# Integrability and regularity of the flow of stochastic differential equations with jumps

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**Abstract**

We derive sufficient conditions for the differentiability of all orders for the flow of stochastic differential equations with jumps, and prove related $L^p$-integrability results for all orders. Our results extend similar results obtained in [Kun04] for first order differentiability and rely on the Burkholder-Davis-Gundy (BDG) inequality for time-inhomogeneous Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}$, for which we provide a new proof.

*Keywords*: Stochastic differential equations with jumps, moment bounds, Poisson random measures, stochastic flows, Markov semigroups.
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## 1 Introduction

In this paper we consider the regularity and integrability of all orders of the flow of Stochastic Differential Equations (SDEs) with jumps of the form

$$dX_t(x) = r(t, X_t(x))dt + \sigma(t, X_t(x))dW_t + \int_{-\infty}^{\infty} g(t, X_{t^-}(x), y)\big(N(dt, dy) - \nu_t(dy)dt\big),$$

$$(1.1)$$

with $X_0(x) = x$, where $X_{t^-}$ stands for $\lim_{s \nearrow t} X_s$, $g : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a deterministic measurable function and $(W_t)_{t \in \mathbb{R}_+}$, $N(dt, dy)$ are a standard Brownian motion

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and a Poisson random measure on $\mathbb{R}^+ \times \mathbb{R}$ with compensator $\nu_t(dy)dt$, generating a filtration $(\mathcal{F}_t)_{t \geq 0}$.

In the diffusion case, the smoothness of the solution flow $x \mapsto X_t(x)$ of stochastic differential equations of the form

$$dX_t(x) = r\big(t, X_t(x)\big) \ dt + \sigma\big(t, X_t(x)\big) \ dW_t \tag{1.2}$$

with $X_0(x) = x$, where $r : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are deterministic co-efficients, has been studied in [Kun84], [Kun90]. In Theorem II.4.4 of [Kun84] and Theorem 4.6.5 of [Kun90] it is shown that $x \mapsto X_t(x)$ is $k$ times continuously differentiable when the SDE coefficients of (1.2) are $\mathcal{C}^k$ functions with globally Lipschitz derivatives. Such results have been proved in the jump-diffusion case in [Kun04] in the case of first order differentiability, however, the extension to higher orders of differentiability is not trivial and requires us to use the framework of [BGJ87].

Our proofs rely on the Burkholder-Davis-Gundy (BDG) inequality, which states that for any martingale $(M_t)_{t \in \mathbb{R}_+}$ and for all $p \geq 1$ we have

$$\mathbb{E}\big[\big|M_t^*\big|^p\big] \leq C_p \mathbb{E}\big[[M, M]_t^{p/2}\big], \qquad t \in \mathbb{R}_+, \tag{1.3}$$

where $M_t^* = \sup_{s \in [0,t]} |M_s|$, with

$$C_p = (10p)^{p/2} \ \text{for} \ 1 \leq p < 2, \ C_2 = 2^p, \ \text{and} \ C_p = p^p \left(\frac{e}{2}\right)^{p/2} \ \text{for} \ p > 2, \tag{1.4}$$

cf. e.g. Theorem 4.2.12 of [Bic02] or Theorem 48 in Chapter IV of [Pro04]. When $p = 2$ we have

$$\mathbb{E}\left[\sup_{s \in [0,t]} |M_s|^p\right] \leq C_p \mathbb{E}\big[\langle M, M \rangle_t^{p/2}\big], \qquad t \in \mathbb{R}_+, \tag{1.5}$$

which implies the bound

$$\mathbb{E}\left[\sup_{s \in [0,t]} |M_s|^p\right] \leq C_p \big(\mathbb{E}[\langle M, M \rangle_t]\big)^{p/2}, \quad t \in \mathbb{R}_+,$$

for $p \in [1, 2]$. However, (1.5) does not extend to any $p > 2$, see e.g. Remark 357 page 384 of [Sit05]. For this reason we use Kunita's BDG inequality for jump processes, see

Theorem 2.11 in [Kun04], which is recovered under a form similar to Corollary 2.14 in [Hau11], see Lemma 2.1 and Corollary 2.2 below. This also extends related results obtained in the case of a standard Poisson process, in [HK05] see Corollary 1 and Lemma 1 therein, with application to the numerical solution of SDEs.

We proceed by deriving moment bounds for the solutions of SDEs with jumps of the form (1.1) in Theorem 3.1. Similar bounds have been obtained in Theorem 3.2 of [Kun04], however, here we work with random $(\mathcal{F}_t)_{t \geq 0}$-adapted coefficients and under weaker integrability conditions. Other moment bounds for SDEs with jumps have been derived using (1.3) in various works, see for example Lemma 1 in [ZWL15] or Lemma 2.2 in [ZZ16]. However, those approaches rely on the incorrect assumption that (1.5) holds for any $p \geq 1$. Nevertheless, (1.5) is valid for $p = 2$, and in this case it has been used in [ZZ16] to derive bounds on $\mathbb{E}\left[\sup_{s \in [0,t]} |M_s|^p\right]$ for $p > 1$, see the proof of Theorem 2.1 therein and also [PW19] for an application of Kunita's BDG inequality to SIR population models for any $p > 2$.

The proofs of Proposition 4.1 on the existence of the flow derivatives and of Theorem 5.1 on their $L^p$ integrability rely on Theorems 6-29 and 6-44 of [BGJ87]. For this reason, in Sections 4 and 5 we will assume that the compensators $\nu_t(dy)$, $t \in \mathbb{R}_+$, in (1.1) are dominated by a (deterministic) measure $\eta$ on $\mathbb{R}$, i.e.

$$\nu_t(A) \leq \eta(A), \quad A \in \mathcal{B}(\mathbb{R}), \tag{1.6}$$

where $\mathcal{B}(\mathbb{R})$ denotes the Borel $\sigma$-algebra of $\mathbb{R}$, in addition to the following Assumption $(A_n)$, see $(A'\text{-}r)$ page 60 in [BGJ87].

**Assumption $(A_n)$:** *For every $t \in \mathbb{R}_+$, the functions $r(t, \cdot) : \mathbb{R} \to \mathbb{R}$, $\sigma(t, \cdot) : \mathbb{R} \to \mathbb{R}$ and $g(t, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are $\mathcal{C}^n$-differentiable and there is a constant $C > 0$ such that*

$$\left|\frac{\partial^k r}{\partial x^k}(t, x)\right| \leq C, \quad \left|\frac{\partial^k \sigma}{\partial x^k}(t, x)\right| \leq C, \quad \left|\frac{\partial^{k+l} g}{\partial x^k \partial y^l}(t, x, y)\right| \leq C,$$

*for all $k, l = 1, \ldots, n$ with $1 \leq k + l \leq n$, and a function $\theta \in \bigcap_{q \geq 2} L^q(\mathbb{R}, \eta)$ such that*

$$\left|\frac{\partial^k g}{\partial x^k}(t, x, y)\right| \leq C\theta(y), \qquad x, y \in \mathbb{R}, \quad k = 1, \ldots, n. \tag{1.7}$$

Although the results of [BGJ87] are stated for time-homogeneous SDE coefficients in (1.1), they remain valid under our time-inhomogeneous Assumption $(A_n)$. For this, we note that Theorems 6-20, 6-24, 6-29 and 6-44 in [BGJ87] all rely on Lemma 5.1 page 44 therein, which extends to the time-inhomogeneous case thanks to the domination condition (1.6), see Lemma A.14 of [BJ83] and Theorem 2.1 in [Bic81].

Under (1.6) and Assumption $(A_n)$, in Theorem 5.1 we provide sufficient conditions for the flow derivative

$$X_t^{(n)}(x) := \frac{\partial^n X_t}{\partial x^n}(x) \tag{1.8}$$

to exist and belong to $L^p(\Omega)$ uniformly in $(x,t) \in \mathbb{R} \times [0,T]$, i.e.,

$$\sup_{x \in \mathbb{R}} \mathbb{E}\left[ \sup_{t \in [0,T]} \left| X_t^{(n)}(x) \right|^p \right] < \infty,$$

for given orders of derivation $n \geq 1$ and of integrability $p \geq 1$.

Flow regularity results up to order four of differentiability have also been obtained in [PT97] based on a different version of the BDG inequality for Lévy processes with stochastic integrands depending only on time (see Lemma 4.1 page 409 of [PT97]), with application to the convergence of the Euler method.

As a consequence of (1.8), when $f(x) := (x - K)^+$ is the (Lipschitz) payoff function of a European call option, we can also express the Delta, or first derivative of the option price with respect to the underlying price $x$ as

$$\frac{\partial P_t f}{\partial x}(x) = \mathbb{E}\left[ \mathbf{1}_{\{X_t(x) \geq K\}} X_t^{(1)}(x) \right], \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R}.$$

More generally, given the transition semigroup $(P_t)_{t \geq 0}$ of $(X_t(x))_{t \geq 0}$, defined as

$$P_t f(x) := \mathbb{E}\big[ f(X_t(x)) \big], \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R},$$

we deduce that for any $f \in \mathcal{C}_b^\infty(\mathbb{R})$ the function $x \mapsto P_t f(x)$ is $\mathcal{C}^\infty$, with

$$\frac{\partial^n P_t f}{\partial x^n}(x) = \sum_{\pi \in \Pi[n]} \mathbb{E}\left[ \frac{\partial^{|\pi|} f}{\partial x^{|\pi|}}(x, X_t(x)) \prod_{B \in \pi} X_t^{(|B|)}(x) \right], \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R},$$

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by the Faà di Bruno formula, where the sum over $\pi$ runs in the set $\Pi[n]$ of all partitions of $\{1, 2, \ldots, n\}$, the product over $B \in \pi$ runs in all blocks $B$ in the partition $\pi$, and $|A|$ stands for the cardinality of the set $A$. The moment bounds obtained in this paper are also applied to the derivation of distance estimates between jump-diffusion processes in [BP19].

We proceed as follows. In Section 2 we derive two versions of the BDG inequality with jumps, similarly to Theorem 2.11 [Kun04] and to Corollary 2.14 of [Hau11], and we show that they can be unified in Corollary 2.2. This is followed by moment bounds for SDEs in Section 3. In Section 4 we deal with the flow derivatives $X_t^{(n)}(x)$ by noting that they satisfy an affine SDE, for which moment bounds can be obtained from Theorem 3.1, see Proposition 4.1. Next, in Section 5 we present our result on moment bounds for flow derivatives, see Theorem 5.1.

## 2 Burkholder-Davis-Gundy inequality with jumps

Our moment bounds rely on a version of the BDG inequality (1.3) which uses the compensator $\langle M, M \rangle$ of $M$ instead of its bracket $[M, M]$. Consider the compensated Poisson stochastic integral process

$$K_t := \int_0^t \int_{-\infty}^{\infty} g_s(y) \left( N(ds, dy) - \nu_s(dy)ds \right), \qquad t \in \mathbb{R}_+, \tag{2.1}$$

of the predictable integrand $(g_s(y))_{(s,y) \in \mathbb{R}_+ \times \mathbb{R}}$, where $N$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with compensator $\nu_t(dy)dt$. When $p = 2$, the BDG inequality (1.3) shows that

$$\mathbb{E}\left[ \left( K_t^* \right)^2 \right] \leq 2\mathbb{E}\left[ \int_0^t \int_{-\infty}^{\infty} \left( g_s(y) \right)^2 N(ds, dy) \right] = 2\mathbb{E}\left[ \int_0^t \int_{-\infty}^{\infty} \left( g_s(y) \right)^2 \nu_s(dy)ds \right],$$

where $K_t^* = \sup_{s \in [0,t]} |K_s|$, since $t \mapsto \int_0^t \int_{-\infty}^{\infty} \left( g_s(y) \right)^2 \left( N(ds, dy) - \nu_s(dy)ds \right)$ is a martingale. In particular, for any $p \in [1, 2]$ we have

$$\begin{aligned} \mathbb{E}\left[ (K_t^*)^p \right] &\leq \left( \mathbb{E}\left[ (K_t^*)^2 \right] \right)^{p/2} \\ &\leq 2^{p/2} \left( \mathbb{E}\left[ \int_0^t \int_{-\infty}^{\infty} (g_s(y))^2 \, \nu_s(dy)ds \right] \right)^{p/2}, \quad t \in \mathbb{R}_+. \end{aligned} \tag{2.2}$$

Lemma 2.1 below extends the BDG inequality to $p > 2$ with explicit bounding constants, in relation to the BDG inequality stated for $1 < p \leq 2$ in Corollary 2.14 of [Hau11].

**Lemma 2.1** *Consider the compensated Poisson stochastic integral process* $(K_t)_{t \in \mathbb{R}_+}$ *in* (2.1) *of a predictable integrand* $(g_s(y))_{(s,y) \in \mathbb{R}_+ \times \mathbb{R}}$. *Then, for all* $p \geq 2$ *we have*

$$\mathbb{E}\big[(K_t^*)^p\big] \leq \frac{2}{p}(40p)^{p/2} \left(\frac{p^2 e}{2}\right)^{p(\log_2 p)/2} \mathbb{E}\left[\int_0^t \int_{-\infty}^{\infty} |g_s(y)|^p \, \nu_s(dy)ds\right] \tag{2.3}$$

$$+ 2^p \sum_{k=1}^{\lceil \log_2 p \rceil - 1} \frac{p^{pk}}{2^k} \left(\frac{e}{2}\right)^{kp/2} \mathbb{E}\left[\left(\int_0^t \int_{-\infty}^{\infty} (g_s(y))^{2^k} \nu_s(dy)ds\right)^{p/2^k}\right], \quad t \in \mathbb{R}_+.$$

*Proof.* For $r \geq 1$ let

$$K_t^{(r)} := \int_0^t \int_{-\infty}^{\infty} (g_s(y))^r \left(N(ds, dy) - \nu_s(dy)ds\right),$$

with $K_t^{(1)} = K_t$, $t \in \mathbb{R}_+$. When $p > 2$, since $x \mapsto |x|^{p/2}$ is convex, (1.3) entails

$$\mathbb{E}\big[((K_t^{(r)})^*)^p\big] \leq p^p \left(\frac{e}{2}\right)^{p/2} \mathbb{E}\left[\left(\int_0^t \int_{-\infty}^{\infty} (g_s(y))^{2r} \, N(ds, dy)\right)^{p/2}\right]$$

$$= p^p \left(\frac{e}{2}\right)^{p/2} \mathbb{E}\left[\left|\int_0^t \int_{-\infty}^{\infty} (g_s(y))^{2r} \left(N(ds, dy) - \nu_s(dy)\right)ds + \int_0^t \int_{-\infty}^{\infty} (g_s(y))^{2r} \nu_s(dy)ds\right|^{p/2}\right]$$

$$\leq 2^{p/2-1} p^p \left(\frac{e}{2}\right)^{p/2} \mathbb{E}\left[\left|\int_0^t \int_{-\infty}^{\infty} (g_s(y))^{2r} \left(N(ds, dy) - \nu_s(dy)ds\right)\right|^{p/2}\right]$$

$$+ 2^{p/2-1} \left(\frac{e}{2}\right)^{p/2} \mathbb{E}\left[\left|\int_0^t \int_{-\infty}^{\infty} (g_s(y))^{2r} \nu_s(dy)ds\right|^{p/2}\right]$$

$$\leq 2^{p/2-1} p^p \left(\frac{e}{2}\right)^{p/2} \mathbb{E}\big[((K_t^{(2r)})^*)^{p/2}\big] + 2^{p/2-1} p^p \left(\frac{e}{2}\right)^{p/2} \mathbb{E}\left[\left(\int_0^t \int_{-\infty}^{\infty} (g_s(y))^{2r} \nu_s(dy)ds\right)^{p/2}\right]. \tag{2.4}$$

The recursive application of the bound (2.4) starting from $K_t^{(1)} = K_t$ yields

$$\mathbb{E}\big[(K_t^*)^p\big] \leq p^{np} \left(\frac{e}{2}\right)^{pn/2} \left(\prod_{j=1}^n 2^{p/2^j-1}\right) \mathbb{E}\big[((K_t^{(2^n)})^*)^{p/2^n}\big]$$

$$+ \sum_{k=1}^n p^{kp} \left(\frac{e}{2}\right)^{pn/2} \left(\prod_{j=1}^k 2^{p/2^j-1}\right) \mathbb{E}\left[\left(\int_0^t \int_{-\infty}^{\infty} (g_s(y))^{2^k} \nu_s(dy)ds\right)^{p/2^k}\right].$$

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Taking $n = \lceil \log_2 p \rceil - 1$, i.e. $p \in (2^n, 2^{n+1}]$, by (1.3) we have

$$
\begin{aligned}
\mathbb{E}\big[\big(\big(K_t^{(2^n)}\big)^*\big)^{p/2^n}\big] &\leq (10p)^{p/2}\mathbb{E}\left[\left(\int_0^t \int_{-\infty}^\infty \big(g_s(y)\big)^{2^{n+1}} N(ds, dy)\right)^{p/2^{n+1}}\right] \\
&\leq (10p)^{p/2}\mathbb{E}\left[\int_0^t \int_{-\infty}^\infty |g_s(y)|^p N(ds, dy)\right] \\
&= (10p)^{p/2}\mathbb{E}\left[\int_0^t \int_{-\infty}^\infty |g_s(y)|^p \nu_s(dy)ds\right],
\end{aligned}
$$

since $t \mapsto \int_0^t \int_{-\infty}^\infty \big(g_s(y)\big)^2 \big(N(ds, dy) - \nu_s(dy)ds\big)$ is a martingale, where we used the fact that

$$
\|a\|_{\ell^2} := \left(\sum_{k=0}^\infty (a_k)^2\right)^{1/2} \leq \left(\sum_{k=0}^\infty (a_k)^q\right)^{1/q} =: \|a\|_{\ell^q}
$$

for any real sequence $(a_k)_{k \in \mathbb{N}}$ and $1 \leq q = p/2^n \leq 2$ as on page 410 after Equation (22) in [PT97] , which allows us to conclude to (2.3). □

From Lemma 2.1 we recover the following version of the Kunita's BDG inequality with jumps, cf. Theorem 2.11 of [Kun04] and Theorem 4.4.23 of [App09].

**Corollary 2.2** *Consider the compensated Poisson stochastic integral process*

$$
K_t := x + \int_0^t u_s \, ds + \int_0^t v_s \, dW_s + \int_0^t \int_{-\infty}^\infty g_s(y) \, \big(N(ds, dy) - \nu_s(dy)ds\big), \quad t \in \mathbb{R}_+,
$$
(2.5)

*of the predictable integrands* $(u_s)_{s \in \mathbb{R}_+}$, $(v_s)_{s \in \mathbb{R}_+}$, $(g_s(y))_{(s,y) \in \mathbb{R}_+ \times \mathbb{R}}$. *Then, for all* $p \geq 2$ *and* $T \in \mathbb{R}_+$ *we have*

$$
\begin{aligned}
&\mathbb{E}\big[(K_T^*)^p\big] \\
&\leq 2^{2p-2}\left(|x|^p + \mathbb{E}\left[\left(\int_0^T |u_t| \, dt\right)^p\right] + C_p\mathbb{E}\left[\left(\int_0^T |v_t|^2 \, dt\right)^{p/2}\right] \right. \\
&\left. + \widetilde{C}_p\mathbb{E}\left[\int_0^T \int_{-\infty}^\infty |g_t(y)|^p \, \nu_t(dy)dt\right] + \widetilde{C}_p\mathbb{E}\left[\left(\int_0^T \int_{-\infty}^\infty |g_t(y)|^2 \, \nu_t(dy)dt\right)^{p/2}\right]\right),
\end{aligned}
$$

*where* $C_p$ *is defined in* (1.4), *and*

$$
\widetilde{C}_p = \frac{2}{p}(40p)^{p/2}\left(\frac{p^2 e}{2}\right)^{p(\log_2 p)/2} + 2^p \sum_{k=1}^{\lceil \log_2 p \rceil - 1} \frac{p^{pk}}{2^k}\left(\frac{e}{2}\right)^{kp/2} \leq 2^p p^{p \log_2 p}\left(2 + \big(10e^{\lceil \log_2 p \rceil}\big)^{p/2}\right).
$$

*Proof.* By the convexity of $\mathbb{R} \ni x \mapsto |x|^p$, we have

$$
\mathbb{E}\left[\sup_{s \in [0,T]} |K_{t^-}|^p\right] \leq 4^{p-1}\left(x^p + \mathbb{E}\left[\sup_{t \in [0,T]} \left|\int_0^t u_s \, ds\right|^p\right] + \mathbb{E}\left[\sup_{t \in [0,T]} \left|\int_0^t v_s \, dW_s\right|^p\right]\right.
$$

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$$+ \mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_0^{t^-} \int_{-\infty}^\infty g_{s^-}(z) \left( N(ds, dz) - \nu_s(dz)ds \right) \right|^p \right].$$

Further, by the (standard) BDG inequality for Brownian stochastic integrals, the Jensen inequality for the uniform measure on $[0, t]$ and the Fubini theorem, we find:

$$\mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_0^t v_s \, dW_s \right|^p \right] \leq C_p \mathbb{E}\left[ \left( \int_0^T |v_s|^2 \, ds \right)^{p/2} \right].$$

Regarding the jump term, by the log-convexity in $p$ of the $L^p$ norms, taking $n = \lceil \log_2 p \rceil - 1$, i.e. $p \in (2^n, 2^{n+1}]$, and $\theta \in (0, 1)$ such that $2^{-k} = (1 - \theta)/p + \theta/2$, we have

$$\mathbb{E}\left[ \left( \int_0^T \int_{-\infty}^\infty (g_t(y))^{2^k} \, \nu_t(dy)dt \right)^{p/2^k} \right]$$

$$\leq \mathbb{E}\left[ \left( \int_0^T \int_{-\infty}^\infty (g_t(y))^p \, \nu_t(dy)dt \right)^{1-\theta} \left( \int_0^T \int_{-\infty}^\infty (g_t(y))^2 \, \nu_t(dy)dt \right)^{\theta p/2} \right]$$

$$\leq \left( \mathbb{E}\left[ \int_0^T \int_{-\infty}^\infty (g_t(y))^p \, \nu_t(dy)dt \right] \right)^{1-\theta} \left( \mathbb{E}\left[ \left( \int_0^T \int_{-\infty}^\infty (g_t(y))^2 \, \nu_t(dy)dt \right)^{p/2} \right] \right)^\theta$$

$$\leq (1-\theta)\mathbb{E}\left[ \int_0^T \int_{-\infty}^\infty (g_t(y))^p \, \nu_t(dy)dt \right] + \theta \mathbb{E}\left[ \left( \int_0^T \int_{-\infty}^\infty (g_t(y))^2 \, \nu_t(dy)dt \right)^{p/2} \right]$$

$$\leq \mathbb{E}\left[ \int_0^T \int_{-\infty}^\infty (g_t(y))^p \, \nu_t(dy)dt \right] + \mathbb{E}\left[ \left( \int_0^T \int_{-\infty}^\infty (g_t(y))^2 \, \nu_t(dy)dt \right)^{p/2} \right], \quad T \in \mathbb{R}_+,$$

$k = 1, \ldots, n$, after using the Hölder inequality and the bound $x^{1-\theta}y^\theta \leq (1 - \theta)x + \theta y$, $x, y \geq 0$. Hence, substituting this bound in (2.3), we obtain

$$\mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_0^{t^-} \int_{-\infty}^\infty g_{s^-}(z) \left( N(ds, dz) - \nu_s(dz)ds \right) \right|^p \right]$$

$$\leq \frac{2^{p+1}}{p} p^{p \log_2 p} (10p)^{p/2} \left( \frac{e}{2} \right)^{p(\log_2 p)/2} \mathbb{E}\left[ \int_0^T \int_{-\infty}^\infty (g_t(y))^p \, \nu_t(dy)dt \right]$$

$$+ 2^p \sum_{k=1}^{\lceil \log_2 p \rceil - 1} \frac{p^{pk}}{2^k} \left( \frac{e}{2} \right)^{kp/2} \mathbb{E}\left[ \left( \int_0^T \int_{-\infty}^\infty (g_t(y))^{2^k} \, \nu_t(dy)dt \right)^{p/2^k} \right]$$

$$\leq \widetilde{C}_p \left( \mathbb{E}\left[ \int_0^T \int_{-\infty}^\infty (g_t(y))^p \, \nu_t(dy)dt \right] + \mathbb{E}\left[ \left( \int_0^T \int_{-\infty}^\infty (g_t(y))^2 \, \nu_t(dy)dt \right)^{p/2} \right] \right).$$

$\square$

The following consequence of Corollary 2.2 recovers Corollary 2.12 in [Kun04] using the Hölder inequality.

**Corollary 2.3** *Consider the compensated Poisson stochastic integral process* $(K_t)_{t\in[0,T]}$ *in* (2.5). *For all* $p \geq 2$ *and* $T \in \mathbb{R}_+$ *we have*

$$\mathbb{E}\big[(K_T^*)^p\big] \leq 2^{2p-2}\left(|x|^p + T^{p-1}\mathbb{E}\left[\int_0^T |u_t|^p \ dt\right] + C_p T^{p/2-1}\mathbb{E}\left[\int_0^T |v_t|^p \ dt\right]\right.$$
$$\left. + \widetilde{C}_p\mathbb{E}\left[\int_0^T \int_{-\infty}^\infty |g_t(y)|^p \ \nu_t(dy)dt\right] + T^{p/2-1}\widetilde{C}_p\mathbb{E}\left[\int_0^T \left(\int_{-\infty}^\infty |g_t(y)|^2 \ \nu_t(dy)\right)^{p/2} dt\right]\right).$$

When the integrand $g_t(y)$ satisfies $|g_t(y)| \leq |f(y)||g_t|$ where $f(y)$ is a deterministic function of $y \in \mathbb{R}$, $(g_t)_{t\in\mathbb{R}_+}$ is an $(\mathcal{F}_t)_{t\geq0}$-adapted process, and $(\nu_t(dy))_{t\in\mathbb{R}_+} = \nu(dy)$, $t \in \mathbb{R}_+$, is the intensity measure of a time-homogeneous Poisson point process, Corollary 2.3 yields

$$\mathbb{E}\big[(K_T^*)^p\big] \leq 2^{2p-2}\left(|x|^p + T^{p-1}\mathbb{E}\left[\int_0^T |u_t|^p \ dt\right] + C_p T^{p/2-1}\mathbb{E}\left[\int_0^T |v_t|^p \ dt\right]\right.$$
$$\left. + \widetilde{C}_p\left(\int_{-\infty}^\infty |f(y)|^p\nu(dy) + T^{p/2-1}\left(\int_{-\infty}^\infty |f(y)|^2 \ \nu(dy)\right)^{p/2}\right)\mathbb{E}\left[\int_0^T |g_t|^p dt\right]\right),$$

which recovers related versions of the BDG inequality with jumps such as Lemma 5.2 of [BC86] which is stated for $p = 2^n$, $n \geq 1$, or Lemma 4.1 of [PT97] which is stated for $p \geq 2$ using a related recursion. We also refer the reader to Lemma A.14 of [BJ83] and to the proof of Theorem 2.1 in [Bic81], or to [LLP80] and [Pra83], for other versions of the BDG inequality with jumps.

# 3 Moment bounds for SDE solutions

In this section, we derive moment bounds for jump-diffusion SDEs, based on the BDG inequality with jumps given in Corollary 2.2.

The following result provides moment bounds in $L^p(\Omega)$, $p \geq 2$, on the solution of SDEs of the form

$$dX_t = a_t(X_t)dt + b_t(X_t)dW_t + \int_{-\infty}^\infty c_{t^-}(z, X_{t^-})\big(N(dt, dz) - \nu_t(dz)dt\big), \qquad (3.1)$$

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whose existence and uniqueness of solutions follows by standard arguments, see e.g. Theorem 3.1 in [Kun04]. In contrast with Theorem 3.2 of [Kun04], we work with random $(\mathcal{F}_t)_{t\geq 0}$-adapted coefficients and under weaker integrability conditions as Condition (3.2) in [Kun04] requires integrability of all orders. We let $\|X\|_\infty$ stand for the $L^\infty(\Omega)$ norm of a random variable $X$.

**Theorem 3.1** *Let $p \geq 2$ and consider the solution $(X_t)_{t\geq 0}$ of the one-dimensional solution of the jump-diffusion SDE* (3.1), *where the coefficients $(a_t(x))_{t\in[0,T]}$, $(b_t(x))_{t\in[0,T]}$, $(c_t(z,x))_{t\in[0,T]}$ are $(\mathcal{F}_t)_{t\geq 0}$-adapted processes such that*

$$|a_t(x) - a_t(y)| \leq a_t|x - y|, \quad |b_t(x) - b_t(y)| \leq b_t|x - y|, \qquad x, y \in \mathbb{R},$$

*where $(a_t(0))_{t\in[0,T]}, (b_t(0))_{t\in[0,T]} \in L^p(\Omega\times[0,T])$, $(a_t)_{t\in[0,T]}, (b_t)_{t\in[0,T]} \in L^p([0,T], L^\infty(\Omega))$, and*

$$|c_t(z,x) - c_t(z,y)| \leq c_t(z)|x - y|, \quad x, y \in \mathbb{R},$$

*where*

$$\int_0^T \Big\| \int_{-\infty}^\infty |c_t(z)|^2 \, \nu_t(dz) \Big\|_\infty^{p/2} \, dt < \infty, \quad \int_0^T \Big\| \int_{-\infty}^\infty |c_t(z)|^p \, \nu_t(dz) \Big\|_\infty \, dt < \infty, \quad (3.2)$$

*and*

$$\mathbb{E}\left[ \int_0^T \Big( \int_{-\infty}^\infty |c_t(z,0)|^2 \, \nu_t(dz) \Big)^{p/2} \, dt \right] < \infty, \; \mathbb{E}\left[ \int_0^T \int_{-\infty}^\infty |c_t(z,0)|^p \, \nu_t(dz) \, dt \right] < \infty.$$
$$(3.3)$$

*Then we have*

$$\mathbb{E}\left[ \sup_{t\in[0,T]} |X_t|^p \right] < C(p,T) < \infty,$$

*where $C(p,T)$ depends on the above norms of $a, b, c, u, v, w$.*

*Proof.* We have

$$\mathbb{E}\left[ \int_0^T |a_t(X_t)|^p dt \right] \leq 2^{p-1} \left( \int_0^T \|a_t\|_\infty^p \mathbb{E}\big[|X_t|^p\big] dt + \mathbb{E}\left[ \int_0^T |a_t(0)|^p dt \right] \right), \quad (3.4)$$

and

$$\mathbb{E}\left[ \int_0^T |b_t(X_t)|^p dt \right] \leq 2^{p-1} \left( \int_0^T \|b_t\|_\infty^p \mathbb{E}\big[|X_t|^p\big] dt + \mathbb{E}\left[ \int_0^T |b_t(0)|^p dt \right] \right). \quad (3.5)$$

Regarding the jump term, we note that

$$\widetilde{C}_p T^{p/2-1} \mathbb{E}\left[\int_0^T \left(\int_{-\infty}^\infty |c_t(z,X_t)|^2\,\nu_t(dz)\right)^{p/2}\,dt\right] + \widetilde{C}_p \mathbb{E}\left[\int_0^T \int_{-\infty}^\infty |c_t(z,X_t)|^p\,\nu_t(dz)\,dt\right]$$

$$\leq 2^p \widetilde{C}_p T^{p/2-1} \mathbb{E}\left[\int_0^T \left(\int_{-\infty}^\infty |c_t(z)|^2 |X_t|^2\,\nu_t(dz) + \int_{-\infty}^\infty |c_t(z,0)|^2\,\nu_t(dz)\right)^{p/2}\,dt\right]$$

$$+ 2^{p-1}\widetilde{C}_p \mathbb{E}\left[\int_0^T \int_{-\infty}^\infty \left(|c_t(z)|^p |X_t|^p + |c_t(z,0)|^p\right)\,\nu_t(dz)\,dt\right]$$

$$\leq 2^{3p/2-1}\widetilde{C}_p T^{p/2-1} \mathbb{E}\left[\int_0^T \left(\left(\int_{-\infty}^\infty |c_t(z)|^2\,\nu_t(dz)\right)^{p/2}|X_t|^p + \left(\int_{-\infty}^\infty |c_t(z,0)|^2\,\nu_t(dz)\right)^{p/2}\right)\,dt\right]$$

$$+ 2^{p-1}\widetilde{C}_p \mathbb{E}\left[\int_0^T \int_{-\infty}^\infty |c_t(z)|^p\,\nu_t(dz)|X_t|^p\,dt\right] + 2^{p-1}\widetilde{C}_p \mathbb{E}\left[\int_0^T \int_{-\infty}^\infty |c_t(z,0)|^p\,\nu_t(dz)\,dt\right]$$

$$\leq 2^{3p/2-1}\widetilde{C}_p T^{p/2-1} \int_0^T \left\|\int_{-\infty}^\infty |c_t(z)|^2\,\nu_t(dz)\right\|_\infty^{p/2} \mathbb{E}\left[X_t^p\right]\,dt \qquad (3.6)$$

$$+ 2^{3p/2-1}\widetilde{C}_p T^{p/2-1} \mathbb{E}\left[\int_0^T \left(\int_{-\infty}^\infty |c_t(z,0)|^2\,\nu_t(dz)\right)^{p/2}\,dt\right]$$

$$+ 2^{p-1}\widetilde{C}_p \int_0^T \left\|\int_{-\infty}^\infty |c_t(z)|^p\,\nu_t(dz)\right\|_\infty \mathbb{E}\left[X_t^p\right]\,dt + 2^{p-1}\widetilde{C}_p \mathbb{E}\left[\int_0^T \int_{-\infty}^\infty |c_t(z,0)|^p\,\nu_t(dz)\,dt\right].$$

Hence by the BDG inequality of Corollary 2.3 and the bounds (3.4), (3.5), (3.6), setting

$$F(T) := 4^{p-1}\left(x^p + (2T)^{p-1}\,\mathbb{E}\left[\int_0^T |a_t(0)|^p dt\right] + 2^{p-1}C_p T^{p/2-1}\,\mathbb{E}\left[\int_0^T |b_t(0)|^p dt\right]\right.$$

$$\left. + 2^{p-1}\widetilde{C}_p\left(2^{p/2}T^{p/2-1}\mathbb{E}\left[\int_0^T \left(\int_{-\infty}^\infty |c_t(z,0)|^2\,\nu_t(dz)\right)^{p/2}\,dt\right] + \mathbb{E}\left[\int_0^T \int_{-\infty}^\infty |c_t(z,0)|^p\,\nu_t(dz)\,dt\right]\right)\right),$$

and

$$G(t) := 4^{p-1}\left((2T)^{p-1}\|a_t\|_\infty^p + 2^{p-1}C_p T^{p/2-1}\|b_t\|_\infty^p\right.$$

$$\left. + 2^{3p/2-1}\widetilde{C}_p T^{p/2-1}\left\|\int_{-\infty}^\infty |c_t(z)|^2\,\nu_t(dz)\right\|_\infty^{p/2} + 2^{p-1}\widetilde{C}_p\left\|\int_{-\infty}^\infty |c_t(z)|^p\,\nu_t(dz)\right\|_\infty\right),$$

$t \in [0,T]$, we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_{t^-}|^p\right] \leq F(T) + \mathbb{E}\left[\int_0^T G(t)|X_t|^p\,dt\right]$$

$$\leq F(T) + \int_0^T G(t)\mathbb{E}\left[\sup_{s\in[0,t]} |X_s|^p\right]\,dt,$$

hence by the Grönwall lemma we find

$$\mathbb{E}\left[\sup_{t\in[0,T]}|X_{t^-}|^p\right] \le C(p,T) := F(T)\exp\left(\int_0^T G(t)\,dt\right), \tag{3.7}$$

which is finite since $(a_t)_{t\in[0,T]}, (b_t)_{t\in[0,T]} \in L^p([0,T], L^\infty(\Omega))$ and (3.2)-(3.3) are in force. Since $X_{t^-} = X_t$ almost surely, the same bound follows for the moment of order $p$ of $X_t$. $\qquad\square$

Theorem 3.1 applies, in particular, to the solution $(X_t)_{t\ge0}$ of the one-dimensional jump-diffusion affine SDE

$$\begin{aligned} dX_t \;=\;& u_t dt + a_t X_t dt + v_t dW_t + b_t X_t dW_t \tag{3.8}\\ & + \int_{-\infty}^\infty w_{t^-}(z)\big(N(dt,dz) - \nu_t(dz)dt\big) + X_{t^-}\int_{-\infty}^\infty c_{t^-}(z)\big(N(dt,dz) - \nu_t(dz)dt\big), \end{aligned}$$

by taking

$$a_t(y) = a_t y + u_t, \quad b_t(y) = b_t y + v_t, \quad c_t(z,y) = c_t(z)y + w_t(z), \tag{3.9}$$

where $u, v$ are in a certain $L^p$ space and $a, b, c$ are in $L^\infty$.

The following uniform version of Theorem 3.1 will be required in Section 4. When the processes $a_\alpha$, $b_\alpha$, $c_\alpha$ all depend on a parameter $\alpha \in A$, the solution $(X_{\alpha,t})_{t\ge0}$ of the corresponding SDE (3.12) below enjoys the following uniform bound.

**Corollary 3.2** *Let $p \ge 2$. Assume that the coefficients $(a_{\alpha,t}(x))_{t\in[0,T]}$, $(b_{\alpha,t}(x))_{t\in[0,T]}$, $(c_{\alpha,t}(z,x))_{t\in[0,T]}$ are $(\mathcal{F}_t)_{t\ge0}$-adapted processes such that*

$$|a_{\alpha,t}(x) - a_{\alpha,t}(y)| \le a_{\alpha,t}|x-y|, \quad |b_{\alpha,t}(x) - b_{\alpha,t}(y)| \le b_{\alpha,t}|x-y|, \qquad x,y \in \mathbb{R},$$

*where $(a_{\alpha,t}(0))_{t\in[0,T]}, (b_{\alpha,t}(0))_{t\in[0,T]} \in L^p(\Omega\times[0,T])$, $(a_{\alpha,t})_{t\in[0,T]}, (b_{\alpha,t})_{t\in[0,T]} \in L^p([0,T], L^\infty(\Omega))$, uniformly in $\alpha \in A$, with*

$$|c_{\alpha,t}(z,x) - c_{\alpha,t}(z,y)| \le c_{\alpha,t}(z)|x-y|, \quad x,y \in \mathbb{R},$$

*with*

$$\sup_{\alpha\in A}\int_0^T \left\|\int_{-\infty}^\infty |c_{\alpha,t}(z)|^2\,\nu_t(dz)\right\|_\infty^{p/2} dt < \infty, \quad \sup_{\alpha\in A}\int_0^T \left\|\int_{-\infty}^\infty |c_{\alpha,t}(z)|^p\,\nu_t(dz)\right\|_\infty dt < \infty, \tag{3.10}$$

*and*

$$\sup_{\alpha \in A} \mathbb{E}\left[\int_0^T \left(\int_{-\infty}^\infty |c_{\alpha,t}(z,0)|^2 \nu_t(dz)\right)^{p/2} dt\right] < \infty, \quad \sup_{\alpha \in A} \mathbb{E}\left[\int_0^T \int_{-\infty}^\infty |c_{\alpha,t}(z,0)|^p \nu_t(dz)\, dt\right] < \infty.$$
(3.11)

*Then, for the solutions $X_\alpha$ of the SDE*

$$dX_{\alpha,t} = a_{\alpha,t}(X_{\alpha,t})dt + b_{\alpha,t}(X_{\alpha,t})dW_t + \int_{-\infty}^\infty c_{\alpha,t^-}(z, X_{\alpha,t^-})\big(N(dt,dz) - \nu_t(dz)dt\big),$$
(3.12)

$\alpha \in A$, *we have*

$$\sup_{\alpha \in A} \mathbb{E}\left[\sup_{t \in [0,T]} |X_{\alpha,t}|^p\right] < C(p,T) < \infty,$$
(3.13)

*where $C(p,T)$ depends on the above norms of $a_\alpha$, $b_\alpha$, $c_\alpha$, which are all assumed to be bounded uniformly in $\alpha \in A$.*

*Proof.* Only the conclusion of the previous proof for Theorem 3.1 is required to be changed. The bound (3.7) still holds true for $X_{\alpha,t}$ with the functions

$$F_\alpha(T) := 4^{p-1}\left(x^p + (2T)^{p-1} \mathbb{E}\left[\int_0^T |a_{\alpha,t}(0)|^p dt\right] + 2^{p-1}C_p T^{p/2-1} \mathbb{E}\left[\int_0^T |b_{\alpha,t}(0)|^p dt\right]\right.$$

$$\left. + 2^{p-1}\widetilde{C}_p \left(2^{p/2}T^{p/2-1} \mathbb{E}\left[\int_0^T \left(\int_{-\infty}^\infty |c_{\alpha,t}(z,0)|^2 \nu_t(dz)\right)^{p/2} dt\right] + \mathbb{E}\left[\int_{-\infty}^\infty |c_{\alpha,t}(z,0)|^p \nu_t(dz)dt\right]\right)\right)$$

*and*

$$G_\alpha(t) := 4^{p-1}\left((2T)^{p-1}\|a_{\alpha,t}\|_\infty^p + 2^{p-1}C_p T^{p/2-1}\|b_{\alpha,t}\|_\infty^p\right.$$

$$\left. + 2^{3p/2-1}\widetilde{C}_p\, T^{p/2-1}\left\|\int_{-\infty}^\infty |c_{\alpha,t}(z)|^2\, \nu_t(dz)\right\|_\infty^{p/2} + 2^{p-1}\widetilde{C}_p\left\|\int_{-\infty}^\infty |c_{\alpha,t}(z)|^p\right)\, \nu_t(dz)\right\|_\infty\right),$$

$t \in [0,T]$. Under the conditions of Corollary 3.2, we have

$$\sup_{\alpha \in A} F_\alpha(T) < \infty, \quad \sup_{\alpha \in A} G_\alpha(t) < \infty, \qquad t \in [0,T],$$

and the conclusion (3.13) follows likewise. □

## 4 Flow derivatives

In this section, we show that the derivatives of the flow $x \mapsto X_t(x)$ of the SDE (1.1) are solutions of an affine SDE.

**Convention.** *Given the gradient* $\nabla_z = \left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_d}\right)$ *and* $F : \mathbb{R}^d \to \mathbb{R}^p$, *we denote* $\nabla_z F(z) = \left(\frac{\partial F_i}{\partial z_j}(z)\right)_{\substack{1 \le i \le p \\ 1 \le j \le d}} \in M_{p,d}(\mathbb{R})$, *and under the identification* $M_{p,d}(\mathbb{R}) \sim \mathbb{R}^{pd}$, *we write*

$$\nabla_z \big(\nabla_z F\big)(x) = \nabla_z^2 F(x) \in M_{pd,d}(\mathbb{R}) \sim \mathbb{R}^{pd^2}.$$

By successive differentiation of

$$r\big(s, X_s(x)\big), \quad \sigma\big(s, X_s(x)\big), \quad g\big(s, X_s(x), y\big)$$

with respect to $x$ and applying Theorem 6-29 of [BGJ87] recursively under Assumption $(A_n)$, we obtain the following result.

**Proposition 4.1** *Assume that* (1.6) *and* $(A_n)$ *hold for some* $n \ge 1$. *Then the flow* $x \mapsto X_t(x)$ *of the solution to the real SDE* (1.1) *is* $n$ *times differentiable on* $\mathbb{R}$ *and, for* $k = 1, \ldots, n$, $X_t^{(k)}(x) := \dfrac{\partial^k X_t}{\partial x^k}(x)$ *is solution of*

$$
\begin{aligned}
dX_t^{(k)}(x) = {}& \left( \sum_{\pi \in \Pi[k]} \frac{\partial^{|\pi|} r}{\partial x^{|\pi|}}\big(t, X_t(x)\big) \prod_{B \in \pi} X_t^{(|B|)}(x) \right) \, dt \\
& + \left( \sum_{\pi \in \Pi[k]} \frac{\partial^{|\pi|} \sigma}{\partial x^{|\pi|}}\big(t, X_t(x)\big) \prod_{B \in \pi} X_t^{(|B|)}(x) \right) \, dW_t \\
& + \int_{-\infty}^{\infty} \left( \sum_{\pi \in \Pi[k]} \frac{\partial^{|\pi|} g}{\partial x^{|\pi|}}\big(t, X_t(x), y\big) \prod_{B \in \pi} X_t^{(|B|)}(x) \right) \big(N(dt, dy) - \nu_t(dy)dt\big), \quad (4.1)
\end{aligned}
$$

*where, again, the sum over* $\pi$ *runs in the set* $\Pi[k]$ *of all partitions of* $\{1, 2, \ldots, k\}$, *and the product over* $B \in \pi$ *runs in all blocks* $B$ *in the partition* $\pi$.

*Proof.* For $n = 0$, $X_t^{(0)}(x) = X_t(x)$ and (4.1) reduces to the SDE (1.1). For $n = 1$, (4.1) is given by Theorem 6-29 in [BGJ87]:

$$
\begin{aligned}
dX_t^{(1)}(x) = {}& X_t^{(1)}(x)\frac{\partial r}{\partial x}(x, X_t(x))dt + X_t^{(1)}(x)\frac{\partial \sigma}{\partial x}(x, X_t(x))dW_t \\
& + X_t^{(1)}(x) \int_{-\infty}^{\infty} \frac{\partial g}{\partial x}(x, X_t(x), y)\big(N(dt, dy) - \nu_t(dy)dt\big). \quad (4.2)
\end{aligned}
$$

We continue the proof by induction on $n \ge 1$. We assume that

$$Z_t^{(n-1)}(x) := \left( X_t(x), \frac{\partial X_t}{\partial x}(x), \ldots, \frac{\partial^{n-1} X_t}{\partial x^{n-1}}(x) \right)^{\mathsf{T}} \in \mathbb{R}^n$$

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is solution of the $n$-dimensional SDE

$$Z_t^{(n-1)}(x) = z^{(n-1)} + \int_0^t r^{(n-1)}(s, Z_s^{(n-1)}) \, ds + \int_0^t \sigma^{(n-1)}(s, Z_s^{(n-1)}) \, dW_s$$

$$+ \int_0^t \int_{-\infty}^{\infty} g^{(n-1)}(t, Z_{s-}^{(n-1)}(x), y) \, (N(ds, dy) - \nu_s(dy) ds) \qquad (4.3)$$

with

$$z_0^{(n-1)} = x, \quad z_1^{(n-1)} = 1 \ \text{ and } \ z_k^{(n-1)} = 0, \qquad k \geq 2,$$

and

a) $r^{(n-1)} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$r^{(n-1)}\left(t, z^{(n-1)}\right) = \left(r_0^{(n-1)}, \ldots, r_{n-1}^{(n-1)}\right)^{\mathsf{T}} = \left(r^{(n-2)}\left(t, z^{(n-2)}\right), \ r_{n-1}^{(n-1)}(t, z^{(n-1)})\right)^{\mathsf{T}},$$

$$(4.4)$$

where

$$r_{n-1}^{(n-1)}\left(s, z^{(n-1)}\right) = \sum_{\pi \in \Pi[n-1]} \frac{\partial^{|\pi|} r}{\partial x^{|\pi|}}\left(s, z^{(n-1)}\right) \prod_{B \in \pi} z_{|B|}^{(n-1)}, \qquad (4.5)$$

b) $\sigma^{(n-1)} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\sigma^{(n-1)}\left(t, z^{(n-1)}\right) = \left(\sigma_0^{(n-1)}, \ldots, \sigma_{n-1}^{(n-1)}\right)^{\mathsf{T}} = \left(\sigma^{(n-2)}\left(t, z^{(n-2)}\right), \ \sigma_{n-1}^{(n-1)}\left(t, z^{(n-1)}\right)\right)^{\mathsf{T}},$$

where

$$\sigma_{n-1}^{(n-1)}\left(s, z^{(n-1)}\right) = \sum_{\pi \in \Pi[n-1]} \frac{\partial^{|\pi|} \sigma}{\partial x^{|\pi|}}\left(s, z^{(n-1)}\right) \prod_{B \in \pi} z_{|B|}^{(n-1)}, \qquad (4.6)$$

c) $g^{(n-1)} : [0, T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is given by

$$g^{(n-1)}(t, z^{(n-1)}, y) = \left(g_0^{(n-1)}, \ldots, g_{n-1}^{(n-1)}\right)^{\mathsf{T}} = \left(g^{(n-2)}(t, z^{(n-2)}, y), \ g_{n-1}^{(n-1)}(t, z^{(n-1)}, y)\right)^{\mathsf{T}},$$

where

$$g_{n-1}^{(n-1)}\left(s, z^{(n-1)}, y\right) = \sum_{\pi \in \Pi[n-1]} \frac{\partial^{|\pi|} g}{\partial x^{|\pi|}}\left(s, z^{(n-1)}, y\right) \prod_{B \in \pi} z_{|B|}^{(n-1)}, \qquad (4.7)$$

where $\Pi[n-1]$ stands for the set of partition of $\{1, \ldots, n-1\}$. We also assume that $X_t^{(k)}(x)$ is solution of the SDE (4.1) for $k = 0, \ldots, n-1$.

Observe first that (4.3) holds for $Z^{(0)} = X(x)$ since in this case (4.3) reduces to (1.1) with

$$r^{(0)}\big(t, z^{(0)}\big) = r\big(t, z^{(0)}\big), \quad \sigma^{(0)}\big(t, z^{(0)}\big) = \sigma\big(t, z^{(0)}\big), \quad g^{(0)}(t, z^{(0)}, y) = g(t, z^{(0)}, y).$$

Next, regarding $Z^{(1)}$, by (1.1) and (4.2) we have

$$Z^{(1)}(t) = \begin{pmatrix} X_t \\ X_t^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} r(s, X_s) \\ \frac{\partial r}{\partial x}(s, X_s)X_t^{(1)} \end{pmatrix} ds + \int_0^t \begin{pmatrix} \sigma(s, X_s) \\ \frac{\partial \sigma}{\partial x}(s, X_s)X_s^{(1)} \end{pmatrix} dW_s$$
$$+ \int_0^t \int_{-\infty}^\infty \begin{pmatrix} g(s, X_{s^-}, y) \\ g(s, X_{s^-}, y)X_{s^-}^{(1)} \end{pmatrix} \big(N(ds, dy) - \nu_s(dy)ds\big)$$

which is (4.3) for $Z^{(1)}$ with

$$r^{(1)}\big(t, z^{(1)}\big) = \begin{pmatrix} r(t, z_0^{(1)}) \\ \frac{\partial r}{\partial x}(t, z_0^{(1)})z_1^{(1)} \end{pmatrix}, \quad \sigma^{(1)}\big(t, z^{(1)}\big) = \begin{pmatrix} \sigma(t, z_0^{(1)}) \\ \frac{\partial \sigma}{\partial x}(t, z_0^{(1)})z_1^{(1)} \end{pmatrix},$$
$$g^{(1)}(t, z^{(1)}, y) = \begin{pmatrix} g(t, z_0^{(1)}, y) \\ \frac{\partial g}{\partial x}(t, z_0^{(1)}, y)z_1^{(1)} \end{pmatrix},$$

corresponding indeed to (4.4)–(4.7) in this case. We now show that $Z^{(n)}(x)$ solves an SDE similar to (4.3), and that $X_t^{(n)}$ is solution to (4.1) for the index $n$. Since $Z^{(n-1)}(x)$ is solution to (4.3), by Theorem 6-29 in [BGJ87], $\nabla_{z^{(n-1)}}Z^{(n-1)}$ is solution of the $M_{n,n}(\mathbb{R})$-valued matrix equation

$$\nabla_{z^{(n-1)}}Z_t^{(n-1)}(x) = I_{n,n} + \int_0^t \nabla_{z^{(n-1)}}r^{(n-1)}(s, Z_s^{(n-1)}(x))\nabla_{z^{(n-1)}}Z_s^{(n-1)}(x) \ ds$$
$$+ \int_0^t \nabla_{z^{(n-1)}}\sigma^{(n-1)}(s, Z_s^{(n-1)}(x))\nabla_{z^{(n-1)}}Z_s^{(n-1)}(x) \ dW_s$$
$$+ \int_0^t \int_{-\infty}^\infty \nabla_{z^{(n-1)}}g^{(n-1)}(s, Z_{s^-}^{(n-1)}(x), y)\nabla_{z^{(n-1)}}Z_{s^-}^{(n-1)}(x) \ \big(N(ds, dy) - \nu_s(dy)ds\big).$$

$$(4.8)$$

With the notation $\nabla_{z^{(n-1)}} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{n-1}} \right)$, extracting the first column for the matrix equality in (4.8) we have

$$\frac{\partial Z_t^{(n-1)}}{\partial x}(x) = (1, 0, \ldots, 0)^\intercal$$
$$+ \int_0^t \nabla_{z^{(n-1)}}r^{(n-1)}(s, Z_s^{(n-1)}(x))\frac{\partial Z_s^{(n-1)}}{\partial x}(x) \ ds + \int_0^t \nabla_{z^{(n-1)}}\sigma^{(n-1)}(s, Z_s^{(n-1)}(x))\frac{\partial Z_s^{(n-1)}}{\partial x}(x) \ dW_s$$

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$$+ \int_0^t \int_{-\infty}^{\infty} \nabla_{z^{(n-1)}} g^{(n-1)}\big(s, Z_{s-}^{(n-1)}(x), y\big) \frac{\partial Z_{s-}^{(n-1)}}{\partial x}(x) \, \big(N(ds, dy) - \nu_s(dy)ds\big). \qquad (4.9)$$

Next, for the leftmost entry in (4.9) we have

$$\frac{\partial^n X_t}{\partial x^n}(x)$$

$$= \int_0^t \left( \frac{\partial r_{n-1}^{(n-1)}}{\partial x}\big(s, Z_s^{(n-1)}(x)\big) \frac{\partial X_s}{\partial x}(x) + \sum_{i=1}^{n-1} \frac{\partial r_{n-1}^{(n-1)}}{\partial z_i^{(n-1)}}\big(s, Z_s^{(n-1)}(x)\big) \frac{\partial^{i+1} X_s}{\partial x^{i+1}}(x) \right) ds$$

$$+ \int_0^t \left( \frac{\partial \sigma_{n-1}^{(n-1)}}{\partial x}\big(s, Z_s^{(n-1)}(x)\big) \frac{\partial X_s}{\partial x}(x) + \sum_{i=1}^{n-1} \frac{\partial \sigma_{n-1}^{(n-1)}}{\partial z_i^{(n-1)}}\big(s, Z_s^{(n-1)}(x)\big) \frac{\partial^{i+1} X_s}{\partial x^{i+1}}(x) \right) dW_s$$

$$+ \int_0^t \int_{-\infty}^{\infty} \left( \frac{\partial g_{n-1}^{(n-1)}}{\partial x}\big(s, Z_{s-}^{(n-1)}(x), y\big) \frac{\partial X_s}{\partial x}(x) + \sum_{i=1}^{n-1} \frac{\partial g_{n-1}^{(n-1)}}{\partial z_i^{(n-1)}}\big(s, Z_{s-}^{(n-1)}(x), y\big) \frac{\partial^{i+1} X_s}{\partial x^{i+1}}(x) \right)$$

$$\big(N(ds, dy) - \nu_s(dy)ds\big).$$
$$(4.10)$$

Putting together (4.3) and (4.10) yields an equation for $Z^{(n)}(x)$ similar to (4.3), and (4.10) proves (4.1) for $X^{(n)}(x) = \frac{\partial^n}{\partial x^n} X(x)$. Indeed, from (4.10) we recover the expressions of $r_n^{(n)}$, $\sigma_n^{(n)}$ and $g_n^{(n)}$ as in (4.5)–(4.7), which achieves the induction. For instance, using the expression (4.5) of $r_{n-1}^{(n-1)}$, we have

$$
\begin{aligned}
r_n^{(n)}\big(s, z^{(n)}\big) &= z_1^{(n)} \frac{\partial r_{n-1}^{(n-1)}}{\partial x}\big(s, z^{(n-1)}\big) + z_{i+1}^{(n)} \sum_{i=1}^{n-1} \frac{\partial r_{n-1}^{(n-1)}}{\partial z_i^{(n-1)}}\big(s, z^{(n-1)}\big) \\
&= z_1^{(n)} \frac{\partial}{\partial x} \left( \sum_{\pi \in \Pi[n-1]} \frac{\partial^{|\pi|} r}{\partial x^{|\pi|}}\big(s, z^{(n)}\big) \prod_{B \in \pi} z_{|B|}^{(n-1)} \right) \\
&\quad + \sum_{i=1}^{n-1} z_{i+1}^{(n)} \frac{\partial}{\partial z_i^{(n-1)}} \left( \sum_{\pi \in \Pi[n-1]} \frac{\partial^{|\pi|} r}{\partial x^{|\pi|}}\big(s, z^{(n)}\big) \prod_{B \in \pi} z_{|B|}^{(n)} \right) \\
&= z_1^{(n)} \sum_{\pi \in \Pi[n-1]} \frac{\partial^{|\pi|+1} r}{\partial x^{|\pi|+1}}\big(s, z^{(n)}\big) \left( \prod_{B \in \pi} z_{|B|}^{(n-1)} \right) \\
&\quad + \sum_{i=1}^{n-1} z_{i+1}^{(n)} \sum_{\pi \in \Pi[n-1]} \frac{\partial^{|\pi|} r}{\partial x^{|\pi|}}\big(s, z^{(n)}\big) \frac{\partial}{\partial z_i^{(n-1)}} \prod_{B \in \pi} z_{|B|}^{(n)}.
\end{aligned}
$$
$$(4.11)$$

In the second term above we have

$$\frac{\partial}{\partial z_i^{(n-1)}} \prod_{B \in \pi} z_{|B|}^{(n)} = \frac{\partial}{\partial z_i^{(n-1)}} \prod_{j=1}^{n-1} (z_j^{(n)})^{\#\{B \in \pi : |B| = j\}}$$

$$= \#\{B \in \pi : |B| = i\}(z_i^{(n)})^{\#\{B \in \pi : |B| = i\} - 1} \prod_{\substack{1 \leq j \leq n \\ j \neq i}} (z_j^{(n)})^{\#\{B \in \pi : |B| = j\}}.$$

(4.12)

We note that $\Pi[n]$ consists of partitions $\pi \in \Pi[n-1]$ with either the addition of $\{n\}$ as a new block, or the completion of an existing block by $\{n\}$. In the latter case, if $\{n\}$ is added to a block of size $j$ the new partition of $\Pi[n]$ obtained in this way will have one block of size $j$ less and one block of size $j+1$ more, and there are $\#\{B \in \pi : |B| = j\}$ such blocks. We conclude that the sums in (4.11) and (4.12) are effectively over $\Pi[n]$, which yields (4.5) for $r_n^{(n)}$, are as follows:

$$r_n^{(n)}\big(s, z^{(n)}\big) = \sum_{\pi \in \Pi[n]} \frac{\partial^{|\pi|} r}{\partial x^{|\pi|}}\big(s, z^{(n-1)}\big) \prod_{B \in \pi} z_{|B|}^{(n-1)}.$$

Similar computations yield also (4.6) and (4.7) and achieves the induction proving Proposition 4.1. $\qquad\square$

# 5 Regularity of stochastic flows

In this section we consider the solution $(X_t(x))_{t \in [0,T]}$ of SDE (1.1), for which Proposition 4.1 gives condition for the differentiability of the flow $x \mapsto X_t(x)$ up to any order $n$. The next Theorem 5.1 deals with the integrability of order $q \geq 2$ of the flow derivatives, based on Corollaries 2.3 and 3.2, see also Theorem 3.3 in [Kun04] which only covers first order differentiability. We let $\|f(\cdot)\|_\infty$ denote the supremum of functions $f$ on $\mathbb{R}$.

**Theorem 5.1** *Let $n \geq 1$ and $q \geq 2$, and assume that (1.6) and $(A_n)$ hold. Then, for all $k = 1, \ldots, n$ we have*

$$\sup_{x \in \mathbb{R}} \mathbb{E}\left[ \sup_{t \in [0,T]} \big|X_t^{(k)}(x)\big|^q \right] < \infty.$$

(5.1)

*Proof.* Since $X^{(n)}$ in (4.1) is expressed in terms of $X^{(k)}$ for $k < n$, deriving a moment of order $q$ for $X^{(n)}$ requires to show the existence of moments of order $q$ of certain products of the $X^{(k)}$. Accordingly, from the Hölder inequality, higher moments of every $X^{(k)}$, $k < n$, are required in our argument, see (5.5) below. By Proposition 4.1,

$X_t^{(k)}(x) := \dfrac{\partial^k X_t}{\partial x^k}(x)$ solves the SDE

$$dX_t^{(k)}(x) = \Big( \sum_{\pi \in \Pi[k]} \frac{\partial^{|\pi|} r}{\partial x^{|\pi|}}\big(x, X_t(x)\big) \prod_{B \in \pi} X_t^{(|B|)}(x) \Big) \, dt$$

$$+ \Big( \sum_{\pi \in \Pi[k]} \frac{\partial^{|\pi|} \sigma}{\partial x^{|\pi|}}\big(x, X_t(x)\big) \prod_{B \in \pi} X_t^{(|B|)}(x) \Big) \, dW_t$$

$$+ \int_{-\infty}^{\infty} \Big( \sum_{\pi \in \Pi[k]} \frac{\partial^{|\pi|} g}{\partial x^{|\pi|}}\big(t, X_t(x), z\big) \prod_{B \in \pi} X_t^{(|B|)}(x) \Big) \, \big( N(dt, dz) - \nu_t(dz)dt \big), \quad (5.2)$$

with $X_0^{(k)}(x) = 0$ for $k \geq 2$, where the sum over $\pi$ runs in the set $\Pi[k]$ of all partitions of $\{1, 2, \ldots, k\}$. In order to prove (5.1) for $k = 1, \ldots, n$ we shall prove

$$\sup_{x \in \mathbb{R}} \mathbb{E} \Big[ \sup_{t \in [0,T]} \big| X_t^{(k)}(x) \big|^{p_k} \Big] < \infty \qquad (5.3)$$

by induction, for the order $p_k$ defined by

$$p_k = q \frac{n!}{k!}. \qquad (5.4)$$

For $k = 1$, (5.2) reduces to the affine equation

$$dX_t^{(1)}(x) = a_t^{(1)} X_t^{(1)}(x) \, dt + b_t^{(1)} \, dW_t + X_{t^-}^{(1)}(x) \int_{-\infty}^{\infty} c_{t^-}^{(1)}(z) \, \big( N(dt, dz) - \nu_t(dz)dt \big),$$

of the form (3.8), with $X_0^{(1)}(x) = 1$ and

$$a_t^{(1)} = \frac{\partial r}{\partial x}\big(t, X_t(x)\big), \quad b_t^{(1)} = \frac{\partial \sigma}{\partial x}\big(t, X_t(x)\big), \quad c_t^{(1)}(y) = \frac{\partial g}{\partial x}\big(t, X_t(x), y\big),$$

see also Theorem 6-29 in [BGJ87]. Since $(a_t^{(1)})_{t \in [0,T]}, (b_t^{(1)})_{t \in [0,T]} \in L^{p_1}([0, T], L^\infty(\Omega))$ under $(A_n)$ and since Conditions (3.2)-(3.3) are satisfied by $c^{(1)}$ with $p = p_1 = n!q$ under (1.7), Corollary 3.2 shows that $X_t^{(1)}(x)$ admits a moment of order $p_1$, uniformly bounded in $t \in [0, T]$, that is (5.3) holds true for $k = 1$.

Further, we assume that (5.3) holds true with order $p_l$ for $l = 1, \ldots, k-1$ and we show that it remains true for the rank $l = k$. We note that (5.2) is thus an affine SDE of the form (3.8) in $X_t^{(k)}(x)$, with the random coefficients

$$a_{x,t}^{(k)} := \frac{\partial r}{\partial x}\big(t, X_t(x)\big),$$

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$$b_{x,t}^{(k)} := \frac{\partial \sigma}{\partial x}\big(t, X_t(x)\big),$$

$$c_{x,t}^{(k)}(z) := \frac{\partial g}{\partial x}\big(t, X_t(x), z\big)$$

$$u_{x,t}^{(k)} := \sum_{\pi \in \Pi[k] \setminus \{1,\ldots,k\}} \frac{\partial^{|\pi|} r}{\partial x^{|\pi|}}\big(t, X_t(x)\big) \prod_{B \in \pi} X_t^{(|B|)}(x),$$

$$v_{x,t}^{(k)} := \sum_{\pi \in \Pi[k] \setminus \{1,\ldots,k\}} \frac{\partial^{|\pi|} \sigma}{\partial x^{|\pi|}}\big(t, X_t(x)\big) \prod_{B \in \pi} X_t^{(|B|)}(x),$$

$$w_{x,t}^{(k)}(z) := \sum_{\pi \in \Pi[k] \setminus \{1,\ldots,k\}} \frac{\partial^{|\pi|} g}{\partial x^{|\pi|}}\big(t, X_t(x), z\big) \prod_{B \in \pi} X_t^{(|B|)}(x),$$

as in (3.9). In order to show that $X_t^{(k)}(x)$ satisfies (5.3), we shall apply Corollary 3.2 for every $p_k$ as in (5.3) to the affine SDE (5.2) written as (3.12) and parameterized by the initial condition $x \in \mathbb{R}$, after checking that $(a_{x,t}^{(k)})_{t \in [0,T]}, (b_{x,t}^{(k)})_{t \in [0,T]} \in L^{p_k}\big([0,T], L^\infty(\Omega \times \mathbb{R})\big)$, $(u_{x,t}^{(k)})_{t \in [0,T]}, (v_{x,t}^{(k)})_{t \in [0,T]} \in L^{p_k}\big([0,T] \times \Omega, L^\infty(\mathbb{R})\big)$, and the conditions (3.10)-(3.11) for $(c_{x,t}^{(k)}(z))_{t \in [0,T]}, (w_{x,t}^{(k)}(z))_{t \in [0,T]}$ hold, as follows.

i) The conditions

$$\sup_{x \in \mathbb{R}} \int_0^T \|a_{x,t}^{(k)}\|_\infty^{p_k} \, dt < \infty, \quad \sup_{x \in \mathbb{R}} \int_0^T \|b_{x,t}^{(k)}\|_\infty^{p_k} \, dt < \infty,$$

follow immediately from Assumption $(A_n)$ on $\partial r / \partial x$ and $\partial \sigma / \partial x$. On the other hand, regarding $c^{(k)}$, the bounds

$$\int_0^T \left\| \int_{-\infty}^\infty |c_{x,t}^{(k)}(z)|^2 \, \nu_t(dz) \right\|_\infty^{p_k/2} dt < \infty \text{ and } \int_0^T \left\| \int_{-\infty}^\infty |c_{x,t}^{(k)}(z)|^{p_k} \, \nu_t(dz) \right\|_\infty dt < \infty$$

follow from (1.7) since $\displaystyle \sup_{x \in \mathbb{R}} \|c_{x,t}^{(k)}(y)\|_\infty \le \left\| \frac{\partial g}{\partial x}(t, \cdot, y) \right\|_\infty$.

ii) We show that $(u_{x,t}^{(k)})_{t \in [0,T]}, (v_{x,t}^{(k)})_{t \in [0,T]} \in L^{p_k}([0,T] \times \Omega)$, uniformly in $x \in \mathbb{R}$. Since the coefficients of (5.2) involve finite sums, we can deal with each summand separately using the convexity of $y \in \mathbb{R} \mapsto |y|^{p_k}$. For all $\pi \in \Pi[k] \setminus \{1, \ldots, k\}$, using $(A_n)$ and Hölder's inequality yields

$$\mathbb{E}\left[ \int_0^T \left| \frac{\partial^{|\pi|} r}{\partial x^{|\pi|}}\big(x, X_t(x)\big) \right|^{p_k} \prod_{B \in \pi} |X_t^{(|B|)}(x)|^{p_k} \, dt \right] \le C \int_0^T \prod_{B \in \pi} \left( \mathbb{E}\big[ |X_t^{(|B|)}(x)|^{p_k |\pi|} \big] \right)^{1/|\pi|} dt.$$

$$\tag{5.5}$$

Since $\pi \in \Pi[k] \setminus \{1, \ldots, k\}$ has at least two blocks, for $B \in \pi$, we have $|B| < k$ and the induction hypothesis (5.3) applies for $X^{(|B|)}$. Additionally, $|\pi| \leq k$, and so

$$p_k|\pi| \leq kp_k = p_{k-1}. \tag{5.6}$$

As a consequence, by the induction hypothesis (5.1) applied to each $X_t^{(|B|)}(x)$, we have

$$\sup_{x \in \mathbb{R}} \mathbb{E}\left[ \sup_{t \in [0,T]} \left| X_t^{(|B|)}(x) \right|^{p_k|\pi|} \right] < \infty \tag{5.7}$$

for each $B \in \pi$ and (5.5) ensures that

$$\sup_{x \in \mathbb{R}} \mathbb{E}\left[ \int_0^T |u_{x,t}^{(k)}|^{p_k}\, dt \right] < \infty,$$

and similarly for $v_{x,t}^{(k)}$ we find:

$$\sup_{x \in \mathbb{R}} \mathbb{E}\left[ \int_0^T |v_{x,t}^{(k)}|^{p_k}\, dt \right] < \infty.$$

iii) Verification of (3.2) for $w_{x,t}^{(k)}(z)$. Again, since the set $\Pi[k]$ of partitions of $\{1, 2, \ldots, k\}$ is finite and $y \in \mathbb{R} \mapsto |y|^2$, $y \in \mathbb{R} \mapsto |y|^{p_k/2}$ are both convex functions, we can deal with each summand separately. For all $\pi \in \Pi[k] \setminus \{1, \ldots, k\}$ we have

$$\mathbb{E}\left[ \int_0^T \left( \int_{-\infty}^\infty \left| \frac{\partial^{|\pi|} g}{\partial x^{|\pi|}}(t, X_t(x), z) \right|^2 \prod_{B \in \pi} \left| X_t^{(|B|)}(x) \right|^2 \nu_t(dz) \right)^{p_k/2} dt \right]$$

$$= \mathbb{E}\left[ \int_0^T \left( \int_{-\infty}^\infty \left| \frac{\partial^{|\pi|} g}{\partial x^{|\pi|}}(t, X_t(x), z) \right|^2 \nu_t(dz) \right)^{p_k/2} \prod_{B \in \pi} \left| X_t^{(|B|)}(x) \right|^{p_k} dt \right]$$

$$\leq \int_0^T \left( \int_{-\infty}^\infty \left\| \frac{\partial^{|\pi|} g}{\partial x^{|\pi|}}(t, \cdot, z) \right\|_\infty^2 \nu_t(dz) \right)^{p_k/2} \prod_{B \in \pi} \mathbb{E}\left[ \left| X_t^{(|B|)}(x) \right|^{p_k|\pi|} \right]^{1/|\pi|} dt$$

$$\leq C \int_0^T \left( \int_{-\infty}^\infty \left\| \frac{\partial^{|\pi|} g}{\partial x^{|\pi|}}(t, \cdot, z) \right\|_\infty^2 \nu_t(dz) \right)^{p_k/2} dt,$$

where the last bound uses (5.7) for the $X_t^{(|B|)}(x)$ with $|B| < k$ due to the induction hypothesis. This final bound is finite under (1.7), which ensures that

$$\sup_{x \in \mathbb{R}} \mathbb{E}\left[ \int_0^T \left( \int_{-\infty}^\infty \left| w_{x,t}^{(k)}(z) \right|^2 \nu_t(dz) \right)^{p_k/2} dt \right] < \infty.$$

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Similarly for (3.3), since $y \in \mathbb{R} \mapsto |y|^2$ and $y \in \mathbb{R} \mapsto |y|$ are both convex functions we have

$$
\begin{aligned}
\mathbb{E}\bigg[ \int_0^T \int_{-\infty}^\infty \Big| \frac{\partial^{|\pi|} g}{\partial x^{|\pi|}}\big(t, X_t(x), z\big) \Big|^{p_k} & \prod_{B \in \pi} \big|X_t^{(|B|)}(x)\big|^{p_k} \nu_t(dz) \ dt \bigg] \\
= \ \mathbb{E}&\bigg[ \int_0^T \int_{-\infty}^\infty \Big| \frac{\partial^{|\pi|} g}{\partial x^{|\pi|}}\big(t, X_t(x), z\big) \Big|^{p_k} \nu_t(dz) \prod_{B \in \pi} \big|X_t^{(|B|)}(x)\big|^{p_k} \ dt \bigg] \\
\leq \ & \int_0^T \int_{-\infty}^\infty \Big\| \frac{\partial^{|\pi|} g}{\partial x^{|\pi|}}(t, \cdot, z) \Big\|_\infty^{p_k} \nu_t(dz) \prod_{B \in \pi} \mathbb{E}\Big[ \big|X_t^{(|B|)}(x)\big|^{p_k |\pi|} \Big]^{1/|\pi|} \ dt \\
\leq \ & C \int_0^T \int_{-\infty}^\infty \Big\| \frac{\partial^{|\pi|} g}{\partial x^{|\pi|}}(t, \cdot, z) \Big\|_\infty^{p_k} \nu_t(dz) \ dt,
\end{aligned}
$$

using (5.7) for $X_t^{(|B|)}(x)$ with $|B| < n$, and we conclude to

$$
\sup_{x \in \mathbb{R}} \mathbb{E}\bigg[ \int_0^T \int_{-\infty}^\infty |w_{x,t}^{(k)}(z)|^{p_k} \ \nu_t(dz) \ dt \bigg] < \infty.
$$

As a consequence, Theorem 3.1 can be applied to (5.2), which yields

$$
\sup_{t \in [0,T]} \mathbb{E}\big[ \big|X_t^{(k)}(x)\big|^{p_k} \big] < \infty,
$$

proving the induction hypothesis (5.3) for index $k$ and with order $p_k$ in (5.4). In particular, this proves Theorem 5.1. $\qquad\square$

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