FKG inequality on the Wiener space via predictable representation

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Abstract

Using the Clark predictable representation formula, we give a proof of the FKG inequality on the Wiener space. Solutions of stochastic differential equations are treated as applications and we recover by a simple argument the covariance inequalities obtained for diffusions processes by several authors.

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P}, \preceq)$ be a probability space equipped with a partial order relation $\preceq$ on Ω. An (everywhere defined) real-valued random variable F on $(\Omega, \mathcal{F}, \mathbb{P}, \preceq)$ is said to be non-decreasing if

$$F(\omega_1) \leq F(\omega_2)$$

for any $\omega_1, \omega_2 \in \Omega$ satisfying $\omega_1 \preceq \omega_2$. The FKG inequality [4] states that if F and G are two square-integrable random functionals which are non-decreasing for the order $\preceq$, then F and G are non-negatively correlated:

$$\mathrm{Cov}\,(F, G) \geq 0.$$

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It is well known that the FKG inequality holds for the standard ordering on $\Omega = \mathbb{R}$, since given $X, Y : \mathbb{R} \to \mathbb{R}$ two non-decreasing functions on $\mathbb{R}$ we have:

$$\begin{aligned}
\mathrm{Cov}\,(X, Y) &= \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} (X(x) - X(y))(Y(x) - Y(y))\mathbb{P}(dx)\mathbb{P}(dy) \\
&= \frac{1}{2}\int_{\{y\leq x\}} (X(x) - X(y))(Y(x) - Y(y))\mathbb{P}(dx)\mathbb{P}(dy) \\
&\quad + \frac{1}{2}\int_{\{x<y\}} (X(y) - X(x))(Y(y) - Y(x))\mathbb{P}(dx)\mathbb{P}(dy) \\
&\geq 0.
\end{aligned}$$

The FKG inequality also holds on $\mathbb{R}^n$ for the pointwise ordering, cf. e.g. Bakry and Michel [2].

On the Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$ with Brownian motion $(W_t)_{t\in\mathbb{R}_+}$, Barbato [3] introduced a weak ordering on continuous functions and proved an FKG inequality for Wiener functionals, with application to diffusion processes.

In this paper we recover the results of [3] under weaker hypotheses via a simple argument. Our approach is inspired by Remark 1.5 stated on the Poisson space in Wu [14], page 432, which can be carried over to the Wiener space by saying that the *predictable representation* of a random variable F as a an Itô integral, obtained via the Clark formula

$$F = \mathbb{E}[F] + \int_0^1 \mathbb{E}[D_t F|\mathcal{F}_t]dW_t,$$

yields the covariance identity

$$\mathrm{Cov}\,(F, G) = \mathbb{E}\left[\int_0^1 \mathbb{E}[D_t F|\mathcal{F}_t]\mathbb{E}[D_t G|\mathcal{F}_t]dt\right], \tag{1.1}$$

where D is the Malliavin gradient expressed as

$$\langle DF, \dot{h}\rangle_{L^2([0,1])} = \frac{d}{d\varepsilon}F(\omega + \epsilon h)_{|\varepsilon=0}. \tag{1.2}$$

From (1.2) we deduce that DF is non-negative when F is non-decreasing, which implies $\mathrm{Cov}\,(F, G) \geq 0$ from (1.1). Applications are given to diffusion processes and in Theorem 3.4 we recover, under weaker hypotheses, the covariance inequality obtained in Theorem 3.2 of [7] and in Theorem 7 of [3].

We proceed as follows. Elements of analysis on the Wiener space and applications to covariance identities are recalled in Sections 2. The FKG inequality and covariance inequalities for diffusions are proved in Section 3. We also show that our method allows us to deal with the discrete case, cf. Section 4.

2 Analysis on the Wiener space

In this section we recall some elements of stochastic analysis on the classical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$ on $\Omega = C_0([0,1];\mathbb{R})$, with canonical Brownian motion $(W_t)_{t\in[0,1]}$ generating the filtration $(\mathcal{F}_t)_{t\in[0,1]}$. Our results extend without difficulty to the Wiener space on $C_0(\mathbb{R}_+;\mathbb{R})$. Let H denote the Cameron-Martin space, i.e. the space of absolutely continuous functions with square-integrable derivative:

$$H = \left\{ h : [0,1] \to \mathbb{R} \ : \ \int_0^1 |\dot{h}(s)|^2 ds < \infty \right\}.$$

Let $I_n(f_n)$, $n \geq 1$, denote the iterated stochastic integral of f_n in the space $L^2_s([0,1]^n)$ of symmetric square-integrable functions in n variables on $[0,1]^n$, defined as

$$I_n(f_n) = n! \int_0^1 \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1,\dots,t_n) dW_{t_1} \cdots dW_{t_n},$$

with the isometry formula

$$\mathbb{E}[I_n(f_n) I_m(g_m)] = n! 1_{\{n=m\}} \langle f_n, g_m \rangle_{L^2([0,1]^n)}.$$

Every $F \in L^2(\Omega)$ admits a unique Wiener chaos expansion

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n)$$

with $f_n \in L^2_s([0,1]^n)$, $n \geq 1$. Let $(e_k)_{k\geq 1}$ denote the dyadic basis of $L^2([0,1])$ given by

$$e_k = 2^{n/2} \mathbf{1}_{\left[\frac{k-2^n}{2^n}, \frac{k+1-2^n}{2^n}\right]}, \qquad 2^n \leq k \leq 2^{n+1}-1, \quad n \in \mathbb{N}.$$

Recall the following two equivalent definitions of the Malliavin gradient D and its domain $\mathrm{Dom}\,(D)$, cf. Lemma 1.2 of [8] and [10]:

a) Finite dimensional approximations. Given $F \in L^2(\Omega)$, let for all $n \in \mathbb{N}$:

$$\mathcal{G}_n = \sigma(I_1(e_{2^n}), \dots, I_1(e_{2^{n+1}-1})),$$

and $F_n = \mathbb{E}[F|\mathcal{G}_n]$, and consider f_n a square-integrable function with respect to the standard Gaussian measure on $\mathbb{R}^{2^n}$, such that

$$F_n = f_n(I_1(e_{2^n}), \dots, I_1(e_{2^{n+1}-1})).$$

Then $F \in \mathrm{Dom}\,(D)$ if and only if f_n belongs for all $n \geq 1$ to the Sobolev space $W^{2,1}(\mathbb{R}^{2^n})$ with respect to the standard Gaussian measure on $\mathbb{R}^{2^n}$, and the sequence

$$D_t F_n := \sum_{i=1}^{2^n} e_{2^n+i-1}(t) \frac{\partial f_n}{\partial x_i}(I_1(e_{2^n}), \dots, I_1(e_{2^{n+1}-1})), \qquad t \in [0,1],$$

converges in $L^2(\Omega \times [0,1])$. In this case we let

$$DF := \lim_{n \to \infty} DF_n.$$

b) Chaos expansions. Let $G \in L^2(\Omega)$ be given by

$$G = \mathbb{E}[G] + \sum_{n=1}^{\infty} I_n(g_n).$$

Then G belongs to $\mathrm{Dom}\,(D)$ if and only if the series

$$\sum_{n=1}^{\infty} n! n \|g_n\|^2_{L^2([0,1]^n)}$$

converges, and in this case,

$$D_t G = g_1(t) + \sum_{n=1}^{\infty} n I_{n-1}(g_n(*, t)), \qquad t \in [0,1].$$

In case (a) above the gradient $\langle DF_n, \dot{h} \rangle_{L^2([0,1])}$, $h \in H$, coincides with the directional derivative

$$\begin{aligned} &\langle DF_n, \dot{h} \rangle_{L^2([0,1])} \\ &= \frac{d}{d\varepsilon} f_n(I_1(e_{2^n}) + \varepsilon \langle e_{2^n}, \dot{h} \rangle_{L^2([0,1])}, \dots, I_1(e_{2^{n+1}-1}) + \varepsilon \langle e_{2^{n+1}-1}, \dot{h} \rangle_{L^2([0,1])})_{|\varepsilon=0} \\ &= \frac{d}{d\varepsilon} F_n(\omega + \epsilon h)_{|\varepsilon=0}, \end{aligned}$$

where the limit exists in $L^2(\Omega)$.

Similarly, the Ornstein-Uhlenbeck semi-group $(P_t)_{t\in\mathbb{R}_+}$ admits the following equivalent definitions, cf. e.g. [9], [12], [13]:

a) Integral representation. For any $F \in L^2(\Omega)$ and $t \in \mathbb{R}_+$, let

$$P_t F(\omega) = \int_\Omega F(e^{-t}\omega + \sqrt{1-e^{-2t}}\tilde{\omega})d\mathbb{P}(\tilde{\omega}), \qquad \mathbb{P}(d\omega) - a.s. \tag{2.1}$$

b) Chaos representation. For any $F \in L^2(\Omega)$ with the chaos expansion

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n),$$

we have

$$P_t F = \mathbb{E}[F] + \sum_{n=1}^{\infty} e^{-nt} I_n(f_n), \qquad t \in \mathbb{R}_+. \tag{2.2}$$

The operator D satisfies the Clark formula, i.e.

$$F = \mathbb{E}[F] + \int_0^1 \mathbb{E}[D_t F|\mathcal{F}_t]dW_t, \qquad F \in \mathrm{Dom}\,(D), \tag{2.3}$$

cf. e.g. [12]. By continuity of the operator mapping $F \in L^2(\Omega)$ to the adapted and square-integrable process $(u_t)_{t\in\mathbb{R}_+}$ appearing in predictable representation

$$F = \mathbb{E}[F] + \int_0^1 u_t dW_t, \tag{2.4}$$

the Clark formula can be extended to any $F \in L^2(\Omega)$ as in the following proposition.

Proposition 2.1. *The operator $F \mapsto (\mathbb{E}[D_t F|\mathcal{F}_t])_{t\in[0,1]}$ extends as a continuous operator on $L^2(\Omega)$.*

Proof. We use the bound

$$\mathbb{E}\left[\int_0^1 (\mathbb{E}[D_t F|\mathcal{F}_t])^2 dt\right] = \mathbb{E}[(F - \mathbb{E}[F])^2] = \mathbb{E}[F^2] - (\mathbb{E}[F])^2 \leq \|F\|^2_{L^2(\Omega)}, \tag{2.5}$$

for $F \in \mathrm{Dom}\,(D)$. □

Moreover, by uniqueness of the predictable representation of $F \in L^2(\Omega)$, an expression of the form

$$F = c + \int_0^1 u_t dW_t$$

where $c \in \mathbb{R}$ and $(u_t)_{t \in \mathbb{R}_+}$ is adapted and square-integrable, implies $u_t = \mathbb{E}[D_t F | \mathcal{F}_t]$, $dt \times d\mathbb{P}$-a.e. The Clark formula and the Itô isometry yield the following covariance identity, cf. Proposition 2.1 of [6].

Proposition 2.2. *For any $F, G \in L^2(\Omega)$ we have*

$$\mathrm{Cov}\,(F, G) = \mathbb{E}\left[\int_0^1 \mathbb{E}[D_t F | \mathcal{F}_t] \mathbb{E}[D_t G | \mathcal{F}_t] dt\right]. \tag{2.6}$$

This identity can be written as

$$\mathrm{Cov}\,(F, G) = \mathbb{E}\left[\int_0^1 \mathbb{E}[D_t F | \mathcal{F}_t] D_t G dt\right], \tag{2.7}$$

provided $G \in \mathrm{Dom}\,(D)$. The following lemma is an immediate consequence of (2.6).

Lemma 2.3. *Let $F, G \in L^2(\Omega)$ such that*

$$\mathbb{E}[D_t F | \mathcal{F}_t] \cdot \mathbb{E}[D_t G | \mathcal{F}_t] \geq 0, \qquad dt \times d\mathbb{P} - a.e.$$

Then F and G are non-negatively correlated:

$$\mathrm{Cov}\,(F, G) \geq 0.$$

If $G \in \mathrm{Dom}\,(D)$, resp. $F, G \in \mathrm{Dom}\,(D)$, the above condition can be replaced by

$$\mathbb{E}[D_t F | \mathcal{F}_t] \geq 0 \quad \text{and} \quad D_t G \geq 0, \qquad dt \times d\mathbb{P} - a.e.,$$

resp.

$$D_t F \geq 0 \quad \text{and} \quad D_t G \geq 0, \qquad dt \times d\mathbb{P} - a.e..$$

As recalled in the introduction, if X is a real random variable and f, g are $\mathcal{C}^1(\mathbb{R})$ functions with non-negative derivatives f', g', then $f(X)$ and $g(X)$ are non-negatively correlated. Lemma 2.3 provides an analog of this result on the Wiener space, replacing the ordinary derivative with the adapted process $(\mathbb{E}[D_t F | \mathcal{F}_t])_{t \in [0,1]}$.

3 FKG inequality on the Wiener space

We consider the order relation introduced in [3].

Definition 3.1. Given $\omega_1, \omega_2 \in \Omega$, we say that $\omega_1 \preceq \omega_2$ if and only if we have

$$\omega_1(t_2) - \omega_1(t_1) \leq \omega_2(t_2) - \omega_2(t_1), \qquad 0 \leq t_1 \leq t_2 \leq 1.$$

The class of non-decreasing functionals with respect to $\preceq$ is larger than that of non-decreasing functionals with respect to the pointwise order on Ω defined by

$$\omega_1(t) \leq \omega_2(t), \qquad t \in [0,1], \quad \omega_1, \omega_2 \in \Omega.$$

Definition 3.2. A random variable $F : \Omega \to \mathbb{R}$ is said to be non-decreasing if

$$\omega_1 \preceq \omega_2 \;\Rightarrow\; F(\omega_1) \leq F(\omega_2), \qquad \mathbb{P}(d\omega_1) \otimes \mathbb{P}(d\omega_2) - a.s.$$

Note that unlike in [3], the above definition allows for almost-surely defined functionals. The next result is the FKG inequality on the Wiener space. It recovers Theorem 4 of [3] under weaker (i.e. almost-sure) hypotheses.

Theorem 3.1. *For any non-decreasing functionals $F, G \in L^2(\Omega)$ we have*

$$\mathrm{Cov}\,(F, G) \geq 0.$$

The proof of this result is a direct consequence of Lemma 2.3 and Proposition 3.3 below.

Lemma 3.2. *For every non-decreasing $F \in \mathrm{Dom}\,(D)$ we have*

$$D_t F \geq 0, \qquad dt \times d\mathbb{P} - a.e.$$

Proof. For $n \in \mathbb{N}$, let π_n denotes the orthogonal projection from $L^2([0,1])$ onto the linear space generated by $(e_k)_{2^n \leq k < 2^{n+1}}$. Consider h in the Cameron-Martin space H and let

$$h_n(t) = \int_0^t [\pi_n \dot{h}](s) ds, \qquad t \in [0,1], \quad n \in \mathbb{N}.$$

Let Λ_n denote the square-integrable and $\mathcal{G}_n$-measurable random variable

$$\Lambda_n = \exp\left(\int_0^1 [\pi_n \dot{h}](s) dW_s - \frac{1}{2} \int_0^1 |[\pi_n \dot{h}](s)|^2 ds \right).$$

From the Cameron-Martin theorem, for all $n \in \mathbb{N}$ and $\mathcal{G}_n$-measurable bounded random variable G_n we have, letting $F_n = \mathbb{E}[F \mid \mathcal{G}_n]$:

$$\begin{aligned}
\mathbb{E}[F_n(\cdot + h_n) G_n] &= \mathbb{E}[\Lambda_n F_n G_n(\cdot - h_n)] \\
&= \mathbb{E}[\Lambda_n \mathbb{E}[F|\mathcal{G}_n] G_n(\cdot - h_n)] \\
&= \mathbb{E}[\mathbb{E}[\Lambda_n F G_n(\cdot - h_n)|\mathcal{G}_n]] \\
&= \mathbb{E}[\Lambda_n F G_n(\cdot - h_n)] \\
&= \mathbb{E}[F(\cdot + h_n) G_n],
\end{aligned}$$

hence

$$F_n(\omega + h_n) = \mathbb{E}[F(\cdot + h_n)|\mathcal{G}_n](\omega), \qquad \mathbb{P}(d\omega) - a.s.$$

If $\dot{h}$ is non-negative, then $\pi_n \dot{h}$ is non-negative by construction hence $\omega \preceq \omega + h_n$, $\omega \in \Omega$, and we have

$$F(\omega) \leq F(\omega + h_n), \qquad \mathbb{P}(d\omega) - a.s.,$$

since from the Cameron-Martin theorem, $\mathbb{P}(\{\omega + h_n \ : \ \omega \in \Omega\}) = 1$. Hence with the notation of Section 2,

$$\begin{aligned} F_n(\omega + h) &= f_n(I_1(e_{2^n}) + \langle e_{2^n}, \dot{h}\rangle_{L^2([0,1])}, \ldots, I_1(e_{2^{n+1}-1}) + \langle e_{2^{n+1}-1}, \dot{h}\rangle_{L^2([0,1])}) \\ &= f_n(I_1(e_{2^n}) + \langle e_{2^n}, \pi_n\dot{h}\rangle_{L^2([0,1])}, \ldots, I_1(e_{2^{n+1}-1}) + \langle e_{2^{n+1}-1}, \pi_n\dot{h}\rangle_{L^2([0,1])}) \\ &= F_n(\omega + h_n) \\ &= \mathbb{E}[F(\cdot + h_n)|\mathcal{G}_n](\omega) \\ &\geq \mathbb{E}[F|\mathcal{G}_n](\omega) \\ &= F_n(\omega), \qquad \mathbb{P}(d\omega) - a.s., \end{aligned}$$

i.e. for any $\varepsilon_1 \leq \varepsilon_2$ and $h \in H$ such that $\dot{h}$ is non-negative we have

$$F_n(\omega + \varepsilon_1 h) \leq F_n(\omega + \varepsilon_2 h),$$

and the smooth function $\varepsilon \mapsto F_n(\omega + \varepsilon h)$ is non-decreasing in ε on $\mathbb{R}$, $\mathbb{P}(d\omega)$-a.s. As a consequence,

$$\langle DF_n, \dot{h}\rangle_{L^2([0,1])} = \frac{d}{d\varepsilon} F_n(\omega + \epsilon h)_{|\varepsilon=0} \geq 0,$$

for all $h \in H$ such that $\dot{h} \geq 0$, hence $DF_n \geq 0$. Taking the limit of $(DF_n)_{n\in\mathbb{N}}$ as n goes to infinity shows that $DF \geq 0$. □

Next, we extend Lemma 3.2 to $F \in L^2(\Omega)$.

Proposition 3.3. *For any non-decreasing functional $F \in L^2(\Omega)$ we have*

$$\mathbb{E}[D_t F|\mathcal{F}_t] \geq 0, \qquad dt \times d\mathbb{P} - a.e.$$

Proof. Assume that $F \in L^2(\Omega)$ is non-decreasing. Then $P_{1/n}F$, $n \geq 1$, is non-decreasing from (2.1), and belongs to Dom (D) from (2.2). From Lemma 3.2 we have

$$D_t P_{1/n} F \geq 0, \qquad dt \times d\mathbb{P} - a.e.,$$

hence

$$\mathbb{E}[D_t P_{1/n} F | \mathcal{F}_t] \geq 0, \qquad dt \times d\mathbb{P} - a.e.$$

Taking the limit as n goes to infinity yields $\mathbb{E}[D_t F | \mathcal{F}_t] \geq 0$, $dt \times d\mathbb{P}$-a.e. from (2.5) and the fact that $P_{1/n}F$ converges to F in $L^2(\Omega)$ as n goes to infinity. $\square$

Conversely it is not difficult to show that if $u \in L^2([0,1])$ is a non-negative deterministic function, then the Wiener integral $\int_0^1 u_t dW_t$ is a non-decreasing functional. Note however that the stochastic integral of a non-negative square-integrable process may not necessarily be a non-decreasing functional. For example, consider $u_t = G 1_{[a,1]}$, $t \in [0,1]$, where $G \in L^2(\Omega, \mathcal{F}_a)$ is non-negative and decreasing, then

$$\int_0^1 u_t dW_t = G(B_1 - B_a)$$

is not non-decreasing.

Example: maximum of Brownian motion.

By Proposition 2.1.3 of [9] the maximum $M = \sup_{0 \leq t \leq 1} W(t)$ of Brownian motion on $[0,1]$ belongs to Dom (D) and satisfies $DM = 1_{[0,\tau]} \geq 0$, where τ is the a.s. unique point where M attains its maximum. Here, M is clearly an increasing functional.

Example: diffusion processes.

Consider the stochastic differential equations

$$\begin{cases} dX_t = b_t(X_t)dt + \sigma_t(X_t)dW_t \\ X_0 = x_0, \end{cases} \tag{3.1}$$

and

$$\begin{cases} d\tilde{X}_t = \tilde{b}_t(\tilde{X}_t)dt + \tilde{\sigma}_t(\tilde{X}_t)dW_t \\ \tilde{X}_0 = x_0, \end{cases} \tag{3.2}$$

where b, $\tilde{b}$, σ, $\tilde{\sigma}$ are functions on $\mathbb{R}_+ \times \mathbb{R}$ satisfying the following global Lipschitz and boundedness conditions, cf. [9], page 99:

(i) $|\sigma_t(x) - \sigma_t(y)| + |b_t(x) - b_t(y)| \leq K|x - y|, \qquad x, y \in \mathbb{R},\ t \in [0,1],$

(ii) $t \mapsto \sigma_t(0)$ and $t \mapsto b_t(0)$ are bounded on $[0,1]$,

for some $K > 0$. Lemma 8 of [3] shows that the solutions $(X_t)_{t\in[0,1]}$, $(\tilde{X}_t)_{t\in[0,1]}$ of (3.1) and (3.2) are increasing functionals when $\sigma(x)$, $\tilde{\sigma}(x)$ are differentiable with Lipschitz derivative in one variable and satisfy uniform bounds of the form

$$0 < \varepsilon \le \sigma(x) \le M < \infty \quad \text{and} \quad 0 < \tilde{\varepsilon} \le \tilde{\sigma}(x) \le \tilde{M} < \infty, \qquad x \in \mathbb{R}.$$

Thus from Proposition 3.3 it satisfies the FKG inequality as in Theorem 7 of [3]. Here the same covariance inequality can be obtained without using the FKG inequality, and under weaker hypotheses.

Theorem 3.4. *Let $s, t \in [0,1]$ and assume that σ, $\tilde{\sigma}$ satisfy the condition*

$$\sigma_r(x)\tilde{\sigma}_r(y) \ge 0, \qquad x, y \in \mathbb{R}, \quad 0 \le r \le s \wedge t.$$

Then we have

$$\mathrm{Cov}\,(f(X_s), g(\tilde{X}_t)) \ge 0, \tag{3.3}$$

for all non-decreasing Lipschitz functions f, g.

Proof. From Proposition 1.2.3 and Theorem 2.2.1 of [9], we have $f(X_s) \in \mathrm{Dom}\,(D)$, $s \in [0,1]$, and

$$D_r f(X_s) = 1_{[0,s]}(r)\sigma_r(X_r) f'(X_s) e^{\int_r^s \alpha_u dW_u + \int_r^s (\beta_u - \frac{1}{2}\alpha_u^2) du}, \tag{3.4}$$

$r, s \in [0,1]$, where $(\alpha_u)_{u\in[0,1]}$ and $(\beta_u)_{u\in[0,1]}$ are uniformly bounded adapted processes. Hence we have

$$\mathbb{E}[D_r X_s \mid \mathcal{F}_r] = 1_{[0,s]}(r)\sigma_r(X_r)\mathbb{E}\left[f'(X_s) e^{\int_r^s \alpha_u dW_u + \int_r^s (\beta_u - \frac{1}{2}\alpha_u^2) du} \Big| \mathcal{F}_r \right],$$

$r, s \in [0,1]$. Similarly we show that $\mathbb{E}[D_r g(\tilde{X}_t) \mid \mathcal{F}_r]$ has the form

$$\mathbb{E}[D_r g(\tilde{X}_t) \mid \mathcal{F}_r] = 1_{[0,t]}(r)\tilde{\sigma}_r(\tilde{X}_r)\mathbb{E}\left[g'(\tilde{X}_t) e^{\int_r^t \tilde{\alpha}_u dW_u + \int_r^t (\tilde{\beta}_u - \frac{1}{2}\tilde{\alpha}_u^2) du} \Big| \mathcal{F}_r \right],$$

$r, t \in [0,1]$, and we conclude the proof from Lemma 2.3. □

Note that (3.3) has also been obtained for $s = t$ and $X = \tilde{X}$ in [7], Theorem 3.2, by semigroup methods. In this case it also follows by applying Corollary 1.4 of [5] in dimension one. The argument of [7] can in fact be extended to recover Theorem 3.4 as above. Also, (3.3) may also hold under local Lipschitz hypotheses on σ and $\tilde{\sigma}$, for example as a consequence of Corollary 4.2 of [1].

4 The discrete case

Let $\Omega = \{-1,1\}^{\mathbb{N}}$ and consider the family $(X_k)_{k\geq 1}$ of independent Bernoulli $\{-1,1\}$-valued random variables constructed as the canonical projections on Ω, under a measure $\mathbb{P}$ such that

$$p_n = \mathbb{P}(X_n = 1) \quad \text{and} \quad q_n = \mathbb{P}(X_n = -1), \quad n \in \mathbb{N}.$$

Let $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$ and

$$\mathcal{F}_n = \sigma(X_0, \ldots, X_n), \qquad n \in \mathbb{N}.$$

Consider the linear gradient operator D defined as

$$D_k F(\omega) = \sqrt{p_k q_k}(F((\omega_i 1_{\{i\neq k\}} + 1_{\{i=k\}})_{i\in\mathbb{N}}) - F(\omega_i 1_{\{i\neq k\}} - 1_{\{i=k\}})_{i\in\mathbb{N}}), \tag{4.1}$$

$k \in \mathbb{N}$. Recall the discrete Clark Formula, cf. Proposition 7 of [11]:

$$F = \mathbb{E}[F] + \sum_{k=0}^{\infty} \mathbb{E}[D_k F | \mathcal{F}_{k-1}] Y_k, \tag{4.2}$$

where

$$Y_k = 1_{\{X_k=1\}} \sqrt{\frac{q_k}{p_k}} - 1_{\{X_k=-1\}} \sqrt{\frac{p_k}{q_k}}, \qquad k \in \mathbb{N},$$

defines a normalized i.i.d. sequence of centered random variables with unit variance. The Clark formula entails the following covariance identity, cf. Theorem 2 of [11]:

$$\mathrm{Cov}\,(F,G) = \mathbb{E}\left[\sum_{k=0}^{\infty} \mathbb{E}[D_k F|\mathcal{F}_{k-1}]\mathbb{E}[D_k G|\mathcal{F}_{k-1}]\right],$$

which yields a discrete time analog of Lemma 2.3.

Lemma 4.1. *Let $F, G \in L^2(\Omega)$ such that*

$$\mathbb{E}[D_k F|\mathcal{F}_{k-1}] \cdot \mathbb{E}[D_k G|\mathcal{F}_{k-1}] \geq 0, \qquad k \in \mathbb{N}.$$

Then F and G are non-negatively correlated:

$$\mathrm{Cov}\,(F,G) \geq 0.$$

According to the next definition, a non-decreasing functional F satisfies $D_k F \geq 0$ for all $k \in \mathbb{N}$.

Definition 4.1. A random variable $F : \Omega \to \mathbb{R}$ is said to be non-decreasing if for all $\omega_1, \omega_2 \in \Omega$ we have

$$\omega_1(k) \le \omega_2(k), \quad \forall k \in \mathbb{N}, \qquad \Rightarrow \ F(\omega_1) \le F(\omega_2).$$

The following result is then immediate from (4.1) and Lemma 4.1, and shows that the FKG inequality holds on Ω.

Proposition 4.2. *If $F, G \in L^2(\Omega)$ are non-decreasing then F and G are non-negatively correlated:*

$$\mathrm{Cov}\,(F, G) \ge 0.$$

Note however that the assumptions of Lemma 4.1 are actually weaker as they do not require F and G to be non-decreasing.

References

[1] E. Alós and C.O. Ewald. A note on the Malliavin differentiability of the Heston volatility. Economics and Business Working Papers Series, No 880, Universitat Pompeu Fabra, 2005.

[2] D. Bakry and D. Michel. Sur les inégalités FKG. In *Séminaire de Probabilités, XXVI*, volume 1526 of *Lecture Notes in Math.*, pages 170–188. Springer, Berlin, 1992.

[3] D. Barbato. FKG inequality for Brownian motion and stochastic differential equations. *Electron. Comm. Probab.*, 10:7–16 (electronic), 2005.

[4] C.M. Fortuin, P.W. Kasteleyn, and J. Ginibre. Correlation inequalities on some partially orderd sets. *Comm. Math. Phys.*, 22:89–103, 1971.

[5] I. Herbst and L. Pitt. Diffusion equation techniques in stochastic monotonicity and positive correlations. *Probab. Theory Related Fields*, 87:275–312, 1991.

[6] C. Houdré and N. Privault. Concentration and deviation inequalities in infinite dimensions via covariance representations. *Bernoulli*, 8(6):697–720, 2002.

[7] Y.Z. Hu. Itô-Wiener chaos expansion with exact residual and correlation, variance inequalities. *J. Theoret. Probab.*, 10(4):835–848, 1997.

[8] D. Nualart. Markov fields and transformations of the Wiener measure. In T. Lindstrøm, B. Øksendal, and A. S. Üstünel, editors, *Proceedings of the Fourth Oslo-Silivri Workshop on Stochastic Analysis*, volume 8 of *Stochastics Monographs*, Oslo, 1993. Gordon and Breach.

[9] D. Nualart. *The Malliavin Calculus and Related Topics.* Probability and its Applications. Springer-Verlag, 1995.

[10] D. Ocone. A guide to the stochastic calculus of variations. In H. Körezlioğlu and A.S. Üstünel, editors, *Stochastic Analysis and Related Topics*, volume 1316 of *Lecture Notes in Mathematics*, Silivri, 1988. Springer-Verlag.

[11] N. Privault and W. Schoutens. Discrete chaotic calculus and covariance identities. *Stochastics and Stochastics Reports*, 72:289–315, 2002.

[12] A.S. Üstünel. *An introduction to analysis on Wiener space*, volume 1610 of *Lecture Notes in Mathematics*. Springer Verlag, 1995.

[13] S. Watanabe. *Lectures on Stochastic Differential Equations and Malliavin Calculus.* Tata Institute of Fundamental Research, 1984.

[14] L. Wu. A new modified logarithmic Sobolev inequality for Poisson point processes and several applications. *Probab. Theory Related Fields*, 118(3):427–438, 2000.