Factorial moments of point processes

Jean-Christophe Breton*
IRMAR - UMR CNRS 6625
Université de Rennes 1
Campus de Beaulieu
F-35042 Rennes Cedex
France

Nicolas Privault†
Division of Mathematical Sciences
School of Physical and Mathematical Sciences
Nanyang Technological University
21 Nanyang Link
Singapore 637371

May 19, 2014

Abstract

We derive joint factorial moment identities for point processes with Papangelou intensities. Our proof simplifies previous combinatorical approaches to the computation of moments for point processes. We also obtain new explicit sufficient conditions for the distributional invariance of point processes with Papangelou intensities under random transformations.

Key words: Point process, Papangelou intensity, factorial moment, moment identity.
Mathematics Subject Classification (2010): 60G57; 60G55; 60H07.

1 Introduction

Consider the compound Poisson random variable

$$\beta_1 Z_{\alpha_1} + \cdots + \beta_p Z_{\alpha_p} \tag{1.1}$$

where $\beta_1, \ldots, \beta_p \in \mathbb{R}$ are constant parameters and $Z_{\alpha_1}, \ldots, Z_{\alpha_p}$ is a sequence of independent Poisson random variables with respective parameters $\alpha_1, \ldots, \alpha_p \in \mathbb{R}_+$. The Lévy-Khintchine formula

$$\mathbb{E}\big[e^{t(\beta_1 Z_{\alpha_1} + \cdots + \beta_p Z_{\alpha_p})}\big] = e^{\alpha_1(e^{\beta_1 t}-1) + \cdots + \alpha_p(e^{\beta_p t}-1)}$$

*jean-christophe.breton@univ-rennes1.fr
†nprivault@ntu.edu.sg

shows that the cumulant of order $k \geq 1$ of (1.1) is given by

$$\alpha_1 \beta_1^k + \cdots + \alpha_p \beta_p^k.$$

As a consequence, the moment of order $n \geq 1$ of (1.1) is given by the Faà di Bruno formula as

$$\mathbb{E}\left[\left(\sum_{i=1}^{p} \beta_i Z_{\alpha_i}\right)^n\right] = \sum_{m=1}^{n} \sum_{P_1 \cup \cdots \cup P_m = \{1,\ldots,n\}} \sum_{i_1,\ldots,i_m=1}^{p} \beta_{i_1}^{|P_1|} \alpha_{i_1} \cdots \beta_{i_m}^{|P_m|} \alpha_{i_m}, \tag{1.2}$$

where the above sum runs over all partitions $P_1, \ldots, P_m$ of $\{1, \ldots, n\}$.

Such cumulant-type moment identities have been extended in [8] to Poisson stochastic integrals of random integrands through the use of the Skorohod integral on the Poisson space, cf. [6], [7]. The construction of the Skorohod integral has been extended to point processes with Papangelou intensities in [9], and in [3], the moment identities of [8] have been extended to point processes with Papangelou intensities via a simpler combinatorial argument based on induction.

In this paper we deal with factorial moments, which are known to be easier to handle than standard moments, cf. for example, the direct relation between factorial moments and the correlation functions of point processes (2.3) below. See also [2] for the use of factorial moments to light-traffic approximations in queueing processes. In the case of random sets we obtain natural factorial moment identities by a direct induction argument, see Proposition 2.1 and Proposition 2.2. The moment indentities of [3] can then be recovered from standard relations between factorial moments and classical moments.

On the other hand, our results allow us to derive new practicable sufficient conditions for the distributional invariance of point processes, with Papangelou intensities, cf. Condition (3.8) in Proposition 3.2. Such conditions are shown to be satisfied on typical examples including transformations acting within the convex hull generated by the point process.

This paper is organized as follows. In Section 2, we derive factorial moment identities for random point measure of random sets in Propositions 2.1 and 2.2, and in Section 3 we apply those identities to point process transformations in Proposition 3.2. In Section 4, we show that the corresponding moment identities can be recovered by combinatorial arguments, cf. Proposition 4.2. In Section 5, we recover some recent results on the invariance of Poisson random measures under interacting transformations, with simplified proofs.

Notation and preliminaries on Papangelou intensities

Let X be a Polish space equipped with a σ-finite measure $\sigma(dx)$. Let Ω^X denote the space of configurations whose elements $\omega \in \Omega^X$ are identified with the Radon point measures $\omega = \sum_{x\in\omega} \epsilon_x$, where ϵ_x denotes the Dirac measure at $x \in X$. A point process is a probability measure P on Ω^X equipped with the σ-algebra $\mathcal{F}$ generated by the topology of vague convergence.

Point processes can be characterized by their Campbell measure C defined on $\mathcal{B}(X) \otimes \mathcal{F}$ by

$$C(A \times B) := \mathbb{E}\left[\int_X \mathbf{1}_A(x)\mathbf{1}_B(\omega \setminus \{x\})\, \omega(dx)\right], \qquad A \in \mathcal{B}(X), \quad B \in \mathcal{F}.$$

Recall the Georgii-Nguyen-Zessin identity

$$\mathbb{E}\left[\int_X u(x,\omega)\omega(dx)\right] = \int_{\Omega^X}\int_X u(x,\omega\cup x)C(dx,d\omega), \tag{1.3}$$

for all measurable function $u : X \times \Omega^X \to \mathbb{R}$ such that both sides of (1.3) make sense. In particular, a Poisson point process with intensity σ is a point process with Campbell measure $C = \sigma \otimes P$, and the Poisson measure with intensity $\sigma(dx)$ will be denoted by π_σ.

In the sequel we will deal with point processes whose Campbell measure $C(dx, d\omega)$ is absolutely continuous with respect to $\sigma \otimes P$, i.e.

$$C(dx, d\omega) = c(x,\omega)\sigma(dx)P(d\omega),$$

where the density $c(x,\omega)$ is called the Papangelou density. In this case the identity (1.3) reads

$$\mathbb{E}\left[\int_X u(x,\omega)\omega(dx)\right] = \mathbb{E}\left[\int_X u(x,\omega\cup x)c(x,\omega)\sigma(dx)\right], \tag{1.4}$$

and $c(x,\omega)=1$ for Poisson point process with intensity σ.

Denoting by Ω_0^X the set of finite configurations in Ω^X we will use the compound Campbell density

$$\hat{c} : \Omega_0^X \times \Omega^X \longrightarrow \mathbb{R}_+$$

which is defined inductively by

$$\hat{c}(\{x_1,\ldots,x_n,y\},\omega) := c(y,\omega)\hat{c}(\{x_1,\ldots,x_n\},\omega\cup\{y\}), \qquad n\geq 0. \tag{1.5}$$

Given $\mathfrak{x}_n=(x_1,\ldots,x_n)\in X^n$, we will use the notation $\varepsilon^+_{\mathfrak{x}_n}$ for the operator

$$(\varepsilon^+_{\mathfrak{x}_n}F)(\omega) = F(\omega\cup\{x_1,\ldots,x_n\}), \qquad \omega\in\Omega,$$

where F is any random variable on Ω^X. With this notation we also have

$$\hat{c}(\mathfrak{x}_n,\omega) = c(x_1,\omega)c(x_2,\omega\cup\{x_1\})c(x_3,\omega\cup\{x_1,x_2\})\cdots c(x_n,\omega\cup\{x_1,\ldots,x_{n-1}\}).$$

In addition, we define the random measure $\hat{\sigma}^n(d\mathfrak{x}_n)$ on X^n by

$$\hat{\sigma}^n(d\mathfrak{x}_n) = \hat{c}(\mathfrak{x}_n,\omega)\sigma^n(d\mathfrak{x}_n) = \hat{c}(\mathfrak{x}_n,\omega)\sigma(dx_1)\cdots\sigma(dx_n),$$

with $\sigma^n(d\mathfrak{x}_n)=\sigma(dx_1)\cdots\sigma(dx_n)$.

Finally, given a (possibly random) set A we let $N(A)(\omega)=\int_X \mathbf{1}_A(x)\ \omega(dx)$ denote the cardinality of $\omega\cap A(\omega)$. Recall that for a Poisson point process with intensity σ, $\omega(A)$ has the Poisson distribution with parameter $\sigma(A)$ for any (non random) $A\in\mathcal{F}$, and $N(A)$ is independent of $N(B)$ whenever $A,B\in\mathcal{F}$ are disjoint.

2 Factorial moments

In this section we deal with the *factorial moment* defined by

$$\mu_n^f(N(A)) := \mathbb{E}[N(A)_{(n)}],$$

where

$$x_{(n)} = x(x-1)\cdots(x-n+1), \qquad x \in \mathbb{R}, \quad n \in \mathbb{N},$$

is the falling factorial product and A is a (possibly random) measurable subset of X.

Proposition 2.1 *Let $A = A(\omega)$ be a random set. For all $n \geq 1$ and sufficiently integrable random variable F, we have*

$$\mathbb{E}\left[F\ N(A)_{(n)}\right] = \mathbb{E}\left[\int_{X^n} \varepsilon_{\mathfrak{x}_n}^+(F\mathbf{1}_{A^n}(x_1,\ldots,x_n))(\omega)\ \hat{\sigma}^n(dx_1,\ldots,dx_n)\right].$$

Proof. We show by induction on $n \geq 1$ that

$$\mathbb{E}\left[F\ N(A)_{(n)}\right] = \mathbb{E}\left[\int_{X^n} \varepsilon_{\mathfrak{x}_n}^+(F\mathbf{1}_A(x_1)\cdots\mathbf{1}_A(x_n))(\omega)\ \hat{\sigma}^n(d\mathfrak{x}_n)\right]. \tag{2.1}$$

Clearly the formula

$$\mathbb{E}[FN(A)] = \mathbb{E}\left[\int_X \varepsilon_x^+(F\mathbf{1}_A(x))(\omega)\ c(x,\omega)\ \sigma(dx)\right]$$

holds at the rank $n = 1$ due to (1.4) applied to $u(x,\omega) = F(\omega)\mathbf{1}_{A(\omega)}(x)$. Next, assuming that (2.1) holds at the rank n, we apply it with F replaced by $F(N(A)-n)$ and get

$$\begin{aligned}
\mathbb{E}\big[F\ N(A)\cdots(N(A)-n)\big] &= \mathbb{E}\left[\int_{X^n} \varepsilon_{\mathfrak{x}_n}^+(F\ (N(A)-n)\mathbf{1}_A(x_1)\cdots\mathbf{1}_A(x_n))(\omega)\ \hat{\sigma}^n(d\mathfrak{x}_n)\right] \\
&= \mathbb{E}\left[\int_{X^n} \varepsilon_{\mathfrak{x}_n}^+(F\ N(A)\mathbf{1}_A(x_1)\cdots\mathbf{1}_A(x_n))(\omega)\ \hat{\sigma}^n(d\mathfrak{x}_n)\right] \\
&\quad -n\mathbb{E}\left[\int_{X^n} \varepsilon_{\mathfrak{x}_n}^+(F\mathbf{1}_A(x_1)\cdots\mathbf{1}_A(x_n))(\omega)\ \hat{\sigma}^n(d\mathfrak{x}_n)\right] \\
&= \mathbb{E}\left[\int_{X^n} N(\varepsilon_{\mathfrak{x}_n}^+(A))(\omega)\varepsilon_{\mathfrak{x}_n}^+(F\mathbf{1}_A(x_1)\cdots\mathbf{1}_A(x_n))(\omega)\ \hat{\sigma}^n(d\mathfrak{x}_n)\right], \qquad (2.2)
\end{aligned}$$

where in (2.2) we used the relation

$$\varepsilon_{\mathfrak{x}_n}^+(N(A))(\omega) \quad = \quad N(\varepsilon_{\mathfrak{x}_n}^+(A))(\omega) + \sum_{i=1}^n \delta_{x_i}(\varepsilon_{\mathfrak{x}_n}^+(A)(\omega))$$

$$= \quad N(\varepsilon_{\mathfrak{x}_n}^+(A))(\omega) + \sum_{i=1}^n \varepsilon_{\mathfrak{x}_n}^+(\mathbf{1}_A(x_i))(\omega).$$

Next, with $\mathfrak{x}_{n+1} = (x_1, \ldots, x_n, x_{n+1})$, recalling that $N(A) = \int_X \mathbf{1}_A(x)\ \omega(dx)$ and applying (1.4) to

$$\begin{aligned} u(x,\omega) &:= \varepsilon_{\mathfrak{x}_n}^+(F\mathbf{1}_A(x_1)\cdots\mathbf{1}_A(x_n)\mathbf{1}_A(x))(\omega)\ \hat{c}(\mathfrak{x}_n,\omega) \\ &= F(\omega\cup\{x_1,\ldots,x_n\})\mathbf{1}_{A(\omega\cup\{x_1,\ldots,x_n\})}(x_1)\cdots\mathbf{1}_{A(\omega\cup\{x_1,\ldots,x_n\})}(x_n) \\ &\qquad \times\mathbf{1}_{A(\omega\cup\{x_1,\ldots,x_n\})}(x)\ \hat{c}(\mathfrak{x}_n,\omega) \end{aligned}$$

for fixed $x_1, \ldots, x_n$ with the relation $\varepsilon_{\mathfrak{x}_{n+1}}^+ = \varepsilon_{x_{n+1}}^+ \circ \varepsilon_{\mathfrak{x}_n}^+$ we find

$$\begin{aligned} &\mathbb{E}\left[F\ N(A)\cdots(N(A)-n)\right] \\ &= \mathbb{E}\left[\int_{X^{n+1}} \varepsilon_{\mathfrak{x}_{n+1}}^+\big(F\mathbf{1}_A(x_1)\cdots\mathbf{1}_A(x_{n+1})\big)(\omega)\ \hat{c}(\mathfrak{x}_n,\omega\cup\{x_{n+1}\})\ c(x_{n+1},\omega)\right. \\ &\qquad\qquad \left.\sigma(dx_1)\cdots\sigma(dx_{n+1})\right] \\ &= \mathbb{E}\left[\int_{X^{n+1}} \varepsilon_{\mathfrak{x}_{n+1}}^+(F\mathbf{1}_A(x_1)\cdots\mathbf{1}_A(x_{n+1}))(\omega)\ \hat{\sigma}^n(d\mathfrak{x}_{n+1})\right], \end{aligned}$$

where on the last line we used (1.5). □

By induction, in the next Proposition 2.2 we also obtain a joint factorial moment identity for a.s. disjoint (random) sets $A_1, \ldots, A_p$. It extends the classical identity

$$\mathbb{E}\left[N(A_1)_{(n_1)}\cdots N(A_p)_{(n_p)}\right] = \int_{A_1^{n_1}\times\cdots\times A_p^{n_p}} \rho_n(x_1,\ldots,x_n)\ \sigma(dx_1)\cdots\sigma(dx_n), \qquad (2.3)$$

for deterministic disjoint sets $A_1, \ldots, A_p$, where $\rho_n(x_1, \ldots, x_n)$ is the correlation function of the point process and $n = n_1 + \cdots + n_p$.

Proposition 2.2 *Let $n = n_1 + \cdots + n_p$ and $A_1(\omega), \ldots, A_p(\omega)$ be measurable and disjoint for almost all $\omega \in \Omega$, then*

$$\mathbb{E}\left[F\ N(A_1)_{(n_1)}\cdots N(A_p)_{(n_p)}\right] = \mathbb{E}\left[\int_{X^n} \varepsilon_{\mathfrak{x}_n}^+(F(\mathbf{1}_{A_1^{n_1}}\otimes\cdots\otimes\mathbf{1}_{A_p^{n_p}})(\mathfrak{x}_n))\ \hat{\sigma}^n(d\mathfrak{x}_n)\right]. \qquad (2.4)$$

Proof. We proceed by induction on $p \geq 1$. For $p = 1$, the identity reduces to that of Proposition 2.1. We assume that the identity holds true for p and show it for $p+1$. Letting $n = n_1 + \cdots + n_p$ and $m = n + n_{p+1}$ we have:

$$\mathbb{E}\left[F\ N(A_1)_{(n_1)}\cdots N(A_{p+1})_{(n_{p+1})}\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\int_{X^n} \varepsilon^+_{\mathfrak{x}_n}\Big(FN(A_{p+1})_{(n_{p+1})}(\mathbf{1}_{A_1^{n_1}}\otimes\cdots\otimes\mathbf{1}_{A_p^{n_p}})(x_1,\dots,x_n)\Big)\,\hat\sigma^n(d\mathfrak{x}_n)\right]\\
&= \mathbb{E}\left[\int_{X^n} N(\varepsilon^+_{\mathfrak{x}_n}(A_{p+1})_{(n_{p+1})})\varepsilon^+_{\mathfrak{x}_n}(F(\mathbf{1}_{A_1^{n_1}}\otimes\cdots\otimes\mathbf{1}_{A_p^{n_p}})(x_1,\dots,x_n))\ \hat\sigma^n(d\mathfrak{x}_n)\right] \qquad (2.5)\\
&= \mathbb{E}\Bigg[\int_{X^n}\int_{X^{n_{p+1}}} \varepsilon^+_{\mathfrak{y}_{n_{p+1}}}\Big(\varepsilon^+_{\mathfrak{x}_n}\Big(F\mathbf{1}_{A_1^{n_1}}\otimes\cdots\otimes\mathbf{1}_{A_p^{n_p}}(x_1,\dots,x_n)\Big)\,\hat c(\{x_1,\dots,x_n\},\omega)\\
&\qquad \mathbf{1}_{\varepsilon^+_{\mathfrak{x}_n}(A_{p+1}^{n_{p+1}})}(y_1,\dots,y_{n_{p+1}})\Big)\ \hat c(\{y_1,\dots,y_{n_{p+1}}\},\omega)\ \sigma(dy_1)\cdots\sigma(dy_{n_{p+1}})\sigma(dx_1)\cdots\sigma(dx_n)\Bigg] \qquad (2.6)\\
&= \mathbb{E}\left[\int_{X^m} \varepsilon^+_{\mathfrak{x}_m}\Big(F\mathbf{1}_{A_1^{n_1}}\otimes\cdots\otimes\mathbf{1}_{A_{p+1}^{n_{p+1}}}(x_1,\dots,x_m)\Big)\,\hat\sigma(dx_1,\dots,dx_m)\right], \qquad (2.7)
\end{aligned}
$$

where in (2.5) we used

$$
\begin{aligned}
\varepsilon^+_{\mathfrak{x}_n}(N(A_{p+1})_{(n_{p+1})}) &= \left(N(\varepsilon^+_{\mathfrak{x}_n}(A_{p+1})) + \sum_{i=1}^n \delta_{x_i}(\varepsilon^+_{\mathfrak{x}_n}(A_{p+1}))\right)_{(n_{p+1})}\\
&= \left(N(\varepsilon^+_{\mathfrak{x}_n}(A_{p+1})) + \sum_{i=1}^n \varepsilon^+_{\mathfrak{x}_n}(\mathbf{1}_{A_{p+1}}(x_i))\right)_{(n_{p+1})}
\end{aligned}
$$

and observe that the contribution of the sum is zero since, for all $1 \le k \le p$ and $1 \le i \le n$, $\mathbf{1}_{A_{p+1}}(x_i)\mathbf{1}_{A_k}(x_i) = 0$. In (2.6), we noted $\mathfrak{y}_{n_{p+1}} = (y_1,\dots,y_{n_{p+1}})$ and used Proposition 2.1 with, for a fixed $\mathfrak{x}_n = (x_1,\dots,x_n)$,

$$
\widetilde F(\omega) = \varepsilon^+_{\mathfrak{x}_n}\big(F(\mathbf{1}_{A_1^{n_1}}\otimes\cdots\otimes\mathbf{1}_{A_p^{n_p}})(x_1,\dots,x_n)\big)(\omega)\ \hat c(\{x_1,\dots,x_n\},\omega))
$$

and the set $\varepsilon^+_{\mathfrak{x}_n}(A_{p+1})$. Finally in (2.7), we used the following consequence of (1.5)

$$
\hat c\big(\{x_1,\dots,x_n\},\omega\cup\{y_1,\dots,y_{n_p}\}\big)\ \hat c\big(\{y_1,\dots,y_{n_p}\},\omega\big) = \hat c\big(\{x_1,\dots,x_n,y_1,\dots,y_{n_p}\},\omega\big)
$$

together with $\varepsilon^+_{\mathfrak{y}_{n_p}}\circ\varepsilon^+_{\mathfrak{x}_n} = \varepsilon^+_{\mathfrak{y}_{n_p}\cup\mathfrak{x}_n}$. □

3 Transformations of point processes

In this section we derive a sufficient condition for the distributional invariance of point processes under random transformations. Consider the finite difference operator

$$
D_xF(\omega) = F(\omega\cup\{x\}) - F(\omega)
$$

where F is any random variable on Ω^X. Note that multiple finite difference operator expresses

$$D_\Theta F = \sum_{\eta \subset \Theta} (-1)^{|\Theta|+1+|\eta|} F(\omega \cup \eta) \tag{3.1}$$

where the summation above holds over all (possibly empty) subset η of Θ. Let $\mathfrak{x}_n = \{x_1, \ldots, x_n\}$, from the relation

$$\begin{aligned} \varepsilon^+_{\mathfrak{x}_n}(u_1(x_1,\omega)\cdots u_n(x_n,\omega)) &= \varepsilon^+_{x_1,\ldots,x_n}(u_1(x_1,\omega)\cdots u_n(x_n,\omega)) \\ &= \sum_{\Theta \subset \{x_1,\ldots,x_n\}} D_\Theta\big(u_1(x_1,\omega)\cdots u_n(x_n,\omega)\big), \end{aligned} \tag{3.2}$$

where $D_\Theta = D_{x_1}\cdots D_{x_l}$ when $\Theta = \{x_1, \ldots, x_l\}$ and from (2.4) we have

$$\begin{aligned} \mathbb{E}\left[F\ N(A_1)_{(n_1)}\cdots N(A_p)_{(n_p)}\right] &= \mathbb{E}\left[\int_{X^n} \varepsilon^+_{\mathfrak{x}_n}(F(\mathbf{1}_{A_1^{n_1}} \otimes \cdots \otimes \mathbf{1}_{A_p^{n_p}})(\mathfrak{x}_n))\ \hat{\sigma}^n(d\mathfrak{x}_n)\right] \\ &= \sum_{\Theta \subset \{x_1,\ldots,x_n\}} \mathbb{E}\left[\int_{X^n} D_\Theta(F(\mathbf{1}_{A_1^{n_1}} \otimes \cdots \otimes \mathbf{1}_{A_p^{n_p}})(\mathfrak{x}_n))\ \hat{\sigma}^n(d\mathfrak{x}_n)\right] \end{aligned} \tag{3.3}$$

for $n_1 + \cdots + n_p = n$ and a.s. disjoint sets $A_1(\omega), \ldots, A_p(\omega)$. The next lemma will be useful in Proposition 3.2 to characterize the invariance of transformations of point processes from (3.3).

Lemma 3.1 *Let $m \geq 1$ and assume that for all $x_1, \ldots, x_m \in X$ the processes $u_i : X \times \Omega^X \longrightarrow \mathbb{R}$, $1 \leq i \leq m$ satisfy the condition*

$$D_{\Theta_1} u_1(x_1,\omega)\cdots D_{\Theta_m} u_m(x_m,\omega) = 0, \tag{3.4}$$

for every family $\{\Theta_1, \ldots, \Theta_m\}$ of (non empty) subsets such that $\Theta_1 \cup \cdots \cup \Theta_m = \{x_1, \ldots, x_m\}$, for all $x_1, \ldots, x_m \in X$ and all $\omega \in \Omega^X$. Then we have

$$D_{x_1}\cdots D_{x_m}\big(u_1(x_1,\omega)\cdots u_m(x_m,\omega)\big) = 0 \tag{3.5}$$

for all $x_1, \ldots, x_m \in X$ and all $\omega \in \Omega^X$.

Proof. It suffices to note that

$$D_{x_1}\cdots D_{x_n}\big(u_1(x_1,\omega)\cdots u_l(x_l,\omega)\big) = \sum_{\Theta_1 \cup \cdots \cup \Theta_l = \{x_1,\ldots,x_n\}} D_{\Theta_1} u_1(x_1,\omega)\cdots D_{\Theta_l} u_l(x_l,\omega), \tag{3.6}$$

where the above sum is not restricted to partitions, but includes all (possibly empty) sets $\Theta_1, \ldots, \Theta_l$ whose union is $\{x_1, \ldots, x_n\}$. □

In the next result, we recover Theorem 5.1 of [3] in a more direct way due to the use of factorial moments, but using a different cyclic type condition. Condition (3.8) below is interpreted by saying that

$$D_{\Theta_1} h_1(\tau(x_1, \omega)) \cdots D_{\Theta_m} h_m(\tau(x_m, \omega)) = 0, \tag{3.7}$$

for any family $h_1, \ldots, h_m$ of bounded real-valued Borel functions on Y. The merit of our condition (3.8) is to be satisfied in a typical example based on convex hull generated by a point process, see page 11.

Proposition 3.2 *Let X, Y be Polish spaces equipped respectively with σ-finite measures σ and μ and let $\tau : X \times \Omega^X \to Y$ be a random transformation such that $\tau(\cdot, \omega) : X \to Y$ maps bijectively σ to μ for all $\omega \in \Omega^X$, i.e.*

$$\sigma \circ \tau(\cdot, \omega)^{-1} = \mu, \qquad \omega \in \Omega^X,$$

and satisfying the condition

$$D_{\Theta_1} \tau(x_1, \omega) \cdots D_{\Theta_m} \tau(x_m, \omega) = 0, \tag{3.8}$$

for every family $\{\Theta_1, \ldots, \Theta_m\}$ of (non empty) subsets such that $\Theta_1 \cup \cdots \cup \Theta_m = \{x_1, \ldots, x_m\}$, for all $x_1, \ldots, x_m \in X$ and all $\omega \in \Omega^X$, $m \geq 1$. Then $\tau_* : \Omega^X \to \Omega^Y$ defined by

$$\tau_* \omega = \sum_{x \in \omega} \epsilon_{\tau(x,\omega)} = \omega \circ \tau(\cdot, \omega)^{-1}, \qquad \omega \in \Omega^X,$$

transforms a point process ξ with Papangelou intensity $c(x, \omega)$ with respect to $\sigma \otimes P$ into a point process on Y with correlation function

$$\rho_\tau(y_1, \ldots, y_n) = \mathbb{E}\big[\hat{c}(\{\tau^{-1}(y_1, \omega), \ldots, \tau^{-1}(y_n, \omega)\}, \omega)\big],$$

$y_1, \ldots, y_n \in Y$, with respect to μ.

Proof. Consider $B_1, \ldots, B_p$ disjoint *deterministic* subsets of Y such that $\mu(B_1), \ldots, \mu(B_p)$ are finite. From interpretation (3.7), Condition (3.8) ensures (3.4) for $u_k(x, \omega) = \mathbf{1}_{B_{i_k}}(\tau(x, \omega))$ and, in turn, Lemma 3.1 shows that

$$D_{x_1} \cdots D_{x_k} \big(\mathbf{1}_{B_{i_1}}(\tau(x_1, \omega)) \cdots \mathbf{1}_{B_{i_k}}(\tau(x_k, \omega))\big) = 0, \qquad x_1, \ldots, x_k \in X, \tag{3.9}$$

for all $i_1, \ldots, i_k \in \{1, \ldots, p\}$ and $\omega \in \Omega$. For $i = 1, \ldots, p$, let $A_i(\omega) = \tau(\cdot, \omega)^{-1}(B_i)$ and $\tau_* N(B_i)$ be the cardinal of $\tau_*\omega \cap B_i$, i.e.

$$\tau_* N(B_i) = \sum_{x \in \omega} \epsilon_{\tau(x,\omega)}(B_i) = \sum_{x \in \omega} \epsilon_x(A_i) = N(A_i).$$

Then applying (3.3) with $F = 1$ and the disjoint random sets $A_i(\omega)$, $i = 1, \ldots, p$, yields

$$\begin{aligned}
&\mathbb{E}\left[\tau_* N(B_1)_{(n_1)} \cdots \tau_* N(B_p)_{(n_p)}\right] = \mathbb{E}\left[N(A_1)_{(n_1)} \cdots N(A_p)_{(n_p)}\right] \\
&= \sum_{\Theta \subset \{x_1, \ldots, x_n\}} \mathbb{E}\left[\int_{X^n} D_\Theta((\mathbf{1}_{A_1^{n_1}} \otimes \cdots \otimes \mathbf{1}_{A_p^{n_p}})(\mathfrak{x}_n))\ \hat{\sigma}^n(d\mathfrak{x}_n)\right] \\
&= \mathbb{E}\left[\int_{X^n} D_{x_1} \cdots D_{x_n}((\mathbf{1}_{B_1^{n_1}} \otimes \cdots \otimes \mathbf{1}_{B_p^{n_p}})(\tau(\mathfrak{x}_n, \omega)))\ \hat{\sigma}^n(d\mathfrak{x}_n)\right] \\
&\quad + \sum_{\Theta \subsetneq \{x_1, \ldots, x_n\}} \mathbb{E}\left[\int_{X^n} D_\Theta((\mathbf{1}_{B_1^{n_1}} \otimes \cdots \otimes \mathbf{1}_{B_p^{n_p}})(\tau(\mathfrak{x}_n, \omega)))\ \hat{\sigma}^n(d\mathfrak{x}_n)\right] \\
&= \sum_{\Theta \subsetneq \{x_1, \ldots, x_n\}} \mathbb{E}\left[\int_{X^n} D_\Theta((\mathbf{1}_{B_1^{n_1}} \otimes \cdots \otimes \mathbf{1}_{B_p^{n_p}})(\tau(\mathfrak{x}_n, \omega)))\ \hat{\sigma}^n(d\mathfrak{x}_n)\right] \qquad (3.10)
\end{aligned}$$

$n_1 + \cdots + n_p = n \geq 1$, where $\tau(\mathfrak{x}_n, \omega)$ stands for $\big(\tau(x_1, \omega), \ldots, \tau(x_n, \omega)\big)$ and where we used (3.9). Next, without loss of generality the generic term of (3.10) can be reduced to the term with $\Theta = \{x_1, \ldots, x_{n-1}\}$ and using (3.6), we have

$$\begin{aligned}
&\mathbb{E}\left[\int_{X^n} D_{x_1} \cdots D_{x_{n-1}}((\mathbf{1}_{B_1^{n_1}} \otimes \cdots \otimes \mathbf{1}_{B_p^{n_p}})(\tau(\mathfrak{x}_n, \omega)))\hat{\sigma}^n(d\mathfrak{x}_n)\right] \\
&= \mathbb{E}\Bigg[\int_X \int_{X^{n-1}} D_{x_1} \cdots D_{x_{n-1}}((\mathbf{1}_{B_1^{n_1}} \otimes \cdots \otimes \mathbf{1}_{B_p^{n_p}})\big(\tau(\mathfrak{x}_{n-1}), y_n, \omega\big) \\
&\qquad \hat{c}(\{x_1, \ldots, x_{n-1}, \tau^{-1}(y_n, \omega)\}, \omega))\ \sigma^{n-1}(d\mathfrak{x}_{n-1})\mu(dy_n)\Bigg]
\end{aligned}$$

with the change of variable $y_n = \tau(x_n, \omega)$. Finally, by applying the above argument recursively we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\tau_* N(B_1)_{(n_1)} \cdots \tau_* N(B_p)_{(n_p)}\right] \\
&= \int_{X^n} (\mathbf{1}_{B_1^{n_1}} \otimes \cdots \otimes \mathbf{1}_{B_p^{n_p}})(\mathfrak{y}_n)\ \mathbb{E}\big[\hat{c}(\{\tau^{-1}(y_1, \omega), \ldots, \tau^{-1}(y_n, \omega)\}, \omega)\big]\ \mu^n(d\mathfrak{y}_n),
\end{aligned}$$

$n_1, \ldots, n_p = n \geq 1$, which recovers the definition of the correlation function of $\tau_*\xi$ (see (2.3)). $\square$

The proof of Proposition 3.2 also shows that if $A_1, \ldots, A_p$ are disjoint random subsets of X such that

$$D_{\Theta_1} \mathbf{1}_{A_1(\omega)}(x_1) \cdots D_{\Theta_m} \mathbf{1}_{A_m(\omega)}(x_m) = 0,$$

for every family $\{\Theta_1, \ldots, \Theta_m\}$ of (non empty) subsets such that $\Theta_1 \cup \cdots \cup \Theta_m = \{x_1, \ldots, x_m\}$, for all $x_1, \ldots, x_m \in X$ and all $\omega \in \Omega^X$, $m \geq 1$, then we have

$$\mathbb{E}\left[N(A_1)_{(n_1)} \cdots N(A_p)_{(n_p)}\right] = \mathbb{E}\left[\int_{X^n} (\mathbf{1}_{A_1^{n_1}} \otimes \cdots \otimes \mathbf{1}_{A_p^{n_p}})(\mathfrak{x}_n) \hat{\sigma}^n(d\mathfrak{x}_n)\right],$$

$n_1 + \cdots + n_p = n$.

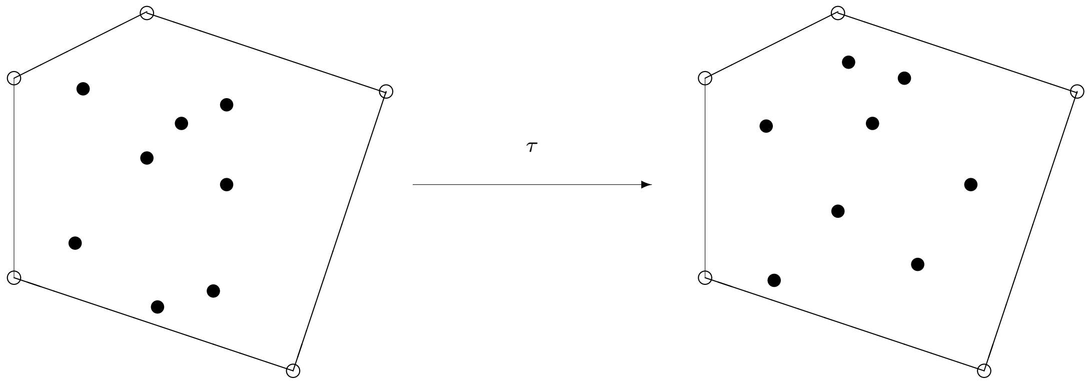
Example

Here we take $X = \mathbb{R}^d$ with norm $\| \cdot \|$ and we consider an example of transformation satisfying Condition (3.8), defined conditionally on the extremal vertices $\omega_e \subset \omega$ of the convex hull of $\omega \cap B(0, 1)$.

Denote by $\mathcal{C}(\omega)$ the convex hull of ω, and by $\mathring{\mathcal{C}}(\omega)$ its interior, we consider a mapping $\tau : X \times \Omega^X \longrightarrow X$ such that for all $\omega \in \Omega$, $\tau(\cdot, \omega) : X \longrightarrow X$ leaves $X \setminus \mathring{\mathcal{C}}(\omega_e)$ invariant (thus including the extremal vertices ω_e of $\mathcal{C}(\omega_e)$) while the points inside $\mathring{\mathcal{C}}(\omega_e)$ are shifted depending on the data of ω_e, i.e. we have

$$\tau(x, \omega) = \begin{cases} \tau(x, \omega_e), & x \in \mathring{\mathcal{C}}(\omega_e), \\ \\ x, & x \in X \setminus \mathring{\mathcal{C}}(\omega_e). \end{cases} \tag{3.11}$$

As shown in Proposition 3.3 below, such a transformation τ satisfies Condition (3.8). The next figure shows an example of behaviour such a transformation, with a finite set of points for simplicity of illustration.



Proposition 3.3 *The mapping $\tau : X \times \Omega^X \longrightarrow X$ given in (3.11) satisfies Condition (3.8).*

Proof. Let $x_1, \ldots, x_m \in X$. Clearly, we can assume that some x_i lies outside of $\mathcal{C}(\omega) = \mathcal{C}(\omega_e)$, otherwise

$$\begin{aligned} D_{x_i}\tau(x_j, \omega) &= \tau(x_j, \omega \cup \{x_i\}) - \tau(x_j, \omega) = \tau(x_j, (\omega \cup \{x_i\})_e) - \tau(x_j, \omega_e) \\ &= \tau(x_j, \omega_e) - \tau(x_j, \omega_e) = 0 \end{aligned}$$

for all $i, j = 1, \ldots, m$. Similarly, we can assume that $\mathcal{C}(\omega \cup \{x_1, \ldots, x_m\})$ has at least one extremal point $x_i \in \{x_1, \ldots, x_m\}$.

Now we have

$$\tau(x_i, \omega \cup \eta) = \tau(x_i, \omega) = x_i$$

for all $\eta \subset \{x_1, \ldots, x_m\}$, hence

$$D_\Theta \tau(x_i, \omega) = 0,$$

for all $\Theta \subset \{x_1, \ldots, x_m\}$, due to the following consequence of (3.1)

$$\begin{aligned} D_\Theta \tau(x_i, \omega) &= \sum_{\eta \subset \Theta} (-1)^{|\Theta|+1-|\eta|} \tau(x_i, \omega \cup \eta) \\ &= \tau(x_i, \omega) \sum_{\eta \subset \Theta} (-1)^{|\Theta|+1-|\eta|} \\ &= \tau(x_i, \omega \cup \eta)(1-1)^{|\Theta|+1} \\ &= 0, \end{aligned}$$

where the summation above holds over all (possibly empty) subset η of Θ. As a consequence, one factor of (3.8) necessarily vanishes. $\square$

4 Moment identities

As a consequence of the our factorial moment identities, we recover the moment identities recently derived for random integrands in [8] for Poisson stochastic integrals and extended in to point processes in [3]. Let

$$S(n,k) = \frac{1}{k!} \sum_{d_1+\cdots+d_k=n} \frac{n!}{d_1!\cdots d_k!} \tag{4.1}$$

denote the Stirling number of the second kind, i.e. the number of partitions of a set of n objects into k non-empty subsets, cf. also Relation (3) page 2 of [1]. As a consequence we recover the following elementary moment identity from Proposition 2.1.

Lemma 4.1 *Let $A_1 = A_2(\omega), \ldots, A_p = A_p(\omega)$ be disjoint random sets. We have*

$$\begin{aligned}
&\mathbb{E}\big[F N(A_1)^{n_1} \ldots N(A_p)^{n_p}\big] \\
&= \sum_{k_1=0}^{n_1} \cdots \sum_{k_p=0}^{n_p} S(n_1,k_1)\ldots S(n_p,k_p) \\
&\qquad \mathbb{E}\left[\int_{X^{k_1+\cdots+k_p}} \varepsilon^+_{\mathfrak{x}_{k_1+\cdots+k_p}} \big(F(\mathbf{1}_{A_1^{k_1}} \otimes \cdots \otimes \mathbf{1}_{A_p^{k_p}})(\mathfrak{x}_{k_1+\cdots+k_p})\big)\ \hat{\sigma}^{k_1+\cdots+k_p}(d\mathfrak{x}_{k_1+\cdots+k_p})\right].
\end{aligned}$$

As a particular case, for a random set $A = A(\omega)$, we have

$$\mathbb{E}\left[F\ N(A)^n\right] = \sum_{k=0}^{n} S(n,k) \mathbb{E}\left[\int_{X^k} \varepsilon^+_{\mathfrak{x}_k}(F\mathbf{1}_A(x_1)\cdots\mathbf{1}_A(x_k))\ \hat{\sigma}(dx_1,\ldots,dx_k)\right].$$

Proof. Using the classical identity

$$x^n = \sum_{k=0}^{n} S(n,k)\ x(x-1)\cdots(x-k+1),$$

cf. e.g. [5] or page 72 of [4], we have

$$\begin{aligned}
&\mathbb{E}\big[F N(A_1)^{n_1} \ldots N(A_p)^{n_p}\big] \\
&= \mathbb{E}\left[F\left(\sum_{k_1=0}^{n_1} S(n_1,k_1) N(A_1)_{(k_1)}\right) \cdots \left(\sum_{k_p=0}^{n_p} S(n_p,k_p) N(A_p)_{(k_p)}\right)\right] \\
&= \sum_{k_1=1}^{n_1} \cdots \sum_{k_p=0}^{n_p} S(n_1,k_1)\ldots S(n_p,k_p) \mathbb{E}\Big[F N(A_1)_{(k_1)} \ldots N(A_p)_{(k_p)}\Big]
\end{aligned}$$

$$= \sum_{k_1=0}^{n_1} \cdots \sum_{k_p=0}^{n_p} S(n_1, k_1) \dots S(n_p, k_p)$$

$$\mathbb{E}\left[\int_{X^{k_1+\cdots+k_p}} \varepsilon^+_{\mathfrak{x}_{k_1+\cdots+k_p}} \left(F(\mathbf{1}_{A_1^{k_1}} \otimes \cdots \otimes \mathbf{1}_{A_p^{k_p}})(\mathfrak{x}_{k_1+\cdots+k_p}) \right) \hat{\sigma}^{k_1+\cdots+k_p}(d\mathfrak{x}_{k_1+\cdots+k_p}) \right]$$

where we use (2.4) in the last line. □

More generally, Lemma 4.1 allows us to recover the following moment identity, cf. Theorem 3.1 of [3], and Proposition 3.1 of [8] for the Poisson case.

Proposition 4.2 *Let $u : X \times \Omega^X \longrightarrow \mathbb{R}$ be a (measurable) process. We have*

$$\mathbb{E}\left[\left(\int_X u(x, \omega) \, \omega(dx) \right)^n \right] = \sum_{k=1}^{n} \sum_{B_1^n, \dots, B_k^n} \mathbb{E}\left[\int_{X^k} \varepsilon^+_{\mathfrak{x}_k} \left(u(x_1, \cdot)^{|B_1^n|} \cdots u(x_k, \cdot)^{|B_k^n|} \right) \hat{\sigma}(d\mathfrak{x}_k) \right] \tag{4.2}$$

where the sum runs over the partitions $B_1^n, \dots, B_k^n$ of $\{1, \dots, n\}$, for any $n \geq 1$ such that all terms are integrable.

Proof. First we establish (4.2) for simple processes of the form $u(x, \omega) = \sum_{i=1}^p F_i(\omega) \mathbf{1}_{A_i(\omega)}(x)$ with a.s. disjoint random sets $A_i(\omega)$, $1 \leq i \leq p$. Applying Lemma 4.1 inductively we have

$$\mathbb{E}\left[\left(\sum_{i=1}^p F_i \int_X \mathbf{1}_{A_i}(x) \, \omega(dx) \right)^n \right] = \mathbb{E}\left[\left(\sum_{i=1}^p F_i N(A_i) \right)^n \right]$$

$$= \sum_{\substack{n_1+\cdots+n_p=n \\ n_1,\dots,n_p \geq 0}} \frac{n!}{n_1! \cdots n_p!} \mathbb{E}\left[(F_1 N(A_1))^{n_1} \cdots (F_p N(A_p))^{n_p} \right]$$

$$= \sum_{\substack{n_1+\cdots+n_p=n \\ n_1,\dots,n_p \geq 0}} \frac{n!}{n_1! \cdots n_p!} \sum_{k_1=0}^{n_1} \cdots \sum_{k_p=0}^{n_p} S(n_1, k_1) \cdots S(n_p, k_p)$$

$$\mathbb{E}\Bigg[\int_{X^{k_1+\cdots+k_p}} \varepsilon^+_{\mathfrak{x}_{k_1+\cdots+k_p}} \left(F_1^{n_1} \cdots F_p^{n_p} \mathbf{1}_{A_1^{k_1}} \otimes \cdots \otimes \mathbf{1}_{A_p^{k_p}}(x_1, \dots, x_{k_1+\cdots+k_p}) \right)$$
$$\hat{c}(\{x_1, \dots, x_{k_1+\cdots+k_p}\}, \omega) \, \sigma(dx_1) \cdots \sigma(dx_{k_1+\cdots+k_p}) \Bigg]$$

$$= \sum_{m=0}^{n} \sum_{\substack{n_1+\cdots+n_p=n \\ n_1,\dots,n_p \geq 0}} \frac{n!}{n_1! \cdots n_p!} \sum_{\substack{k_1+\cdots+k_p=m \\ 0 \leq k_1 \leq n_1, \dots, 0 \leq k_p \leq n_p}} S(n_1, k_1) \cdots S(n_p, k_p)$$

$$\mathbb{E}\left[\int_{X^m} \varepsilon^+_{\mathfrak{x}_m} \left(F_1^{n_1} \cdots F_p^{n_p} \mathbf{1}_{A_1^{k_1}} \otimes \cdots \otimes \mathbf{1}_{A_p^{k_p}}(x_1, \dots, x_m) \right) \hat{c}(\{x_1, \dots, x_m\}, \omega) \, \sigma(d\mathfrak{x}_m) \right]$$

$$= \sum_{m=0}^{n} \sum_{\substack{n_1+\cdots+n_p=n \\ n_1,\dots,n_p\ge 0}} \frac{n!}{n_1!\cdots n_p!} \sum_{\substack{I_1\cup\cdots\cup I_p=\{1,\cdots,m\} \\ |I_1|\le n_1,\dots,|I_p|\le n_p}} S(n_1,|I_1|)\cdots S(n_p,|I_p|)\frac{|I_1|!\cdots|I_p|!}{m!}$$

$$\mathbb{E}\left[\int_{X^m} \varepsilon^+_{\mathfrak{x}_m}\left(F_1^{n_1}\cdots F_p^{n_p}\prod_{j\in I_1}\mathbf{1}_{A_1}(x_j)\cdots\prod_{j\in I_p}\mathbf{1}_{A_p}(x_j)\right)\ \hat{c}(\{x_1,\dots,x_m\},\omega)\ \sigma(d\mathfrak{x}_m)\right] \tag{4.3}$$

$$= \sum_{m=0}^{n} \sum_{P_1\cup\cdots\cup P_m=\{1,\dots,n\}} \sum_{i_1,\dots,i_m=1}^{p} \mathbb{E}\left[\int_{X^m} \varepsilon^+_{\mathfrak{x}_m}\left(F_{i_1}^{|P_1|}\mathbf{1}_{A_{i_1}}(x_1)\cdots F_{i_m}^{|P_m|}\mathbf{1}_{A_{i_m}}(x_m)\right)\ \hat{\sigma}(d\mathfrak{x}_m)\right], \tag{4.4}$$

where in (4.3) we made changes of variables in the integral and, in (4.4), we used the combinatorial identity of Lemma 4.3 below with $\alpha_{i,j}=\mathbf{1}_{A_i}(x_j)$, $1\le i\le p, 1\le j\le m$, and $\beta_i=F_i$. The proof is concluded by using the fact that the A_i's in (4.5) are disjoint, as follows:

$$\mathbb{E}\left[\left(\sum_{i=1}^{p} F_i\int_X \mathbf{1}_{A_i}(x)\ \omega(dx)\right)^n\right]$$

$$= \sum_{m=0}^{n} \sum_{P_1\cup\cdots\cup P_m=\{1,\dots,n\}} \mathbb{E}\left[\int_{X^m} \varepsilon^+_{\mathfrak{x}_m}\left(\sum_{i=1}^{p}\left(F_i^{|P_1|}\mathbf{1}_{A_i}(x_1)\right)\cdots\sum_{i=1}^{p}\left(F_i^{|P_m|}\mathbf{1}_{A_i}(x_m)\right)\right)\ \hat{\sigma}(d\mathfrak{x}_m)\right]$$

$$= \sum_{m=0}^{n} \sum_{P_1\cup\cdots\cup P_m=\{1,\dots,n\}} \mathbb{E}\left[\int_{X^m} \varepsilon^+_{\mathfrak{x}_m}\left(\left(\sum_{i=1}^{p}F_i\mathbf{1}_{A_i}(x_1)\right)^{|P_1|}\cdots\left(\sum_{i=1}^{p}F_i\mathbf{1}_{A_i}(x_m)\right)^{|P_m|}\right)\ \hat{\sigma}(d\mathfrak{x}_m)\right]. \tag{4.5}$$

The general case is obtained by approximating $u(x,\omega)$ with simple processes. □

Using (3.2), we can also write

$$\mathbb{E}\left[\left(\int_X u(x,\omega)\ \omega(dx)\right)^n\right] = \sum_{k=1}^{n}\sum_{B_1^n,\dots,B_k^n}\sum_{\Theta\subset\{x_1,\dots,x_k\}} \mathbb{E}\left[\int_{X^k} D_\Theta(u_{x_1}^{|B_1^n|}\cdots u_{x_k}^{|B_k^n|})\ \hat{\sigma}(d\mathfrak{x}_k)\right].$$

The next lemma has been used above in the proof of Proposition 4.2.

Lemma 4.3 *Let $m,n,p\in\mathbb{N}$, $(\alpha_{i,j})_{1\le i\le p, 1\le j\le m}$ and $\beta_1,\dots,\beta_p\in\mathbb{R}$. We have*

$$\sum_{\substack{n_1+\cdots+n_p=n \\ n_1,\dots,n_p\ge 0}} \frac{n!}{n_1!\cdots n_p!} \sum_{\substack{I_1\cup\cdots\cup I_p=\{1,\dots,m\} \\ |I_1|\le n_1,\dots,|I_p|\le n_p}} S(n_1,|I_1|)\cdots S(n_p,|I_p|)\times$$

$$\frac{|I_1|!\cdots|I_p|!}{m!}\beta_1^{n_1}\Big(\prod_{j\in I_1}\alpha_{1,j}\Big)\cdots\beta_p^{n_p}\Big(\prod_{j\in I_p}\alpha_{p,j}\Big)$$

$$= \sum_{P_1\cup\cdots\cup P_m=\{1,\ldots,n\}}\sum_{i_1,\ldots,i_m=1}^{p}\beta_{i_1}^{|P_1|}\alpha_{i_1,1}\cdots\beta_{i_m}^{|P_m|}\alpha_{i_m,m}. \tag{4.6}$$

Proof. Observe that (4.1) ensures

$$S(n,|I|)\beta^n\Big(\prod_{j\in I}\alpha_j\Big) = \sum_{\bigcup_{a\in I}P_a=\{1,\ldots,n\}}\prod_{j\in I}\left(\alpha_j\beta^{|P_j|}\right)$$

for all α_j, $j\in I$, $\beta\in\mathbb{R}$, $n\in\mathbb{N}$. We have

$$\sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\geq 0}}\frac{n!}{n_1!\cdots n_p!}\sum_{\substack{I_1\cup\cdots\cup I_p=\{1,\ldots,m\}\\ |I_1|\leq n_1,\ldots,|I_p|\leq n_p}}S(n_1,|I_1|)\cdots S(n_p,[I_p|)$$
$$\frac{|I_1|!\cdots|I_p|!}{m!}\beta_1^{n_1}\big(\prod_{j\in I_1}\alpha_{1,j}\Big)\cdots\beta_p^{n_p}\Big(\prod_{j\in I_p}\alpha_{p,j}\Big)$$
$$= \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\geq 0}}\frac{n!}{n_1!\cdots n_p!}\sum_{\substack{I_1\cup\cdots\cup I_p=\{1,\ldots,m\}\\ |I_1|\leq n_1,\ldots,|I_p|\leq n_p}}\frac{|I_1|!\cdots|I_p|!}{m!}$$
$$\Bigg(\sum_{\bigcup_{a\in I_1}P_a^1=\{1,\ldots,n_1\}}\prod_{j_1\in I_1}\left(\alpha_{1,j_1}\beta_1^{|P_{j_1}^1|}\right)\Bigg)\cdots\Bigg(\sum_{\bigcup_{a\in I_p}P_a^p=\{1,\ldots,n_p\}}\prod_{j_p\in J_p}\left(\alpha_{p,j_p}\beta_p^{|P_{j_p}^p|}\right)\Bigg)$$
$$= \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\geq 0}}\frac{n!}{n_1!\cdots n_p!}\sum_{\substack{I_1\cup\cdots\cup I_p=\{1,\ldots,m\}\\ |I_1|\leq n_1,\ldots,|I_p|\leq n_p}}\sum_{\bigcup_{a\in I_1}P_a^1=\{1,\ldots,n_1\}}\cdots\sum_{\bigcup_{a\in I_p}P_a^p=\{1,\ldots,n_p\}}$$
$$\frac{|I_1|!\cdots|I_p|!}{m!}\prod_{l=1}^{p}\prod_{j_l\in I_l}\left(\alpha_{l,j_l}\beta_l^{|P_{j_l}^l|}\right)$$
$$= \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\geq 0}}\frac{n!}{n_1!\cdots n_p!}\sum_{\substack{I_1\cup\cdots\cup I_p=\{1,\ldots,m\}\\ |I_1|\leq n_1,\ldots,|I_p|\leq n_p}}\sum_{\bigcup_{a\in I_1}P_a^1=\{1,\ldots,n_1\}}\cdots\sum_{\bigcup_{a\in I_p}P_a^p=\{1,\ldots,n_p\}}$$
$$\frac{|I_1|!\cdots|I_p|!}{m!}\prod_{l=1}^{p}\prod_{j_l\in I_l}\alpha_{l,j_l}\prod_{l=1}^{p}\prod_{j_l\in I_l}\beta_l^{|P_{j_l}^l|}$$
$$= \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\geq 0}}\frac{n!}{n_1!\cdots n_p!}\sum_{\substack{k_1+\cdots+k_p=m\\ 1\leq k_1\leq n_1,\ldots,1\leq k_p\leq n_p}}\sum_{i_1,\ldots,i_m=1}^{p}$$
$$\sum_{P_1^1\cup\cdots\cup P_{k_1}^1=\{1,\ldots,n_1\}}\cdots\sum_{P_{k_1+\cdots+k_{p-1}+1}^p\cup\cdots\cup P_{k_1+\cdots+k_p}^p=\{1,\ldots,n_p\}}\prod_{j=1}^{m}\left(\alpha_{i_j,j}\beta_{i_j}^{|P_j^{i_1}|+\cdots+|P_j^{i_m}|}\right)$$

$$= \sum_{P_1\cup\cdots\cup P_m=\{1,\ldots,n\}} \sum_{i_1,\ldots,i_m=1}^{p} \beta_{i_1}^{|P_1|}\alpha_{i_1,1}\cdots\beta_{i_m}^{|P_m|}\alpha_{i_m,m},$$

by a reindexing of the summations and the fact that the reunions of the partitions $P_1^j,\ldots,P_{|I_j|}^j$, $1\le j\le p$, of disjoint p subsets of $\{1,\ldots,m\}$ run the partition of $\{1,\ldots,m\}$ when we take into account the choice of the p subsets and the possible length k_j, $1\le j\le p$, of the partitions. □

Note that the combinatorial result of Lemma 4.3 can also be shown in a probabilistic way when $\alpha_{i,j}=\alpha_i$, $1\le i\le p$, $1\le j\le m$. Next we have

$$\begin{aligned}
&\sum_{m=0}^{n}\lambda^m \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\ge 0}} \frac{n!}{n_1!\cdots n_p!} \sum_{\substack{k_1+\cdots+k_p=m\\ k_1\le n_1,\ldots,k_p\le n_p}} S(n_1,k_1)\cdots S(n_p,k_p)\beta_1^{n_1}\alpha_1^{k_1}\cdots\beta_p^{n_p}\alpha_p^{k_p}\\
&= \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\ge 0}} \frac{n!}{n_1!\cdots n_p!} \sum_{k_1=0}^{n_1} S(n_1,k_1)(\lambda\alpha_1)^{k_1}\cdots\sum_{k_p=0}^{n_p} S(n_p,k_p)(\lambda\alpha_p)^{k_p}\beta_1^{n_1}\cdots\beta_p^{n_p}\\
&= \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\ge 0}} \frac{n!}{n_1!\cdots n_p!}\beta_1^{n_1}\cdots\beta_p^{n_p}\mathbb{E}[Z_{\lambda\alpha_1}^{n_1}\cdots Z_{\lambda\alpha_p}^{n_p}]\\
&= \mathbb{E}\left[\left(\sum_{i=1}^{p}\beta_i Z_{\lambda\alpha_i}\right)^n\right]\\
&= \sum_{m=0}^{n}\lambda^m \sum_{P_1\cup\cdots\cup P_m=\{1,\ldots,n\}} \sum_{i_1,\ldots,i_m=1}^{p} \beta_{i_1}^{|P_1|}\alpha_{i_1}\cdots\beta_{i_m}^{|P_m|}\alpha_{i_m},
\end{aligned}\tag{4.7}$$

since by the relation (1.2) between standard and factorial moments the moment of order n_i of $Z_{\lambda\alpha_i}$ is given by

$$\mathbb{E}\left[Z_{\lambda\alpha_i}^{n_i}\right] = \sum_{k=0}^{n_i} S(n_i,k)(\lambda\alpha_i)^k.$$

The above relation (4.7) being true for all λ, this implies (4.6) for this choice of $\alpha_{i,j}$'s.

5 Poisson case

In the Poisson case, we have $c(x,\omega)=1$ and the results of the previous sections specialize immediately to new factorial moment identities for Poisson point processes with intensity $\sigma(dx)$. For any random set $A = A(\omega)$ and sufficiently integrable random variable F, we have

$$\mathbb{E}\left[F\ N(A)_{(n)}\right] = \mathbb{E}\left[\int_{X^n} \varepsilon^+_{\mathfrak{x}_n}(F\mathbf{1}_A(x_1)\cdots\mathbf{1}_A(x_n))\ \sigma(dx_1)\cdots\sigma(dx_n)\right],$$

$n \geq 1$. For all almost surely disjoint random sets $A_i(\omega)$, $1 \leq i \leq p$, and sufficiently integrable random variable F, we have

$$\begin{aligned}&\mathbb{E}\left[F\ N(A_1)_{(n_1)}\cdots N(A_p)_{(n_p)}\right]\\ &= \mathbb{E}\left[\int_{X^n} \varepsilon^+_{\mathfrak{x}_n}(F(\mathbf{1}_{A_1^{n_1}}\otimes\cdots\otimes\mathbf{1}_{A_p^{n_p}})(x_1,\ldots,x_n))\ \sigma(dx_1)\cdots\sigma(dx_n)\right],\end{aligned}$$

with $n = n_1 + \cdots + n_p$. In addition, we have the following proposition whose proof is similar to that of Proposition 3.2 although it cannot be obtained as a direct consequence of Proposition 3.2 and is not available in the (non-Poisson) point process setting.

Proposition 5.1 *Consider $A_1(\omega), \ldots, A_p(\omega)$ a.s. disjoint random sets such that $\sigma(A_i(\omega))$ is deterministic, $i = 1, \ldots, p$, and*

$$D_{\Theta_1}\mathbf{1}_{A_i(\omega)}(x_1)\cdots D_{\Theta_m}\mathbf{1}_{A_i(\omega)}(x_m) = 0, \tag{5.1}$$

for every family $\{\Theta_1, \ldots, \Theta_m\}$ of (non empty) subsets such that $\Theta_1 \cup \cdots \cup \Theta_m = \{x_1, \ldots, x_m\}$, all $x_1, \ldots, x_m \in X$, all $\omega \in \Omega^X$. Then the family

$$\big(N(A_1), \ldots, N(A_p)\big)$$

is a vector of independent Poisson random variables with parameters $\sigma(A_1), \ldots, \sigma(A_p)$.

Proof. Let $n = n_1 + \cdots + n_p$. Under Condition (5.1), Lemma 3.1 and (3.5) show that

$$D_{x_1}\cdots D_{x_k}\big(\mathbf{1}_{A_{i_1}(\omega)}(x_1)\cdots\mathbf{1}_{A_{i_k}(\omega)}(x_k)\big) = 0, \qquad x_1, \ldots, x_k \in X, \tag{5.2}$$

for all $i_1, \ldots, i_k \in \{1, \ldots, n\}$ and $\omega \in \Omega$. Since in addition $\sigma(A_i)$ is deterministic, $i = 1, \ldots, p$, then by (3.3) with $F = 1$ we obtain

$$\mathbb{E}\left[N(A_1)_{(n_1)}\cdots N(A_p)_{(n_p)}\right] = \sum_{\Theta\subset\{x_1,\ldots,x_n\}} \mathbb{E}\left[\int_{X^n} D_\Theta\big((\mathbf{1}_{A_1^{n_1}}\otimes\cdots\otimes\mathbf{1}_{A_p^{n_p}})(\mathfrak{x}_n)\big)\ \sigma^n(d\mathfrak{x}_n)\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\int_{X^n} D_{x_1}\cdots D_{x_n}\big((\mathbf{1}_{A_1^{n_1}}\otimes\cdots\otimes\mathbf{1}_{A_p^{n_p}})(\mathfrak{x}_n)\big)\ \sigma^n(d\mathfrak{x}_n)\right]\\
&\quad + \sum_{\Theta\subsetneq\{x_1,\ldots,x_n\}} \mathbb{E}\left[\int_{X^n} D_\Theta\big((\mathbf{1}_{A_1^{n_1}}\otimes\cdots\otimes\mathbf{1}_{A_p^{n_p}})(\mathfrak{x}_n)\big)\ \sigma^n(d\mathfrak{x}_n)\right]\\
&= \sum_{\Theta\subsetneq\{x_1,\ldots,x_n\}} \mathbb{E}\left[\int_{X^n} D_\Theta\big((\mathbf{1}_{A_1^{n_1}}\otimes\cdots\otimes\mathbf{1}_{A_p^{n_p}})(\mathfrak{x}_n)\big)\ \sigma^n(d\mathfrak{x}_n)\right] \qquad (5.3)
\end{aligned}
$$

using (5.2). Next, without loss of generality the generic, term of (5.3) can be reduced to the term with $\Theta = \{x_1, \ldots, x_{n-1}\}$ and using (3.5), we have

$$
\begin{aligned}
&\mathbb{E}\left[\int_{X^n} D_{x_1}\cdots D_{x_{n-1}}\big((\mathbf{1}_{A_1^{n_1}}\otimes\cdots\otimes\mathbf{1}_{A_p^{n_p}})(\mathfrak{x}_n)\big)\ \sigma^n(d\mathfrak{x}_n)\right]\\
&= \mathbb{E}\left[\int_{X^n} \sum_{\Theta_1\cup\cdots\cup\Theta_n=\{x_1,\ldots,x_{n-1}\}} \prod_{k=1}^{p}\prod_{j=1}^{n_k} D_{\Theta_{n_1+\cdots+n_{k-1}+j}}\mathbf{1}_{A_k}(x_{n_1+\cdots+n_{k-1}+j})\ \sigma^n(d\mathfrak{x}_n)\right]\\
&= \mathbb{E}\Bigg[\int_{X^{n-1}} \sum_{\Theta_1\cup\cdots\cup\Theta_n=\{x_1,\ldots,x_{n-1}\}} \prod_{k=1}^{p-1}\prod_{j=1}^{n_k} D_{\Theta_{n_1+\cdots+n_{k-1}+j}}\mathbf{1}_{A_k}(x_{n_1+\cdots+n_{k-1}+j}) \qquad (5.4)\\
&\quad \times \prod_{j=1}^{n_p-1} D_{\Theta_{n_1+\cdots+n_{p-1}+j}}\mathbf{1}_{A_p}(x_{n_1+\cdots+n_{p-1}+j}) \int_X D_{\Theta_n}\mathbf{1}_{A_p}(x_n)\ \sigma(dx_n)\ \sigma^{n-1}(d\mathfrak{x}_{n-1})\Bigg]\\
&= \mathbb{E}\Bigg[\int_{X^{n-1}} \sum_{\Theta_1\cup\cdots\cup\Theta_{n-1}=\{x_1,\ldots,x_{n-1}\}} \prod_{k=1}^{p-1}\prod_{j=1}^{n_k} D_{\Theta_{n_1+\cdots+n_{k-1}+j}}\mathbf{1}_{A_k}(x_{n_1+\cdots+n_{k-1}+j})\\
&\quad \times \prod_{j=1}^{n_p-1} D_{\Theta_{n_1+\cdots+n_{p-1}+j}}\mathbf{1}_{A_p}(x_{n_1+\cdots+n_{p-1}+j})\sigma(A_p)\ \sigma^{n-1}(d\mathfrak{x}_{n-1})\Bigg] \qquad (5.5)\\
&= \sigma(A_p)\mathbb{E}\left[\int_{X^{n-1}} D_{x_1}\cdots D_{x_{n-1}}((\mathbf{1}_{A_1^{n_1}}\otimes\cdots\otimes\mathbf{1}_{A_p^{n_p-1}})(\mathfrak{x}_{n-1}))\ \sigma^{n-1}(d\mathfrak{x}_{n-1})\right]
\end{aligned}
$$

where (5.5) comes from the fact that in (5.4) only the term with $\Theta_n = \emptyset$ is not zero since

$$
\int_X D_\Theta \mathbf{1}_{A_p}(x)\ \sigma(dx) = D_\Theta\left(\int_X \mathbf{1}_{A_p}(x)\sigma(dx)\right) = D_\Theta\big(\sigma(A_p)\big) = 0
$$

using $\sigma(A_p)$ is deterministic. Finally, by applying the above argument recursively we obtain

$$
\begin{aligned}
\mathbb{E}\left[N(A_1)_{(n_1)}\cdots N(A_p)_{(n_p)}\right] &= \mathbb{E}\left[\int_{X^n} (\mathbf{1}_{A_1^{n_1}}\otimes\cdots\otimes\mathbf{1}_{A_p^{n_p}})(\mathfrak{x}_n)\sigma^n(d\mathfrak{x}_n)\right]\\
&= \sigma(A_1)^{n_1}\cdots\sigma(A_p)^{n_p},
\end{aligned}
$$

$n_1, \ldots, n_p \geq 1$, which characterizes the Poisson distribution with parameters

$$(\sigma(A_1), \ldots, \sigma(A_p)).$$

□

As a consequence of Proposition 5.1 we recover Theorem 3.3 in [7] on the invariance for Poisson measures when $(X, \sigma) = (Y, \mu)$. Note that Condition (5.6), which is interpreted as in (3.7) above, is actually a stregthening of Condition (3.8) in [7] and fills a gap in the proof of Theorem 3.3 therein, based on the fact that $D_x u(\omega) = 0$ does not imply $D_\Theta u(\omega) = 0$ when $x \in \Theta_0$.

Theorem 5.2 *Let $\tau : X \times \Omega^X \to Y$ be a random transformation such that $\tau(\cdot, \omega) : X \to Y$ maps σ to μ for all $\omega \in \Omega^X$, i.e.*

$$\sigma \circ \tau(\cdot, \omega)^{-1} = \mu, \qquad \omega \in \Omega^X,$$

and satisfying the condition

$$D_{\Theta_1} \tau(x_1, \omega) \cdots D_{\Theta_m} \tau(x_m, \omega) = 0, \tag{5.6}$$

for every family $\{\Theta_1, \ldots, \Theta_m\}$ of (non empty) subsets such that $\Theta_1 \cup \cdots \cup \Theta_m = \{x_1, \ldots, x_m\}$, all $x_1, \ldots, x_m \in X$, all $\omega \in \Omega^X$, and all $i = 1, \ldots, p$. Then $\tau_* : \Omega^X \to \Omega^Y$ defined by

$$\tau_* \omega = \sum_{x \in \omega} \epsilon_{\tau(x, \omega)} = \omega \circ \tau(\cdot, \omega)^{-1}, \qquad \omega \in \Omega^X,$$

maps π_σ to π_μ, i.e. $\tau_* \pi_\sigma$ is the Poisson measure π_μ with intensity $\mu(dy)$ on Y.

Proof. For any family $B_1, \ldots, B_p$ of disjoint measurable subsets of Y with finite measure, we let $A_i(\omega) = \tau^{-1}(B_i, \omega) \subset X$, i.e. $\mathbf{1}_{A_i}(\cdot) = \mathbf{1}_{B_i} \circ \tau(\cdot, \omega)$, $i = 1, \ldots, p$, and by Proposition 5.1, we find that

$$\omega \longmapsto (\tau_* \omega(B_1), \ldots, \tau_* \omega(B_p)) = (\omega(A_1), \ldots, \omega(A_p))$$

is a vector of independent Poisson random variables with parameters $\mu(A_1), \ldots, \mu(A_p)$ since $\sigma(A_i(\omega)) = \sigma(\tau^{-1}(B_i, \omega)) = \mu(B_i)$ is deterministic, $i = 1, \ldots, p$, and (5.1) comes from the following consequence of (5.6):

$$D_{\Theta_1} \mathbf{1}_{A_{i_1}(\omega)}(x_1) \cdots D_{\Theta_m} \mathbf{1}_{A_{i_m}(\omega)}(x_m)$$

$$
\begin{aligned}
&= D_{\Theta_1}\mathbf{1}_{B_{i_1}}(\tau(x_1,\omega))\cdots D_{\Theta_m}\mathbf{1}_{B_{i_m}}(\tau(x_m,\omega)) \\
&= 0.
\end{aligned}
$$

□

The example of random transformation given on page 11 at the end of Section 3 also satisfies Condition (5.6) in Theorem 5.2.

References

[1] M. Bernstein and N. J. A. Sloane. Some canonical sequences of integers. *Linear Algebra Appl.*, 226/228:57–72, 1995.

[2] B. Błaszczyszyn. Factorial moment expansion for stochastic systems. *Stochastic Process. Appl.*, 56(2):321–335, 1995.

[3] L. Decreusefond and I. Flint. Moment formulae for general point processes. Preprint arXiv:1211.4811, 2012, to appear in J. Funct. Anal.

[4] B. Fristedt and L. Gray. *A modern approach to probability theory.* Probability and its Applications. Birkhäuser Boston Inc., Boston, MA, 1997.

[5] T. Gerstenkorn. Relations between the crude, factorial and inverse factorial moments. In *Transactions of the ninth Prague conference on information theory, statistical decision functions, random processes, Vol. A (Prague, 1982)*, pages 247–252. Reidel, Dordrecht, 1983.

[6] N. Privault. Moment identities for Poisson-Skorohod integrals and application to measure invariance. *C. R. Math. Acad. Sci. Paris*, 347:1071–1074, 2009.

[7] N. Privault. Invariance of Poisson measures under random transformations. *Ann. Inst. H. Poincaré Probab. Statist.*, 48(4):947–972, 2012.

[8] N. Privault. Moments of Poisson stochastic integrals with random integrands. *Probability and Mathematical Statistics*, 32(2):227–239, 2012.

[9] G. L. Torrisi. Point processes with Papangelou conditional intensity: from the Skorohod integral to the Dirichlet form. *Markov Process. Related Fields*, 19(2):195–248, 2013.