Explicit stochastic analysis of Brownian motion and point measures on Riemannian manifolds

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Abstract

The gradient and divergence operators of stochastic analysis on Riemannian manifolds are expressed using the gradient and divergence of the flat Brownian motion. By this method we obtain the almost-sure version of several useful identities that are usually stated under expectations. The manifold-valued Brownian motion and random point measures on manifolds are treated successively in the same framework, and stochastic analysis of the Brownian motion on a Riemannian manifold turns out to be closely related to classical stochastic calculus for jump processes. In the setting of point measures we introduce a damped gradient that was lacking in the multidimensional case.

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1 Introduction

The $\mathbb{R}^d$-valued Brownian motion on the Wiener space $(W, \mathcal{F}^W, \mu)$ gathers many properties that are important in stochastic analysis. In particular, most definitions of gradient and divergence operators on the Wiener space coincide with the gradient and divergence on Fock space via the Wiener chaos isomorphism. In non-Gaussian settings the situation is different since there exists several reasonable choices for a gradient and divergence. Each family of operators carries a part of the interesting properties of its Gaussian counterpart, so that a choice has to be made when dealing with a specific problem. In the case of Brownian motion on a manifold it has been established, cf. [5], [10], that at least three gradient coexist as unbounded operators from $L^2(W;\mathbb{R})$ to $L^2(W\times\mathbb{R}_+;\mathbb{R}^d)$:

- the classical gradient D defined from the flat $\mathbb{R}^d$-valued Brownian motion, or from chaos expansions,

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- the gradient $\hat{D}$ associated to the manifold-valued Brownian motion,

- the damped gradient $\tilde{D}$ of [10] which is defined from $\hat{D}$.

The interest of the damped gradient $\tilde{D}$ is that it extends the interesting properties of D to the setting of manifolds: for example its adjoint extends the stochastic integral and $\tilde{D}$ allows to compute the kernel of the Clark formula. (These properties are not satisfied by $\hat{D}$). In the case of point measures on $\mathbb{R}^d$ or on Riemannian manifolds, there is a gradient D defined from chaos expansions (which is identified to a finite difference operator), and a gradient $\hat{D}$, defined in [1], [3] by infinitesimal shifts of configurations. The gradient $\tilde{D}$ which has been defined in the particular case of a Poisson process on $\mathbb{R}_+$, cf. [6], plays in fact the role of damped gradient and we will extend its definition to the case of Riemannian manifolds.
We obtain explicit formulas linking the gradients D, $\hat{D}$ and $\tilde{D}$, both in the continuous and in the jump cases. Our calculations are explicit in the sense that they exclusively use the Fock gradient and divergence of the flat case. The presentation of those two different settings are made as similar as possible to each other, but they present many important differences. Our method shows that several useful identities in expectations are in fact the consequences of more precise identities that hold in the almost-sure sense, cf. Remarks 1 and 2. Quantum stochastic differentials are used to reformulate some expressions and show that the difficulties of stochastic calculus on manifolds are similar to that of the Poisson case. We also treat the anticipating integral on manifolds with an explicit definition that differs from that of [8], [9], but leads to the same operators. In the one-dimensional case, the Wiener and Poisson constructions are based on the same Fock space over $L^2(\mathbb{R}_+; \mathbb{R})$, and then they become more similar to each other.
This paper is organized as follows. Sect. 2 recalls the algebraic tools of Fock space, gradient and divergence that can be defined without referring to a probability measure, and will be in use in both the Wiener and Poisson cases. After stating some notation relative to d-dimensional Brownian motion in Sect. 3 (see [13], [14], [16]) we study in Sect. 4 the differentiation of Wiener functionals with respect to a general class of not necessarily quasi-invariant transformations that include Euclidean motions as particular cases. These transformations are called morphisms because they are compatible with the pointwise product of random variables. Sect. 5 is devoted to the manifold-valued Wiener case. The differential calculus with respect to random

morphisms is applied to obtain an explicit expression for the gradient on a Riemannian manifold in terms of the flat gradient and divergence. The damped gradient of [10] and its application to the Clark formula are treated by the same method. By duality we obtain an explicit construction of Skorohod type anticipating integrals, and certain results are translated in the language of quantum stochastic calculus. The Poisson case is considered in Sections 6, 7 and 8. We proceed in the same way, in particular in Sect. 7 we study the differential calculus with respect to shifts of configurations without quasi-invariance assumptions, and we introduce a damped gradient which was lacking in the case of Riemannian manifolds. We try to keep the notation as close as possible to that of the Wiener case, but the constructions are in fact significantly different.

2 Algebraic preliminaries

In this section we introduce the algebraic tools that are in use in both the Poisson and Wiener cases. Let $\Gamma(H) = \bigoplus H^{\circ n}$ denote the completed symmetric Fock space on a Hilbert space H, where "$\circ$" denotes the completed symmetric tensor product. Let "$\otimes$" and "$\hat{\otimes}$" respectively denote the algebraic and completed tensor products. Let $\mathcal{S}$ denote the algebraic Fock space over H, i.e. the vector space generated by $f_1 \circ \cdots \circ f_n$, $f_1, \ldots, f_n \in H$, $n \geq 1$, and let $\mathcal{U}$ denote the space generated by $F \otimes g$, $F \in \mathcal{S}$, $g \in H$. Let $D : \Gamma(H) \longrightarrow \Gamma(H)\hat{\otimes}H$ and $\delta : \Gamma(H)\hat{\otimes}H \longrightarrow \Gamma(H)$ denote the classical unbounded gradient and divergence operators defined on $\mathcal{S}$ and $\mathcal{U}$, that satisfy $Df^{\circ n} = nf^{\circ(n-1)} \otimes f$ and $\delta(f^{\circ n} \otimes g) = f^{\circ n} \circ g$, $f, g \in H$, $n \geq 0$. Let $\mathrm{Dom}(D)$ and $\mathrm{Dom}(\delta)$ denote the respective domains of the closed extensions of D and δ. If $H = L^2(X; \mathbb{R}^d, \lambda)$, where (X, λ) is a measure space, the Hudson-Parthasarathy quantum stochastic integrals of $h \in L^2(X; \mathbb{R}^d, \lambda)$ are operators that act on $\mathcal{S}$, cf. [2] and references therein, and are defined as follows:

$$\int_X \langle h(x), da_x^- \rangle F = \int_X \langle D_x F, h(x) \rangle d\lambda(x),$$
$$\int_X \langle h(x), da_x^+ \rangle F = \delta(F \otimes h), \quad F \in \mathcal{S}, \quad h \in H,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$. Let $q(x) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $x \in X$, denote a family of bounded endomorphisms of $\mathbb{R}^d$, such that $q(\cdot) : L^2(X; \mathbb{R}^d) \longrightarrow L^2(X; \mathbb{R}^d)$ is bounded. The operator $\int_X q(x) da_x^\circ$ is defined as

$$\int_X q(x) da_x^\circ F = \delta(q(\cdot) D.F), \quad F \in \mathcal{S}.$$

3 The flat Wiener case

Let $W = \mathcal{C}_0(\mathbb{R}_+;\mathbb{R}^d)$ denote the space of continuous $\mathbb{R}^d$-valued functions starting at 0, with Wiener measure μ. Let $H = L^2(\mathbb{R}_+;\mathbb{R}^d)$ and let $(B(t))_{t\in\mathbb{R}_+}$ denote the $\mathbb{R}^d$-valued Brownian motion on W, generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Adaptedness conditions in the Wiener case will always refer to this filtration. We denote by $I_n(f_n)$ the multiple stochastic integral with respect to $(B(t))_{t\in\mathbb{R}_+}$ of a symmetric function of n variables $f_n \in \hat{L}^2(\mathbb{R}_+^n;\mathbb{R}^d) \simeq H^{\circ n}$. We also denote the first order stochastic integral of an adapted process $h \in L^2(W\times\mathbb{R}_+;\mathbb{R}^d)$ by

$$I_1(h) = \int_0^\infty \langle h(t), dB(t)\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of $\mathbb{R}^d$. Let $\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$ denote an algebra which is dense in $L^2(\mathbb{R}_+;\mathbb{R}^d)$, e.g. $\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) = \mathcal{C}_c^\infty(\mathbb{R}_+;\mathbb{R}^d)$.

Definition 3.1.1 *Let*

$$\mathcal{S}(W;\mathbb{R}) = \left\{F = f(I_1(u_1),\dots,I_1(u_n)) \ : \ u_1,\dots,u_n \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d),\ f\in\mathcal{C}_b^\infty(\mathbb{R}^n;\mathbb{R}),\ n\geq 1\right\},$$

and

$$\mathcal{U}(W\times\mathbb{R}_+;\mathbb{R}^d) = \left\{\sum_{k=1}^{k=n} F_k\int_0^\cdot u_k(s)ds \ : \ F_1,\dots,F_n\in\mathcal{S}(W;\mathbb{R}),\ u_1,\dots,u_n\in\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)\right\}$$

In the Wiener interpretation of $\Gamma(H)$, D satisfies

$$\int_0^\infty \langle D_tF, \dot{h}(t)\rangle dt = \lim_{\varepsilon\to 0}\frac{F(\omega+\varepsilon h)-F(\omega)}{\varepsilon}, \quad F\in\mathcal{S}(W;\mathbb{R}),$$

for deterministic $h \in \mathcal{U}(W\times\mathbb{R}_+;\mathbb{R}^d)$, and we have

$$D_tF = \sum_{i=1}^{i=n} u_i(t)\partial_i f(I_1(u_1),\dots,I_1(u_n)), \quad F\in\mathcal{S}(W;\mathbb{R}).$$

In particular, D has the derivation property:

$$D_t(FG) = FD_tG + GD_tF, \quad t\in\mathbb{R}_+, \tag{3.1.1}$$

for $F,G\in\mathrm{Dom}(D)$ such that $FDG,\ GDF \in L^2(W\times\mathbb{R}_+;\mathbb{R}^d)$, and by duality this implies the relation

$$F\delta(u) = \delta(uF) + \int_0^\infty \langle D_tF, u(t)\rangle dt, \tag{3.1.2}$$

for $F\in\mathrm{Dom}(D)$ and $u\in\mathrm{Dom}(\delta)$ such that $F\delta(u)\in L^2(W;\mathbb{R})$. We state explicitly Relations (3.1.1) and (3.1.2) because they will be repeatedly used in the sequel.

4 Differential calculus with respect to random morphisms on Wiener space

In this section we remain in the flat Wiener case. The stochastic calculus of variations usually works by perturbations of Brownian motion by infinitesimal shifts in directions of the Cameron-Martin space or by Euclidean motions, cf. [10]. We consider a more general class of infinitesimal transformations of Brownian motion and its associated differential calculus, cf. Prop. 4.1.2. Let $U : \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) \longrightarrow L^2(W \times \mathbb{R}_+;\mathbb{R}^d)$ be an operator such that $Uf \in L^2(W \times \mathbb{R}_+;\mathbb{R}^d)$ is adapted for all $f \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$. The operator U will extended by linearity to the algebraic tensor product $\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) \otimes \mathcal{S}(W;\mathbb{R})$, in this case Uf is not necessarily adapted if $f \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) \otimes \mathcal{S}(W;\mathbb{R})$. Let $(h(t))_{t\in\mathbb{R}_+} \in L^2(W \times \mathbb{R}_+;\mathbb{R}^d)$ be a square-integrable process.

Definition 4.1.1 *We let the transformation*

$$\Lambda(U,h) : \mathcal{S}(W;\mathbb{R}) \longrightarrow L^2(W \times \mathbb{R}_+;\mathbb{R}^d)$$

be defined as

$$\Lambda(U,h)F = f\left(I_1(Uu_1) + \int_0^\infty \langle u_1(t), h(t)\rangle dt, \ldots, I_1(Uu_n) + \int_0^\infty \langle u_n(t), h(t)\rangle dt\right),$$

for $F \in \mathcal{S}(W;\mathbb{R})$ of the form

$$F = f(I_1(u_1), \ldots, I_1(u_n)), \quad u_1, \ldots, u_n \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d), \quad f \in \mathcal{C}_b^\infty(\mathbb{R}^n;\mathbb{R}).$$

In the particular case where $U : \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) \longrightarrow L^2(W\times\mathbb{R}_+;\mathbb{R}^d)$ is given as $[Uf](t) = V(t)f(t)$, $t \in \mathbb{R}_+$, by an adapted family of random endomorphisms $V(t) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $t \in \mathbb{R}_+$, this definition states that $\Lambda(U,h)F$ is the evaluation of F on the perturbed process of differential $V(t)^*dB(t) + h(t)dt$ instead of $dB(t)$, where $V(t)^* : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ denotes the dual of $V(t) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $t \in \mathbb{R}_+$. In [7], a class of transformations called Euclidean motions, is considered. In this case, the operator $V(t) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is chosen to be an isometry and h is adapted, so that $\Lambda(U,h)$ becomes well-defined by quasi-invariance of the Wiener measure. We are going to show that this hypothesis is not needed in order to define $\Lambda(U,h)$ on the space $\mathcal{S}(W;\mathbb{R})$ of smooth functionals. For this we need to show that the definition of $\Lambda(U,h)F$ is independent of the particular representation

$$F = f(I_1(u_1), \ldots, I_1(u_n)), \qquad u_1, \ldots, u_n \in H,$$

chosen for $F \in \mathcal{S}(W;\mathbb{R})$.

Proposition 4.1.1 *Let $F, G \in \mathcal{S}(W;\mathbb{R})$ be written as*

$$F = f\left(I_1(u_1), \ldots, I_1(u_n)\right), \quad u_1, \ldots, u_n \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d), \ f \in \mathcal{C}^1(\mathbb{R}^n;\mathbb{R}),$$

and

$$G = g\left(I_1(v_1), \ldots, I_1(v_m)\right), \quad v_1, \ldots, v_m \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d), \ g \in \mathcal{C}^1(\mathbb{R}^m;\mathbb{R}).$$

If $F = G$ μ-a.s. then $\Lambda(U,h)F = \Lambda(U,h)G$, μ-a.s.

Proof. Let $e_1, \ldots, e_k \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$ be orthonormal vectors that generate $u_1, \ldots, u_n$, $v_1, \ldots, v_m$. Let $u_i(\cdot) = \sum_{j=1}^{j=k} \alpha_i^j e_j(\cdot)$ and $v_i(\cdot) = \sum_{j=1}^{j=k} \beta_i^j e_j(\cdot)$ be the expressions of $u_1, \ldots, u_n, v_1, \ldots, v_m$ in the basis $e_1, \ldots, e_k$. Then F and G are also represented as $F = \tilde{f}\left(I_1(e_1), \ldots, I_1(e_k)\right)$, and $G = \tilde{g}\left(I_1(e_1), \ldots, I_1(e_k)\right)$, where the functions $\tilde{f}$ and $\tilde{g}$ are defined by

$$\tilde{f}(x_1, \ldots, x_k) = f\left(\sum_{j=1}^{j=k} \alpha_1^j x_j, \ldots, \sum_{j=1}^{j=k} \alpha_n^j x_j\right), \quad x_1, \ldots, x_k \in \mathbb{R},$$

and

$$\tilde{g}(x_1, \ldots, x_k) = f\left(\sum_{j=1}^{j=k} \beta_1^j x_j, \ldots, \sum_{j=1}^{j=k} \beta_n^j x_j\right), \quad x_1, \ldots, x_k \in \mathbb{R}.$$

Since $F = G$ and $I_1(e_1), \ldots, I_1(e_k)$ are independent, we have $\tilde{f} = \tilde{g}$ *a.e.*, hence everywhere, and by linearity,

$$\Lambda(U,h)F = \tilde{f}\left(I_1(Ue_1) + \int_0^\infty \langle e_1(t), h(t)\rangle dt, \ldots, I_1(Ue_k) + \int_0^\infty \langle e_k(t), h(t)\rangle dt\right),$$

and

$$\Lambda(U,h)G = \tilde{g}\left(I_1(Ue_1) + \int_0^\infty \langle e_1(t), h(t)\rangle dt, \ldots, I_1(Ue_k) + \int_0^\infty \langle e_k(t), h(t)\rangle dt\right),$$

hence $\Lambda(U,h)F = \Lambda(U,h)G$.

□

Moreover, $\Lambda(U,h)$ is linear and multiplicative:

$$\Lambda(U,h)f(F_1, \ldots, F_n) = f(\Lambda(U,h)F_1, \ldots, \Lambda(U,h)F_n),$$

$F_1, \ldots, F_n \in \mathcal{S}(W;\mathbb{R})$, $f \in \mathcal{C}_b^1(\mathbb{R}^n;\mathbb{R})$, for this reason we use the term "morphism" for $\Lambda(U,h) : \mathcal{S}(W;\mathbb{R}) \longrightarrow L^2(W;\mathbb{R})$.

Definition 4.1.2 *Let $(U_\varepsilon)_{\varepsilon\in[0,1]}$ be a family of linear operators*

$$U_\varepsilon : \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) \longrightarrow L^2(W\times\mathbb{R}_+;\mathbb{R}^d),$$

such that

- $U_0 : \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) \longrightarrow L^2(W\times\mathbb{R}_+;\mathbb{R}^d)$ *is the identity of* $\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$, *i.e. we have* $U_0 f = f$, μ*-a.s.*, $f\in\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$.

- *for any* $f\in\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$, $U_\varepsilon f\in L^2(W\times\mathbb{R}_+;\mathbb{R}^d)$ *and is adapted,* $\forall\varepsilon\in\mathbb{R}$,

- *the family* $(U_\varepsilon)_{\varepsilon\in[0,1]}$ *admits a derivative at* $\varepsilon=0$, *i.e. there exists an operator*

$$\mathcal{L} : \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) \longrightarrow L^2(W\times\mathbb{R}_+;\mathbb{R}^d),$$

  such that $((U_\varepsilon f - f)/\varepsilon)_{\varepsilon\in[0,1]}$ *converges in* $L^2(W\times\mathbb{R}_+;\mathbb{R}^d)$ *to* $\mathcal{L}f = (\mathcal{L}_t f)_{t\in\mathbb{R}_+}$ *as* ε *goes to zero,* $\forall f\in\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$.

Let $(h_\varepsilon)_{\varepsilon\in[0,1]}\subset L^2(W\times\mathbb{R}_+;\mathbb{R}^d)$ *be a family of adapted processes, continuous in* $L^2(W\times\mathbb{R}_+;\mathbb{R}^d)$ *with respect to* $\varepsilon\in[0,1]$.

The operator $\mathcal{L}$ is extended by linearity to $\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)\otimes\mathcal{S}(W;\mathbb{R})$. The family $(U_\varepsilon)_{\varepsilon\in[0,1]}$ needs not be a semigroup. The above assumptions imply that $\mathcal{L}DF\in \mathrm{Dom}(\delta)$, $F\in\mathcal{S}(W;\mathbb{R})$, since from (3.1.2):

$$\delta(\mathcal{L}DF) = \sum_{i=1}^{i=n}\partial_i f(I_1(u_1),\ldots,I_1(u_n))\delta(\mathcal{L}u_i) - \sum_{i,j=1}^{n}(u_i,\mathcal{L}u_j)_{L^2(\mathbb{R}_+;\mathbb{R}^d)}\partial_i\partial_j f(I_1(u_1),\ldots,I_1(u_n)),$$

for $F = f(I_1(u_1),\ldots,I_1(u_n))$. We now compute on $\mathcal{S}(W;\mathbb{R})$ the derivative at $\varepsilon=0$ of one-parameter families

$$\Lambda(U_\varepsilon,\varepsilon h_\varepsilon) : \mathcal{S}(W;\mathbb{R}) \longrightarrow L^2(W;\mathbb{R}), \quad \varepsilon\in\mathbb{R},$$

of transformations of Brownian functionals. We define the linear operator trace : $H\otimes H\longrightarrow\mathbb{R}$ on the algebraic tensor product $H\otimes H$ as

$$\mathrm{trace}\ u\otimes v = (u,v)_H, \quad u,v\in H.$$

Proposition 4.1.2 *For* $F\in\mathcal{S}(W;\mathbb{R})$, *we have in* $L^2(W;\mathbb{R})$*:*

$$\frac{d}{d\varepsilon}\Lambda(U_\varepsilon,\varepsilon h_\varepsilon)F_{|\varepsilon=0} = \int_0^\infty\langle h_0(t),D_tF\rangle dt + \delta(\mathcal{L}DF) + \mathrm{trace}\ (\mathrm{Id}_H\otimes\mathcal{L})DDF. \quad (4.1.1)$$

Proof. Let $A : \mathcal{S}(W;\mathbb{R}) \longrightarrow \mathcal{S}(W;\mathbb{R})$ be defined by

$$AF = \delta(\mathcal{L}DF) + \text{trace }(\mathrm{Id}_H \otimes \mathcal{L})DDF + \int_0^\infty \langle h_0(t), D_t F\rangle dt, \quad F \in \mathcal{S}(W;\mathbb{R}).$$

For $F = I_1(u)$, $u \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$, we have

$$\begin{aligned}
\frac{d}{d\varepsilon}\Lambda(U_\varepsilon, \varepsilon h_\varepsilon)F_{|\varepsilon=0} &= \int_0^\infty \langle h_0(t), u(t)\rangle dt + I_1(\mathcal{L}u) \\
&= \int_0^\infty \langle h_0(t), D_t F\rangle dt + \delta(\mathcal{L}DF) + \text{trace }(\mathrm{Id}_H \otimes \mathcal{L})DDF \\
&= AF
\end{aligned}$$

since $DDF = 0$. Using (3.1.1) and (3.1.2), we have for $F_1, \ldots, F_n \in \mathcal{S}(W;\mathbb{R})$ and $f \in \mathcal{C}_b^\infty(\mathbb{R}^n;\mathbb{R})$:

$$\begin{aligned}
&Af(F_1, \ldots, F_n) = \\
&\delta(\mathcal{L}Df(F_1, \ldots, F_n)) + \text{trace }(\mathrm{Id}_H \otimes \mathcal{L})DDf(F_1, \ldots, F_n) + \int_0^\infty \langle h_0(t), D_t f(F_1, \ldots, F_n)\rangle dt \\
&= \sum_{i=1}^{i=n} \delta\left(\partial_i f(F_1, \ldots, F_n)\mathcal{L}DF_i\right) \\
&\quad + \sum_{i=1}^{i=n} \partial_i f(F_1, \ldots, F_n) \text{trace }(\mathrm{Id}_H \otimes \mathcal{L})DDF_i \\
&\quad + \sum_{i,j=1}^{i=n} \partial_i \partial_j f(F_1, \ldots, F_n) \int_0^\infty \langle \mathcal{L}_s DF_i, D_s F_j\rangle ds \\
&\quad + \sum_{i=1}^{i=n} \partial_i f(F_1, \ldots, F_n) \int_0^\infty \langle h_0(t), D_t F_i\rangle dt \\
&= \sum_{i=1}^{i=n} \partial_i f(F_1, \ldots, F_n)\delta(\mathcal{L}DF_i) + \sum_{i=1}^{i=n} \partial_i f(F_1, \ldots, F_n) \text{trace }(\mathrm{Id}_H \otimes \mathcal{L})DDF_i \\
&\quad + \sum_{i=1}^{i=n} \partial_i f(F_1, \ldots, F_n) \int_0^\infty \langle h_0(t), D_t F_i\rangle dt \\
&= \sum_{i=1}^{i=n} \partial_i f(F_1, \ldots, F_n) \left(\delta(\mathcal{L}DF_i) + \text{trace }(\mathrm{Id}_H \otimes \mathcal{L})DDF_i + \int_0^\infty \langle h_0(t), D_t F_i\rangle dt\right) \\
&= \sum_{i=1}^{i=n} \partial_i f(F_1, \ldots, F_n) AF_i.
\end{aligned}$$

Hence for $F_1 = I_1(u_1), \ldots, F_n = I_1(u_n) \in \mathcal{S}(W;\mathbb{R})$ and $f \in \mathcal{C}_b^\infty(\mathbb{R}^n;\mathbb{R})$:

$$\begin{aligned}
Af(F_1, \ldots, F_n) &= \sum_{i=1}^{i=n} \partial_i f(F_1, \ldots, F_n) AF_i \\
&= \sum_{i=1}^{i=n} \partial_i f(F_1, \ldots, F_n) \left(\frac{d}{d\varepsilon}\Lambda(U_\varepsilon, \varepsilon h_\varepsilon)F_i\right)_{|\varepsilon=0} \\
&= \left(\frac{d}{d\varepsilon}\Lambda(U_\varepsilon, \varepsilon h_\varepsilon) f(F_1, \ldots, F_n)\right)_{|\varepsilon=0}.
\end{aligned}$$

Consequently, (4.1.1) holds on $\mathcal{S}(W;\mathbb{R})$.

□

Corollary 4.1.1 *If $\mathcal{L} : L^2(\mathbb{R}_+;\mathbb{R}^d) \longrightarrow L^2(W \times \mathbb{R}_+;\mathbb{R}^d)$ is antisymmetric as an endomorphism of $L^2(\mathbb{R}_+;\mathbb{R}^d)$, μ-a.s., we have in $L^2(W;\mathbb{R})$:*

$$\frac{d}{d\varepsilon}\Lambda(U_\varepsilon, \varepsilon h_\varepsilon)F_{|\varepsilon=0} = \int_0^\infty \langle h_0(t), D_t F\rangle dt + \delta(\mathcal{L}DF), \quad F \in \mathcal{S}(W;\mathbb{R}).$$

Proof. Since $\mathcal{L}$ is antisymmetric, we have for any symmetric tensor $u \otimes u \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) \otimes \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$:

$$\text{trace } (\mathrm{Id}_H \otimes \mathcal{L}) u \otimes u = \text{trace } u \otimes \mathcal{L}u = \langle u, \mathcal{L}u\rangle_H = -\langle \mathcal{L}u, u\rangle_H = 0.$$

Hence the term trace $(\mathrm{Id}_H \otimes \mathcal{L})DDF$ of Prop. 4.1.2 vanishes μ-a.s. since DDF is a symmetric tensor.

□

This corollary is in particular valid if U_ε is given as $[U_\varepsilon f](t) = V_\varepsilon f(t)$, from an adapted process of isometries $V_\varepsilon(t) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ of the form $V_\varepsilon(t) = \exp(\varepsilon \mathcal{L}_t) \in O(d)$, where $\mathcal{L}_t \in \mathrm{so}(\mathbb{R}^d)$, μ-a.s., $t \in \mathbb{R}_+$. If $h_\varepsilon = 0$, the well-known statement

$$E\left[\frac{d}{d\varepsilon}\Lambda(U_\varepsilon, 0)F_{|\varepsilon=0}\right] = \frac{d}{d\varepsilon}E\left[\Lambda(U_\varepsilon, 0)F\right]_{|\varepsilon=0} = \frac{d}{d\varepsilon}E[F]_{|\varepsilon=0} = 0,$$

which follows from the invariance of the Wiener measure under isometries, is given here a more precise meaning since we have

$$E\left[\frac{d}{d\varepsilon}\Lambda(U_\varepsilon, 0)F_{|\varepsilon=0}\right] = E[\delta(\mathcal{L}DF)] = 0, \quad F \in \mathcal{S}(W;\mathbb{R}).$$

Until the end of this paper we assume that $\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) = L^2(\mathbb{R}_+;\mathbb{R}^d)$. Conversely we can show the following.

Proposition 4.1.3 *If $F \mapsto \delta(\mathcal{L}DF)$ has the derivation property on $\mathcal{S}(W;\mathbb{R})$, and if $\mathcal{L} : L^2(\mathbb{R}_+;\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}_+;\mathbb{R}^d)$ is continuous, μ-a.s., then*

$$\mathcal{L} : L^2(\mathbb{R}_+;\mathbb{R}^d) \longrightarrow L^2(W \times \mathbb{R}_+;\mathbb{R}^d)$$

is antisymmetric as an endomorphism of $L^2(\mathbb{R}_+;\mathbb{R}^d)$, μ-a.s.

Proof. We have

$$\begin{aligned}
&F\delta(\mathcal{L}DG) + G\delta(\mathcal{L}DF) - \delta(\mathcal{L}D(FG)) \\
&= \delta(F\mathcal{L}DG) + \delta(G\mathcal{L}DF) - \delta(\mathcal{L}D(FG)) + \int_0^\infty \langle D_t F, \mathcal{L}_t DG\rangle dt \\
&\quad + \int_0^\infty \langle D_t G, \mathcal{L}_t DF\rangle dt \\
&= \int_0^\infty \langle D_t F, \mathcal{L}_t DG\rangle dt + \int_0^\infty \langle D_t G, \mathcal{L}_t DF\rangle dt,
\end{aligned}$$

hence the derivation property implies

$$\int_0^\infty \langle D_t F, \mathcal{L}_t DG\rangle dt + \int_0^\infty \langle D_t G, \mathcal{L}_t DF\rangle dt = 0, \quad \mu - a.s.$$

Choosing $F = I_1(f)$ and $G = I_1(g)$ in the first Wiener chaos, which obtain

$$\int_0^\infty \langle f(t), \mathcal{L}_t g\rangle dt + \int_0^\infty \langle g(t), \mathcal{L}_t f\rangle dt = 0, \quad \mu - a.s., \ \forall f, g \in L^2(\mathbb{R}_+; \mathbb{R}^d).$$

If $\{e_k\}_{k\in\mathbb{N}}$ denotes a complete orthonormal subset of $L^2(\mathbb{R}_+; \mathbb{R}^d)$, we have

$$\int_0^\infty \langle e_k(t), \mathcal{L}_t e_l\rangle dt + \int_0^\infty \langle e_l(t), \mathcal{L}_t e_k\rangle dt = 0, \ \ \forall k, l \in \mathbb{N}, \ \mu - a.s.,$$

hence from the continuity assumption on $\mathcal{L}$:

$$\int_0^\infty \langle f(t), \mathcal{L}_t g\rangle dt + \int_0^\infty \langle g(t), \mathcal{L}_t f\rangle dt = 0, \ \ \forall f, g \in L^2(\mathbb{R}_+; \mathbb{R}^d), \ \mu - a.s. \quad \square$$

5 Stochastic analysis on a Riemannian manifold: the Wiener case

5.1 Gradient $\hat{D} : L^2(W; \mathbb{R}) \longrightarrow L^2(W \times \mathbb{R}_+; \mathbb{R})$ on a Riemannian manifold

We refer to [8], [9], [10], [12] for the notation and results recalled in this subsection. Let M be a Riemannian manifold of dimension d, and let $O(M)$ denote the bundle of orthonormal frames over M. Let $(m_0, r_0) \in O(M)$, i.e. $m_0 \in M$ and $r_0 : \mathbb{R}^d \longrightarrow T_{m_0}M$ is an isometry onto $T_{m_0}M$. We identify $T_{m_0}M$ to $\mathbb{R}^d$, i.e. given $u \in T_{m_0}M$ and $v \in \mathbb{R}^d$ we write $u = v$ if and only if $u = r_0 v$. We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $T_{m_0}M$, or equivalently the canonical scalar product on $\mathbb{R}^d$. The Levi-Civita parallel transport defines d canonical horizontal vector fields $A_1, \ldots, A_d$ on $O(M)$, and the Stratonovich stochastic differential equation

$$\begin{cases} dr(t) = \sum_{i=1}^{i=d} A_i(r(t)) \circ dB^i(t), \quad t \in \mathbb{R}_+, \\ r(0) = (m_0, r_0), \end{cases}$$

defines a $O(M)$-valued process $(r(t))_{t\in\mathbb{R}_+}$. Let $\pi : O(M) \longrightarrow M$ denote the canonical projection, and let $\gamma(t) = \pi(r(t))$, $t \in \mathbb{R}_+$. Then $(\gamma(t))_{t\in\mathbb{R}_+}$ is a Brownian motion on M and the Itô parallel transport along $(\gamma(t))_{t\in\mathbb{R}_+}$ is defined as

$$t_{t\leftarrow 0} = r(t)r_0^{-1} : T_{\gamma(0)}M \longrightarrow T_{\gamma(t)}M, \quad t \in \mathbb{R}_+.$$

Let $\mathbb{P}(M) = \mathcal{C}_{m_0}(\mathbb{R}_+; M)$ denote the set of continuous paths on M starting at m_0, let

$$\begin{aligned} I \ : \ & \mathcal{C}_0(\mathbb{R}_+; \mathbb{R}^d) \longrightarrow \mathcal{C}_{m_0}(\mathbb{R}_+; M) \\ & (\omega(t))_{t\in\mathbb{R}_+} \mapsto I(\omega) = (\gamma(t))_{t\in\mathbb{R}_+} \end{aligned}$$

denote the Itô map, and let ν denote the image measure of the Wiener measure μ by I. In the sequel we will endow $\mathbb{P}(M)$ with the following σ-algebra.

Definition 5.1.1 *Let $\mathcal{F}^P$ denote the σ-algebra on $\mathbb{P}(M)$ generated by subsets of the form*

$$\{\gamma \in \mathbb{P}(M) \ : \ (\gamma(t_1), \ldots, \gamma(t_n)) \in B_1 \times \cdots \times B_n\},$$

where $0 \leq t_1 < \cdots < t_n$, $B_1, \ldots, B_n \in \mathcal{B}(M)$, $n \geq 1$.

The σ-algebra $\mathcal{F}^P$ is smaller than the σ-algebra defined by I on $\mathbb{P}(M)$.

Definition 5.1.2 *Let*

$$\begin{aligned} & \mathcal{S}(\mathbb{P}(M); \mathbb{R}) \\ & \quad = \ \{F = f(\gamma(t_1), \ldots, \gamma(t_n)) \ : \ f \in \mathcal{C}_b^\infty(M^n; \mathbb{R}), \ 0 \leq t_1 \leq \cdots \leq t_n \leq 1, \ n \geq 1\}, \end{aligned}$$

and

$$\begin{aligned} & \mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d) \\ & \quad = \ \left\{ \sum_{k=1}^{k=n} F_k \int_0^\cdot u_k(s)ds \ : \ F_1, \ldots, F_n \in \mathcal{S}(\mathbb{P}(M); \mathbb{R}), \ u_1, \ldots, u_n \in L^2(\mathbb{R}_+; \mathbb{R}^d), \ n \geq 1 \right\} \end{aligned}$$

Every element of $\mathcal{S}(\mathbb{P}(M); \mathbb{R})$ is a functional on $\mathbb{P}(M)$, and defines a functional $F \circ I$ on W. In order to simplify the notation we will often write F instead of $F \circ I$, for random variables and stochastic processes. In the following, the space $L^2(\mathbb{P}(M), \mathcal{F}^P, \nu)$ will be simply denoted by $L^2(\mathbb{P}(M))$.

Proposition 5.1.1 *The spaces $\mathcal{S}(\mathbb{P}(M); \mathbb{R})$ and $\mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ are dense in $L^2(\mathbb{P}(M); \mathbb{R})$ and in $L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ respectively.*

Proof. Let $\mathcal{D}$ denote the algebra generated by the sets

$$\{\gamma \in \mathbb{P}(M) \ : \ (\gamma(t_1), \ldots, \gamma(t_n)) \in E_1 \times \cdots \times E_n\},$$

where $0 \leq t_1 < \cdots < t_n$, $E_1, \ldots, E_n \in \mathcal{B}(M)$, $n \geq 1$, and let $\mathcal{W}$ denote the set of $A \in \mathcal{F}^P$ for which there exists a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ such that $(1_{A_n})_{n\in\mathbb{N}}$ converges in $L^2(\mathbb{P}(M), \nu)$ to 1_A. Then $\mathcal{W}$ is a monotone class hence it is equal to $\mathcal{F}^P$. Moreover, for any Borel subset A of M there exists a uniformly bounded sequence in $\mathcal{C}_c^\infty(M)$ converging a.e. on M to 1_A, hence $\mathcal{S}(\mathbb{P}(M); \mathbb{R})$ is dense in $L^2(\mathbb{P}(M); \mathbb{R})$. The density of $\mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ in $L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ follows similarly. □

The following definition can be found in [10].

Definition 5.1.3 *Let $\hat{D} : L^2(\mathbb{P}(M); \mathbb{R}) \longrightarrow L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ be the gradient operator defined as*

$$\hat{D}_t F = \sum_{i=1}^{i=n} t_{0\leftarrow t_i} \nabla_i^M f(\gamma(t_1), \ldots, \gamma(t_n)) 1_{[0,t_i]}(t), \quad t \in \mathbb{R}_+,$$

for $F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R})$ of the form $F = f(\gamma(t_1), \ldots, \gamma(t_n))$, where ∇_i^M denotes the gradient on M applied to the i-th variable of f.

5.2 Explicit expression of the gradient $\hat{D}$

In this section we put together the definition 5.1.3 of $\hat{D} : L^2(\mathbb{P}(M); \mathbb{R}) \longrightarrow L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ by variational calculus, cf. [10], and the result of Sect. 4 in order to obtain an explicit relation between the gradient $\hat{D}$ and the operators D and δ.
Corollary 4.1.1 is stated for $F \in \mathcal{S}(W; \mathbb{R})$, i.e. for polynomial functionals in single stochastic integrals on the flat Wiener space. In order to work on $\mathbb{P}(M)$ we need to be able to consider smooth functionals of $(\gamma(t))_{t\in\mathbb{R}_+}$, which are no longer given by functions of single stochastic integrals. Therefore, before proceeding further we need to extend Corollary 4.1.1 from $F \in \mathcal{S}(W; \mathbb{R})$ to $F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R})$. In the following proposition we assume that U_ε is given by $V_\varepsilon(t) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $t \in \mathbb{R}_+$, as $[U_\varepsilon f](t) = V_\varepsilon [f(t)]$, $t \in \mathbb{R}_+$.

Proposition 5.2.1 *Let $V_\varepsilon(\cdot) : W \times \mathbb{R}_+ \longrightarrow O(d)$ and $\mathcal{L}(\cdot) : W \times \mathbb{R}_+ \longrightarrow so(d)$ be adapted processes satisfying Def. 4.1.2, with $V_\varepsilon(t) = \exp(\varepsilon \mathcal{L}_t)$, $t \in \mathbb{R}_+$, $\varepsilon \in \mathbb{R}$. Let $h \in L^2(\mathbb{R}_+; L^\infty(W; \mathbb{R}^d))$ be adapted and such that $\varepsilon \mapsto \Lambda(U_\varepsilon^{-1}, 0)h$ is continuous in $L^2(W \times \mathbb{R}_+; \mathbb{R}^d)$. Then we have in $L^2(W; \mathbb{R})$:*

$$\frac{d}{d\varepsilon} \Lambda(U_\varepsilon, \varepsilon h) F_{|\varepsilon=0} = \int_0^\infty \langle h(t), D_t F\rangle dt + \delta(\mathcal{L} DF), \quad F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R}). \qquad (5.2.1)$$

Proof. Since $V_\varepsilon(t) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is isometric, $t \in \mathbb{R}_+$, $\Lambda(U_\varepsilon, \varepsilon h)F$ and $\Lambda(U_\varepsilon^{-1}, 0)h$ are well-defined by quasi-invariance of the Wiener measure. Moreover, the definition of $\Lambda(U_\varepsilon, \varepsilon h)$ extends to $G = g(I_1(u_1), \ldots, I_1(u_n))$, with adapted $u_1, \ldots, u_n \in L^2(W \times \mathbb{R}_+; \mathbb{R}^d)$, as

$$\Lambda(U_\varepsilon, \varepsilon h)G = g\left(I_1(U_\varepsilon u_1) + \varepsilon \int_0^\infty \langle u_1(t), h(t)\rangle dt, \ldots, I_1(U_\varepsilon u_n) + \varepsilon \int_0^\infty \langle u_n(t), h(t)\rangle dt\right).$$

By invariance of the Wiener measure under Euclidean transformations we have for $G \in \mathcal{S}(W; \mathbb{R})$:

$$E\left[G\Lambda(U_\varepsilon, \varepsilon h)F\right] = E\left[(\Lambda(U_\varepsilon^{-1}, 0)G)\Lambda(\mathrm{Id}_H, \varepsilon\Lambda(U_\varepsilon^{-1}, 0)h)F\right],$$

since $[\Lambda(U_\varepsilon, 0)]^{-1} = \Lambda(U_\varepsilon^{-1}, 0)$ and

$$\Lambda(U_\varepsilon^{-1}, 0)\Lambda(U_\varepsilon, \varepsilon h) = \Lambda(\mathrm{Id}_H, \varepsilon\Lambda(U_\varepsilon^{-1}, 0)h).$$

From the Girsanov theorem applied to the shift $\varepsilon\Lambda(U_\varepsilon^{-1}, 0)h$, we obtain

$$\begin{aligned} E\left[G\Lambda(U_\varepsilon, \varepsilon h)F\right] &= E\Big[F \exp\left(\varepsilon \int_0^\infty \Lambda(U_\varepsilon^{-1}, 0)h(t)dB(t) - \frac{1}{2}\varepsilon^2 \|\Lambda(U_\varepsilon^{-1}, 0)h\|^2_{L^2(\mathbb{R}_+;\mathbb{R}^d)}\right) \\ &\quad \times\Lambda(\mathrm{Id}_H, -\varepsilon\Lambda(U_\varepsilon^{-1}, 0)h)\Lambda(U_\varepsilon^{-1}, 0)G\Big] \\ &= E\Big[F \exp\left(\varepsilon \int_0^\infty \Lambda(U_\varepsilon^{-1}, 0)h(t)dB(t) - \frac{1}{2}\varepsilon^2 \|\Lambda(U_\varepsilon^{-1}, 0)h\|^2_{L^2(\mathbb{R}_+;\mathbb{R}^d)}\right) \\ &\quad \times\Lambda(U_\varepsilon^{-1}, -\varepsilon V_\varepsilon\Lambda(U_\varepsilon^{-1}, 0)h)G\Big]. \end{aligned} \tag{5.2.2}$$

Th. 2.2.1 of [10] and Prop. 3.5.3 of [10] show that for $F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R})$,

$$\varepsilon \mapsto \Lambda(U_\varepsilon, \varepsilon h)F$$

is differentiable in $L^2(W; \mathbb{R})$ at $\varepsilon = 0$. We have

$$\begin{aligned} &\frac{\exp\left(\varepsilon \int_0^\infty \Lambda(U_\varepsilon^{-1}, 0)h(t)dB(t) - \frac{1}{2}\varepsilon^2 \|\Lambda(U_\varepsilon^{-1}, 0)h\|^2_{L^2(\mathbb{R}_+;\mathbb{R}^d)}\right) - 1}{\varepsilon} \\ &= \exp\left(\varepsilon \int_0^\infty \Lambda(U_\varepsilon^{-1}, 0)h(t)dB(t)\right) \frac{\exp\left(-\frac{1}{2}\varepsilon^2 \|\Lambda(U_\varepsilon^{-1}, 0)h\|^2_{L^2(\mathbb{R}_+;\mathbb{R}^d)}\right) - 1}{\varepsilon} \\ &\quad + \frac{\exp\left(\varepsilon \int_0^\infty \Lambda(U_\varepsilon^{-1}, 0)h(t)dB(t)\right) - 1}{\varepsilon}, \end{aligned}$$

which is bounded in $L^2(W; \mathbb{R})$, uniformly in $\varepsilon \in [0, 1]$, since $\|h\|_{L^2(\mathbb{R}_+;L^\infty(W;\mathbb{R}^d))} < \infty$. Moreover,

$$\frac{\Lambda(U_\varepsilon^{-1}, -\varepsilon V_\varepsilon\Lambda(U_\varepsilon^{-1}, 0)h)G - G}{\varepsilon}$$

is bounded in $L^2(W;\mathbb{R})$, uniformly in $\varepsilon \in [-1,1]\setminus\{0\}$ (from Taylor's formula it is sufficient to check this fact for $G = I_1(u)$, $u \in L^2(\mathbb{R}_+;\mathbb{R}^d)$). Hence we can differentiate at $\varepsilon = 0$ under the expectations in (5.2.2) and apply Cor. 4.1.1 to $G \in \mathcal{S}(W;\mathbb{R})$ and $\Lambda(U_\varepsilon^{-1}, -\varepsilon\Lambda(U_\varepsilon,0)h)$:

$$\begin{aligned}
E\left[G\frac{d}{d\varepsilon}\Lambda(U_\varepsilon,\varepsilon h)F_{|\varepsilon=0}\right] &= -E\left[F\left(\int_0^\infty \langle h(t), D_tG\rangle dt + \delta(\mathcal{L}DG)\right)\right] \\
&\quad +E\left[FG\int_0^\infty h(t)dB(t)\right] \\
&= E\left[G\delta(\mathcal{L}DF)\right] - E\left[F\int_0^\infty \langle h(t), D_tG\rangle dt\right] + E\left[FG\delta(h)\right] \\
&= E\left[G\delta(\mathcal{L}DF)\right] + E\left[F\delta(hG)\right] \\
&= E\left[G\int_0^\infty \langle h(t), D_tF\rangle dt + G\delta(\mathcal{L}DF)\right], \quad G \in \mathcal{S}(W;\mathbb{R}),
\end{aligned}$$

which implies (5.2.1) by density of $\mathcal{S}(W;\mathbb{R})$ in $L^2(W;\mathbb{R})$.

□

Given an adapted vector field $(Z(t))_{t\in\mathbb{R}_+}$ on M with $Z(t) \in T_{\gamma(t)}M$, $t \in \mathbb{R}_+$, we let $z(t) = t_{0\leftarrow t}Z(t)$, $t \in \mathbb{R}_+$, and assume that $\dot{z}(t)$ exists, $\forall t \in \mathbb{R}_+$. Let

$$\nabla Z(t) = \lim_{\varepsilon\to 0} \frac{t_{t\leftarrow t+\varepsilon}Z(t+\varepsilon) - Z(t)}{\varepsilon}.$$

Then

$$\dot{z}(t) = t_{0\leftarrow t}\nabla Z(t), \quad t \in \mathbb{R}_+.$$

let Ω_r denote the curvature tensor of M and let $\mathrm{ric}_r : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ denote the Ricci tensor, at the frame $r \in O(M)$, and let the process $(\hat{z}(t))_{t\in\mathbb{R}_+}$ be defined by

$$\begin{cases} \dot{\hat{z}}(t) = \dot{z}(t) + \frac{1}{2}\mathrm{ric}_{r(t)}z(t), \quad t \in \mathbb{R}_+, \\ \hat{z}(0) = 0. \end{cases} \tag{5.2.3}$$

As a consequence of Prop. 5.2.1 we obtain the following expression of $\hat{D}$, which has some similarity with Th. 2.3.8 and Th. 2.6 of [7], and a simpler proof.

Corollary 5.2.1 *Assume that the Ricci curvature of M is uniformly bounded, and let $z \in \mathcal{U}(\mathbb{P}(M)\times\mathbb{R}_+;\mathbb{R}^d)$ be adapted. We have*

$$\int_0^\infty \langle \hat{D}_tF, \dot{z}(t)\rangle dt = \int_0^\infty \langle D_tF, \dot{\hat{z}}(t)\rangle dt + \delta(q(\cdot,z)D.F), \tag{5.2.4}$$

$F \in \mathcal{S}(\mathbb{P}(M);\mathbb{R})$, *where $q(t,z) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is defined as*

$$q(t,z) = -\int_0^t \Omega_{r(s)}(\circ dB(s), z(s)), \quad t \in \mathbb{R}_+.$$

Proof. We let $V_\varepsilon(t) = \exp(\varepsilon q(t,z))$, $t \in \mathbb{R}_+$, $\varepsilon \in \mathbb{R}$. Then from Prop. 3.5.3 of [10] we have

$$\int_0^\infty \langle \hat{D}F, \dot{z}(t)\rangle dt = \frac{d}{d\varepsilon}\Lambda(U_\varepsilon, \varepsilon\dot{\hat{z}})F_{|\varepsilon=0}.$$

Since the Ricci curvature of M is bounded, we have $\dot{\hat{z}} \in L^2(\mathbb{R}_+; L^\infty(W;\mathbb{R}))$ from (5.2.3). Moreover, from Th. 2.2.1 of [10], $\varepsilon \mapsto \Lambda(U_\varepsilon, 0)r(t)$ is differentiable in $L^2(W;\mathbb{R})$, hence continuous, $\forall t \in \mathbb{R}_+$. Consequently, from (5.2.3) and by construction of $\mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$, $\varepsilon \mapsto \Lambda(U_\varepsilon, 0)\dot{\hat{z}}$ is continuous in $L^2(W \times \mathbb{R}_+; \mathbb{R}^d)$ and we can apply Prop. 5.2.1 to obtain (5.2.4).

□

Remark 1 *Since $E[\delta(\mathcal{L}DF)] = 0$, Th. 2.3.2 of [10] (which follows from the invariance of the Wiener measure under Euclidean transformations) is also explained from Prop. 5.2.1 by taking expectations in the identity (5.2.4) that holds in the almost-sure sense.*

If $u \in \mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ is written as $u = \sum_{i=1}^{i=n} G_i z_i$, z_i deterministic, $G_i \in \mathcal{S}(\mathbb{P}(M);\mathbb{R})$, $i = 1, \ldots, n$, we define trace $q(t, D_t u) \in \mathbb{R}^d$ as

$$\text{trace } q(t, D_t u) \quad = \quad \sum_{i=1}^{i=n} q(t, z_i) D_t G_i.$$

Given $u \in \mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ written as $u = \sum_{i=1}^{i=n} G_i z_i$, z_i deterministic, $G_i \in \mathcal{S}(\mathbb{P}(M);\mathbb{R})$, $i = 1, \ldots, n$, we let

$$\hat{u} = \sum_{i=1}^{i=n} G_i \hat{z}_i.$$

The following proposition extends Cor. 5.2.1 to non-adapted processes $u \in \mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$.

Theorem 5.2.1 *We have for $u \in \mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ and $F \in \mathcal{S}(\mathbb{P}(M);\mathbb{R})$:*

$$\int_0^\infty \langle \hat{D}_t F, \dot{u}(t)\rangle dt = \int_0^\infty \langle D_t F, \dot{\hat{u}}(t)\rangle dt + \delta(q(\cdot, u)D.F) - \int_0^\infty \langle D_t F, \text{trace } q(t, D_t u)\rangle dt.$$

Proof. For $u = zG$, $G \in \mathcal{S}(\mathbb{P}(M);\mathbb{R})$ and deterministic $z \in \mathcal{U}(W \times \mathbb{R}_+; \mathbb{R}^d)$, we have

$$\begin{aligned}\int_0^\infty \langle \hat{D}_t F, G\dot{z}(t)\rangle dt &= \int_0^\infty \langle \hat{D}_t(FG), \dot{z}(t)\rangle dt - F\int_0^\infty \langle \hat{D}_t G, \dot{z}(t)\rangle dt \\ &= \int_0^\infty \langle D_t(FG), \dot{\hat{z}}(t)\rangle dt + \delta(q(\cdot, z)D.(FG))\end{aligned}$$

$$\begin{aligned}
& -F\int_0^\infty \langle D_t G, \dot{\hat{z}}(t)\rangle dt - F\delta(q(\cdot,z)D.G) \\
= & \int_0^\infty \langle D_t(FG), \dot{\hat{z}}(t)\rangle dt + \delta(Fq(\cdot,z)D.G) \\
& +\delta(Gq(\cdot,z)D.F) - F\int_0^\infty \langle D_t G, \dot{\hat{z}}(t)\rangle dt - F\delta(q(\cdot,z)D.G) \\
= & \int_0^\infty \langle D_t F, G\dot{\hat{z}}(t)\rangle dt + \delta(Gq(\cdot,z)D.F) - \int_0^\infty \langle D_t F, q(t,z)D_t G\rangle dt \\
= & \int_0^\infty \langle D_t F, \dot{\hat{u}}(t)\rangle dt + \delta(q(\cdot,u)D.F) - \int_0^\infty \langle D_t F, \text{trace } q(t, D_t u)\rangle dt. \quad \square
\end{aligned}$$

In [10], $r(t)\hat{D}_t F \in T_{\gamma(t)}M$ is considered instead of $\hat{D}_t F \in T_{\gamma(0)}M$, with the relation

$$\langle t_{t\leftarrow 0}\hat{D}_t F, \nabla Z(t)\rangle_{T_{\gamma(t)}M} = \langle \hat{D}_t F, \dot{z}(t)\rangle,$$

where $(Z(t))_{t\in\mathbb{R}_+}$ denotes an adapted vector field on M with $Z(t) \in T_{\gamma(t)}M$, $t \in \mathbb{R}_+$.

5.3 Inversion of $z \mapsto \hat{z}$

This subsection recalls the inversion of $z \mapsto \hat{z}$ by the method of variation of constants described in Sect. 3.7 of [10]. Let $\text{Id}_{\gamma(t)}$ denote the identity of $T_{\gamma(t)}M$. We have

$$\dot{z}(t) = \dot{\hat{z}}(t) + \frac{1}{2}\text{ric}_{r(t)}\tilde{z}(t), \quad t \in \mathbb{R}_+,$$

where $(\tilde{z}(t))_{t\in\mathbb{R}_+}$ is defined as

$$\tilde{z}(t) = \int_0^t Q_{t,s}\dot{z}(s)ds, \quad t \in \mathbb{R}_+,$$

and $Q_{t,s} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ satisfies

$$\frac{dQ_{t,s}}{dt} = -\frac{1}{2}\text{ric}_{r(t)}Q_{t,s}, \quad Q_{s,s} = \text{Id}_{\gamma(0)}, \quad 0 \le s \le t.$$

Also, let the process $(\hat{Z}(t))_{t\in\mathbb{R}_+}$ be defined by

$$\begin{cases} \nabla\hat{Z}(t) = \nabla Z(t) + \frac{1}{2}\text{Ric}_{\gamma(t)}Z(t), \quad t \in \mathbb{R}_+, \\ \hat{Z}(0) = 0, \end{cases}$$

with $\hat{z}(t) = \tau_{0\leftarrow t}\hat{Z}(t)$, $t \in \mathbb{R}_+$. In order to invert $Z \mapsto \hat{Z}$, let

$$\tilde{Z}(t) = \int_0^t R_{t,s}\nabla Z(s)ds, \quad t \in \mathbb{R}_+,$$

where $R_{t,s} : T_{\gamma(s)}M \longrightarrow T_{\gamma(t)}M$ is defined by the equation

$$\nabla_t R_{t,s} = -\frac{1}{2}\text{Ric}_{\gamma(t)}R_{t,s}, \quad R_{s,s} = \text{Id}_{\gamma(s)}, \quad 0 \le s \le t,$$

∇_t denotes the covariant derivative along $(\gamma(t))_{t\in\mathbb{R}_+}$, and $\mathrm{Ric}_m : T_mM \longrightarrow T_mM$ denotes the Ricci tensor at $m \in M$, with the relation

$$\mathrm{ric}_{r(t)} = t_{0\leftarrow t} \circ \mathrm{Ric}_{\gamma(t)} \circ t_{t\leftarrow 0}.$$

Then we have

$$\begin{cases} \nabla Z(t) = \nabla \tilde{Z}(t) + \frac{1}{2}\mathrm{Ric}_{\gamma(t)}\tilde{Z}(t), & t \in \mathbb{R}_+, \\ Z(0) = 0. \end{cases}$$

5.4 Expression of the damped gradient $\tilde{D}$

The damped gradient $\tilde{D} : L^2(\mathbb{P}(M);\mathbb{R}) \longrightarrow L^2(\mathbb{P}(M) \times \mathbb{R}_+;\mathbb{R}^d)$ has been defined in [10].

Definition 5.4.1 *The damped gradient $\tilde{D}$ is defined as*

$$\tilde{D}_tF = \sum_{i=1}^{i=n} 1_{[0,t_i]}(t)Q^*_{t_i,t}t_{0\leftarrow t_i}\nabla^M_i f(\gamma(t_1),\ldots,\gamma(t_n)), \quad t \in \mathbb{R}_+,$$

for $F \in \mathcal{S}(\mathbb{P}(M);\mathbb{R})$ of the form $F = f(\gamma(t_1),\ldots,\gamma(t_n))$, where $Q^*_{t,s} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ denotes the adjoint of $Q_{t,s} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $0 \le s < t$.

We also have

$$\tilde{D}_tF = \sum_{i=1}^{i=n} 1_{[0,t_i]}(t)t_{0\leftarrow t}R^*_{t_i,t}\nabla^M_i f(\gamma(t_1),\ldots,\gamma(t_n)), \quad t \in \mathbb{R}_+,$$

where $R^*_{t_i,t} : T_{\gamma(t_i)} \longrightarrow T_{\gamma(t)}$ is the adjoint of $R_{t_i,t} : T_{\gamma(t)} \longrightarrow T_{\gamma(t_i)}$. For completeness we state the following proposition.

Proposition 5.4.1 *We have for $z \in \mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+;\mathbb{R}^d)$:*

$$\int_0^\infty \langle \tilde{D}_tF, \dot{z}(t)\rangle dt = \int_0^\infty \langle \hat{D}_tF, \dot{\tilde{z}}(t)\rangle dt, \quad F \in \mathcal{S}(\mathbb{P}(M);\mathbb{R}). \tag{5.4.1}$$

Proof. We compute

$$\begin{aligned} \int_0^\infty \langle \tilde{D}_tF, \dot{z}(t)\rangle dt &= \sum_{i=1}^{i=n} \int_0^{t_i} \langle Q^*_{t_i,s}t_{0\leftarrow t_i}\nabla^M_i f(\gamma(t_1),\ldots,\gamma(t_n)), \dot{z}(s)\rangle ds \\ &= \sum_{i=1}^{i=n} \int_0^{t_i} \langle t_{0\leftarrow t_i}\nabla^M_i f(\gamma(t_1),\ldots,\gamma(t_n)), Q_{t_i,s}\dot{z}(s)\rangle dt \\ &= \int_0^\infty \langle \hat{D}_tF, \dot{\tilde{z}}(t)\rangle dt, \quad F \in \mathcal{S}(\mathbb{P}(M);\mathbb{R}). \quad \square \end{aligned}$$

We also have

$$\int_0^\infty \langle \tilde{D}_tF, \dot{\tilde{z}}(t)\rangle dt = \int_0^\infty \langle \hat{D}_tF, \dot{z}(t)\rangle dt, \quad F \in \mathcal{S}(\mathbb{P}(M);\mathbb{R}).$$

We now give an explicit expression of the damped gradient $\tilde{D}$ in terms of D and δ.

Corollary 5.4.1 *If $z \in \mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ is deterministic,*

$$\int_0^\infty \langle \tilde{D}_t F, \dot{z}(t) \rangle dt = \int_0^\infty \langle D_t F, \dot{z}(t) \rangle dt + \delta(q(\cdot, \tilde{z}) D.F), \quad F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R}).$$

Proof. We use Relation (5.4.1) and Cor. 5.2.1.

□

In the anticipating case we have:

Corollary 5.4.2 *We have for $u \in \mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$:*

$$\int_0^\infty \langle \tilde{D}_t F, \dot{u}(t) \rangle dt = \int_0^\infty \langle D_t F, \dot{u}(t) \rangle dt + \delta(q(\cdot, u) D.F) - \int_0^\infty \langle D_t F, \text{trace } q(t, D_t \tilde{u}) \rangle dt.$$

Proof. We use Relation (5.4.1) and Th. 5.2.1.

□

In [10], the damped gradient is chosen as $t_{t \leftarrow 0} \tilde{D} F : L^2(\mathbb{P}(M); \mathbb{R}) \longrightarrow L^2(\mathbb{P}(M) \times \mathbb{R}_+; TM)$ instead of $\tilde{D} F : L^2(\mathbb{P}(M); \mathbb{R}) \longrightarrow L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$, and it satisfies

$$\langle t_{t \leftarrow 0} \tilde{D}_t F, \nabla Z(t) \rangle_{T_{\gamma(t)} M} = \langle \tilde{D}_t F, \dot{z}(t) \rangle, \quad t \in \mathbb{R}_+.$$

5.5 Anticipating stochastic integration

The stochastic integral of the adapted vector field $(Z(s))_{s \in \mathbb{R}_+} : \mathbb{R}_+ \longrightarrow TM$ is defined as

$$\int_0^\infty \langle \nabla Z(s), d\hat{\gamma}(s) \rangle_{T_{\gamma(s)} M} = \int_0^\infty \langle \dot{\hat{z}}(s), dB(s) \rangle,$$

cf. (3.3.1) of [10]. The following is an explicit formulation for the operator defined in [9].

Definition 5.5.1 *We define the operator $\hat{\delta} : L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d) \longrightarrow L^2(\mathbb{P}(M); \mathbb{R})$ as*

$$\hat{\delta}(\dot{u}) = \delta(\dot{\hat{u}}) - \delta(\text{trace } q(\cdot, D.u)), \quad u \in \mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d).$$

According to the definition of trace $q(\cdot, D.u)$ we have for any deterministic $z \in \mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$:

$$\hat{\delta}(F \dot{z}) = \delta(F \dot{\hat{z}}) - \delta(q(\cdot, z) D.F), \quad F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R}). \tag{5.5.1}$$

Moreover, if $\dot{z} \in L^2(W \times \mathbb{R}_+; \mathbb{R}^d)$ is adapted, then

$$\hat{\delta}(\dot{z}) = \delta(\dot{\hat{z}}) = \int_0^\infty \langle \dot{\hat{z}}(s), dB(s) \rangle. \tag{5.5.2}$$

Proposition 5.5.1 *The operators* $\hat{\delta} : L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d) \longrightarrow L^2(\mathbb{P}(M); \mathbb{R})$ *and* $\hat{D} : L^2(\mathbb{P}(M); \mathbb{R}) \longrightarrow L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ *are closable and mutually adjoint:*

$$E\left[\int_0^\infty \langle \hat{D}_t F, \dot{u}(t)\rangle dt\right] = E[F\hat{\delta}(\dot{u})], \tag{5.5.3}$$

$u \in \mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$, $F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R})$, *and* $\hat{\delta}$ *satisfies*

$$\hat{\delta}(F\dot{u}) = F\hat{\delta}(\dot{u}) - \int_0^\infty \langle \hat{D}_t F, \dot{u}(t)\rangle dt. \tag{5.5.4}$$

Proof. We have from Def. 5.5.1 and Th. 5.2.1:

$$\begin{aligned} E[F\hat{\delta}(\dot{u})] &= E[F\delta(\dot{\hat{u}})] - E[F\delta(\text{trace } q(\cdot, D.u))] \\ &= E\left[\int_0^\infty \langle D_t F, \dot{\hat{u}}(t)\rangle dt\right] - E\left[\int_0^\infty \langle D_t F, \text{trace } q(t, D_t u))\rangle dt\right] \\ &= E\left[\int_0^\infty \langle \hat{D}_t F, \dot{u}(t)\rangle dt\right], \end{aligned}$$

hence (5.5.3). On the other hand, (5.5.4) follows by duality and the derivation property of $\hat{D}$, or by direct computation from

$$\begin{aligned} \hat{\delta}(F\dot{u}) &= \delta(F\dot{\hat{u}}) - \delta(\text{trace } q(\cdot, D.(Fu))) \\ &= F\delta(\dot{\hat{u}}) - \int_0^\infty \langle D_t F, \dot{\hat{u}}(t)\rangle dt - \delta(F\text{trace } q(\cdot, D.u)) - \delta(\text{trace } q(\cdot, uD.F)) \\ &= F\delta(\dot{\hat{u}}) - \int_0^\infty \langle D_t F, \dot{\hat{u}}(t)\rangle dt + \int_0^\infty \langle D_t F, \text{trace } q(t, D_t u)\rangle dt \\ &\quad -F\delta(\text{trace } q(\cdot, D.u)) - \delta(\text{trace } q(\cdot, uD.F)) \\ &= F\hat{\delta}(\dot{u}) - \int_0^\infty \langle \hat{D}_t F, \dot{u}(t)\rangle dt. \end{aligned}$$

The closability follows from the density property Prop. 5.1.1.

□

We denote by $\text{Dom}(\hat{D})$ and $\text{Dom}(\hat{\delta})$ the closed domains of $\hat{D} : L^2(\mathbb{P}(M); \mathbb{R}) \longrightarrow L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ and $\hat{\delta} : L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d) \longrightarrow L^2(\mathbb{P}(M); \mathbb{R})$. Relation (5.5.4) is in fact the definition chosen for $\hat{\delta}$ in [9] in the absence of chaos expansions. We now turn to the definition of the damped Skorohod type anticipating integral.

Definition 5.5.2 *We define the operator* $\tilde{\delta} : L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d) \longrightarrow L^2(\mathbb{P}(M); \mathbb{R})$ *as*

$$\tilde{\delta}(\dot{u}) = \delta(\dot{u}) - \delta(\text{trace } q(\cdot, D.\tilde{u})), \quad u \in \mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d). \tag{5.5.5}$$

We have for deterministic $z \in \mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$:

$$\tilde{\delta}(F\dot{z}) = \delta(F\dot{z}) - \delta(q(\cdot, \tilde{z})D.F), \quad F \in \mathcal{S}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}). \tag{5.5.6}$$

Proposition 5.5.2 *The operators* $\tilde{\delta} : L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d) \longrightarrow L^2(\mathbb{P}(M); \mathbb{R})$ *and* $\tilde{D} : L^2(\mathbb{P}(M); \mathbb{R}) \longrightarrow L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ *are closable and mutually adjoint:*

$$E\left[\int_0^\infty \langle \tilde{D}_t F, \dot{u}(t)\rangle dt\right] = E[F\tilde{\delta}(\dot{u})], \tag{5.5.7}$$

$u \in \mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$, $F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R})$, *with*

$$\tilde{\delta}(F\dot{u}) = F\tilde{\delta}(\dot{u}) - \int_0^\infty \langle \tilde{D}_t F, \dot{u}(t)\rangle dt. \tag{5.5.8}$$

Proof. We use the relations $\tilde{\delta}(\dot{u}) = \hat{\delta}(\dot{\tilde{u}})$ and

$$\int_0^\infty \langle \tilde{D}_t F, \dot{u}(t)\rangle dt = \int_0^\infty \langle \hat{D}_t F, \dot{\tilde{u}}(t)\rangle dt, \quad u \in \mathcal{U}(\mathbb{P}(M)\times\mathbb{R}_+; \mathbb{R}^d),\ F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R}). \quad \square$$

We denote by $\mathrm{Dom}(\tilde{D})$ and $\mathrm{Dom}(\tilde{\delta})$ the closed domains of $\tilde{D} : L^2(\mathbb{P}(M); \mathbb{R}) \longrightarrow L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ and $\tilde{\delta} : L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d) \longrightarrow L^2(\mathbb{P}(M); \mathbb{R})$.

Proposition 5.5.3 *The operators* δ *and* $\tilde{\delta}$ *both coincide with the stochastic integral with respect to* $(B(t))_{t\in\mathbb{R}_+}$ *on the adapted processes in* $L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$*:*

$$\tilde{\delta}(u) = \delta(u) = \int_0^\infty \langle u(s), dB(s)\rangle.$$

Proof. The relation $\delta(u) = \int_0^\infty \langle u(s), dB(s)\rangle$, for adapted $u \in L^2(W \times \mathbb{R}_+; \mathbb{R}^d)$, is well-known, cf. [11]. Given an adapted process $u \in \mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ written as $u = Fz$, where $F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R})$ is an $\mathcal{F}_t$-measurable functional and $\dot{z} = x1_{[t,t+a]}$, $t, a \in \mathbb{R}_+$, $x \in \mathbb{R}^d$, we have trace $q(s, D_s u) = q(s, z)D_s F = 0$, $s \in \mathbb{R}_+$, from the chaos expansion of F, hence $\delta(q(\text{trace }(\cdot, D.u))) = 0$ and $\delta(u) = \tilde{\delta}(u)$. This relation extends to square-integrable adapted processes by linearity and density.

$\square$

5.6 Clark formula for D and $\tilde{D}$

It has been shown in [10] in the continuous case and in [15] in the Poisson case that the Clark formula can be expressed with a damped gradient as well as with the flat gradient D. In this subsection we show that this result is also obtained by our method.

Proposition 5.6.1 *For* $F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R})$*, the processes* $DF, \tilde{D}F \in L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ *have the same adapted projections, i.e.*

$$E[D_t F \mid \mathcal{F}_t] = E[\tilde{D}_t F \mid \mathcal{F}_t], \quad t \in \mathbb{R}_+, \quad F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R}). \tag{5.6.1}$$

Proof. Let $F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R})$. From Prop. 5.5.3, given any square-integrable adapted process $u \in L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ we have $\tilde{\delta}(u) = \delta(u)$, hence by duality,

$$E\left[\int_0^\infty \langle u(t), D_t F\rangle dt\right] = E\left[\int_0^\infty \langle u(t), \tilde{D}_t F\rangle dt\right], \quad F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R}),$$

which proves (5.6.1).

□

This implies that the Clark formula has two expressions, since the adapted projections of D and $\tilde{D}$ coincide.

Proposition 5.6.2 *Let $F \in \mathrm{Dom}(D) \cap \mathrm{Dom}(\tilde{D})$. We have*

$$F = E[F] + \int_0^\infty \langle E[D_t F \mid \mathcal{F}_t], dB(t)\rangle = E[F] + \int_0^\infty \langle E[\tilde{D}_t F \mid \mathcal{F}_t], dB(t)\rangle.$$

Proof. This is a consequence of the classical Clark-Ocone formula for Brownian motion on W and of Prop. 5.6.1.

□

The interest in the damped gradient is also that from (5.5.7) it gives a more natural expression to the formula the integration by parts formula of [5]:

$$E\left[\int_0^\infty \langle \hat{D}_t F, \dot{u}(t)\rangle dt\right] = E\left[F \int_0^\infty \langle \dot{\hat{u}}(t), dB(t)\rangle\right], \tag{5.6.2}$$

where $F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R})$ and $u \in \mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ is adapted, i.e. (5.5.3) can be written as

$$E\left[\int_0^\infty \langle \tilde{D}_t F, \dot{u}(t)\rangle dt\right] = E\left[F \int_0^\infty \langle \dot{u}(t), dB(t)\rangle\right], \tag{5.6.3}$$

$u \in \mathcal{U}(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$, $F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R})$.

5.7 Quantum stochastic differentials

By analogy we define quantum stochastic integrals on the Riemannian manifold M to be the operators

$$\int_0^\infty \langle h(s), d\hat{a}_s^-\rangle F = \int_0^\infty \langle \hat{D}_s F, h(s)\rangle ds,$$
$$\int_0^\infty \langle h(s), d\hat{a}_s^+\rangle F = \hat{\delta}(hF), \quad F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R}),$$

where $h \in L^2(\mathbb{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ is adapted. The following relations are reformulations of Cor. 5.2.1 and (5.5.1).

Proposition 5.7.1 *We have the following relations between quantum stochastic differentials:*

$$\langle \dot{z}(t), d\hat{a}_t^- \rangle = \langle \dot{\hat{z}}(t), da_t^- \rangle + q(t,z) da_t^\circ, \tag{5.7.1}$$

and

$$\langle \dot{z}(t), d\hat{a}_t^+ \rangle = \langle \dot{\hat{z}}(t), da_t^+ \rangle + q^*(t,z) da_t^\circ. \tag{5.7.2}$$

Using the antisymmetry of $q(t,z) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, (5.7.2) is rewritten as

$$\langle \dot{z}(t), d\hat{a}_t^+ \rangle = \langle \dot{\hat{z}}(t), da_t^+ \rangle - q(t,z) da_t^\circ.$$

These relations imply

$$\langle \dot{z}(t), d\hat{a}_t^- + d\hat{a}_t^+ \rangle = \langle \dot{\hat{z}}(t), da_t^- + da_t^+ \rangle = \langle \dot{\hat{z}}(t), dB_t \rangle,$$

which are reformulations of (5.5.4) and (3.1.2). Similarly, the following proposition reformulates Cor. 5.4.1 and (5.5.6) respectively.

Proposition 5.7.2 *The "damped" quantum stochastic differentials satisfy*

$$\langle \dot{z}(t), d\tilde{a}_t^- \rangle = \langle \dot{z}(t), da_t^- \rangle + q(t,\tilde{z}) da_t^\circ,$$

and

$$\langle \dot{z}(t), d\tilde{a}_t^+ \rangle = \langle \dot{z}(t), da_t^+ \rangle - q(t,\tilde{z}) da_t^\circ.$$

We have

$$\langle \dot{z}(t), d\tilde{a}_t^- + d\tilde{a}_t^+ \rangle = \langle \dot{z}(t), da_t^- + da_t^+ \rangle = \langle \dot{z}(t), dB_t \rangle,$$

i.e.

$$d\tilde{a}_t^- + d\tilde{a}_t^+ = da_t^- + da_t^+ = dB_t$$

which is a reformulation of (5.5.8) and (3.1.2).

6 The flat Poisson case

In what follows we will deal with the counterpart of the above construction, when the M-valued Brownian motion $(\gamma(t))_{t\geq 0}$ is replaced by random point measures γ on the Riemannian manifold M. Let Ω denote the configuration space on the Riemannian manifold M, that is the set of Radon measures on M of the form $\sum_{i=1}^{i=n} \epsilon_{x_i}$ with $(x_i)_{i=1}^{i=n} \subset M$, $x_i \neq x_j$ $\forall i \neq j$, $n \in \mathbb{N} \cup \{\infty\}$, where ϵ_x denotes the Dirac measure at $x \in M$. The configuration space Ω is endowed with the vague topology and its associated σ-algebra, cf. [1]. Let σ be a diffuse Radon measure on M, let P denote the Poisson measure with intensity σ on Ω, and let ∇^M and div^M denote the gradient and divergence on M. We assume that σ is the volume element of M, under which div^M and ∇^M are adjoint, and that $\int_M \mathrm{div}^M Z(x)\sigma(dx) = 0$, $\forall Z \in \mathcal{C}_c^\infty(M;TM)$. We denote by T_xM the tangent space at $x \in M$ (in this setting there is no parallel transport). Let $H = L^2(M;\mathbb{R},\sigma)$, and let $I_n(f_n)$ denote the multiple stochastic integral with respect to $(\gamma(dx)-\sigma(dx))$ of a symmetric function of n variables $f_n \in \hat{L}^2(M^n) \simeq H^{\circ n}$. The identification of $f_n \in \hat{L}^2(M^n) \simeq H^{\circ n}$ to $I_n(f_n)$ provides an isometric isomorphism between $\Gamma(H)$ and $L^2(\Omega;\mathbb{R})$. We have in particular $\delta(u) = \int_M u(x)(\gamma(dx)-\sigma(dx))$, $u \in L^2(M;\mathbb{R})$. Let $\mathcal{S}(M;\mathbb{R})$ denote an algebra of compactly supported functions which is dense in $L^2(M;\mathbb{R})$, e.g. $\mathcal{S}(M;\mathbb{R}) = \mathcal{C}_c^\infty(M;\mathbb{R})$, and let

$$\mathcal{S}_0(M;\mathbb{R}) = \left\{ u \in \mathcal{S}(M;\mathbb{R}) \ : \ \int_M u(x)\sigma(dx) = 0 \right\}.$$

Definition 6.1.1 *Let*

$$\mathcal{S}(\Omega;\mathbb{R}) = \{ f(I_1(u_1),\ldots,I_1(u_n)) \ : \ u_1,\ldots,u_n \in \mathcal{S}_0(M;\mathbb{R}), \ f \in \mathcal{C}_b^\infty(\mathbb{R}^n;\mathbb{R}), \ n \geq 1 \},$$

$$\begin{aligned} &\mathcal{U}(\Omega \times M;\mathbb{R}) \\ &= \left\{ \sum_{k=1}^{k=n} F_k u_k \ : \ F_1,\ldots,F_n \in \mathcal{S}(\Omega;\mathbb{R}), \ u_1,\ldots,u_n \in \mathcal{S}_0(M;\mathbb{R}), \ i=1,\ldots,n, \ n \geq 1 \right\}, \end{aligned}$$

and

$$\mathcal{U}(\Omega \times M;TM) = \left\{ \sum_{k=1}^{k=n} F_k u_k \ : \ F_1,\ldots,F_n \in \mathcal{S}(\Omega;\mathbb{R}), \ u_1,\ldots,u_k \in \mathcal{C}_c^\infty(M;TM) \right\}.$$

In the Poisson interpretation of $\Gamma(H)$, D is a finite difference operator:

$$D_xF(\gamma) = F(\gamma + (1-\gamma(\{x\}))\delta_x) - F(\gamma), \tag{6.1.1}$$

hence

$$D_x(FG) = FD_xG + GD_xF + D_xFD_xG, \qquad x \in M, \tag{6.1.2}$$

and by duality this shows that

$$F\delta(u) = \delta(uF) + \int_M u(x)D_xF\sigma(dx) + \delta(uDF), \tag{6.1.3}$$

$F \in \mathcal{S}(\Omega;\mathbb{R})$, $u \in \mathcal{U}(\Omega \times M;\mathbb{R})$, which are the analogs of (3.1.1) and (3.1.2).

7 Differential calculus and morphisms on configuration spaces

Let $U : \mathcal{S}(M;\mathbb{R}) \longrightarrow \mathcal{S}(M;\mathbb{R})$ denote a deterministic mapping.

Definition 7.1.1 *We let the transformation* $\Lambda(U)$ *be defined as*

$$\Lambda(U)F(\gamma) = f\left(\int_M Uu_1(x)\gamma(dx), \ldots, \int_M Uu_n(x)\gamma(dx)\right)$$

for $F \in \mathcal{S}(\Omega;\mathbb{R})$ *of the form* $F(\gamma) = f\left(\int_M u_1(x)\gamma(dx), \ldots, \int_M u_n(x)\gamma(dx)\right)$.

We have

$$\Lambda(U)F = f\left(I_1(Uu_1) + \int_M Uu_1 d\sigma, \ldots, I_1(Uu_n) + \int_M Uu_n d\sigma\right).$$

Due to the smoothness of $F \in \mathcal{S}(W;\mathbb{R})$, no additional hypothesis is required on U. If U is given by a measurable mapping $V : M \longrightarrow M$, as $[Uf](x) = f(V(x))$, $x \in M$, then $(\Lambda(U)F)(\gamma)$ is the evaluation of F at the configuration γ whose points have been shifted by V, i.e. $\Lambda(U)F(\gamma) = F(V^*\gamma)$, where $V^*\gamma$ denotes the image measure of γ by $V : M \longrightarrow M$.

Definition 7.1.2 *Let* $(U_\varepsilon)_{\varepsilon\in\mathbb{R}}$ *be a family of linear operators*

$$U_\varepsilon : \mathcal{S}(M;\mathbb{R}) \longrightarrow L^2(M;\mathbb{R}),$$

preserving compact sets, and such that

- $U_0 : \mathcal{S}(M;\mathbb{R}) \longrightarrow \mathcal{S}(M;\mathbb{R})$ *is the identity of* $\mathcal{S}(M;\mathbb{R})$.

- *the family* $(U_\varepsilon)_{\varepsilon\in[0,1]}$ *admits a derivative at* $\varepsilon = 0$, *i.e. there exists a linear operator*

  $$\mathcal{L} : \mathcal{S}(M;\mathbb{R}) \longrightarrow L^2(M;\mathbb{R}),$$

  such that $((U_\varepsilon f - f)/\varepsilon)_{\varepsilon\in[0,1]}$ *converges in* $L^2(M;\mathbb{R})$ *to* $\mathcal{L}f = (\mathcal{L}_t f)_{t\in\mathbb{R}_+}$ *as* ε *goes to zero,* $f \in \mathcal{S}(M;\mathbb{R})$.

Examples of such operators can be constructed by shifts of configurations points via a flow of diffeomorphisms on M. (The operator $\mathcal{L}$ is naturally extended to $\mathcal{S}(\Omega \times M;\mathbb{R})$).

Proposition 7.1.1 *For* $F \in \mathcal{S}(\Omega;\mathbb{R})$, *we have in* $L^2(\Omega;\mathbb{R})$:

$$\frac{d}{d\varepsilon}\Lambda(U_\varepsilon)F_{|\varepsilon=0} = \int_M \mathcal{L}D_x F\sigma(dx) + \delta(\mathcal{L}DF). \tag{7.1.1}$$

Proof. The steps of the proof are the same as in Prop. 4.1.2. Let $A : \mathcal{S}(\Omega;\mathbb{R}) \longrightarrow \mathcal{S}(\Omega;\mathbb{R})$ be defined as

$$AF = \delta(\mathcal{L}DF) + \int_M \mathcal{L}D_xF\sigma(dx), \quad F \in \mathcal{S}(\Omega;\mathbb{R}).$$

For $F = \int_M u(x)\gamma(dx)$, $u \in \mathcal{C}_c^\infty(M;\mathbb{R})$, we have $\Lambda(U_\varepsilon)F = \int_M U_\varepsilon u(x)\gamma(dx)$ and

$$\frac{d}{d\varepsilon}\Lambda(U_\varepsilon)F_{|\varepsilon=0} = \int_M \mathcal{L}u(x)\gamma(x) = \delta(\mathcal{L}u) + \int_M \mathcal{L}D_xF\sigma(dx) = AF.$$

We show that A is a derivation operator. Using (6.1.2), (6.1.3) and the fact that $\mathcal{L}$ is a derivation operator on $\mathcal{S}(M;\mathbb{R})$, we have:

$$\begin{aligned}
&\delta(\mathcal{L}D(FG)) = \delta(\mathcal{L}D(FG)) \\
&\quad = \delta(F\mathcal{L}DG + G\mathcal{L}DF + \mathcal{L}(DFDG)) \\
&\quad = F\delta(\mathcal{L}DG) + G\delta(\mathcal{L}DF) + \delta(DF\mathcal{L}DG) + \delta(DG\mathcal{L}DF) \\
&\qquad - \int_M D_xF\mathcal{L}D_xG\sigma(dx) - \int_M D_xG\mathcal{L}D_xF\sigma(dx) - \delta(\mathcal{L}DGDF) - \delta(\mathcal{L}DFDG) \\
&\quad = F\delta(\mathcal{L}DG) + G\delta(\mathcal{L}DF) - \int_M D_xF\mathcal{L}D_xG\sigma(dx) - \int_M D_xG\mathcal{L}D_xF\sigma(dx).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_M \mathcal{L}D_x(FG)\sigma(dx) &= G\int_M \mathcal{L}D_xF\sigma(dx) + F\int_M \mathcal{L}D_xG\sigma(dx) \\
&\quad + \int_M \mathcal{L}(D_xFD_xG)\sigma(dx) \\
&= G\int_M \mathcal{L}D_xF\sigma(dx) + F\int_M \mathcal{L}D_xG\sigma(dx) \\
&\quad + \int_M D_xG\mathcal{L}D_xF\sigma(dx) + \int_M D_xF\mathcal{L}D_xG\sigma(dx).
\end{aligned}$$

Hence

$$\begin{aligned}
A(FG) &= \delta(\mathcal{L}D(FG)) + \int_M \mathcal{L}D_x(FG)\sigma(dx) \\
&= F\left(\delta(\mathcal{L}DG) + \int_M \mathcal{L}D_xG\sigma(dx)\right) + G\left(\delta(\mathcal{L}DF) + \int_M \mathcal{L}D_xF\sigma(dx)\right) \\
&= FAG + GAF,
\end{aligned}$$

which extends as

$$Af(F_1,\ldots,F_n) = \sum_{i=1}^{i=n} \partial_i f(F_1,\ldots,F_n)AF_i$$

to polynomial f and successively to $f \in \mathcal{C}_c^\infty(\mathbb{R}^n;\mathbb{R})$ and $f \in \mathcal{C}_b^\infty(\mathbb{R}^n;\mathbb{R})$. Hence if $F_1 = \int_M u_1(x)\gamma(dx),\ldots,F_n = \int_M u_n(x)\gamma(dx)$, we have

$$Af(F_1,\ldots,F_n) = \sum_{i=1}^{i=n} \partial_i f(F_1,\ldots,F_n)AF_i = \sum_{i=1}^{i=n} \partial_i f(F_1,\ldots,F_n)\left(\frac{d}{d\varepsilon}\Lambda(U_\varepsilon)F_i\right)_{|\varepsilon=0}$$

$$= \left(\frac{d}{d\varepsilon} \Lambda(U_\varepsilon) f(F_1, \ldots, F_n) \right)_{|\varepsilon=0},$$

which implies that (7.1.1) holds on $\mathcal{S}(\Omega; \mathbb{R})$.

□

If $(U_\varepsilon)_{\varepsilon \in [0,1]}$ is given as $[U_\varepsilon f](x) = f(V_\varepsilon(x))$, $x \in M$, by a family of measurable mappings $V_\varepsilon : M \longrightarrow M$, then $\mathcal{L}$ is the vector field on M defined as

$$\mathcal{L} f(x) = \lim_{\varepsilon \to 0} \frac{f(V_\varepsilon(x)) - f(x)}{\varepsilon}, \quad x \in M, \ f \in \mathcal{C}_c^\infty(M; \mathbb{R}).$$

If $V_\varepsilon : M \longrightarrow M$ preserves the measure σ, then

$$E\left[\frac{d}{d\varepsilon} \Lambda(U_\varepsilon) F_{|\varepsilon=0} \right] = \frac{d}{d\varepsilon} E\left[\Lambda(U_\varepsilon) F \right]_{|\varepsilon=0} = \frac{d}{d\varepsilon} E[F]_{|\varepsilon=0} = 0,$$

since the Poisson measure is invariant under the shift $V_\varepsilon : M \longrightarrow M$. Here this identity is interpreted as

$$E\left[\frac{d}{d\varepsilon} \Lambda(U_\varepsilon, 0) F_{|\varepsilon=0} \right] = E[\delta(\mathcal{L} D F)] = 0, \quad F \in \mathcal{S}(\Omega; \mathbb{R}).$$

8 Stochastic analysis of point measures on a Riemannian manifold

8.1 Gradient $\hat{D} : L^2(\Omega) \longrightarrow L^2(\Omega \times M; TM)$

The study of variational calculus for jump processes has been started in [4]. The gradient $\hat{D}$ is defined in [1] and [3]. It is TM-valued and defined for $F \in \mathcal{S}(\Omega; \mathbb{R})$ as

$$\hat{D}_x F = \sum_{i=1}^{i=n} \partial_i f \left(\int_M u_1(y) \gamma(dy), \ldots, \int_M u_n(y) \gamma(dy) \right) \nabla^M u_i(x), \quad x \in M,$$

with $F = f\left(\int_M u_1(x) \gamma(dx), \ldots, \int_M u_n(x) \gamma(dx) \right)$, $\quad f \in \mathcal{C}_b^\infty(\mathbb{R}^n; \mathbb{R})$. The vector field $Z \in \mathcal{C}_c^\infty(M; TM)$ defines a flow $(\phi_\varepsilon^Z)_{\varepsilon \in [0,1]} : M \longrightarrow M$ on M and we have from [1]

$$\int_M \langle \hat{D}_x F, Z(x) \rangle_{T_x M} \gamma(dx) = \lim_{\varepsilon \to 0} \frac{F(\phi_\varepsilon^Z \gamma) - F(\gamma)}{\varepsilon}.$$

8.2 Explicit expression of the gradient $\hat{D}$

In the following we assume that $\mathcal{S}(M; \mathbb{R}) = \mathcal{C}_c^\infty(M; \mathbb{R})$. From the result of Sect. 7 we obtain the expression of $\hat{D}$ in terms of D and δ. If $Z \in \mathcal{C}_c^\infty(M; TM)$ denotes a smooth vector field on M, we let $\hat{Z} : M \longrightarrow \mathbb{R}$ be defined as

$$\hat{Z}(x) = \mathrm{div}^M Z(x), \quad x \in M.$$

Theorem 8.2.1 *We have for $F \in \mathcal{S}(\Omega;\mathbb{R})$ and $Z \in \mathcal{C}_c^\infty(M;TM)$:*

$$\int_M \langle \hat{D}_x F, Z(x)\rangle_{T_xM}\gamma(dx) = \int_M \hat{Z}(x) D_x F \sigma(dx) + \delta(q(\cdot,Z)DF), \tag{8.2.1}$$

where $q(x,Z)$ is the derivation operator associated to $Z \in \mathcal{C}_c^\infty(M;TM)$, i.e.

$$q(x,Z)\Phi = \langle \nabla^M \Phi(x), Z(x)\rangle_{T_xM}, \quad x \in M, \ \Phi \in \mathcal{C}_c^\infty(M;\mathbb{R}).$$

Proof. We define $U_\varepsilon : \mathcal{C}_c^\infty(M) \longrightarrow \mathcal{C}_c^\infty(M)$ by $U_\varepsilon f = f \circ \phi_\varepsilon^Z$, $f \in \mathcal{C}_c^\infty(M;\mathbb{R})$, with $\mathcal{S}(M;\mathbb{R}) = \mathcal{C}_c^\infty(M;\mathbb{R})$. Then $\mathcal{L}u(x) = q(x,Z)u$, $u \in \mathcal{C}_c^\infty(M;\mathbb{R})$ and from Prop. 7.1.1,

$$\int_M \langle \hat{D}_x F, Z(x)\rangle_{T_xM}\gamma(dx) = \int_M \mathcal{L} D_x F \sigma(dx) + \delta(\mathcal{L}DF),$$

where $\mathcal{L}$ is the derivative at $\varepsilon = 0$ of $(\phi_t^Z)_{t\in\mathbb{R}}$, i.e. $\mathcal{L}D_xF = \langle \nabla^M D_x F, Z(x)\rangle_{T_xM}$, $x \in M$. Hence

$$\int_M \langle \hat{D}_x F, Z(x)\rangle_{T_xM}\gamma(dx) = \int_M D_x F \mathrm{div}^M Z(x)\sigma(dx) + \delta(q(\cdot,Z)DF). \quad \square$$

The following remark is the Poisson analog of Remark 1.

Remark 2 *Taking expectations in the almost sure identity (8.2.1) we obtain the equality of Th. 5.2 in [1], since $E[\delta(\mathcal{L}DF)] = 0$.*

The adjoint $q^*(x,Z)$ of $q(x,Z)$ is given as

$$\begin{aligned} q^*(x,Z)u &= \mathrm{div}^M(Z(x)u(x)) = u(x)\mathrm{div}^M(Z(x)) - \langle \nabla^M u(x), Z(x)\rangle_{T_xM} \\ &= u(x)\hat{Z}(x) - q(x,Z)u, \quad u \in \mathcal{C}_c^\infty(M;\mathbb{R}), \quad Z \in \mathcal{C}_c^\infty(M;TM), \end{aligned}$$

and the duality relation between $q(x,Z)$ and $q^*(x,Z)$ is

$$\int_M \langle u(x), q^*(x,Z)v\rangle_{T_xM}\sigma(dx) = \int_M \langle v(x), q(x,Z)u\rangle_{T_xM}\sigma(dx), \quad u,v \in \mathcal{C}_c^\infty(M;\mathbb{R}).$$

For $u \in \mathcal{U}(\Omega \times M;TM)$, the processes $(q(x,D.u(\cdot)))_{x\in M}$ and $(q^*(x,D.u(\cdot)))_{x\in M}$ are naturally defined by linearity, and will be denoted by $q(\cdot,D.u(\cdot))$ and $q^*(\cdot,D.u(\cdot))$. Given $u \in \mathcal{U}(\Omega\times M;TM)$ written as $u = \sum_{i=1}^{i=n} G_i Z_i$, $G_i \in \mathcal{S}(\Omega;\mathbb{R})$, $Z_i \in \mathcal{C}_c^\infty(M;TM)$, $i = 1,\ldots,n$, we let $\hat{u} = \sum_{i=1}^{i=n} G_i \mathrm{div}^M Z_i$. The following result extends Th. 8.2.1 to random processes.

Theorem 8.2.2 *We have for $u \in \mathcal{U}(\Omega \times M;TM)$:*

$$\begin{aligned} \int_M \langle \hat{D}_x F, u(x)\rangle_{T_xM}\gamma(dx) &= \int_M D_x F \hat{u}(x)\sigma(dx) + \delta(q(\cdot,u)DF) \\ &\quad + \int_M q(x,D.u(\cdot))DF\sigma(dx) + \delta(q(\cdot,D.u)DF). \end{aligned}$$

Proof. Let $u = GZ$, with $Z \in \mathcal{C}_c^\infty(M;TM)$ and $G \in \mathcal{S}(\Omega;\mathbb{R})$. We have

$$\begin{aligned}\int_M \langle \hat{D}_x F, GZ(x)\rangle_{T_xM}\gamma(dx) &= G\int_M D_xF\hat{Z}(x)\sigma(dx) + G\delta(q(\cdot,Z)DF)\\ &= \int_M D_xF\hat{u}(x)\sigma(dx) + \delta(Gq(\cdot,Z)DF)\\ &\quad + \int_M D_xGq(x,Z)DF\sigma(dx) + \delta(D.Gq(\cdot,Z)DF)\\ &= \int_M D_xF\hat{u}(x)\sigma(dx) + \delta(q(\cdot,u)DF)\\ &\quad + \int_M q(x,D.u(\cdot))DF\sigma(dx) + \delta(q(\cdot,D.u(\cdot))DF). \quad \square\end{aligned}$$

8.3 Inversion of $Z \mapsto \hat{Z}$

Given $\Phi \in \mathcal{S}_0(M;\mathbb{R})$ (such that $\int_M \Phi(x)\sigma(dx) = 0$), the inversion of $Z \mapsto \hat{Z}$ consists in the determination of a vector field $\tilde{\Phi} : M \longrightarrow TM$ such that

$$\mathrm{div}^M \tilde{\Phi} = \Phi.$$

This is possible in particular if M is a compact manifold, since the Laplacian $L = \mathrm{div}^M\nabla^M$ is negative and symmetric on $\mathcal{C}_c^\infty(M;\mathbb{R})$. In this case, $L : \mathcal{C}_c^\infty(M;\mathbb{R}) \longrightarrow \mathcal{C}_c^\infty(M;\mathbb{R})$ is invertible, so that we can let

$$\tilde{\Phi} = \nabla^M L^{-1}\Phi.$$

Let $G : M \times M \longrightarrow \mathbb{R}$ denote the Green function associated to L^{-1}, such that

$$L^{-1}u(x) = \int_M G(x,y)u(y)\sigma(dy), \quad x \in M,\ u \in \mathcal{C}_c^\infty(M;\mathbb{R}).$$

Let $\varepsilon : M \times M \longrightarrow TM$ be defined as

$$\varepsilon(x,y) = \nabla_x^M G(x,y), \quad \sigma(dx) - a.e.,\ \sigma(dy) - a.e.$$

Then $\tilde{\Phi}$ can be defined as

$$\tilde{\Phi}(x) = \nabla^M L^{-1}\Phi(x) = \int_M \varepsilon(x,y)\Phi(y)\sigma(dy), \quad x \in M,\ \Phi \in \mathcal{S}_0(M;\mathbb{R}).$$

In the case where $M = \mathbb{R}_+$ and σ is the Lebesgue measure, then $G(x,y) = x \wedge y$ and $\varepsilon(x,y) = 1_{[0,y]}(x)$, or indifferently $\varepsilon(x,y) = -1_{[x,\infty[}(y)$, $x, y \in \mathbb{R}_+$, since we are working on $\mathcal{S}_0(\mathbb{R}_+;\mathbb{R})$.

8.4 Expression of the damped gradient $\tilde{D}$

In this subsection we complete the operators D, $\hat{D}$ with a damped gradient $\tilde{D}$ which is linked to the Clark formula and stochastic integration.

Definition 8.4.1 *We define the damped gradient $\tilde{D} : L^2(\Omega; \mathbb{R}) \longrightarrow L^2(\Omega \times M; \mathbb{R})$ on $\mathcal{S}(\Omega; \mathbb{R})$ as*

$$\tilde{D}_x F = \sum_{i=1}^{i=n} \int_M \langle \nabla^M u_i(y), \varepsilon(x,y) \rangle_{T_y M} \gamma(dy) \partial_i f \left(\int_M u_1 d\gamma, \dots, \int_M u_n d\gamma \right), \qquad (8.4.1)$$

with $F = f\left(\int_M u_1(x)\gamma(dx), \dots, \int_M u_n(x)\gamma(dx)\right)$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n; \mathbb{R})$.

In other terms,

$$\tilde{D}_y F = \int_M \langle \hat{D}_x F, \varepsilon(x,y) \rangle_{T_x M} \gamma(dx), \quad y \in M.$$

Proposition 8.4.1 *The damped gradient satisfies*

$$\int_M \Phi(x) \tilde{D}_x F \sigma(dx) = \int_M \langle \hat{D}_x F, \tilde{\Phi}(x) \rangle_{T_x M} \gamma(dx), \quad \Phi \in \mathcal{S}_0(M; \mathbb{R}).$$

Proof. We have

$$\begin{aligned}
\int_M \tilde{D}_y F \Phi(y) \sigma(dy) &= \int_M \int_M \langle \hat{D}_x F, \varepsilon(x,y) \rangle_{T_x M} \gamma(dx) \Phi(y) \sigma(dy) \\
&= \int_M \langle \hat{D}_x F, \int_M \Phi(y) \varepsilon(x,y) \sigma(dy) \rangle_{T_x M} \gamma(dx) \\
&= \int_M \langle \hat{D}_x F, \tilde{\Phi}(x) \rangle_{T_x M} \gamma(dx). \quad \square
\end{aligned}$$

Corollary 8.4.1 *For $\Phi \in \mathcal{S}_0(M; \mathbb{R})$ we have*

$$\int_M \tilde{D}_x F \Phi(x) \sigma(dx) = \int_M D_x F \Phi(x) \sigma(dx) + \delta(q(\cdot, \tilde{\Phi}) DF), \quad F \in \mathcal{S}(\Omega; \mathbb{R}).$$

Proof. We apply Prop. 8.2.1 and the relation

$$\int_M \tilde{D}_x F \Phi(x) \sigma(dx) = \int_M \langle \hat{D}_x F, \tilde{\Phi}(x) \rangle_{T_x M} \gamma(dx). \quad \square$$

The following corollary is the extension of Cor. 8.4.1 to the random case.

Corollary 8.4.2 *The damped gradient satisfies for $\Phi \in \mathcal{U}(\Omega \times M; \mathbb{R})$:*

$$\begin{aligned}
\int_M \tilde{D}_x F \Phi(x) \sigma(dx) &= \int_M \Phi(x) D_x F \sigma(dx) + \delta(q(\cdot, \tilde{\Phi}) DF) \\
&\quad + \int_M q(x, D.\tilde{\Phi}(\cdot)) DF \sigma(dx) + \delta(q(\cdot, D.\tilde{\Phi}(\cdot)) DF), \quad F \in \mathcal{S}(\Omega; \mathbb{R}).
\end{aligned}$$

Proof. Similarly to the above, we apply Th. 8.2.2 with the relation

$$\int_M \Phi(x)\tilde{D}_x F \sigma(dx) = \int_M \langle \hat{D}_x F, \tilde{\Phi}(x)\rangle_{T_xM} \gamma(dx). \quad \square$$

In the particular case $M = \mathbb{R}_+$, we have from (8.4.1)

$$\tilde{D}_x F = -\sum_{i=1}^{i=n} \int_0^\infty u_i'(y) 1_{[0,x]}(y)\gamma(dy)\partial_i f\left(\int_0^\infty u_1(s)\gamma(ds), \ldots, \int_0^\infty u_n(s)\gamma(ds)\right),$$

for $F = f\left(\int_0^\infty u_1(s)\gamma(ds), \ldots, \int_0^\infty u_n(s)\gamma(ds)\right)$. If $(T_n)_{n\geq 1}$ denotes the jump times of the point measure $\gamma(dx)$ on $\mathbb{R}_+$, i.e.

$$\gamma(dx) = \sum_{k=1}^\infty \epsilon_{T_k}(dx),$$

then

$$\tilde{D}_t F = -\sum_{i=1}^{i=n}\sum_{k=1}^\infty u_i'(T_k) 1_{[0,T_k]}(t)\partial_i f\left(\int_0^\infty u_1(s)\gamma(ds), \ldots, \int_0^\infty u_n(s)\gamma(ds)\right),$$

i.e.

$$\tilde{D}_t f(T_1, \ldots, T_n) = -\sum_{i=1}^{i=n} 1_{[0,T_i]}(t)\partial_i f(T_1, \ldots, T_n), \quad t \in \mathbb{R}_+,$$

and $\tilde{D}$ becomes the gradient of [6]:

$$\int_0^\infty \tilde{D}_t F \Phi(t)dt = \lim_{\varepsilon\to 0} \frac{f\left(T_1 - \varepsilon\int_0^{T_1}\Phi(s)ds, \ldots, T_n - \varepsilon\int_0^{T_n}\Phi(s)ds\right) - f(T_1, \ldots, T_n)}{\varepsilon},$$

for $F = f(T_1, \ldots, T_n)$, since $\tilde{\Phi}(t) = -\int_0^t \Phi(s)ds$, $t \in \mathbb{R}_+$.

8.5 Anticipating stochastic integration

In this subsection we study successively two different Skorohod type anticipating integrals $\hat{\delta} : L^2(\Omega\times M; TM) \longrightarrow L^2(\Omega;\mathbb{R})$ and $\tilde{\delta} : L^2(\Omega\times M;\mathbb{R}) \longrightarrow L^2(\Omega;\mathbb{R})$ that are the respective adjoints of the gradient $\hat{D}$ and of the damped gradient $\tilde{D}$.

Definition 8.5.1 *We define the operator* $\hat{\delta} : L^2(\Omega\times M; TM) \longrightarrow L^2(\Omega;\mathbb{R})$ *as*

$$\hat{\delta}(u) = \delta(\hat{u}) + \delta(q^*(\cdot, D.u(\cdot))), \quad u \in \mathcal{U}(\Omega\times M; TM), \tag{8.5.1}$$

where $q^*(x, D.u(\cdot)) = \mathrm{div}^M(D_x u(x))$, $x \in M$.

We have for $F \in \mathcal{S}(\Omega;\mathbb{R})$ and $Z \in \mathcal{C}_c^\infty(M; TM)$:

$$\hat{\delta}(FZ) = \delta(F\hat{Z}) + \delta(q^*(\cdot, ZDF)) = \delta(F\hat{Z}) + \delta(\hat{Z}DF) - \delta(q(\cdot, Z)DF), \tag{8.5.2}$$

since

$$q^*(x, ZDF) = D_x F\hat{Z}(x) - q(x, Z)DF, \quad Z \in \mathcal{C}_c^\infty(M;\mathbb{R}). \tag{8.5.3}$$

Proposition 8.5.1 *The operators $\hat{\delta}$ and $\hat{D}$ are mutually adjoint:*

$$E\left[\int_M \langle \hat{D}_x F, u(x)\rangle_{T_xM}\gamma(dx)\right] = E[F\hat{\delta}(u)], \tag{8.5.4}$$

$u \in \mathcal{U}(\Omega \times M; TM)$, $F \in \mathcal{S}(\Omega; \mathbb{R})$, *and $\hat{\delta}$ satisfies*

$$\hat{\delta}(Fu) = F\hat{\delta}(u) - \int_M \langle \hat{D}_x F, u(x)\rangle_{T_xM}\gamma(dx), \tag{8.5.5}$$

$u \in \mathcal{U}(\Omega \times M; TM)$, $F \in \mathcal{S}(\Omega; \mathbb{R})$.

Proof. We have

$$\begin{aligned}
E[F\hat{\delta}(u)] &= E[F\delta(\hat{u})] + E[F\delta(q^*(\cdot, D.u(\cdot)))] \\
&= E[F\delta(\hat{u})] + E\left[\int_M q(x, D_x u(x))DF\sigma(dx)\right] \\
&= E\left[\int_M \langle D_x F, \hat{u}(x)\rangle_{T_xM}\sigma(dx)\right] + E\left[\int_M q(x, D_x u(x))DF\sigma(dx)\right] \\
&= E\left[\int_M \langle \hat{D}_x F, u(x)\rangle_{T_xM}\gamma(dx)\right],
\end{aligned}$$

from Th. 8.2.2. On the other hand, (8.5.5) follows from the fact that $\hat{D}$ is a derivation operator, or from (8.5.3) and the following calculation:

$$\begin{aligned}
\hat{\delta}(F\dot{u}) &= \delta(F\dot{\hat{u}}) + \delta(q^*(\cdot, D.(Fu(\cdot)))) \\
&= F\delta(\dot{\hat{u}}) - \int_M D_x F\hat{u}(x)\sigma(dx) - \delta(\hat{u}(\cdot)D.F) \\
&\quad + \delta(Fq^*(\cdot, Z(\cdot)D.G)) + \delta(q^*(\cdot, D.FZ(\cdot)D.G)) + \delta(q^*(\cdot, u(\cdot)D.F)) \\
&= F\delta(\dot{\hat{u}}) - \int_M D_x F\hat{u}(x)\sigma(dx) - \delta(\hat{u}DF) \\
&\quad + F\delta(q^*(\cdot, D.u(\cdot))) - \int_M D_x Fq^*(x, D.u(\cdot))\sigma(dx) - \delta(D.Fq^*(\cdot, D.u(\cdot))) \\
&\quad + \delta(D.Fq^*(\cdot, D.u(\cdot))) - \delta(q(\cdot, D.u)DF) \\
&\quad + \delta(D.F\hat{u}(\cdot)) - \delta(q(\cdot, u)DF) \\
&= F\hat{\delta}(\dot{u}) - \delta(q(\cdot, D.GZ(\cdot))DF) - \delta(q(\cdot, u)DF) \\
&\quad - \int_M D_x F\hat{u}(x)\sigma(dx) - \int_M D_x Gq(x, Z)DF\sigma(dx). \\
&= F\hat{\delta}(\dot{u}) - \delta(q(\cdot, D.u)DF) - \delta(q(\cdot, u)DF) \\
&\quad - \int_M D_x F\hat{u}(x)\sigma(dx) - \int_M q(x, D_x u)DF\sigma(dx). \quad \square
\end{aligned}$$

If $u \in \mathcal{C}_c^\infty(M; TM)$ then (8.5.4) can be written as

$$E\left[\int_M \langle \hat{D}_x F, u(x)\rangle_{T_xM}\gamma(dx)\right] = E\left[F\int_M \hat{u}(x)(\gamma(dx) - \sigma(dx))\right], \tag{8.5.6}$$

$u \in \mathcal{C}(M; TM)$, $F \in \mathcal{S}(\Omega; \mathbb{R})$, which becomes the Poisson analog of the integration by parts formula (5.6.2) of [5].

Definition 8.5.2 *We define the operator* $\tilde{\delta} : L^2(\Omega \times M; \mathbb{R}) \longrightarrow L^2(\Omega; \mathbb{R})$ *as*

$$\tilde{\delta}(\Phi) = \delta(\Phi) + \delta(q^*(\cdot, D.\tilde{\Phi}(\cdot))), \quad \Phi \in \mathcal{U}(\Omega \times M; \mathbb{R}).$$

We have

$$\tilde{\delta}(F\Phi) = \delta(F\Phi) + \delta(\Phi(\cdot)D.F) - \delta(q(\cdot, \tilde{\Phi})DF), \quad F \in \mathcal{S}(\Omega; \mathbb{R}),$$

for deterministic $\Phi \in \mathcal{S}_0(M; \mathbb{R})$.

Proposition 8.5.2 *The operators* $\tilde{\delta}$ *and* $\tilde{D}$ *are mutually adjoint:*

$$E\left[\int_M \Phi(x)\tilde{D}_x F \sigma(dx)\right] = E[F\tilde{\delta}(\Phi)],$$

$\Phi \in \mathcal{U}(\Omega \times M; \mathbb{R})$*,* $F \in \mathcal{S}(\Omega; \mathbb{R})$*, and* $\tilde{\delta}$ *satisfies*

$$\tilde{\delta}(F\Phi) = F\delta(\Phi) - \int_M \tilde{D}_x F \Phi(x)\sigma(dx). \tag{8.5.7}$$

Proof. We use Prop. 8.5.1 and the relation $\tilde{\delta}(\Phi) = \hat{\delta}(\tilde{\Phi})$, $\Phi \in \mathcal{U}(\Omega \times M; \mathbb{R})$. □

Due to the above duality relations and to the density of $\mathcal{S}(\Omega; \mathbb{R})$ in $L^2(\Omega; \mathbb{R})$ of $\mathcal{U}(\Omega \times M; \mathbb{R})$ in $L^2(\Omega \times M; \mathbb{R})$ and of $\mathcal{U}(\Omega \times M; TM)$ in $L^2(\Omega \times M; TM)$, the operators $\hat{D}$, $\tilde{D}$, $\hat{\delta}$ and $\tilde{\delta}$ are closable. Their domains are denoted by $\mathrm{Dom}(\hat{D})$, $\mathrm{Dom}(\tilde{D})$, $\mathrm{Dom}(\hat{\delta})$ and $\mathrm{Dom}(\tilde{\delta})$. In order to deal with stochastic integration we choose M of the form $\mathbb{R}_+ \times X$ with volume element $dt \times d\sigma$, and take $H = L^2(\mathbb{R}_+ \times X; \mathbb{R}) \simeq L^2(\mathbb{R}_+; L^2(X; \mathbb{R}))$. The filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is generated by $\gamma \mapsto \gamma([0, s] \times A)$, $0 \leq s \leq t$, $A \in \mathcal{B}(X)$, and the stochastic integral of $\mathcal{F}_t$-adapted processes in $L^2(\Omega \times \mathbb{R}_+ \times X; \mathbb{R})$ is defined by the isometry formula

$$E\left[\left(\int_0^\infty \int_X u(s, x)(\gamma(ds, dx) - ds\sigma(dx))\right)^2\right] = E\left[\int_0^\infty \int_X u^2(s, x) ds \sigma(dx)\right]. \tag{8.5.8}$$

Proposition 8.5.3 *The operators* δ *and* $\tilde{\delta}$ *coincide with the stochastic integral with respect to* $\gamma(ds, dx) - ds\sigma(dx)$ *on the adapted processes in* $L^2(\Omega \times M; \mathbb{R}^d)$.

Proof. Given an adapted process $\Phi \in \mathcal{U}(\Omega \times M; \mathbb{R}^d)$ written as $\Phi = Fz$, with $F \in \mathcal{S}(\Omega; \mathbb{R})$ an $\mathcal{F}_t$-measurable functional and $z \in \mathcal{C}_c^\infty([t, \infty[\times X)$, we have $z(s, x)D_{s,x}F = 0$, $(s, x) \in M$, and $q^*(s, x, D.\tilde{\Phi}(\cdot)) = \mathrm{div}^M(z(s, x)D_{s,x}F) = 0$, $(s, x) \in M$. Hence $\delta(\Phi) = \tilde{\delta}(\Phi) = \int_0^\infty \int_X \Phi(s, x)(\gamma(ds, dx) - ds\sigma(dx))$ from Def. 8.5.2 and (6.1.3). This relation extends to the adapted processes in $L^2(\Omega \times M; \mathbb{R}^d)$ by linearity, density, closability and from the isometry formula (8.5.8). □

8.6 Clark formula

In this subsection we show that as a consequence of Prop. 8.5.3, the Clark formula has two expressions, depending on the type of gradient used.

Proposition 8.6.1 *For $F \in \mathcal{S}(\Omega; \mathbb{R})$, the processes DF and $\tilde{D}F$ have the same adapted projections, i.e.*

$$E\left[D_{s,x}F \mid \mathcal{F}_t\right] = E\left[\tilde{D}_{s,x}F \mid \mathcal{F}_t\right], \quad ds \times \sigma(dx) - a.e., \quad t \in \mathbb{R}_+. \tag{8.6.1}$$

Proof. This proof is similar to its counterpart in the continuous case (Prop. 8.6.1). Let $F \in \mathcal{S}(\Omega; \mathbb{R})$. Given any square-integrable adapted process u we have $\tilde{\delta}(u) = \delta(u)$ from Prop. 8.5.3 and by duality,

$$E\left[\int_0^\infty \int_X u(s,x) D_{s,x} F ds\sigma(dx)\right] = E\left[\int_0^\infty \int_X u(s,x) \tilde{D}_{s,x} F ds\sigma(dx)\right],$$

$F \in \mathcal{S}(\mathbb{P}(M); \mathbb{R})$, hence (8.6.1).

□

The Clark formula has two expressions.

Proposition 8.6.2 *Let $F \in \mathrm{Dom}(D) \cap \mathrm{Dom}(\tilde{D})$. We have*

$$\begin{aligned} F &= E[F] + \int_0^\infty \int_X E[D_{t,x}F \mid \mathcal{F}_t](\gamma(dt,dx) - dt\sigma(dx)) \\ &= E[F] + \int_0^\infty \int_X E[\tilde{D}_{t,x}F \mid \mathcal{F}_t](\gamma(dt,dx) - dt\sigma(dx)). \end{aligned}$$

Proof. We write the chaos expansion of F:

$$\begin{aligned} F &= E[F] + \sum_{n=1}^\infty n! \int_{\mathbb{R}_+ \times X} \int_{[0,s_n] \times X} \cdots \int_{[0,s_2] \times X} \\ &\quad f_n(s_1, x_1, \ldots, s_n, x_n)(\gamma(ds_1, dx_1) - ds_1\sigma(dx_1)) \cdots (\gamma(ds_n, dx_n) - ds_n\sigma(dx_n)) \\ &= E[F] + \sum_{n=1}^\infty n \int_0^\infty \int_X I_{n-1}(f_n(*; s, x) 1_{\{* \in ([0,t] \times X)^{n-1}\}})(\gamma(ds,dx) - ds\sigma(dx)) \\ &= E[F] + \int_0^\infty \int_X E[D_{t,x}F \mid \mathcal{F}_t](\gamma(dt,dx) - dt\sigma(dx)), \end{aligned}$$

and apply Prop. 8.6.1 and Prop. 8.5.3.

□

8.7 Quantum stochastic differentials

In the Poisson case we define quantum stochastic integrals on the Riemannian manifold M as

$$\int_M \langle Z(x), d\hat{a}_x^- \rangle F = \int_M \langle \hat{D}_x F, Z(x) \rangle \gamma(dx),$$
$$\int_M \langle Z(x), d\hat{a}_x^+ \rangle F = \hat{\delta}(ZF), \quad Z \in L^2(M; TM), \quad F \in \mathcal{S}(\Omega; \mathbb{R}).$$

The following proposition reformulates (8.2.1) and (8.5.1).

Proposition 8.7.1 *We have the following relations between quantum stochastic differentials:*

$$Z(x) d\hat{a}_x^- = \hat{Z}(x) da_x^- + q(x, Z) da_x^\circ,$$

and

$$Z(x) d\hat{a}_x^+ = \hat{Z}(x) da_x^+ + q^*(x, Z) da_x^\circ.$$

The last relation can be written as

$$Z(x) d\hat{a}_x^+ = \hat{Z}(x) da_x^+ + \hat{Z}(x) da_x^\circ - q(x, Z) da_x^\circ$$

since $q^*(x, Z) = \hat{Z}(x) - q(x, Z)$, $x \in M$. The following result reformulates Cor. 8.4.1 and Def. 8.5.2.

Proposition 8.7.2 *The "damped" quantum stochastic differentials satisfy*

$$\Phi(x) d\tilde{a}_x^- = \Phi(x) da_x^- + q(x, \tilde{\Phi}) da_x^\circ,$$

and

$$\Phi(x) d\tilde{a}_x^+ = \Phi(x) da_x^+ + q^*(x, \tilde{\Phi}) da_x^\circ,$$

$x \in M$.

From the expression of $q^*(x, u)$ we obtain

$$\Phi(x) d\tilde{a}_x^+ = \Phi(x) da_x^+ + \Phi(x) da_x^\circ - q(x, \tilde{\Phi}) da_x^\circ$$

and

$$\Phi(x)(d\tilde{a}_x^- + d\tilde{a}_x^+) = \Phi(x)(da_x^- + da_x^+ + da_x^\circ) = \Phi(x)(\gamma(dx) - \sigma(dx)),$$

which is a reformulation of (8.5.7) and (6.1.3). In particular,

$$\gamma(dx) - \sigma(dx) = d\tilde{a}_x^- + d\tilde{a}_x^+,$$

whereas

$$Z(x)(d\hat{a}_x^- + d\hat{a}_x^+) = \hat{Z}(x)(da_x^- + da_x^+ + da_x^\circ) = \hat{Z}(x)(\gamma(dx) - \sigma(dx)),$$

which is a reformulation of (8.5.5).

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