DEVIATION INEQUALITIES FOR EXPONENTIAL JUMP-DIFFUSION PROCESSES

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ABSTRACT. In this note we obtain deviation inequalities for the law of exponential jump-diffusion processes at a fixed time. Our method relies on convex concentration inequalities obtained by forward/backward stochastic calculus. In the pure jump and pure diffusion cases, it also improves on classical results obtained by direct application of Gaussian and Poisson bounds.

1. INTRODUCTION

Deviation inequalities for random variables admitting a predictable representation have been obtained by several authors. When $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion and $(\eta_t)_{t\in\mathbb{R}_+}$ an adapted process, using the time change

$$t \mapsto \tau(t) := \int_0^t |\eta_s|^2 ds \tag{1}$$

on Brownian motion yields the bound

$$\mathbb{P}\left(\int_0^\infty \eta_t dW_t \geq x\right) \leq \exp\left(-\frac{x^2}{2\Sigma^2}\right), \qquad x > 0, \tag{2}$$

provided

$$\Sigma^2 := \left\| \int_0^\infty |\eta_t|^2 dt \right\|_\infty < \infty. \tag{3}$$

On the other hand, if $(Z_t)_{t\in\mathbb{R}_+}$ is a point process with random intensity $(\lambda_t)_{t\in\mathbb{R}_+}$ and $(U_t)_{t\in\mathbb{R}_+}$ is an adapted process, we have the inequality

$$\mathbb{P}\left(\int_0^\infty U_{t^-}(dZ_t - \lambda_t dt) \geq x\right) \quad \leq \quad \exp\left(-\frac{x}{2\beta}\log\left(1 + \frac{\beta}{\Lambda}x\right)\right), \tag{4}$$

$x > 0$, provided $U_t \leq \beta$ a.s. for some constant $\beta > 0$ and

$$\Lambda := \left\| \int_0^\infty |U_t|^2 \lambda_t dt \right\|_\infty < \infty,$$

cf. [1], [5] when $(Z_t)_{t\in\mathbb{R}_+}$ is a Poisson process, and [4] for the mixed point process-diffusion case. Note that although $(Z_t)_{t\in\mathbb{R}_+}$ becomes a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ under the time change

$$t \mapsto \int_0^t \lambda_s ds,$$

when $(U_t)_{t\in\mathbb{R}_+}$ is non-constant the inequality (4) can not be recovered from a Poisson deviation bound in the same way as (2) is obtained from a Gaussian deviation bound.

In this paper we consider linear stochastic differential equations of the form

$$\frac{dS_t}{S_{t^-}} = \sigma_t dW_t + J_{t^-}(dZ_t - \lambda_t dt) \tag{5}$$

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where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, $(Z_t)_{t\in\mathbb{R}_+}$ is a point process of (stochastic) intensity λ_t. Here the processes $(W_t)_{t\in\mathbb{R}_+}$ and $(Z_t)_{t\in\mathbb{R}_+}$ may not be independent, but they are adapted to a same filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, and $(\sigma_t)_{t\in\mathbb{R}_+}$, $(J_t)_{t\in\mathbb{R}_+}$ are sufficiently integrable $\mathcal{F}_t$-adapted processes.

Clearly the above deviation inequalities (2) and (4) require some boundedness on the integrand processes $(\eta_t)_{t\in\mathbb{R}_+}$ and $(U_t)_{t\in\mathbb{R}_+}$, and for this reason they do not apply directly to the solution $(S_t)_{t\in\mathbb{R}_+}$ of (5), since the processes $(\eta_t)_{t\in[0,T]} = (\sigma_t S_t)_{t\in[0,T]}$ and $(U_t)_{t\in[0,T]} = (J_t S_t)_{t\in[0,T]}$ are not in $L^\infty(\Omega, L^2([0,T]))$. This is consistent with the fact that when σ_t is a deterministic function, S_T has a log-normal distribution which is not compatible with a Gaussian tail.

In this paper we derive several deviation inequalities for exponential jump-diffusion processes $(S_t)_{t\in\mathbb{R}_+}$ of the form (5). Our results rely on the following proposition, cf. [2], Corollary 5.2, and Theorem 1.1 below.

Let $(S_t^*)_{t\in\mathbb{R}_+}$ be the solution of

$$\frac{dS_t^*}{S_{t^-}^*} = \sigma^*(t)d\hat{W}_t + J^*(t)(d\hat{N}_t - \lambda^*(t)dt)$$

where $(\hat{W}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, $(\hat{N}_t)_{t\in\mathbb{R}_+}$ is a Poisson process of (deterministic) intensity $\lambda^*(t)$, which are assumed to be mutually independent, while $\sigma^*(t)$ and $J^*(t)$ are deterministic functions with $J^*(t) \geq 0$, $t \in \mathbb{R}_+$.

Theorem 1.1. *Assume that one of the following conditions is satisfied:*

(i) $-1 < J_t \leq J^*(t)$, *$dPdt$-a.e. and*

$$|\sigma_t| \leq |\sigma^*(t)|, \qquad J_t^2\lambda_t \leq |J^*(t)|^2\lambda^*(t), \qquad dPdt - a.e.$$

(ii) $-1 < J_t \leq 0 \leq J^*(t)$, *$dPdt$-a.e. and*

$$|\sigma_t|^2 + J_t^2\lambda_t \leq |\sigma^*(t)|^2 + |J^*(t)|^2\lambda^*(t), \qquad dPdt - a.e.$$

(iii) $0 \leq J_t \leq J^*(t)$, *$dPdt$-a.e.,* $J_t^2\lambda_t \leq |J^*(t)|^2\lambda^*(t)$, *$dPdt$-a.e., and*

$$|\sigma_t|^2 + J_t^2\lambda_t \leq |\sigma^*(t)|^2 + |J^*(t)|^2\lambda^*(t), \qquad dPdt - a.e.$$

Then we have

$$\mathbb{E}[\phi(S_t) \mid S_0 = x] \leq \mathbb{E}[\phi(S_t^*) \mid S_0^* = x], \qquad x > 0, \quad t \in \mathbb{R}_+, \tag{6}$$

for all convex function ϕ such that ϕ' is convex.

Note that in the continuous case $J = 0$, Relation (6) can be recovered by the Doob stopping time theorem and Jensen's inequality applied to the time change (1) since $\tilde{W}_t := t/2 + \log(S_{\tau^{-1}(t)}/S_0)$ is a standard Brownian motion with respect to a time-changed filtration $(\tilde{\mathcal{F}})_{t\in\mathbb{R}_+}$, and letting $X_t := S_0 e^{\tilde{W}_t - t/2}$, $t \in \mathbb{R}_+$, we have

$$\begin{aligned} E[\phi(S_T)] &= E\left[\phi\left(X_{\tau(T)}\right)\right] \\ &= E\left[\phi\left(E\left[X_{\int_0^T |\sigma^*(s)|^2 ds} \middle| \tilde{\mathcal{F}}_{\int_0^T |\sigma_s|^2 ds}\right]\right)\right] \\ &\leq E\left[E\left[\phi\left(X_{\int_0^T |\sigma^*(s)|^2 ds}\right) \middle| \tilde{\mathcal{F}}_{\int_0^T |\sigma_s|^2 ds}\right]\right] \\ &= E\left[\phi\left(X_{\int_0^T |\sigma^*(s)|^2 ds}\right)\right] \\ &= E[\phi(S_T^*)]. \end{aligned} \tag{7}$$

However this time change argument does not apply to the jump-diffusion case, and in addition in the pure jump case it cannot be used when $(J_t)_{t\in\mathbb{R}_+}$ is non-constant.

We refer to [3] for deviation inequalities for exponential stable processes when the number of jumps is a.s. infinite.

2. Deviation bounds

We begin with a result in the pure jump case, i.e. when $\sigma_t = 0$, $dPdt$-a.e., and let

$$g(u) = 1 + u \log u - u, \qquad u > 0.$$

Let $(S_t)_{t\in\mathbb{R}_+}$ denote the solution of (5) with $S_0 = 1$.

Theorem 2.1. *Assume that $\sigma_t = 0$, $dPdt$-a.e., and that*

$$-1 < J_t \leq K, \qquad dPdt - a.e.,$$

for some $K \geq 0$, and let

$$\Lambda_t = \int_0^t \left\| J_s^2 \lambda_s \right\|_\infty ds, \qquad t \in \mathbb{R}_+.$$

Then for any $T > 0$ and all $x \geq \frac{\Lambda_T}{K}\left(\frac{\beta}{K}(1+K)^2 - 1\right)$ we have

$$\begin{aligned}\mathbb{P}(\log S_T \geq x) &\leq \exp\left(-\frac{\Lambda_T}{K^2} g\left(\frac{K}{\beta}\left(1 + \frac{Kx}{\Lambda_T}\right)\right)\right) \\ &\leq \exp\left(-\frac{1}{2}\left(\frac{x}{\beta} + \frac{\Lambda_T}{K^2}\left(\frac{K}{\beta} - 1\right)\right)\log\left(\frac{K}{\beta}\left(1 + \frac{Kx}{\Lambda_T}\right)\right)\right), \end{aligned} \tag{8}$$

where $\beta = \log(1+K)$.

Proof. Let $J^*(t) = K$, $t \in \mathbb{R}_+$,

$$\lambda^*(t) = \frac{1}{K^2}\left\| J_t^2 \lambda_t \right\|_\infty, \qquad 0 \leq t \leq T,$$

and denote by $S_t^* = e^{-\Lambda_t/K}(1+K)^{N_t^*}$, $t \in \mathbb{R}_+$, the solution of

$$\frac{dS_t^*}{S_{t^-}^*} = K(dN_t^* - \lambda^*(t)dt), \tag{9}$$

with $S_0^* = 1$, where $(N_t^*)_{t\in\mathbb{R}_+}$ is a Poisson process with deterministic intensity $(\lambda^*(t))_{t\in\mathbb{R}_+}$. Under the above hypotheses, Theorem 1.1$-i$) yields the inequality

$$\begin{aligned} y^\alpha \mathbb{P}(S_T \geq y) &\leq \mathbb{E}[(S_T^*)^\alpha] \\ &= e^{-\alpha\Lambda_T/K}\mathbb{E}\left[\left((1+K)^{N_T^*}\right)^\alpha\right] \\ &= e^{-\alpha\Lambda_T/K} e^{\Lambda_T((1+K)^\alpha - 1)/K^2}, \end{aligned} \tag{10}$$

for the convex function $y \mapsto y^\alpha$ with convex derivative, $\alpha \geq 2$, hence

$$\mathbb{P}(\log S_T \geq x) \leq \exp\left(\frac{\Lambda_T}{K^2}\left((1+K)^\alpha - 1\right) - \alpha\frac{\Lambda_T}{K} - \alpha x\right). \tag{11}$$

The minimum in $\alpha \geq 0$ in the above bound is obtained at

$$\alpha^* = \frac{1}{\beta}\log\left(\frac{K}{\beta}\left(1 + \frac{Kx}{\Lambda_T}\right)\right),$$

which is greater than 2 if and only if

$$x \geq \frac{\Lambda_T}{K}\left(\frac{\beta}{K}(1+K)^2 - 1\right). \tag{12}$$

Hence for all x satisfying (12) we have

$$\begin{aligned}\mathbb{P}(\log S_T \geq x) &\leq \exp\left(\frac{\Lambda_T}{K^2}((1+K)^{\alpha^*} - 1) - \alpha^*\left(x + \frac{\Lambda_T}{K}\right)\right) \\ &= \exp\left(-\frac{\Lambda_T}{K^2} g\left(\frac{K}{\beta}\left(1 + \frac{Kx}{\Lambda_T}\right)\right)\right),\end{aligned}$$

and Relation (8) follows from the classical inequality

$$\frac{1}{2}u\log(1+u) \leq g(1+u), \qquad u > 0.$$

□

Note that an application of the classical Poisson bound (4) only yields

$$\begin{aligned}
&\mathbb{P}(\log S_T \geq x) \\
&= \mathbb{P}\left(\int_0^T \log(1+J_{t^-})dZ_t - \int_0^T J_{t^-}\lambda_t dt \geq x\right) \\
&= \mathbb{P}\left(\int_0^T \log(1+J_{t^-})d(Z_t - \lambda_t dt) \geq x + \int_0^T J_{t^-}\lambda_t dt - \int_0^T \log(1+J_{t^-})\lambda_t dt\right) \\
&\leq \mathbb{P}\left(\int_0^T \log(1+J_{t^-})d(Z_t - \lambda_t dt) \geq x\right) \\
&\leq \exp\left(-\frac{x}{2\beta}\log\left(1+\beta\frac{x}{\tilde{\Lambda}_T}\right)\right), \qquad x > 0,
\end{aligned}$$

provided

$$J_t \leq K \quad \text{and} \quad \int_0^T |\log(1+J_t)|^2\lambda_t dt \leq \tilde{\Lambda}_T, \qquad a.s.,$$

which is worse than (8) even in the deterministic case since $1 < K/\beta \to \infty$ as $K \to \infty$, and $\tilde{\Lambda}_T \leq \Lambda_T$.

Theorem 2.1 admits a generalization to the case of a continuous component when the jumps J_t have constant sign.

Theorem 2.2. *Assume that*

$$-1 < J_t \leq 0, \quad dPdt - a.e., \quad or \quad 0 \leq J_t \leq K, \quad dPdt - a.e.,$$

for some $K > 0$, and assume that

$$\Lambda_T := \int_0^T \left\| |\sigma_t|^2 + J_t^2\lambda_t \right\|_\infty dt < \infty.$$

Then for all $x \geq \frac{\Lambda_T}{K}\left(\frac{\beta}{K}(1+K)^2 - 1\right)$ we have

$$\mathbb{P}(\log S_T \geq x) \leq \exp\left(-\frac{\Lambda_T}{K^2}g\left(\frac{K}{\beta}\left(1+\frac{Kx}{\Lambda_T}\right)\right)\right) \tag{13}$$

$$\leq \exp\left(-\frac{1}{2}\left(\frac{x}{\beta} + \frac{\Lambda_T}{K^2}\left(\frac{K}{\beta} - 1\right)\right)\log\left(\frac{K}{\beta}\left(1+\frac{Kx}{\Lambda_T}\right)\right)\right), \tag{14}$$

where $\beta = \log(1+K)$.

Proof. We repeat the proof of Theorem 2.1, replacing the use of Theorem 1.1−i) by Theorem 1.1−ii) and Theorem 1.1−iii), and by defining $\lambda^*(t)$ as

$$\lambda^*(t) = \frac{1}{K^2}\left\| |\sigma_t|^2 + J_t^2\lambda_t \right\|_\infty, \qquad 0 \leq t \leq T.$$

□

Letting $K \to 0$ in (13) or (14) we obtain the following Gaussian deviation inequality in the negative jump case with a continuous component.

Theorem 2.3. *Assume that* $-1 < J_t \leq 0$, $dPdt$*-a.e., and let*

$$\Sigma_T^2 = \int_0^T \left\| |\sigma_t|^2 + J_t^2 \lambda_t \right\|_\infty dt < \infty, \qquad T > 0.$$

Then we have

$$\mathbb{P}(\log S_T \geq x) \leq \exp\left(-\frac{(x + \Sigma_T^2/2)^2}{2\Sigma_T^2} \right), \qquad x \geq 3\Sigma_T^2/2. \tag{15}$$

Proof. Although this result follows from Theorem 2.2 by taking $K \to 0$, we show that it can also be obtained from Theorem 1.1. Let

$$|\sigma^*(t)|^2 = \left\| |\sigma_t|^2 + J_t^2 \lambda_t \right\|_\infty, \qquad 0 \leq t \leq T,$$

and denote by $S_t^* = \exp\left(\int_0^t \sigma^*(s) dW_s - \frac{1}{2} \int_0^t |\sigma^*(s)|^2 ds \right)$, $t \in \mathbb{R}_+$, the solution of

$$\frac{dS_t^*}{S_t^*} = \sigma^*(t) dW_t, \tag{16}$$

with initial condition $S_0^* = 1$. By the Tchebychev inequality and Theorem 1.1$-ii)$ applied for $K = 0$, for all positive nondecreasing convex functions $\phi : \mathbb{R} \to \mathbb{R}$ with convex derivative we have

$$\phi(y) \mathbb{P}(S_T \geq y) \leq \mathbb{E}[\phi(S_T^*)]. \tag{17}$$

Applying this inequality to the convex function $t \mapsto y^\alpha$ for fixed $\alpha \geq 2$, we obtain

$$\begin{aligned} y^\alpha \mathbb{P}(S_T \geq y) &\leq \mathbb{E}[(S_T^*)^\alpha] \\ &= \exp\left(\alpha(\alpha - 1)\Sigma_T^2/2 \right), \end{aligned}$$

hence

$$\mathbb{P}(S_T \geq e^x) \leq \exp\left(-\alpha x + \alpha(\alpha - 1)\Sigma_T^2/2 \right), \qquad x \geq 0, \quad \alpha \geq 2. \tag{18}$$

The function

$$\alpha \mapsto -\alpha x + \alpha(\alpha - 1)\Sigma_T^2/2$$

attains its minimum over $\alpha \geq 2$ at

$$\alpha^* = \frac{1}{2} + \frac{x}{\Sigma_T^2}, \qquad x \geq 3\Sigma_T^2/2,$$

which yields (15). □

In the pure diffusion case with $J = 0$ and $(\sigma_t)_{t \in \mathbb{R}_+}$ deterministic, the bound (15) can be directly obtained from

$$\begin{aligned} \mathbb{P}\left(\log S_T \geq x \right) &= \mathbb{P}\left(\exp\left(\int_0^T \sigma_t dW_t - \frac{1}{2} \int_0^T |\sigma_t|^2 dt \right) \geq e^x \right) \\ &= \mathbb{P}\left(\exp\left(W_{\Sigma_T^2} - \frac{1}{2} \Sigma_T^2 \right) \geq e^x \right) \\ &\leq \exp\left(-\frac{(x + \Sigma_T^2/2)^2}{2\Sigma_T^2} \right), \qquad x > 0. \end{aligned} \tag{19}$$

On the other hand, when $J = 0$ and $(\sigma_t)_{t \in \mathbb{R}_+}$ is an adapted process, the bound (2) only yields

$$\begin{aligned} \mathbb{P}(\log S_T \geq x) &= \mathbb{P}\left(\int_0^T \sigma_t dW_t - \frac{1}{2} \int_0^T |\sigma_t|^2 dt \geq x \right) \\ &\leq \mathbb{P}\left(\int_0^T \sigma_t dW_t \geq x \right) \\ &\leq e^{-x^2/(2\Sigma_T^2)}, \qquad x > 0, \end{aligned} \tag{20}$$

which is worse than (15) and (19) by a factor $\exp(x/2+\Sigma_T^2/8)$. In this case the argument of Theorem 2.3 can be based on (7) instead of using Theorem 1.1.

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