^{-y-x/(\sigma^2 y)}\mathtt{W}_{\frac{p+1}{2},p/2-m}\left(\frac{2x}{\sigma^2 y}\right)dy\\
&\quad = 2\frac{(2x)^{p/2}}{\sigma^p m!}K_{p-2m}\left(\sqrt{\frac{8x}{\sigma^2}}\right), \qquad 0\le m<p/2.
\end{aligned}
$$

Next from Fubini's theorem and Relation (7.629.2), page 828 of [12], we have

$$
\begin{aligned}
&\int_0^\infty e^{-xz}z^{-(p+1)/2}e^{-1/(\sigma^2 z)}\mathtt{W}_{\frac{p+1}{2},is/2}\left(\frac{2}{\sigma^2 z}\right)dz\\
&\qquad = x^{(p-1)/2}\int_0^\infty e^{-y-x/(\sigma^2 y)}y^{\frac{-p-1}{2}}\mathtt{W}_{\frac{p+1}{2},is/2}\left(\frac{2x}{\sigma^2 y}\right)dy\\
&\qquad = \sqrt{\frac{8}{\sigma^2}}x^{p/2}K_{is}\left(\sqrt{\frac{8x}{\sigma^2}}\right).
\end{aligned}
$$

We conclude the proof by Corollary 2.2 and the uniqueness of the Laplace transform.

□

The knowledge of the density $\Psi_p(t,z)$ allows in principle one to compute the prices of Asian options of the form

$$
E[\phi(\mathcal{A}_T)] = \int_0^\infty \phi(z)\Psi_p(t,z)dz,
$$

where ϕ is a sufficiently integrable payoff function.

We close this section with an application to bond pricing in the Dothan model in which the short term interest rate process $(r_t)_{t\in\mathbb{R}_+}$ is given by

$$dr_t = (1-p)\frac{1}{2}\sigma^2 r_t dt + \sigma r_t dB_t,$$

where $p\sigma/2$ identifies to the market price of risk, which is usually non-negative. Using Lemma 4.3 below, the zero-coupon bond price

$$P(t,T) = F_p(T-t, r_t) = E\left[\exp\left(-\int_t^T r_s ds\right) \Big| r_t\right]$$

in the Dothan model [10] is given, from Corollary 2.2 and Relation (1.3), by

$$F_p(\tau, r) =$$
$$\frac{(2r)^{p/2}}{2\pi^2\sigma^p}\int_0^\infty \sin\left(\frac{\sqrt{8r}}{\sigma}\sinh a\right)\int_0^\infty u e^{-\sigma^2(p^2+u^2)\tau/8}\cosh\left(\frac{\pi u}{2}\right)\left|\Gamma\left(-\frac{p}{2}+i\frac{u}{2}\right)\right|^2 \sin(ua)duda$$
$$+\frac{(2r)^{p/2}}{\sigma^p}\sum_{0\leq m<p/2}\frac{2(p-2m)}{m!(p-m)!}e^{\sigma^2 m(m-p)\tau/2}K_{p-2m}\left(\frac{\sqrt{8r}}{\sigma}\right), \qquad (2.6)$$

for all $p\in\mathbb{R}$, $r>0$, and $\tau>0$. The above formula extends Corollary 3.2 of [17] to $p\geq 0$, and it corrects the zero coupon bond price given in [10], page 64, cf. also [5] page 63, which is only valid for $p\in(0,2]$, cf. [17].

3 Integral representations and Bessel functions

In this section we prove the extension (1.9) of Relation (1.8) to $p>0$ using asymptotic expansions, combinatorial and integrability arguments, as well as analytic continuation.

Consider the modified Bessel function of the first kind, defined for all $p\in\mathbb{C}$ as

$$I_p(x) = \sum_{l=0}^\infty \frac{1}{l!(l+p)!}\left(\frac{x}{2}\right)^{p+2l}, \qquad x\in\mathbb{R}, \qquad (3.1)$$

cf. Relation (9.6.10), page 375 of [1], where $x! = \Gamma(x+1)\in(0,\infty]$, $x\in\mathbb{R}$, denotes the generalized factorial and $z\mapsto 1/\Gamma(z)$ extends by continuity to all $z\in\mathbb{C}$. Recall that $I_p(x)$ is analytic in both $x \in \mathbb{C}$ and $p \in \mathbb{C}$, and from (3.1) we have in particular the equivalence

$$I_p(x) = \frac{1}{\Gamma(p+1)} \left(\frac{x}{2}\right)^p + o(x^p), \qquad x \to 0, \tag{3.2}$$

for $p > 0$.

Lemma 3.1 *Let $n \in \mathbb{N}$ and $p \in \mathbb{R}$ such that $p < 2n+2$. We have*

$$\sum_{k=0}^{n} \frac{p-2k}{k!(p-k)!} I_{-(p-2k)}(y) = \frac{\sin(p\pi)}{\pi} \left(\frac{2}{y}\right)^p + o(y^\varepsilon), \qquad y \to 0, \tag{3.3}$$

for all $\varepsilon \in (0, 2n+2-p)$.

Proof. From (3.1) we have, for all $p \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\left(\frac{y}{2}\right)^p \sum_{k=0}^{n} \frac{p-2k}{k!(p-k)!} I_{-(p-2k)}(y) &= \sum_{k=0}^{n} \frac{p-2k}{k!(p-k)!} \sum_{l=0}^{\infty} \frac{1}{l!(l-(p-2k))!} \left(\frac{y}{2}\right)^{2k+2l} \\
&= \sum_{m=0}^{\infty} \left(\frac{y}{2}\right)^{2m} \sum_{k=0}^{m \wedge n} \frac{p-2k}{k!(m-k)!(p-k)!(k+m-p)!} \\
&= \frac{\sin(\pi p)}{\pi} \sum_{m=0}^{n} \left(\frac{y}{2}\right)^{2m} \frac{(-1)^m}{m!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (p-2k) \frac{\Gamma(p-k-m)}{\Gamma(p-k+1)} \\
&\quad + \sum_{m=n+1}^{\infty} \left(\frac{y}{2}\right)^{2m} \sum_{k=0}^{n} \frac{p-2k}{k!(m-k)!(p-k)!(k+m-p)!},
\end{aligned}$$

where we used Euler's reflection formula

$$\frac{1}{(k+m-p)!} = (-1)^{k+m} \frac{\sin(\pi p)}{\pi} \Gamma(p-k-m), \qquad k, m \in \mathbb{N}, \tag{3.4}$$

cf. e.g. Relation (6.1.17), page 256 of [1], extended by continuity to all $p \in \mathbb{R}$. Next, using the Pfaff-Sallschütz (or Vandermonde) binomial identity

$$(a+b)_n = \sum_{k=0}^{n} \binom{n}{k} (a)_k (b)_{n-k}$$

for the Pochhammer symbol $(x)_n$ defined for $x > 0$ by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \qquad \text{and} \qquad (-x)_n = (-1)^n \frac{\Gamma(x+1)}{\Gamma(x+1-n)},$$

cf. Theorem 2.2.6 and Remark 2.2.1 of [2], or § 2.7 of [19], we find that for all $m \geq 1$ and $p \in \mathbb{R} \setminus \mathbb{Z}$,

$$\begin{aligned}
&\sum_{k=0}^{m}(-1)^k \binom{m}{k}(p-2k)\frac{\Gamma(p-k-m)}{\Gamma(p-k+1)} \qquad (3.5)\\
&= p\sum_{k=0}^{m}(-1)^k \binom{m}{k}\frac{\Gamma(p-k-m)}{\Gamma(p-k+1)} + 2m\sum_{k=0}^{m-1}(-1)^k \binom{m-1}{k}\frac{\Gamma(p-k-m-1)}{\Gamma(p-k)}\\
&= p\frac{\Gamma(p-2m)}{\Gamma(p+1)}\sum_{k=0}^{m}\binom{m}{k}(-p)_k(p-2m)_{m-k}\\
&\quad +2m\frac{\Gamma(p-2m)}{\Gamma(p)}\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}(1-p)_k(p-2m)_{m-1-k}\\
&= \frac{\Gamma(p-2m)}{\Gamma(p)}\left((-2m)_m + 2m(1-2m)_{m-1}\right)\\
&= \frac{\Gamma(p-2m)}{\Gamma(p)\Gamma(m+1)}\big((-1)^m\Gamma(2m+1) + 2m(-1)^{m-1}\Gamma(2m)\big)\\
&= 0.
\end{aligned}$$

As a consequence, for all $p \in \mathbb{R} \setminus \mathbb{Z}$ we have the identities

$$\sum_{k=0}^{m}\frac{p-2k}{k!(m-k)!(p-k)!(k+m-p)!} = 0, \qquad m \geq 1, \tag{3.6}$$

which extend to all $p \in \mathbb{R}$ by continuity, and

$$\begin{aligned}
\left(\frac{y}{2}\right)^p\sum_{k=0}^{n}\frac{p-2k}{k!(p-k)!}I_{-(p-2k)}(y) &= \frac{\sin(p\pi)}{\pi} \qquad (3.7)\\
&\quad + \sum_{m=n+1}^{\infty}\frac{1}{m!}\left(\frac{y}{2}\right)^{2m}\sum_{k=0}^{n}\binom{m}{k}\frac{p-2k}{(p-k)!(k+m-p)!},
\end{aligned}$$

which is also valid for all $p \in \mathbb{R}$, and proves (3.3) when $p < 2n+2$. □

Lemma 3.1, Relation (3.2) and the identity

$$K_p(x) = \frac{\pi}{2\sin(p\pi)}(I_{-p}(x) - I_p(x)), \qquad p, x \in \mathbb{R}, \tag{3.8}$$

also yield the expansion

$$\sum_{0\leq k<p/2}\frac{p-2k}{k!(p-k)!}K_{-(p-2k)}(y) = \frac{1}{2}\left(\frac{2}{y}\right)^p + o(y^{\varepsilon}), \qquad y \to 0, \tag{3.9}$$

for all $\varepsilon \in (0, (p-2n) \wedge (2n+2-p))$ and $p \in (2n, 2n+2)$, $n \in \mathbb{N}$.

When $p \in \mathbb{N}$, Relations (3.6) and (3.7) yield

$$\sum_{k=0}^{p} \frac{p-2k}{k!(p-k)!} I_{-(p-2k)}(y) = 0, \qquad y \in \mathbb{R}. \tag{3.10}$$

Next from Lemma 3.1 we prove an integrability result which is essential for the application of the Fubini theorem in the proof of Proposition 2.1 and for the proof of Lemma 3.3 below.

Lemma 3.2 *For all $n \in \mathbb{N}$ and $p < 2n+2$ we have*

$$\sup_{s\in\mathbb{R}} \int_0^\infty |K_{is}(y)| \left| \frac{\sin(p\pi)}{\pi} - \left(\frac{y}{2}\right)^p \sum_{k=0}^{n} \frac{p-2k}{k!(p-k)!} I_{-(p-2k)}(y) \right| \frac{dy}{y^{1+p}} < \infty.$$

Proof. For all $\alpha > 0$ there exists a constant $c_\alpha > 0$ such that $\cosh x > c_\alpha x^\alpha$ for all $x > 0$. Hence by Relation (1.5) and Lemma 3.1 we have, for all $s \in \mathbb{R}$ and $\alpha > 1/\varepsilon$,

$$\begin{aligned}
&\int_0^1 |K_{is}(y)| \left| \frac{\sin(p\pi)}{\pi} - \left(\frac{y}{2}\right)^p \sum_{k=0}^{n} \frac{p-2k}{k!(p-k)!} I_{-(p-2k)}(y) \right| \frac{dy}{y^{1+p}} \\
&\quad \leq c \int_0^1 \int_0^\infty e^{-y\cosh x} dx \frac{dy}{y^{1-\varepsilon}} \\
&\quad \leq c \int_0^1 \int_0^\infty e^{-y c_\alpha x^\alpha} dx \frac{dy}{y^{1-\varepsilon}} \\
&\quad = c \frac{\Gamma(1/\alpha)}{\alpha (c_\alpha)^{1/\alpha}} \int_0^1 \frac{dy}{y^{1-\varepsilon+1/\alpha}}, \qquad s \in \mathbb{R}.
\end{aligned}$$

Finally, the bound $|K_{is}(y)| \leq K_0(y)$, $y, s \in \mathbb{R}_+$, that follows from (4.1) below, and the equivalences $K_{is}(y) \simeq e^{-y}\sqrt{\pi/(2y)}$ and $I_{-(p-2k)}(y) \simeq e^y/\sqrt{2\pi y}$ as $y \to \infty$, show that

$$y \to \frac{1}{y^{p+1}} K_{is}(y) \left(\frac{\sin(p\pi)}{\pi} - \left(\frac{y}{2}\right)^p \sum_{k=0}^{n} \frac{p-2k}{k!(p-k)!} I_{-(p-2k)}(y) \right)$$

is integrable on $[1, \infty)$, uniformly in $s \in \mathbb{R}$. □

By (3.9) and the same argument as in the proof of Lemma 3.2, or by Relation (3.1) and the equivalence (3.2), for all $n \in \mathbb{N}$ and $p + is \in \mathbb{C}$ such that $p < 2n+2$ we also have

$$\sup_{s\in\mathbb{R}} \int_0^\infty |K_{is}(y)| \left| 1 - \left(\frac{y}{2}\right)^p \sum_{k=0}^{n} \frac{2(p-2k)}{k!(p-k)!} K_{-(p-2k)}(y) \right| \frac{dy}{y^{1+p}} < \infty. \tag{3.11}$$

In particular, the bound (3.11) shows that

$$\begin{aligned}
\int_0^\infty \int_0^\infty u e^{-u^2 s} \left| \sinh(\pi u) K_{iu}(y) K_{iu}(x) \right| \left| 1 - \left(\frac{x}{2}\right)^p \sum_{0\leq m<p/2} \frac{2(p-2m)}{m!(p-m)!} K_{p-2m}(x) \right| du \frac{dx}{x^{p+1}} \\
\leq C_{p,y} \int_0^\infty u e^{-u^2 s} \sinh(\pi u) du \\
< \infty,
\end{aligned}$$

where $C_{p,y} > 0$ is a constant, as used in connection with the Fubini theorem in the proof of Proposition 2.1 above.

In the next lemma, using analytic continuation, we prove an integral representation formula which will be applied to the proof of Corollary 3.4.

Lemma 3.3 *For all $n \in \mathbb{N}$ and $p + is \in \mathbb{C} \setminus (2\mathbb{N})$ with $p < 2n + 2$, we have*

$$\begin{aligned}
&\int_0^\infty K_{is}(y) \left(\frac{\sin(p\pi)}{\pi} - \left(\frac{y}{2}\right)^p \sum_{k=0}^{n} \frac{p-2k}{k!(p-k)!} I_{-(p-2k)}(y) \right) \frac{dy}{y^{p+1}} \\
&= \frac{\sin(p\pi)}{2^{p+2}\pi} \left| \Gamma\left(-\frac{p}{2} + i\frac{s}{2}\right) \right|^2 - \sum_{k=0}^{n} \frac{2^{-p}(p-2k)}{k!((p-2k)^2+s^2)(p-k)!}.
\end{aligned} \tag{3.12}$$

Proof. First, we check that both sides coincide on $\{p + is \ : \ p < 0, \ s \in \mathbb{R}\}$. This follows from the integral representation (1.8) for $p < 0$, and from the relation

$$\int_0^\infty K_{is}(y) I_{-(p-2k)}(y) \frac{dy}{y} = \frac{1}{(p-2k)^2 + s^2}$$

which is valid for all $s \in \mathbb{R}$ and $p < 2k \leq 2n$, cf. Lemma 4.1 in the Appendix. Next, in order to extend the identity to all $p + is \in \mathbb{C} \setminus (2\mathbb{N})$ with $p < 2n + 2$, we check that both sides of (3.12) are analytic functions on this domain of $\mathbb{C}$. This is clear for the right-hand side, which is analytic in $p + is \in \mathbb{C} \setminus (2\mathbb{N})$. Concerning the left-hand side, the function

$$y \mapsto \frac{1}{y^{p+1}} K_{is}(y) \left(\frac{\sin(p\pi)}{\pi} - \left(\frac{y}{2}\right)^p \sum_{k=0}^{n} \frac{p-2k}{k!(p-k)!} I_{-(p-2k)}(y) \right)$$

is integrable on $\mathbb{R}^+$ for all $p < 2n+2$ and $s \in \mathbb{R}$ by Lemma 3.2, and by dominated convergence the left-hand side of (3.12) is an analytic function on

$$\{p + is \ : \ p < 2n+2, \ s \in \mathbb{R}, \ p \notin 2\mathbb{N}\}.$$

Indeed, the integrand is analytic in $p+is \in \mathbb{C}$ for all fixed y, and its derivative remains bounded locally bounded in p and s by an integrable function, as can be checked from Relation (3.7) and Lemmas 3.1 and 3.2. Consequently, the integral remains differentiable in p and s and the Cauchy-Riemann equations remains satisfied after integration, showing that the integral is an holomorphic function of $p+is$. Finally the identity is extended by analytic continuation to all $p+is \in \mathbb{C} \setminus (2\mathbb{N})$ with $p < 2n+2$, since both sides are analytic functions that coincide on $\{p + is \ : \ p < 0, \ s \in \mathbb{R}\}$, cf. e.g. Theorem 1.2 of [14]. □

Note that (3.10) and (3.12) also imply the identity

$$\sum_{k=0}^{p} \frac{p-2k}{k!((p-2k)^2+s^2)(p-k)!} = 0$$

for all $p \in \mathbb{Z}$ and $s \neq 0$.

Lemma 3.3 shows in particular that for all $p + is \in \mathbb{C} \setminus (2\mathbb{N})$ we have

$$\int_0^\infty K_{is}(y) \left(\frac{\sin(p\pi)}{\pi} - \left(\frac{y}{2}\right)^p \sum_{0 \le k < p/2} \frac{p-2k}{k!(p-k)!} I_{-(p-2k)}(y) \right) \frac{dy}{y^{p+1}} \tag{3.13}$$
$$= \frac{\sin(p\pi)}{2^{p+2}\pi} \left| \Gamma\left(-\frac{p}{2} + i\frac{s}{2}\right) \right|^2 - \sum_{0 \le k < p/2} \frac{2^{-p}(p-2k)}{k!((p-2k)^2+s^2)(p-k)!},$$

by choosing $n \in \mathbb{N}$ such that $2n < p < 2n+2$ when $p > 0$, whereas the formula is known to hold when $p < 0$ by Lemma 4.1 in the Appendix.

Next, we prove the integral representation which is used in the proof of Proposition 2.1 and extends (1.8).

Corollary 3.4 *For all $p+is \in \mathbb{C}$ with $p \notin \mathbb{Z}$ we have*

$$\left|\Gamma\left(-\frac{p}{2}+i\frac{s}{2}\right)\right|^2 = 2\int_0^\infty K_{is}(y)\left(1-\left(\frac{y}{2}\right)^p \sum_{0\le k<p/2} \frac{2(p-2k)}{k!(p-k)!}K_{p-2k}(y)\right)\left(\frac{2}{y}\right)^{p+1} dy. \tag{3.14}$$

Proof. First we note that the integrability follows from the bound (3.11) above. Next, from Relations (3.8), (3.13), and Lemma 4.1 we have, for all $p+is \in \mathbb{C}$, $p \notin \mathbb{Z}$,

$$\begin{aligned}
&\int_0^\infty K_{is}(y)\left(1-\left(\frac{y}{2}\right)^p \sum_{0\le k<p/2} \frac{2(p-2k)}{k!(p-k)!}K_{p-2k}(y)\right)\frac{dy}{y^{p+1}} \\
&= \frac{\pi}{\sin(p\pi)}\int_0^\infty K_{is}(y)\left(\frac{\sin(p\pi)}{\pi} - \left(\frac{y}{2}\right)^p \sum_{0\le k<p/2}\frac{p-2k}{k!(p-k)!}I_{-(p-2k)}(y)\right. \\
&\qquad\qquad \left. + \left(\frac{y}{2}\right)^p \sum_{0\le k<p/2}\frac{p-2k}{k!(p-k)!}I_{p-2k}(y)\right)\frac{dy}{y^{p+1}} \\
&= \frac{\pi}{\sin(p\pi)}\int_0^\infty K_{is}(y)\left(\frac{\sin(p\pi)}{\pi} - \left(\frac{y}{2}\right)^p \sum_{0\le k<p/2}\frac{p-2k}{k!(p-k)!}I_{-(p-2k)}(y)\right)\frac{dy}{y^{p+1}} \\
&\quad + \frac{\pi 2^{-2p}}{\sin(p\pi)}\sum_{0\le k<p/2}\frac{p-2k}{k!(p-k)!}\int_0^\infty K_{is}(y)I_{p-2k}(y)\frac{dy}{y} \\
&= \frac{\pi}{\sin(p\pi)}\int_0^\infty K_{is}(y)\left(\frac{\sin(p\pi)}{\pi} - \left(\frac{y}{2}\right)^p \sum_{0\le k<p/2}\frac{p-2k}{k!(p-k)!}I_{-(p-2k)}(y)\right)\frac{dy}{y^{p+1}} \\
&\quad + \frac{\pi 2^{-p}}{\sin(p\pi)}\sum_{0\le k<p/2}\frac{p-2k}{k!((-(p-2k))^2+s^2)(p-k)!} \\
&= 2^{-p-2}\left|\Gamma\left(-\frac{p}{2}+i\frac{s}{2}\right)\right|^2 - \frac{\pi 2^{-p}}{\sin(p\pi)}\sum_{0\le k<p/2}\frac{p-2k}{k!((p-2k)^2+s^2)(p-k)!} \\
&\quad + \frac{\pi 2^{-p}}{\sin(p\pi)}\sum_{0\le k<p/2}\frac{p-2k}{k!((-(p-2k))^2+s^2)(p-k)!} \\
&= 2^{-p-2}\left|\Gamma\left(-\frac{p}{2}+i\frac{s}{2}\right)\right|^2.
\end{aligned}$$

□

Corollary 3.4 can be extended by continuity to all $p+is \in \mathbb{C}$ with $p \notin 2\mathbb{N}$, although this is not necessary for the proof of Proposition 2.1. The extension to all $p+is \in \mathbb{C}\setminus(2\mathbb{N})$

should be more difficult, for example when $p = 0$ and $s \neq 0$, Relation (3.14) reads

$$|\Gamma(is/2)|^2 = 4 \int_0^\infty K_{is}(y) \frac{dy}{y},$$

but the function $y \mapsto K_{is}(y)/y$ does not appear to be in $L^1(\mathbb{R}_+)$.

4 Appendix

In this appendix we give the proof of some basic identities that have been used in this paper. In particular, the second part of Lemma 4.1 provides a simplified version of Relation (6.576.5) page 684 of [12] in a particular case.

Lemma 4.1 *For all $p < 0$ and $s \in \mathbb{R}$ we have the integral representations*

$$\left|\Gamma\left(-\frac{p}{2} + i\frac{s}{2}\right)\right|^2 = 4 \int_0^\infty K_{is}(y) \left(\frac{2}{y}\right)^p \frac{dy}{y} \qquad \text{and} \qquad \frac{1}{p^2 + s^2} = \int_0^\infty K_{is}(y) I_{-p}(y) \frac{dy}{y}.$$

Proof. From the integral representation of Bessel functions

$$K_{is}(y) = \frac{1}{2} \left(\frac{y}{2}\right)^{is} \int_0^\infty x^{-is-1} e^{-x-y^2/(4x)} dx, \tag{4.1}$$

cf. [20] page 183, and the Fubini theorem, we have

$$\begin{aligned}
\int_0^\infty K_{is}(t) \frac{dt}{t^{1+p}} &= 2^{-is-1} \int_0^\infty t^{-1-p+is} \int_0^\infty e^{-x-t^2/(4x)} x^{-is-1} dx dt \\
&= 2^{-is-1} \int_0^\infty x^{-is-1} e^{-x} \int_0^\infty t^{-1-p+is} e^{-t^2/(4x)} dt dx \\
&= 2^{-2-p} \int_0^\infty x^{-is-1} e^{-x} x^{-p/2+is/2} \int_0^\infty u^{-1-p/2+is/2} e^{-u} du dx \\
&= 2^{-2-p} \Gamma\left(-\frac{p}{2} + i\frac{s}{2}\right) \int_0^\infty x^{-1-p/2-is/2} e^{-x} dx \\
&= 2^{-2-p} \Gamma\left(-\frac{p}{2} + i\frac{s}{2}\right) \Gamma\left(-\frac{p}{2} - i\frac{s}{2}\right).
\end{aligned}$$

By the equivalence (3.2) and the above argument, $y \mapsto K_{is}(y) I_{-p}(y)/y$ is integrable on $\mathbb{R}_+$ for all $p < 0$, and we have

$$\begin{aligned}
\int_0^\infty K_{is}(y) I_{-p}(y) \frac{dy}{y} &= 2^{-is-1} \int_0^\infty y^{-1+is} I_{-p}(y) \int_0^\infty e^{-x-y^2/(4x)} x^{-is-1} dx dy \\
&= 2^{-is-1} \int_0^\infty y^{-1+is} \int_0^\infty e^{-x-y^2/(4x)} x^{-is-1} I_{-p}(y) dx dy
\end{aligned}$$

$$
\begin{aligned}
&= 2^{-is-1}\int_0^\infty x^{-is-1}e^{-x}\int_0^\infty y^{-1+is}e^{-y^2/(4x)}I_{-p}(y)dydx \\
&= 2^{-2}\int_0^\infty x^{-is-1/2}e^{-x}\int_0^\infty z^{-1+is/2}e^{-z}I_{-p}(2\sqrt{zx})dzdx \\
&= 2^{-2}\frac{\Gamma\left(-\frac{p}{2}+i\frac{s}{2}\right)}{\Gamma(1-p)}\int_0^\infty x^{-p/2-is/2-1}e^{-x}\Phi\left(-\frac{p}{2}+i\frac{s}{2},1-p,x\right)dx \\
&= \frac{2^{-2}\Gamma\left(-\frac{p}{2}+i\frac{s}{2}\right)\Gamma\left(-\frac{p}{2}-i\frac{s}{2}\right)}{\Gamma\left(1-\frac{p}{2}+i\frac{s}{2}\right)\Gamma\left(1-\frac{p}{2}-i\frac{s}{2}\right)} \\
&= \frac{1}{p^2+s^2}.
\end{aligned}
$$

In the above calculations we have used the relations

$$
\Phi\left(-\frac{p}{2}+i\frac{s}{2},1-p,x\right)=\frac{\Gamma(1-p)}{\Gamma\left(-\frac{p}{2}+i\frac{s}{2}\right)}x^{p/2}\int_0^\infty e^{-z}z^{-1+is/2}I_{-p}(2\sqrt{xz})dz,
$$

cf. (6.643-2) and (9.220-2), pages 709 and 1024 of [12], and

$$
\int_0^\infty z^{-p/2-is/2-1}e^{-z}\Phi\left(-\frac{p}{2}+i\frac{s}{2},1-p,z\right)dz=\frac{\Gamma(1-p)\Gamma\left(-\frac{p}{2}-i\frac{s}{2}\right)}{\Gamma\left(1-\frac{p}{2}-i\frac{s}{2}\right)\Gamma\left(1-\frac{p}{2}-i\frac{s}{2}\right)},
$$

cf. (7.621-7) page 822 of [12], where $\Phi(a,b,z)$ is the degenerate hypergeometric function, which are valid for all $p<0$ and $s\in\mathbb{R}$. □

In the next proposition we compute the expression of the heat kernel used in the proof of Relation (1.6).

Proposition 4.2 *For $t>0$ and $x,y\in\mathbb{R}$, the kernel $q_t(x,y)$ of $\exp(tH)$ is given by*

$$
q_t(x,y)=\frac{2}{x\pi^2}\int_0^\infty ue^{-u^2t}\sinh(\pi u)K_{iu}(x)K_{iu}(y)du. \tag{4.2}
$$

Proof. We start by computing the Green kernel associated to the operator $(H-p^2)^{-1}$, and for simplicity it suffices to consider the case $p\geq 0$. From [4], page 115, the Green function associated to

$$
\tilde{H}_p:=\frac{\partial^2}{\partial x^2}+\frac{2p+1}{x}\frac{\partial}{\partial x}-1
$$

is given by

$$
\tilde{G}(x,y;p)=x^{p+1}y^{-p}I_p(x)K_p(y),\qquad 0<x\leq y.
$$

Next, noting that for all smooth functions f on $\mathbb{R}$ we have

$$(\tilde{H}_p f)(y) = y^{-p-2}\left(H - p^2\right)(y^p f(y)),$$

we get

$$G(x, y; p) = y^p x^{-p-2}\tilde{G}(x, y; p) = \frac{1}{x} I_p(x) K_p(y).$$

On the other hand, from the relation

$$I_p(\min(x, y))K_p(\max(x, y)) = \frac{2}{\pi^2}\int_0^\infty \frac{u}{u^2 + p^2}\sinh(\pi u) K_{iu}(y) K_{iu}(x) du,$$

$p \in \mathbb{R}$, $(x, y) \in \mathbb{R}^2$, cf. [13] page 228, we get

$$G(x, y; p) = \frac{2}{x\pi^2}\int_0^\infty \frac{u}{u^2 + p^2}\sinh(\pi u) K_{iu}(x) K_{iu}(y) du. \tag{4.3}$$

By inversion of the Laplace transform

$$G(x, y; p) = -\int_0^\infty e^{-p^2 t} q_t(x, y) dt, \qquad x, y \in \mathbb{R},$$

that follows from

$$\left(H - p^2\right)^{-1} = -\int_0^\infty e^{-p^2 t} e^{tH} dt,$$

and using the Fubini theorem, we get

$$\begin{aligned} q_t(x, y) &= \frac{i}{2\pi}\int_{-i\infty}^{+i\infty} e^{zt} G(x, y; \sqrt{z}) dz \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{izt} G(x, y; \sqrt{iz}) dz \\ &= \frac{1}{x\pi^3}\int_{-\infty}^{\infty} e^{izt}\int_0^\infty \frac{u}{u^2 + iz}\sinh(\pi u) K_{iu}(x) K_{iu}(y) du dz \\ &= \frac{1}{x\pi^3}\int_0^\infty u \sinh(\pi u) K_{iu}(x) K_{iu}(y)\int_{-\infty}^{\infty} \frac{e^{izt}}{u^2 + iz} dz du, \end{aligned}$$

and the last integral can be computed as the Fourier transform

$$\int_{-\infty}^{\infty} \frac{e^{izt}}{u^2 + iz} dz = 2\pi e^{-u^2 t}, \qquad u > 0.$$

□

The next lemma has been used in the proof of Relation (2.6) above.

Lemma 4.3 *We have the integral representation*

$$K_{i\mu}(z) = \frac{1}{\sinh(\pi\mu/2)} \int_0^{+\infty} \sin(z \sinh t) \sin(\mu t) dt, \qquad z \in \mathbb{R}_+, \quad \mu \in \mathbb{R}. \tag{4.4}$$

Proof. For all $\mu \geq 0$ we have

$$K_{i\mu}(z) = \frac{1}{2} e^{\mu\pi/2} \int_{-\infty}^{+\infty} e^{-iz \sinh t - i\mu t} dt$$

and

$$K_{i\mu}(z) = \frac{1}{2} e^{-\mu\pi/2} \int_{-\infty}^{+\infty} e^{-iz \sinh t + i\mu t} dt,$$

cf. page 182 of [20], which leads to

$$\left(2e^{\mu\pi/2} - 2e^{-\mu\pi/2}\right) K_{i\mu}(z) = \int_{-\infty}^{+\infty} e^{-iz \sinh t} \left(e^{i\mu t} - e^{-i\mu t}\right) dt,$$

and

$$K_{i\mu}(z) = \frac{i}{2 \sinh(\pi\mu/2)} \int_{-\infty}^{+\infty} e^{-iz \sinh t} \sin(\mu t) dt.$$

Finally, (4.4) is obtained by separation of the real and imaginary parts in the latter integral. □

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