Euclidean quantum mechanics in the momentum representation

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Abstract

A time reversible probabilistic representation of solutions of the (Euclidean) Schrödinger equation in momentum representation is constructed using Lévy processes and bridges. Each diffusion in the position representation is associated with a jump diffusion in the momentum space. Our method can be looked upon as a rigorous version of Feynman's path integral approach. Several examples are studied.

Key words: Quantum mechanics, momentum representation, path integrals, Lévy processes, time reversal.
Classification: 81S20, 60J25.

1 Introduction and notation

Feynman's path integral approach to quantum mechanics can be regarded as an informal reinterpretation of this theory in intrinsically stochastic terms. More than 50 years after its creation [9], this approach has proved to be deep enough to provide basic insights into an amazing list of physical models, far beyond what could be anticipated originally. However, it is only "informal" because the probability measures on the various path spaces underlying Feynman's approach do not make any mathematical sense. Of course, a number of mathematical counterparts of such informal probability measures have been known, and used with profit, for a very long time. The basic one, for configuration representation, is as old as quantum theory itself: it is Wiener measure, induced by Brownian motion. Its original (sample) path space, also named after Wiener, is of the form:

$$\Omega_0 = \{\omega \in C(\mathbb{R}_+;\mathbb{R}^d) \ : \ \omega(0) = 0\},$$

and its relation with quantum theory appears in the famous Feynman-Kac formula [18]

$$(e^{-\frac{t}{\hbar}H}\chi^*)(q) = \int_{\Omega_0} \chi^*(\omega(t)+q)e^{-\frac{1}{\hbar}\int_0^t V(\omega(r)+q)dr}dP_W(\omega) \tag{1.1}$$

where $H = -\frac{\hbar^2}{2}\Delta + V$ is a lower bounded Hamiltonian observable on $L^2(\mathbb{R}^d)$, $V(q)$ a scalar potential, χ^* belongs to the dense domain $\mathcal{D}(H)$ of H in $L^2(\mathbb{R})$, $\hbar$ is the Planck constant, and P_W is the Wiener measure.

It is often more appropriate to consider path spaces on a compact time interval $[0,t]$ instead of $\mathbb{R}_+$, for instance

$$\Omega^{q,t} = \{\omega \in C([0,t];\mathbb{R}^d) \ : \ \omega(t) = q\}.$$

Then the Feynman-Kac formula becomes

$$(e^{-\frac{t}{\hbar}H}\chi^*)(q) = \int_{\Omega^{q,t}} \chi^*(\omega(0))e^{-\frac{1}{\hbar}\int_0^t V(\omega(r))dr}dP_W(\omega). \tag{1.2}$$

Denoting by $\eta_t^* = e^{-\frac{t}{\hbar}H}\chi^*$ the l.h.s. of (1.2), its r.h.s. is a legitimate path integral representation of the solution of the Cauchy problem in $L^2(\mathbb{R}^d)$:

$$\begin{cases} -\hbar\frac{\partial\eta_t^*}{\partial t} = H\eta_t^* \\ \eta_0^*(q) = \chi^*(q), \end{cases} \tag{1.3}$$

regarded as counterpart of Feynman's one for Schrödinger equation [10], resulting informally from the "Wick rotation" $t \to -it$ in (1.3). The representation (1.2) is the Euclidean (or imaginary time) viewpoint of Feynman's formula for the wave function ψ, too often considered as the only rigorous one. In relation with the physicists standard manipulations of paths integrals [26], however, the representation (1.2) is not so convincing. Indeed, in Feynman's framework, the configuration representation is only one of those where the path integral approach should apply successfully. What are the associated probabilistic counterparts of path integrals in momentum or energy representation, for example ? We will really be able to claim that the mathematical content of Feynman's approach is under control when a general probabilistic construction valid in any representation, and providing at once the existence of all underlying probability measures, will be available. This is, of course, still far from being the case.

Such a general construction would, presumably, also be a great advance to Stochastic Analysis itself [22] since it would provide as well new relations (Euclidean counterparts of quantum unitary transformations) between stochastic processes, or measures, usually regarded as unrelated in probability theory.

A probabilistic counterpart of Feynman's formula in configuration representation, distinct from (1.2), has been introduced in the mid-eighties (cf. [34], [6] and references therein). It is founded on the elementary observation that the Feynman-Kac formula is just a conditional expectation, a concept never defined in quantum mechanics where only absolute expectations appear, and in a very specific manner. Precisely, for the quantum system with the same Hamiltonian as before,

$$\int_A \psi_t(q)\overline{\psi}_t(q)dq, \tag{1.4}$$

is interpreted as the (unconditional or absolute) probability measure, for this system, to be in the Borelian A at time t, where $\overline{\psi}$ denotes the complex conjugate of the wave function ψ. Of course, the product form of the density in (1.4) is in fact independent on the representation.

Since various probabilistic interpretations of the solutions of (1.3) as conditional expectations are available, (1.4) suggests to look, among those, for a special class of diffusion processes whose absolute probability density is the product of positive solutions η_t^* of (1.3) and of positive solutions η_t of the equation

$$\begin{cases} \hbar\frac{\partial\eta_t}{\partial t} = H\eta_t \\ \\ \eta_v(q) = \chi(q), \end{cases} \tag{1.5}$$

adjoint to (1.3) with respect to the time parameter on a compact time interval, say $t \in I = [r, v]$. Such diffusions are indeed well defined and it can be shown that their qualitative properties are much closer to what is needed to understand Feynman's approach than those of (1.2), cf. [6], [7]. In particular, they are time-inhomogeneous but still time-reversible. The above mentioned product form of their probability at the time t becomes a mathematical expression of their reversibility (and their Markovian character, actually) since (1.5) can be interpreted as the time reversal of (1.3). In fact, the qualitative properties of these processes are so close to the quantum ones that they allow to understand, for example, fresh aspects of quantum symmetries in Hilbert space, generally ignored [20], [6].

The purpose of this paper is to show that the structure of this probabilistic construction* is preserved in momentum representation, suggesting that this structure is, somehow, independent of the representation. If this is indeed the case, a global mathematical picture of Feynman's path integral approach should be accessible, with a number of exciting consequences, both at the conceptual level of quantum physics and in the infinite dimensional analysis context of (Euclidean) Quantum Field Theory. In this infinite dimensional context, the need for such a unification has been known for a long time in physics and in mathematics (cf., for instance, [26], [13]).

As observed by Feynman [10], § 5.1, it is expected that in momentum representation, the underlying stochastic processes belong to a special class of jump processes. In other words, and in contrast with the configuration representation, the elements ω of the momentum path space cannot be made continuous, in general, but at best continuous on the right.

The organization of our paper is as follows. Section 2 is devoted to a summary of the relations between the Lévy-Khintchine representations, Lévy processes and pseudo-differential operators. Then we shall see how to construct the two adjoint equations corresponding, in momentum representation, to (1.3) and (1.5). What plays the role of the Hamiltonian operator H in (1.3) and (1.5), is now a pseudo-differential operator, denoted by $\hat{H}$, whose explicit form depends, of course, on the scalar potential V in (1.1).

Section 3 describes the construction of the new class of reversible diffusions with jumps $\hat{z}_t$, $t \in I$ (our "momentum process") whose absolute probability density

*EQM, or Euclidean Quantum Mechanics.

at any time $t \in I$ is a product $\hat{\eta}_t^* \hat{\eta}_t$ of positive solutions of the two adjoint equations above. One way to describe them is to give the two stochastic integro-differential equations solved by $\hat{z}_t$, $t \in I$, whose coefficients depend exclusively on the two positive solutions $\hat{\eta}_t^*$ and $\hat{\eta}_t$ on I (Euclidean counterparts of the momentum wave function and its complex conjugate). The dual aspect of this construction is fundamental. In particular, two families of σ-algebras (or "filtrations") are necessary, here, for the description of $\hat{z}_t$ in $I = [r, v]$. Let us recall that traditional constructions of stochastic processes require only one such filtration, the nondecreasing family $(\mathscr{P}_t)$, generated by $\hat{z}_t$, $t \in I$, and describing its past (more precisely the set of all events whose occurrence can result from the observation of $\hat{z}_s$, $r \leq s \leq t$). The information about $\mathscr{P}_t$ is contained in the solutions $\hat{\eta}_t^*$ of the Cauchy problem for $\hat{H}$, given a positive initial condition $\hat{\eta}_r^*$. But we will need as well a non-increasing filtration $\mathscr{F}_t$, describing the future of $\hat{z}_t$ on I. This will correspond to the information included in the solution $\hat{\eta}_t$ of the adjoint PDE, for a positive final condition $\hat{\eta}_v$. Another aspect of this duality can be expressed by saying that the resulting Markovian process $\hat{z}_t$, $t \in I = [r, v]$, is built from the data of two positive probability densities at the boundary ∂I of our time interval, say π_r and π_v, instead of the traditional initial probability density and a transition function. Our data of π_v is not contradictory, however, because $\hat{z}_t$ is the Markovian representative of a wider class of processes built from $\{\pi_r, \pi_v\}$, called Bernstein (or "local Markov", or "reciprocal", cf. [6], [7]) and satisfying the property

$$E[f(\hat{z}_t) \mid \mathscr{P}_s \vee \mathscr{F}_u] = E[f(\hat{z}_t) \mid \hat{z}_s, \hat{z}_u], \quad s < t < u, \tag{1.6}$$

for all f bounded measurable, where $E[\cdot \mid \mathcal{A}]$ denotes the conditional expectation given a σ-algebra $\mathcal{A}$. In other words, the knowledge of all the past $\mathscr{P}_s$ and the future $\mathscr{F}_u$ of the process is irrelevant to compute the conditional expectations; only the boundary values $\hat{z}_s$, $\hat{z}_u$ matter. Property (1.6) is more general than the Markov property but this one is sufficient for quantum physics, as shown by the product form of the integrand of (1.4).

Sections 4 and 5 describe the dynamics of the momentum process $\hat{z}_t$, $t \in I$. Here, again, we follow Feynman's approach. His (unpublished) Princeton PhD thesis was entitled "The least action principle in quantum mechanics" and this principle can be regarded as a stationary phase method in infinite dimension [1]. In our probabilistic context some methods of controlled Markov processes can be adapted to our purpose. They provide a (stochastic) action functional, and in particular a Lagrangian, whose critical points are precisely the diffusion process with jumps constructed in Section 3. The method exploits the maximum principle for a class of PDEs which can be interpreted as quantum deformations of the classical Hamilton-Jacobi equation in momentum representation. The equations of motion of Section 5 are the ones solved by the critical points of the action functional and provide the probabilistic version of the quantum Heisenberg equations in momentum representation. In this sense, the structure of the present construction is, indeed, the same as the one of our reinterpretation of Feynman's approach in configuration representation. In particular, the study of the symmetries of this framework (not done here) should have interesting surprises in store as in the configuration representation [6], [32]. A number of explicit examples illustrate our construction.

2 Lévy processes and pseudo-differential operators

We refer to the survey [15] and to the references therein for the notions summarized in this section. Let $V : \mathbb{R}^d \longrightarrow \mathbb{C}$ with $V(0) \geq 0$ and such that e^{-tV} is continuous positive definite. The function V admits the Lévy-Khintchine representation

$$V(q) = a + i\langle c, q\rangle + \frac{1}{2}\|q\|^2 - \int_{\mathbb{R}^d\setminus\{0\}} (e^{-i\langle q,k\rangle} + i\langle q, k\rangle 1_{\{|k|\leq 1\}} - 1)\nu(dk), \tag{2.1}$$

where $k \in \mathbb{R}^d$ is called the wave vector, $a, c \in \mathbb{R}^d$, $\langle \cdot, \cdot\rangle$ denotes the scalar product in $\mathbb{R}$ with norm $\|\cdot\|$, and ν is a Lévy measure on $\mathbb{R}^d\setminus\{0\}$, i.e. ν satisfies $\int_{\mathbb{R}^d}(\|k\|^2 \wedge 1)\nu(dk) < \infty$. See e.g. [3] and [28] for background on Lévy processes. A stochastic process ξ_t, $t \geq 0$, defined on a probability space $(\Omega, \mathscr{P}, P)$ is called a Lévy process if it has right continuous paths starting from the origin and its increments are independent and stationary:

a) $P(\xi_0 = 0) = 1$,

b) for all $0 \leq s \leq t$, $\xi_t - \xi_s$ is independent of $\mathscr{P}_s \subset \mathscr{P}$, the past filtration generated by all ξ_r, $r \leq s$,

c) for all $0 \leq s \leq t$, $\xi_t - \xi_s$ is equal in distribution to ξ_{t-s}.

The Lévy process ξ_t of characteristic exponent $V(q)$ is defined by

$$E\left[e^{-\frac{i}{\hbar}\langle \xi_t, q\rangle}\right] = e^{-\frac{t}{\hbar}V(q)}, \quad q \in \mathbb{R}^d, \tag{2.2}$$

so that, from now on, the Lévy process ξ_t will have the units of the momentum $p = \hbar k$ and V, of course, the ones of a (potential) energy. Lévy processes form a large class of Markov processes, including the two most commonly used in Mathematical Physics: Brownian motion and the Poisson process.

The $\mathbb{R}^d$-valued process ξ_t, $t \in \mathbb{R}_+$, admits the following canonical ("Lévy-Itô") decomposition:

$$\xi_t = ct + W_t^\hbar + \int_0^t \int_{\{|k|\geq 1\}} k\mu(dk, ds) + \int_0^t \int_{\{|k|\leq 1\}} k(\mu(dk, ds) - \nu(dk)ds), \tag{2.3}$$

where $W_t^\hbar = \hbar^{1/2} W_t$ is a Brownian motion with variance $\hbar$, $\mu(dk, ds)$ is a Poisson random measure (or "canonical jump measure") counting the jumps $\Delta\xi_s = \xi_s - \xi_{s^-}$ (where $\xi_{s^-} = \lim_{r \nearrow s} \xi_r$), namely

$$\mu(dk, ds) = \sum_{\Delta\xi_s \neq 0} \delta_{(\Delta\xi_s, s)}(dk, ds).$$

Notice that the jump process in (2.3) is independent on $W_t^\hbar$. The first Stieltjes integral in the decomposition (2.3) describes the sum of all large jumps (of size bigger than one) up to time t. It is called a "compound Poisson process" and is of bounded variation, but may have no finite moments. The Poisson random measure $\mu(dk, ds)$ is determined by its compensator

$$\nu(dk)ds := E[\mu(dk, ds)]. \tag{2.4}$$

As a function of t, the jump process

$$M_t = \int_0^t \int_{\{|k|\leq 1\}} k(\mu(dk, ds) - \nu(dk)ds) \tag{2.5}$$

in (2.3) (the "compensated sum of small jumps") is a $\mathscr{P}_t$-martingale, i.e. it satisfies $E[|M_t|] < \infty$, $t \geq 0$, and

$$E[M_t \mid \mathscr{P}_s] = M_s, \qquad a.s.,\ 0 \leq s \leq t. \tag{2.6}$$

Taking the (absolute) expectation, the martingale property (2.6) implies that $t \mapsto E[M_t]$ is constant. A physically interesting example of martingale follows from the definition (2.2) of the characteristic exponent of ξ_t, indeed

$$\exp\left(-\frac{1}{\hbar}(iq\xi_t + tV(q))\right), \qquad t > 0, \tag{2.7}$$

is a $\mathscr{P}_t$-martingale. It has been shown in Euclidean Quantum Mechanics that most martingales play, in point of fact, the role of constant of the motion (cf. [6], [20] and references therein).

Remarks

1) If $\hbar = 0$ the only possible term of unbounded variation paths in (2.3) is the process (2.5) of small jumps. A criterion for bounded variation paths is

$$\int_{\mathbb{R}^d} (|k| \wedge 1)\nu(dk) < \infty, \tag{2.8}$$

and, in this case, M_t can be decomposed as

$$M_t = \int_0^t \int_{\{|k|\le 1\}} k\mu(dk, ds) - t\int_{\{|k|\le 1\}} k\nu(dk), \qquad t > 0.$$

Condition (2.8) may be verified even when $\nu(\mathbb{R}^d) = \infty$, i.e. when there are infinitely many jumps in any compact time interval. It follows also clearly from (2.1)-(2.3) that the paths $t \mapsto \xi_t$ are continuous if and only $\nu = 0$.

2) A formal expression of the Lévy-Itô decomposition (2.3), used later on when ν is symmetric, is

$$\xi_t = ct + W_t^\hbar + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} k\mu(dk, ds) + \int_0^t \int_{\{|k|\le 1\}} k\mu(dk, ds), \tag{2.9}$$

but it should be stressed that although the integrals in (2.3) of large and small jumps are convergent, the last term in (2.9) does not make sense in general.

3) From (2.1) and (2.3) it is clear that, in such a framework, the Brownian motion W_t (in the purely diffusive case) has the characteristic exponent $V(q) = \frac{1}{2}\|q\|^2$. We could say as well that $W_t^\hbar$ corresponds to $V(q) = \frac{\hbar}{2}\|q\|^2$ but the first version is more natural for our purpose.

4) Compound Poisson process.

Let Z_n, $n \in \mathbb{N}$, denote a sequence of independent identically distributed $\mathbb{R}^d$-valued random variables with common probability law μ_Z on $\mathbb{R}^d \setminus \{0\}$, and let N_t be a Poisson process with intensity $\lambda > 0$, independent of all the Z_n, $n \in \mathbb{N}$. This means that $N_0 = 0$ and

$$P(N_t = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \qquad n \in \mathbb{N}.$$

Notice that the compensated Poisson process is $\tilde{N}_t = N_t - \lambda t$, so that $E[\tilde{N}_t] = 0$ and $E[\tilde{N}_t^2] = \lambda t$. The compound Poisson process is defined as

$$\xi_t = Z_1 + \cdots + X_{N_t}, \qquad t \in \mathbb{R}_+.$$

It is a Lévy process with Lévy measure $\nu = \lambda \mu_Z$. The Lévy-Khintchine representation reduces here to

$$V(q) = \lambda \int_{\mathbb{R}^d \setminus \{0\}} (e^{-iqk} - 1) \mu_Z(dk).$$

The paths of ξ_t are piecewise constant, and discontinuities occur only at random ("waiting") times

$$T_n = \inf\{t \geq 0 \ : \ \xi_t = n\}, \qquad n \geq 1,$$

with $T_0 = 0$, and the jump sizes are random within the range of the Z_n. For example, reducing $\mu_Z(dk)$, in the above Lévy-Khintchine formula, to $\delta_1(dk)$, we recover the above elementary Poisson process $\xi_t = N_t$, with jumps of size $+1$ at each T_n. Those T_n are gamma distributed (see e.g. [19]), i.e. their probability density has the form

$$\lambda e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!} 1_{[0,\infty)}(s).$$

5) One of the main difficulties in handling Lévy processes is that they can easily be of unbounded variation, i.e.

$$\sum_{0 \leq s \leq t} |\Delta \xi_s| = \infty, \qquad a.s. \tag{2.10}$$

However it is always true that

$$\sum_{0\leq s\leq t} |\Delta\xi_s|^2 < \infty,$$

a.s., and this second property allows to control the problems due to (2.10), see e.g. [2].

Let now $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform

$$\mathcal{F}u(p) = \frac{1}{(2\pi\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{i}{\hbar}\langle p,q\rangle} u(q)dq, \qquad u \in \mathcal{S}(\mathbb{R}^d),$$

and its inverse

$$\mathcal{F}^{-1}v(q) = \frac{1}{(2\pi\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}\langle p,q\rangle} v(p)dp, \qquad v \in \mathcal{S}(\mathbb{R}^d).$$

Given any (measurable) classical observable $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that $q \mapsto f(p,q)$ is continuous with polynomial growth, the pseudo-differential operator with symbol $f(p,q)$ is defined as

$$\begin{aligned} f(p, i\hbar\nabla)u(p) &= \frac{1}{(2\pi\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{i}{\hbar}\langle p,q\rangle} f(p,q)\mathcal{F}^{-1}u(q)dq \\ &= \mathcal{F}(f(p,\cdot)\mathcal{F}^{-1}u(\cdot))(p), \quad p \in \mathbb{R}^d, \quad u \in \mathcal{S}(\mathbb{R}^d). \end{aligned}$$

In particular, the pseudo-differential operator $V(i\hbar\nabla)$, for V as in (2.1), satisfies

$$V(i\hbar\nabla)u(p) = \\ au(p) - \hbar\langle c, \nabla u\rangle - \frac{\hbar^2}{2}\Delta u - \int_{\mathbb{R}^d} (u(p+\hbar k) - u(p) - \langle \hbar k, \nabla u(p)\rangle 1_{\{|k|\leq 1\}})\nu(dk).$$

Let $(P_t)_{t\in\mathbb{R}_+}$ denote the Markov semi-group associated with the Lévy process $(\xi_t)_{t\in\mathbb{R}_+}$. This means that

$$\begin{aligned} P_t u(p) &= E[u(p+\xi_t)] = E[(\mathcal{F}\mathcal{F}^{-1}u)(p+\xi_t)] \\ &= \frac{1}{(2\pi\hbar)^{d/2}} \int_{\mathbb{R}^d} E\left[\exp\left(-\frac{i}{\hbar}\langle \xi_t, q\rangle - \frac{i}{\hbar}\langle p,q\rangle\right)\right] \mathcal{F}^{-1}u(q)dq \\ &= \frac{1}{(2\pi\hbar)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{t}{\hbar}V(q) - \frac{i}{\hbar}\langle p,q\rangle\right) \mathcal{F}^{-1}u(q)dq \end{aligned}$$

$$= \exp\left(-\frac{t}{\hbar}V(i\hbar\nabla)\right)u(p), \qquad u \in \mathcal{S}(\mathbb{R}^d), \quad t \in \mathbb{R}_+,$$

hence the infinitesimal generator of the Markovian semigroup of $(\xi_t)_{t\in\mathbb{R}_+}$ is $-\frac{1}{\hbar}V(i\hbar\nabla)$. We are only interested in real-valued scalar potentials $V(q)$, i.e. the following conditions are assumed to be satisfied from now on:

H1) $c = 0$, and

H2) ν is symmetric with respect to $k \mapsto -k$.

According to (2.1)-(2.3), H1 says that our basic Lévy process has no constant drift and from H2 its measure ν is invariant under time reversal, as it should be for any Hamiltonian observable in the class considered now. We shall also assume that the parameter a is zero, i.e. $V(0) = 0$, without loss of generality from the physical point of view, since the energy is defined up to an additive constant. Hence, we shall restrict ourselves to V compatible with the Lévy-Khintchine representation

$$V(q) = \frac{1}{2}\|q\|^2 - \int_{\mathbb{R}^d\setminus\{0\}} (e^{-i\langle q,k\rangle} - 1)\nu(dk), \tag{2.11}$$

and therefore for all $u \in \mathcal{S}(\mathbb{R})$,

$$V(i\hbar\nabla)u(p) = -\frac{\hbar^2}{2}\Delta u(p) - \int_{\mathbb{R}^d} (u(p+k) - u(p))\nu(dk).$$

Notice that the conditions H1 and H2 imply that we could write as well

$$V(q) = \frac{1}{2}\|q\|^2 - \int_0^\infty (\cos(qk) - 1)\,\nu(dk).$$

Then we would have

$$V(i\hbar\nabla)u(p) = -\frac{\hbar^2}{2}\Delta u(p) - \frac{1}{2}\int_0^\infty (u(p+\hbar k) - 2u(p) + u(p-\hbar k))\nu(dk). \tag{2.12}$$

Consider any Hamiltonian H of the form

$$H = -\frac{\hbar^2}{2}\Delta + V(q) \tag{2.13}$$

and the associated Schrödinger equation in the position representation

$$i\hbar\frac{\partial\Phi}{\partial t}(q,t) = H\Phi(q,t) = -\frac{1}{2}\hbar^2\Delta\Phi(q,t) + V(q)\Phi(q,t).$$

The Euclidean version of this equation results from the substitution $t \mapsto it$:

$$\hbar \frac{\partial \eta_t}{\partial t}(q) = H\eta_t(q) = -\frac{1}{2}\hbar^2 \Delta \eta_t(q) + V(q)\eta_t(q), \qquad t \in [r, v]. \tag{2.14}$$

The equation adjoint to (2.14) is given by the substitution $t \mapsto -it$:

$$-\hbar \frac{\partial \eta_t^*}{\partial t}(q) = H\eta_t^*(q) = -\frac{1}{2}\hbar^2 \Delta \eta_t^*(q) + V(q)\eta_t^*(q), \qquad t \in [r, v]. \tag{2.15}$$

Let us define the Hamiltonian $\hat{H} = \mathcal{F} H \mathcal{F}^{-1}$ in momentum representation, i.e.

$$\begin{aligned}
\hat{H}\hat{\eta}_t(p) &= \mathcal{F}(H\mathcal{F}^{-1}\hat{\eta}_t)(p) \\
&= \frac{1}{2}\|p\|^2 \hat{\eta}_t(p) + \mathcal{F}(V\mathcal{F}^{-1}\hat{\eta}_t)(p) \\
&= \frac{1}{2}\|p\|^2 \hat{\eta}_t(p) + V(i\hbar\nabla)\hat{\eta}_t(p) \\
&= \frac{1}{2}\|p\|^2 \hat{\eta}_t(p) - \frac{\hbar^2}{2}\Delta\hat{\eta}_t(p) - \int_{\mathbb{R}^d} (\hat{\eta}_t(p + \hbar k) - \hat{\eta}_t(p))\nu(dk). \qquad (2.16)
\end{aligned}$$

Using this, the momentum representation of the Euclidean system described by (2.14) becomes

$$\hbar \frac{\partial \hat{\eta}_t}{\partial t}(p) = \hat{H}\hat{\eta}_t(p) = \frac{1}{2}\|p\|^2 \hat{\eta}_t(p) + V(i\hbar\nabla)\hat{\eta}_t(p), \tag{2.17}$$

and η, $\hat{\eta}$ are linked by the relation $\hat{\eta}_t = \mathcal{F}\eta_t$. Similarly for equation (2.15):

$$-\hbar \frac{\partial \hat{\eta}_t^*}{\partial t}(p) = \hat{H}\hat{\eta}_t^*(p) = \frac{1}{2}\|p\|^2 \hat{\eta}_t^*(p) + V(i\hbar\nabla)\hat{\eta}_t^*(p). \tag{2.18}$$

We call (somewhat improperly) integral kernel the kernel $\hat{h}(t, p, u, dl)$, $0 < s < t < u$, $p, l \in \mathbb{R}^d$, associated with:

$$e^{-\frac{1}{\hbar}(u-t)\hat{H}} f(p) = \int_{\mathbb{R}^d} f(l)\hat{h}(t, p, u, dl).$$

In particular, for $r \leq t \leq v$, we have

$$\hat{\eta}_t^*(p) = e^{-\frac{1}{\hbar}(t-r)\hat{H}} \hat{\eta}_r^*(p) = \int_{\mathbb{R}^d} \hat{\eta}_r^*(i)\hat{h}(r, i, t, p)\lambda(di),$$

and

$$\hat{\eta}_t(p) = e^{-\frac{1}{\hbar}(v-t)\hat{H}} \hat{\eta}_v(p) = \int_{\mathbb{R}^d} \hat{\eta}_v(m)\hat{h}(t, p, v, m)\lambda(dm).$$

The following proposition shows how to compute the kernel $\hat{h}(t, p, u, dl)$ starting from the law μ_t of the Lévy process ξ_t at time t.

Proposition 2.1 *For any* $0 < t < v$ *and* $p, l \in \mathbb{R}$ *we have*

$$h(t,p,u,dl) = \alpha(u-t,p,l)\mu_{u-t}(-p+dl), \tag{2.19}$$

where

$$\alpha(u-t,p,l) = E\left[e^{-\frac{1}{2\hbar}\int_0^{u-t}\|p+\xi_\tau\|^2 d\tau} \middle| \xi_{u-t} = l-p \right],$$

and $\mu_{u-t}(-p+dl)$ *denotes the image measure of* μ_{u-t} *under* $l \mapsto l+p$.

Proof. Since $-\frac{1}{\hbar}V(i\hbar\nabla)$ is the generator of $(\xi_t)_{t\in[0,u]}$, we have from the Feynman-Kac formula for Markov processes (see e.g. § III.19 of [27]):

$$\begin{aligned} e^{-\frac{1}{\hbar}(u-t)\hat{H}} f(p) &= E\left[f(\xi_u) e^{-\frac{1}{2\hbar}\int_t^u \|\xi_\tau\|^2 d\tau} \middle| \xi_t = p \right] \\ &= E\left[f(p+\xi_{u-t}) e^{-\frac{1}{2\hbar}\int_0^{u-t}\|p+\xi_\tau\|^2 d\tau} \right], \quad t < u, \end{aligned}$$

where we used the stationarity of the increments of ξ_t. So, by definition of the integral kernel $\hat{h}(t,p,u,dl)$ and the Feynman-Kac formula,

$$\begin{aligned} &\int_{\mathbb{R}^d} \hat{h}(t,p,u,dl) f(l) = e^{-\frac{1}{\hbar}(u-t)\hat{H}} f(p) \\ &= E\left[f(\xi_{u-t}+p) e^{-\frac{1}{2\hbar}\int_0^{u-t}\|p+\xi_\tau\|^2 d\tau} \right] \\ &= \int_{\mathbb{R}^d} E\left[f(\xi_{u-t}+p) e^{-\frac{1}{2\hbar}\int_0^{u-t}\|p+\xi_\tau\|^2 d\tau} \mid \xi_{u-t} = l \right] \mu_{u-t}(dl) \\ &= \int_{\mathbb{R}^d} f(p+l) E\left[e^{-\frac{1}{2\hbar}\int_0^{u-t}\|p+\xi_\tau\|^2 d\tau} \mid \xi_{u-t} = l \right] \mu_{u-t}(dl) \\ &= \int_{\mathbb{R}^d} f(l) E\left[e^{-\frac{1}{2\hbar}\int_0^{u-t}\|p+\xi_\tau\|^2 d\tau} \mid \xi_{u-t} = l-p \right] \mu_{u-t}(-p+dl). \end{aligned}$$

Consequently we obtain (2.19).

□

We now consider various one-dimensional examples.

<u>Examples</u>

1. $V(q) = (1-\cos(\alpha q))$, $\alpha > 0$.

This case corresponds to

$$\nu(dk) = \frac{1}{2}(\delta_\alpha(dk) + \delta_{-\alpha}(dk)).$$

in (2.1). By (2.16),

$$\hat{H}u(p) = \frac{1}{2}\|p\|^2 u(p) - \frac{1}{2}\Delta_\alpha u(p),$$

where Δ_α denotes the discretized Laplace operator

$$\Delta_\alpha u(p) = \frac{u(p+\alpha) - 2u(p) + u(p-\alpha)}{\alpha^2}.$$

The random jump measure is of the form

$$\mu(dk, ds) = \sum_{n=1}^{\infty} \delta_{(\alpha, T_n^1)}(dk, ds) + \delta_{(-\alpha, T_n^2)}(dk, ds)$$

where $(T_k^1)_{k\geq 1}$ and $(T_k^2)_{k\geq 1}$ are two independent sequences of Poissonian waiting times, with the same intensity $1/(2\alpha^2)$. The Lévy-Itô decomposition reduces here to the compensated sum of jumps. After introduction of ν and μ as above we find

$$\xi_t = \int_0^t \int_{\mathrm{R}} p(\mu(dk, ds) - \nu(dk)ds) = \alpha(N_t^1 - N_t^2),$$

where $(N_t^1)_{t\in\mathrm{R}_+}$ and $(N_t^2)_{t\in\mathrm{R}_+}$ are independent standard Poisson processes with intensity $1/(2\alpha^2)$ since (2.5) reduces to a sum, for $k = \pm\alpha$, of all jumps at the waiting times T_n^i, $i = 1, 2$, up to the time t. As an illustration, let us check the property (2.4) of the compensator, in this special case. For any integrable f, we have, since T_n^1 and T_n^2 have the same distribution (cf. Remark 2):

$$\begin{aligned}
&E\left[\int_0^t \int_{\mathrm{R}} f(k)\mu(dk, ds)\right] \\
&= E\left[\sum_{k=1}^{\infty} \int_0^t \int_{\mathrm{R}} f(k)\delta_{(\alpha, T_k^1)}(dk, ds) + \int_0^t \int_{\mathrm{R}} f(k)\delta_{(-\alpha, T_k^2)}(dk, ds)\right] \\
&= E\left[\sum_{k=1}^{\infty} f(\alpha)1_{\{T_k^1 \leq t\}} + f(-\alpha)1_{\{T_k^1 \leq t\}}\right] \\
&= (f(\alpha) + f(-\alpha)) \sum_{k=1}^{\infty} P(T_k^1 \leq t)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\alpha^2}(f(\alpha)+f(-\alpha))\sum_{k=1}^{\infty}\int_0^t e^{-s/(2\alpha^2)}\frac{(s/(2\alpha^2))^{k-1}}{(k-1)!}ds \\
&= \frac{t}{2\alpha^2}(f(\alpha)+f(-\alpha)) \\
&= \int_0^t\int_{\mathrm{R}} f(k)\nu(dk)ds,
\end{aligned}
$$

hence $E[\mu(dk,ds)]=\nu(dk)ds$, as claimed.

Here we can compute

$$
\begin{aligned}
e^{-\frac{1}{\hbar}t\hat{H}}f(p) &= E\left[f(p+\alpha(N_t^1-N_t^2))e^{-\frac{1}{2\hbar}\int_0^t\|p+\alpha(N_\tau^1-N_\tau^2)\|^2d\tau}\right] \\
&= e^{-t}\sum_{k,l\geq 0}\left(\frac{1}{2}\right)^{k+l} f(p+\alpha(k-l))\int_0^t\int_0^{t_k}\cdots\int_0^{t_2}\int_0^t\int_0^{s_l}\cdots\int_0^{s_2} \\
&\quad e^{-\frac{1}{2\hbar}\int_0^t\|p+\alpha(\sum_{i=1}^k i1_{[t_i,t]}(\tau)-\sum_{i=j}^l i1_{[s_j,t]}(\tau))\|^2d\tau}ds_1\cdots ds_l dt_1\cdots dt_k.
\end{aligned}
$$

It is generally hard to solve difference equations like (2.16) for $\hat{H}$ given before. As an illustration, let us observe that the solution of its time independent version, namely the stationary equation associated with (2.17):

$$
\frac{p^2}{2}u(p)-\frac{1}{2}(u(p+1)+u(p-1)-2u(p))=Eu(p),
$$

is given, for $u(0)=u_0$ and $u(1)=u_1$, by:

$$
\begin{aligned}
u(p) &= -\frac{(-2)^{p+1}(au_1-u_1\sqrt{a^2-4}+a^2u_0-au_0\sqrt{a^2-4}-2u_0)}{(a-\sqrt{a^2-4})^{p+2}\sqrt{a^2-4}} \\
&\quad+\frac{(-2)^{p+1}(au_1+u_1\sqrt{a^2-4}+a^2u_0+au_0\sqrt{a^2-4}-2u_0)}{(a+\sqrt{a^2-4})^{p+2}\sqrt{a^2-4}} \\
&\quad+\frac{(a+\sqrt{a^2-4})^{p+1}+(a-\sqrt{a^2-4})^{p+1}}{(-2)^{p+1}(a+2)^2}+\frac{a-(a+2)(p^2+8p+12)}{(a+2)^2},
\end{aligned}
$$

with $a=E+2$ (computation done using the command `rsolve` in Maple).

2. $V(q)=\frac{1}{2}\|q\|^2$.

This is the case $\nu(dk)=0$ in (2.1), and

$$
\hat{H}u(p)=\frac{1}{2}\|p\|^2u(p)-\frac{\hbar^2}{2}\Delta u(p).
$$

Clearly, the Lévy-Itô decomposition reduces to $\xi_t = W_t^\hbar$, i.e. the underlying Lévy process is a Brownian motion (with variance $\hbar$). Notice that this case can be regarded as the limit $\alpha \searrow 0$ of Example 1.

3. $V(q) = c\|q\|^\alpha$, $c > 0$, $\alpha \in (0, 2]$.

Then

$$\nu(dk) = \begin{cases} 0 & \text{when } \alpha = 2 \text{ (this is the case in Example 2)} \\ \frac{c_\alpha}{|k|^{1+\alpha}} dk, & \text{when } \alpha \neq 2, \end{cases}$$

where $c_\alpha > 0$ depends on the value of α. The associated Lévy process is called a "stable process of order α" (and α its "index of stability"). Recall that by (2.2), the existence of moments of order up to $n \geq 1$ for ξ_t implies the n-th differentiability of $V(q)$ at $q = 0$, the converse being true for moments of even order, see e.g. [31], Theorem 1, page 278. In particular, α-stable distributions on $\mathbb{R}$ have finite mean if and only if $\alpha > 1$.

Those processes are important because of their rotational invariance (in dimension $d > 1$) and also because they are self-similar. Recall that a stochastic process Z_t is self-similar with Hurst index $H > 0$ if, $\forall a \geq 0$, Z_{at} and $a^H Z_t$, $t \geq 0$, have same finite-dimensional distributions. For example, since $W_{at} = a^{1/2} W_t$, $a \geq 0$, (a scaling property which can be traced back to one of the "isovectors" of the symmetry group of the classical heat equation (1.5) for the free Hamiltonian $H_0 = -\frac{\hbar^2}{2}\Delta$, cf. [20]), the Brownian motion in (2.3) is self-similar, with Hurst index 1/2. We refer to [8] for more about self-similar processes. Coming back to the stable Lévy process ξ_t of order α, the Hamiltonian (2.16) becomes

$$\hat{H}u(p) = \frac{1}{2}\|p\|^2 u(p) - c\hbar^\alpha \Delta^{\alpha/2} u(p),$$

involving the fractional power of the Laplacian, a highly non-local operator, often linked with relativistic Hamiltonians, cf. [14].

4. $V(q) = \frac{1}{a^2} \log\cosh aq$, $a > 0$.

The Lévy measure is of the form

$$\nu(dk) = \frac{1}{2a^2}\frac{dk}{k\sinh(k\pi/(2a))},$$

see e.g. [5], [12][†], [29]. Notice that in the limit $a \searrow 0$ this measure reduces to 0, and the potential V becomes the harmonic one of Example 2. The associated process ξ_t is called Meixner process ([23]), and its Lévy-Itô decomposition has no diffusive part. Its expectation and variance are respectively given by $E[\xi_t] = 0$, $E[\xi_t^2] = t$, $t > 0$. The corresponding Hamiltonian is given by

$$\hat{H}u(p) = \frac{1}{2}\|p\|^2 u(p) - \frac{1}{2a^2}\int_{\mathbb{R}^d\setminus\{0\}} (u(p+\hbar k) - u(p))\frac{dk}{k\sinh(k\pi/(2a))}.$$

5. $V(q) = \frac{1}{a^2}\log(1 + a^2 q^2)$, $a > 0$.

   The Lévy measure

   $$\nu(dk) = \frac{1}{a^2|k|}e^{-|k|/a}dk$$

   is the one of the (symmetric) variance gamma process (cf., for instance, [21]), whose expectation and variance are, respectively, $E[\xi_t] = 0$, $E[\xi_t^2] = 2t$, $t > 0$. Notice that when the parameter a is large the exponential decay of $\nu(dk)$ is lower around zero and so the probability of large jumps increases. According to Remark 2, the paths of the variance gamma process are of bounded variation. This process has no continuous martingale component but is a pure jump process with infinite number of jumps in any compact time interval. Of course the associated Hamiltonian operator is given by

   $$\hat{H}u(p) = \frac{1}{2}\|p\|^2 u(p) - \frac{1}{a^2}\int_{\mathbb{R}\setminus\{0\}} (u(p+\hbar k) - u(p))\frac{e^{-|k|/a}}{|k|}dk.$$

   As in Example 4, this case reduces to the harmonic one when $a \searrow 0$.

In the sequel we make the absolute continuity hypothesis, for $t < u$, $p, l \in \mathbb{R}^d$

$$h(t,p,u,dl) = h(t,p,u,l)\lambda(dl), \qquad \lambda(dp) - a.e. \tag{2.20}$$

This condition is satisfied, for example, if:

[†]Note that in Eq. (2) of [12] as well as in Eq. (4) of [5], cosh should be replaced by sinh.

- the law of ξ_t, $t > 0$, has a density with respect to Lebesgue measure, e.g. in the case of stable processes (see Example 3) and for Lévy processes with Brownian component, or

- $\mu_{t-s}(-j + dp)$ has a density with respect to $\lambda(dp)$, $\lambda(dj)$-a.e. In particular this will follow from Proposition 2.1 if λ is absolutely continuous under the translation $p \mapsto j + p$, $\lambda(dj)$-a.e., and μ_{t-s} is absolutely continuous with respect to λ:

$$\mu_{t-s}(dp) = \mu_{t-s}(p)\lambda(dp).$$

  This is the case in particular for the symmetric Poisson process of Example 1 with $\lambda(dp) = \sum_{n=-\infty}^{\infty} \delta_{\alpha\times n}(dp)$.

Note that we have $h(s, j, t, p) = h(t, p, s, j)$, since $\hat{H}$ that is symmetric with respect to λ.

3 Momentum representation and Bernstein-Lévy processes

This section summarizes the existence results for Bernstein processes established in [24]. Let $\hat{\eta}_r^*, \hat{\eta}_v : \mathbb{R}^d \to \mathbb{R}_+$ be two λ-a.e. strictly positive initial and final conditions of (1.3) and (1.5), respectively, such that for some $t \in I = [r, v]$, and therefore for any such t,

$$\int_{\mathbb{R}^d} \hat{\eta}_t^*(p)\hat{\eta}_t(p)\lambda(dp) = 1. \tag{3.1}$$

As explained in the introduction, this relation will be interpreted as a Euclidean version of Born's probabilistic interpretation of the wave function in momentum representation. More precisely, and since no specific relation between $\hat{\eta}_t^*$ and $\hat{\eta}_t$ is needed for the last identity to make sense, (3.1) will be regarded as the Euclidean counterpart of the time invariance of any transition amplitudes in Feynman's approach. So the following result shows, in particular, that (quite in contrast with the quantum case !) this Euclidean (Born) probabilistic interpretation of the wave function in momentum representation is mathematically justifiable:

Theorem 3.1 *([24]) There exists a $\mathbb{R}^d$-valued process $(\hat{z}_t)_{t\in[r,v]}$ whose probability density at time t with respect to λ is precisely given by the product*

$$\rho_t(p) = \hat{\eta}_t^*(p)\hat{\eta}_t(p).$$

This process $(\hat{z}_t)_{t\in[r,v]}$ is both forward and backward Markovian, with forward transition probability kernel, for $r < s < t < u < v$ and $j, p, l \in \mathbb{R}^d$, given by

$$\hat{\mathrm{p}}(t,p,u,l) = \frac{\hat{\eta}_u(l)}{\hat{\eta}_t(p)}\hat{h}(t,p,u,l), \tag{3.2}$$

and backward transition probability kernel

$$\hat{\mathrm{p}}^*(s,j,t,p) = \frac{\hat{\eta}_s^*(j)}{\hat{\eta}_t^*(p)}\hat{h}(s,j,t,p). \tag{3.3}$$

In particular, the initial and final laws of $(\hat{z}_t)_{t\in[r,v]}$ are of the form

$$\pi_r(di) = \hat{\eta}_r(i)\hat{\eta}_r^*(i)\lambda(di) \quad \text{and} \quad \pi_v(dm) = \hat{\eta}_v(m)\hat{\eta}_v^*(m)\lambda(dm).$$

In fact, if $h(s,j,t,p)$ is continuous in (j,p) and strictly positive for all $0 < s < t$, Theorem 1 of [4] (see also Theorem 3.2 of [17], and Theorem 3.4 of [34]) state that given any two strictly positive probability densities $\pi_r(i)$ and $\pi_v(m)$, it is indeed possible to find two strictly positive functions $\hat{\eta}_r^*, \hat{\eta}_v : \mathbb{R}^d \to \mathbb{R}_+$ such that

$$\pi_r(i) = \hat{\eta}_r^*(i)\hat{\eta}_r(i), \qquad \pi_v(m) = \hat{\eta}_v(m)\hat{\eta}_v^*(m).$$

A posteriori, $\hat{\eta}_r^*$ and $\hat{\eta}_v$ can, therefore, be interpreted as (positive) initial and final boundary conditions of the two underlying adjoint equations (1.3) and (1.5). The resulting process $(\hat{z}_t)_{t\in[r,v]}$ is a (Markovian) Bernstein process (cf. [7] for example). As observed in (1.6) this means in particular that

$$P(\hat{z}_t \in dp \mid \mathscr{P}_s \vee \mathscr{F}_u) = P(\hat{z}_t \in dp \mid \hat{z}_s, \hat{z}_u),$$

and that the joint law $P(\hat{z}_r \in A,\ \hat{z}_v \in B)$, for A, B two Borelians of $\mathbb{R}^d$, has the special form

$$P(\hat{z}_r \in A,\ \hat{z}_v \in B) = \int_{A\times B} \hat{\eta}_r^*(i)h(r,i,v,m)\hat{\eta}_v(m)\lambda(di)\lambda(dm).$$

Conversely, we also have a uniqueness result, i.e. if $(\hat{z}_t)_{t\in[r,v]}$ is a Markovian Bernstein process with Bernstein kernel $h(s,j,t,dp,u,l) = P(\hat{z}_t \in dp \mid \hat{z}_s = j, \hat{z}_u = l)$ (cf. [7], [24]) such that

$$h(s,j,t,dp,u,l)h(s,j,u,l) = h(s,j,t,p)h(t,p,u,l)\lambda(dp),$$

$s \le t \le u$ and $j,p,l \in \mathbb{R}^d$, then there exist positive density functions $\hat{\eta}^*_r(i)$ and $\hat{\eta}_v(m)$ such that

$$P(\hat{z}_r \in A,\ \hat{z}_v \in B) = \int_{A\times B} \hat{\eta}^*_r(i)h(r,i,v,m)\hat{\eta}_v(m)\lambda(di)\lambda(dm),$$

see Theorem 7.1 in [24]. The processes $(\hat{z}_t)_{t\in[r,v]}$ resulting from the construction of Theorem 3.1 can be regarded as generalizations of the usual concept of Markovian bridges (cf., for example, [6] in the more familiar case of diffusion processes), which corresponds to Dirac boundary conditions at the boundary of the time interval $[r,v]$. In contrast, here, we allow for any (λ-a.e. strictly positive) probability densities π_r and π_v. We shall give a description of Markovian bridges of Theorem 3.1 in terms of forward and backward stochastic integro-differential equations driven by the Lévy process $(\xi_t)_{t\in[r,v]}$ of § 2. We assume that the conditions given in page 434 of [16] are fulfilled:

H3) the functions

$(t,p) \mapsto \int_{\mathrm{R}^d} (1 \wedge \|k\|^2) \frac{\hat{\eta}_t(p+\hbar k)}{\hat{\eta}_t(p)} \nu(dk),$

$(t,p) \mapsto \int_{\{|k|\le 1\}} k \frac{\hat{\eta}_t(p+\hbar k)-\hat{\eta}_t(p)}{\hat{\eta}_t(p)} \nu(dk),$

$(t,p) \mapsto \nabla \log \hat{\eta}_t(p),$

A*) as well as

$(t,p) \mapsto \int_{\mathrm{R}^d} (1 \wedge \|k\|^2) \frac{\hat{\eta}^*_t(p-\hbar k)}{\hat{\eta}^*_t(p)} \nu(dk),$

$(t,p) \mapsto \int_{\{|k|\le 1\}} k \frac{\hat{\eta}^*_t(p-\hbar k)-\hat{\eta}^*_t(p)}{\hat{\eta}^*_t(p)} \nu(dk),$

$(t,p) \mapsto \nabla \log \hat{\eta}^*_t(p),$

are bounded on compacts of $\mathbf{R}_+ \times \mathbf{R}^d$, Let us recall that, for stochastic equations, the analog of classical solutions of integro-differential equations is called a strong solution (in this case the solution is a process). It requires, indeed, strong regularity conditions on the coefficients of the equation, for example Lipschitz conditions for stochastic differential equations. When more general coefficients are needed, there is a concept of weak solution, where one looks for a unique process with the proper set of finite-dimensional distributions (cf. [16] for instance). Then the solution is a probability measure. Let z_{t^-}, resp. z_{t^+}, denote the left, respectively the right, limit of z_t at $t \in [r, v]$.

Proposition 3.2 *The process $(\hat{z}_t)_{t\in[r,v]}$ is solution, in the weak sense and with respect to the forward filtration $(\mathscr{P}_t)_{t\in[r,v]}$, of*

$$d\hat{z}_t = dW_t^{\hbar} + \int_{\mathbb{R}^d} k\mu(dk, dt) + \hbar\nabla \log \hat{\eta}_t(\hat{z}_{t^-})dt, \tag{3.4}$$

under a probability P for which $W_t^{\hbar}$ is a (forward) Brownian motion with variance $\hbar$, and $\mu(dk, ds)$ is the canonical point measure with compensator $\frac{\hat{\eta}_t(\hat{z}_{t^-}+\hbar k)}{\hat{\eta}_t(\hat{z}_{t^-})}\nu(dk)dt$.

In terms of backward differentials, $(\hat{z}_t)_{t\in[r,v]}$ solves weakly, with respect to the decreasing filtration $(\mathscr{F}_t)_{t\in[r,v]}$:

$$d_*\hat{z}_t = d_*W_t^{\hbar *} + \int_{\mathbb{R}^d} k\mu_*(dk, dt) - \hbar\nabla \log \hat{\eta}_t^*(\hat{z}_{t^+})dt, \tag{3.5}$$

where $W_t^{\hbar *}$ denotes a backward Brownian motion with variance $\hbar$, and $\mu_*(dk, dt)$ is the backward Poisson random measure with compensator $\frac{\hat{\eta}_t^*(\hat{z}_{t^+}-\hbar k)}{\hat{\eta}_t^*(\hat{z}_{t^+})}\nu(dk)dt$. In (3.4)-(3.5), $(\hat{z}_t)_{t\in[r,v]}$ represents the process (associated with our system in the momentum representation) whose probability density is of the form $\rho_t(p)$, as in Theorem 3.1. Let us define for $f \in \mathcal{S}(\mathbb{R}^d)$ and $g : \mathbb{R}^d \mapsto (0, \infty)$ the integro differential operator $\mathcal{L}_g$ by

$$\mathcal{L}_g f(p) = \frac{\hbar}{2}\Delta f(p) + \frac{1}{\hbar}\int_{\mathbb{R}^d} (f(p + \hbar k) - f(p))\frac{g(p + \hbar k)}{g(p)}\nu(dk) + \hbar\langle \nabla \log g(p), \nabla f(p)\rangle. \tag{3.6}$$

The proof of Proposition 3.2 relies on the following lemma, (cf. [24]), which shows that the process $(\hat{z}_t)_{t\in I}$ constructed in Theorem 3.1 has respectively $\mathcal{L}_{\hat{\eta}_t}$ and $-\mathcal{L}_{\hat{\eta}_t^*}$ as forward and backward infinitesimal generators. The knowledge of the generators of $(\hat{z}_t)_{t\in I}$ provides the forward and backward representations (3.4), (3.5) of $(\hat{z}_t)_{t\in I}$

as weak solutions of stochastic integro-differential equations, using Theorem 13.58, Theorem 14.80 of [16], p. 438 and p. 481, and references therein.

Proposition 3.3 *The kernels* $\hat{\mathrm{p}}(t,p,u,l)$ *and* $\hat{\mathrm{p}}^*(s,j,t,p)$ *defined in* (3.2) *and* (3.3) *satisfy the partial integro-differential (Kolmogorov forward or Fokker-Planck) equations*

$$\frac{\partial \hat{\mathrm{p}}}{\partial u}(t,p,u,l) = (\mathcal{L}_{\hat{\eta}_u})_l^\dagger \hat{\mathrm{p}}(t,p,u,l) \tag{3.7}$$

where $\dagger$ *denotes the adjoint,* $\mathcal{L}_{\hat{\eta}_s}$ *is the forward generator given by*

$$\mathcal{L}_{\hat{\eta}_s} f = \frac{\hbar}{2}\Delta f(p) + \frac{1}{\hbar}\int_{\mathrm{R}^d}(f(p+\hbar k) - f(p))\frac{\hat{\eta}_t(p+\hbar k)}{\hat{\eta}_t(p)}\nu(dk) + \hbar\langle\nabla\log\hat{\eta}_t(p),\nabla f(p)\rangle, \tag{3.8}$$

and

$$\frac{\partial \hat{\mathrm{p}}^*}{\partial s}(s,j,t,p) = -(\mathcal{L}_{\hat{\eta}_s^*})_j^\dagger \hat{\mathrm{p}}(s,j,t,p),$$

where $-\mathcal{L}_{\hat{\eta}_s^*}$ *is the backward generator given by*

$$-\mathcal{L}_{\hat{\eta}_s^*} f = -\frac{\hbar}{2}\Delta f(p) - \frac{1}{\hbar}\int_{\mathrm{R}^d}(f(p+\hbar k) - f(p))\frac{\hat{\eta}_t^*(p+\hbar k)}{\hat{\eta}_t^*(p)}\nu(dk) - \hbar\langle\nabla\log\hat{\eta}_t^*(p),\nabla f(p)\rangle. \tag{3.9}$$

Let the forward and backward derivative operators D_t and D_t^* be defined informally, on two appropriate domains of real-valued functions f, in terms of the Hamiltonian H and two positive solutions $\hat{\eta}_t$, $\hat{\eta}_t^*$ of (2.17) and (2.18) by

$$D_t f = \frac{1}{\hat{\eta}_t}\left(\frac{\partial}{\partial t} - \frac{1}{\hbar}\hat{H}\right)(\hat{\eta}_t f)$$

and

$$D_t^* f = \frac{1}{\hat{\eta}_t^*}\left(\frac{\partial}{\partial t} + \frac{1}{\hbar}\hat{H}\right)(\hat{\eta}_t^* f).$$

Proposition 3.3 is proved using the following decompositions of D_t and D_t^*, which are straightforward to verify:

$$D_t = \frac{\partial}{\partial t} + \mathcal{L}_{\hat{\eta}_t} \quad \text{and} \quad D_t^* = \frac{\partial}{\partial t} - \mathcal{L}_{\hat{\eta}_t^*}. \tag{3.10}$$

Let us observe that for constant f, in particular, these two derivative operators are zero by definition. But many non trivial functions $f : \mathbb{R}^d \times I \to \mathbb{R}$ in these domains have the same property. For instance let $(\tilde{\eta}_t)_{t\in[r,v]}$ denote a positive solution of (2.17) on I, distinct from $(\hat{\eta}_t)_{t\in[r,v]}$. Then clearly we have $D_t f = 0$ as well when $f = \tilde{\eta}_t/\hat{\eta}_t$.

4 Variational characterization and Lagrangian

In this section we use the approach to stochastic control for jump processes of [30], [11], to obtain a variational characterization of the Markovian Bernstein processes (or reversible diffusions with jumps) considered before. We consider the stochastic control problem $\inf_{\hat{S}} J(t,p;\hat{S})$ with action functional

$$J(t,p;\hat{S}) = E_{(t,p)}\left[\int_t^v \mathscr{L}(\hat{z}(s),\hat{S}_s)ds - \hbar\log\hat{\eta}_v(\hat{z}(v))\right], \tag{4.1}$$

where $E_{(t,p)}$ denotes the conditional expectation given $\{\hat{z}_t = p\}$, the infimum is taken on the set of control variables made of all measurable scalar functions $\hat{S}_\cdot : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$, and the integrand $\mathscr{L}(p,\hat{S}_s)$ is defined as

$$\begin{aligned}
\mathscr{L}(p,\hat{S}_s) &= \mathcal{L}_{\exp(\hat{S}_s/\hbar)}\hat{S}_s(p) + e^{-\hat{S}_s(p)/\hbar}\hat{H}e^{\hat{S}_s(p)/\hbar} \\
&= \frac{1}{2}\|p\|^2 + \frac{1}{2}\left\|\nabla\hat{S}_s(p)\right\|^2 \\
&\quad + \int_{\mathrm{R}^d}\left(1+\left(\frac{\hat{S}_s(p+\hbar k)-\hat{S}_s(p)}{\hbar}-1\right)e^{(\hat{S}_s(p+\hbar k)-\hat{S}_s(p))/\hbar}\right)\nu(dk) \\
&= \frac{1}{2}\|p\|^2 + \frac{1}{2}\left\|\nabla\hat{S}_s(p)\right\|^2 + \int_{\mathrm{R}^d}\frac{\hat{S}_s(p+\hbar k)-\hat{S}_s(p)}{\hbar}e^{(\hat{S}_s(p+\hbar k)-\hat{S}_s(p))/\hbar}\nu(dk) \\
&\quad - \int_{\mathrm{R}^d}\left(e^{(\hat{S}_s(p+\hbar k)-\hat{S}_s(p))/\hbar}-1\right)\nu(dk).
\end{aligned}$$

In particular, when $\hat{S}_s = \hbar\log\hat{\eta}_s$,

$$\begin{aligned}
\mathscr{L}(p,\hbar\log\hat{\eta}_t) &= \hbar\mathcal{L}_{\hat{\eta}_t}\log\hat{\eta}_t(p) + \frac{1}{\hat{\eta}_t(p)}\hat{H}\hat{\eta}_t(p) \\
&= \hbar\mathcal{L}_{\hat{\eta}_t}\log\hat{\eta}_t(p) + \frac{\hbar}{\hat{\eta}_t(p)}\frac{\partial}{\partial t}\hat{\eta}_t(p) \\
&= \hbar\mathcal{L}_{\hat{\eta}_t}\log\hat{\eta}_t(p) + \hbar\frac{\partial}{\partial t}\log\hat{\eta}_t(p) \\
&= \hbar D_t\log\hat{\eta}_t(p).
\end{aligned}$$

In other terms we have

$$\begin{aligned}
\mathscr{L}(p,\hbar\log\hat{\eta}_t) &= \frac{1}{2}\|p\|^2 + \frac{\hbar^2}{2}\|\nabla\log\hat{\eta}_t(p)\|^2 \\
&\quad + \int_{\mathrm{R}^d}(\log\hat{\eta}_t(p+\hbar k)-\log\hat{\eta}_t(p))\frac{\hat{\eta}_t(p+\hbar k)}{\hat{\eta}_t(p)}\nu(dk)
\end{aligned}$$

$$-\int_{\mathbb{R}^d} \frac{\hat\eta_t(p+\hbar k)-\hat\eta_t(p)}{\hat\eta_t(p)}\nu(dk).$$

Now let us observe that

$$D_t p = \mathcal{L}_{\hat\eta_t} p = \hbar\nabla\log\hat\eta_t(p) + \int_{\mathbb{R}^d} k\frac{\hat\eta_t(p+\hbar k)}{\hat\eta_t(p)}\nu(dk).$$

From now on, we will denote by $L(p, D_t p)$ and call Lagrangian, the integrand $\mathscr{L}(p, \hbar\log\hat\eta_t)$ of the action functional, when re-expressed in terms of the variables p, $D_t p$. We have, using (2.12) and the expression of $D_t p$:

$$\begin{aligned} L(p, D_t p) &= \frac{1}{2}\|p\|^2 + \hbar\left\langle\frac{\nabla\hat\eta_t}{\hat\eta_t}, D_t p\right\rangle + \frac{V(i\hbar\nabla)\hat\eta_t}{\hat\eta_t} + \frac{\hbar^2}{2}\frac{\Delta\hat\eta_t}{\hat\eta_t} - \frac{\hbar^2}{2}\left\|\frac{\nabla\hat\eta_t}{\hat\eta_t}\right\|^2 \\ &+ \int_{\mathbb{R}^d}(\log\hat\eta_t(p+\hbar k) - \log\hat\eta_t(p))\frac{\hat\eta_t(p+\hbar k)}{\hat\eta_t(p)}\nu(dk) \\ &-\hbar\int_{\mathbb{R}^d}\langle k, \nabla\log\hat\eta_t(p)\rangle\frac{\hat\eta_t(p+\hbar k)}{\hat\eta_t(p)}\nu(dk). \end{aligned}$$

By Taylor's formula applied inside the integral term we obtain the Lagrangian

$$\begin{aligned} L(p, D_t p) &= \frac{1}{2}\|p\|^2 + \hbar\left\langle\frac{\nabla\hat\eta_t}{\hat\eta_t}, D_t p\right\rangle + \frac{V(i\hbar\nabla)\hat\eta_t}{\hat\eta_t} \\ &+ \frac{\hbar^2}{2}\Delta\log\hat\eta_t(p)\int_{\mathbb{R}^d}(1+\|k\|^2+o(\hbar^3))\frac{\hat\eta_t(p+\hbar k)}{\hat\eta_t(p)}\nu(dk). \end{aligned} \tag{4.2}$$

Let us observe that the action functional for $\hat S_s = \hbar\log\hat\eta_s$ can be expressed in various equivalent ways:

$$\begin{aligned} J(t,p;\hat S) &= E_{(t,p)}\left[\int_t^v D_s\hat S_s(\hat z_s)ds - \hbar\log\hat\eta_v(\hat z(v))\right] \\ &= E_{(t,p)}\left[\int_t^v \mathscr{L}(\hat\eta_t(\hat z_s), \hat S_s)ds\right] - E_{(t,p)}\left[\hbar\log\hat\eta_v(\hat z_v)\right] \\ &= E_{(t,p)}\left[\hat S_v(\hat z_v) - \hat S_t(p) - \hbar\log\hat\eta_v(\hat z(v))\right] \\ &= E_{(t,p)}\left[\int_t^v \circ d\hat S_s(\hat z_s) - \hbar\log\hat\eta_v(\hat z(v))\right] \\ &= E_{(t,p)}\left[\int_t^v \nabla\hat S_s(\hat z_s)\circ d\hat z_s + \int_t^v\int_{\mathbb{R}^d}(\hat S_s(\hat z_{s^-}+\hbar k) - \hat S_s(\hat z_{s^-}))\mu(dk,ds)\right. \\ &\left. + \int_t^v \frac{\partial}{\partial s}\hat S_s(\hat z_s)ds - \hbar\log\hat\eta_v(\hat z(v))\right] \\ &= E_{(t,p)}\left[\int_t^v \nabla\hat S_s(\hat z_s)\circ d\hat z_s + \int_t^v\int_{\mathbb{R}^d}\hat S_s(\hat z_{s^-}+\hbar k)\nu(dk)ds\right. \end{aligned}$$

$$+\int_t^v \frac{\partial}{\partial s}\hat{S}_s(\hat{z}_s)ds - \hbar \log \hat{\eta}_v(\hat{z}(v))\Bigg],$$

where we have used the extension of Itô's calculus to our diffusions with jumps. The symbol $\circ$ denotes the Fisk-Stratonovich differential (cf. e.g. [25]). Let us show that the diffusion processes with jumps constructed before can also be regarded as minima of a stochastic action functional associated with the starting $\hat{H}$. The infima are taken on all measurable functions $\hat{S}_t : \mathbb{R}^d \to \mathbb{R}$.

Proposition 4.1 *The dynamic programming equation with final boundary condition*

$$\frac{\partial A_t}{\partial t}(p) + \min_{\hat{S}_t}\left(\mathcal{L}_{\exp(\hat{S}/\hbar)}A_t(p) + L(p,\hat{S}_t)\right) = 0, \quad A_v = -\hbar \log \hat{\eta}_v, \tag{4.3}$$

associated with the action functional (4.1) *is equivalent to the Hamilton-Jacobi-Bellman equation*

$$\begin{aligned}\frac{\partial A_t}{\partial t}(p) &= -\frac{1}{2}\|p\|^2 - \frac{\hbar}{2}\Delta A_t(p) + \frac{1}{2}\|\nabla A_t(p)\|^2 \\ &\quad + \int_{\mathrm{R}^d}(e^{-\hbar^{-1}(A_t(p+\hbar k)-A_t(p))} - 1)d\nu(k), \quad A_v = -\hbar\log\hat{\eta}_v,\end{aligned} \tag{4.4}$$

with solution $A_t = -\hbar \log \hat{\eta}_t$, $r \le t \le v$. Moreover, in (4.3)*, the minimum in $\hat{S}_t$ is attained on $\hat{S}_t = \hbar \log \hat{\eta}_t$.*

Proof. Given $\hat{S}_t$ and A_t two suitable functions, let us define

$$\begin{aligned}F(p,k) &= \frac{A_t(p+\hbar k) - A_t(p)}{\hbar}e^{(\hat{S}_t(p+\hbar k)-\hat{S}_t(p))/\hbar} + \frac{\hat{S}_t(p+\hbar k) - \hat{S}_t(p)}{\hbar}e^{(\hat{S}_t(p+\hbar k)-\hat{S}_t(p))/\hbar} \\ &\quad -e^{(\hat{S}_t(p+\hbar k)-\hat{S}_t(p))/\hbar} + e^{-(A_t(p+\hbar k)-A_t(p))/\hbar}.\end{aligned}$$

We have

$$\begin{aligned}&\mathscr{L}(p,\hat{S}_t) + \mathcal{L}_{\exp(\hat{S}_t/\hbar)}A_t(p) - e^{A_t(p)/\hbar}\hat{H}e^{-A_t(p)/\hbar} \\ &\quad = \mathcal{L}_{\exp(\hat{S}_t/\hbar)}(A_t(p) + \hat{S}_t(p)) + e^{-\hat{S}_t(p)/\hbar}\hat{H}e^{\hat{S}_t(p)/\hbar} - e^{A_t(p)/\hbar}\hat{H}e^{-A_t(p)/\hbar} \\ &\quad = \hbar\int_{\mathrm{R}^d}F(p,k)\nu(dk) + \frac{1}{2}\|\nabla A_t(p) + \nabla\hat{S}_t(p)\|^2 \\ &\quad \ge \hbar\int_{\mathrm{R}^d}F(p,k)\nu(dk),\end{aligned}$$

Now, for all $a > 0$,

$$\min_{x\in\mathrm{R}}(xa + a\log a - a + e^{-x}) = 0,$$

hence taking $x=(A_t(p+\hbar k)-A_t(p))/\hbar$ and $a=e^{(\hat{S}_t(p+\hbar k)-\hat{S}_t(p))/\hbar}$, we have $F(p,k)\geq 0$, and

$$\mathscr{L}(p,\hat{S}_t)+\mathcal{L}_{\exp(\hat{S}_t/\hbar)}A_t(p)-e^{A_t(p)/\hbar}\hat{H}e^{-A_t(p)/\hbar}\geq 0.$$

the minimum (zero) being attained for $\hat{S}_t=-A_t$, i.e.:

$$\min_{\hat{S}_t}\left(L(p,\hat{S}_t)+\mathcal{L}_{\exp(\hat{S}_t/\hbar)}A_t\right)=e^{A_t(p)/\hbar}\hat{H}e^{-A_t(p)/\hbar}.$$

The dynamic programming equation (4.3) can be formulated as

$$\frac{\partial A_t}{\partial t}+e^{A_t/\hbar}\hat{H}e^{-A_t/\hbar}=0,$$

and its solution is $A_t=-\hbar\log\hat{\eta}_t$. Finally, from the relation

$$\frac{1}{\hbar}\Delta A_t=-e^{A_t/\hbar}\Delta e^{-A_t/\hbar}+\frac{1}{\hbar^2}\left\|\nabla A_t\right\|^2,$$

we have

$$\begin{aligned}
&e^{A_t(p)/\hbar}\hat{H}e^{-A_t(p)/\hbar}\\
&= e^{A_t(p)/\hbar}\left(\frac{1}{2}\|p\|^2e^{-A_t(p)/\hbar}-\frac{\hbar}{2}\Delta e^{-A_t(p)/\hbar}-\int_{\mathrm{R}^d}(e^{-(A_t(p+\hbar k)-A_t(p))/\hbar})\nu(dk)\right)\\
&= \frac{1}{2}\|p\|^2+\frac{\hbar}{2}\Delta A_t-\frac{1}{2}\|\nabla A_t\|^2-\int_{\mathrm{R}^d}(e^{-\hbar^{-1}(A_t(p+\hbar k)-A_t(p))}-1)\nu(dk),
\end{aligned}$$

which yields (4.4). □

In the backward case we consider the action functional time reversed of (4.1):

$$J^*(t,p;\hat{S}^*)=E_{(t,p)}\left[\int_r^t\mathscr{L}(\hat{z}(s),\hat{S}^*)ds-\hbar\log\hat{\eta}_r^*(\hat{z}(r))\right]. \tag{4.5}$$

Similarly when $\hat{S}_s=\hbar\log\hat{\eta}_s^*$ we verify that

$$\mathscr{L}(p,\hbar\log\hat{\eta}_s^*)=-\hbar D_s^*\log\hat{\eta}_s^*(p),$$

where we have used (3.10) and the backward generator (3.9). Then

$$\begin{aligned}
&J^*(t,p;\hat{S}^*)\\
&= -E_{(t,p)}\left[\int_r^t D_s^*\hat{S}_s^*(\hat{z}_s^*)ds+\hbar\log\hat{\eta}_r^*(\hat{z}(r))\right]
\end{aligned}$$

$$
\begin{aligned}
&= E_{(t,p)}\left[\hat{S}^*_r(\hat{z}_r) - \hat{S}^*_t(\hat{z}_t) - \hbar \log \hat{\eta}^*_r(\hat{z}(r))\right] \\
&= -E_{(t,p)}\left[\int_r^t \circ d\hat{S}^*_s(\hat{z}^*_s) + \hbar \log \hat{\eta}^*_r(\hat{z}(r))\right] \\
&= -E_{(t,p)}\left[\int_r^t \nabla \hat{S}^*_s(\hat{z}_s) \circ d_* \hat{z}^*_s + \int_r^t \int_{\mathrm{R}^d} (\hat{S}^*_s(\hat{z}_{s+} + \hbar k) - \hat{S}^*_s(\hat{z}_{s+}))\mu_*(dk, ds)\right. \\
&\quad \left. + \int_r^t \frac{\partial}{\partial s}\hat{S}^*_s(\hat{z}^*_s) ds + \hbar \log \hat{\eta}^*_r(\hat{z}(r))\right] \\
&= -E_{(t,p)}\left[\int_r^t \nabla \hat{S}^*_s(\hat{z}_s) \circ d_* \hat{z}^*_s + \int_r^t \int_{\mathrm{R}^d} \hat{S}^*_s(\hat{z}_{s+} + \hbar k)\nu(dk) ds\right. \\
&\quad \left. + \int_r^t \frac{\partial}{\partial s}\hat{S}^*_s(\hat{z}^*_s) ds + \hbar \log \hat{\eta}^*_r(\hat{z}(r))\right].
\end{aligned}
$$

With respect to the underlying filtration $\mathscr{F}_t$, the Lagrangian now takes the form

$$
\begin{aligned}
L(p, D^*_t p) &= \frac{1}{2}\|p\|^2 - \hbar \left\langle \frac{\nabla \hat{\eta}^*_t}{\hat{\eta}^*_t}, D^*_t p \right\rangle + \frac{V(i\hbar\nabla)\hat{\eta}^*_t}{\hat{\eta}^*_t} \\
&\quad - \frac{\hbar^2}{2}\Delta \log \hat{\eta}^*_t(p) \int_{\mathrm{R}^d} (1 + \|k\|^2 + o(\hbar^3)) \frac{\hat{\eta}^*_t(p + \hbar k)}{\hat{\eta}^*_t(p)} \nu(dk),
\end{aligned} \tag{4.6}
$$

and the following backward version of Proposition 4.1 holds true:

Proposition 4.2 *The backward dynamic programming equation with initial boundary condition*

$$
\frac{\partial A^*_t}{\partial t}(p) + \min_{\hat{S}^*_t} \left(\mathcal{L}_{\exp(\hat{S}^*/\hbar)} A^*_t(p) + L(p, \hat{S}^*_t)\right) = 0, \qquad A^*_r = -\hbar \log \hat{\eta}^*_r, \tag{4.7}
$$

associated with (4.5), *is equivalent to the backward Hamilton-Jacobi-Bellman equation*

$$
\begin{aligned}
\frac{\partial A^*_t}{\partial t}(p) &= \frac{1}{2}\|p\|^2 + \frac{\hbar}{2}\Delta A^*_t(p) - \frac{1}{2}\|\nabla A^*_t(p)\|^2 \\
&\quad - \int_{\mathrm{R}^d} (e^{-\hbar^{-1}(A^*_t(p+\hbar k) - A^*_t(p))} - 1) d\nu(k), \quad A^*_r = -\hbar \log \hat{\eta}^*_r,
\end{aligned} \tag{4.8}
$$

with solution $A^*_t = -\hbar \log \hat{\eta}^*_t$, $r \leq t \leq v$. *Moreover, in* (4.7), *the minimum in* $\hat{S}^*_t$ *is attained at* $\hat{S}^*_t(p) = \hbar \log \hat{\eta}^*_t(p)$.

Proof. Of course, Propositions 4.1 and 4.2 will not be formally used. The key point is that $\hat{z}_t$, $t \in I$, can be regarded as critical point of a stochastic variational principle. Proposition 4.1 is sufficient since $\hat{z}_t$, $t \in I$, is time reversible in the above mentioned sense. But it is illustrative to show that its critical property takes two slightly different forms with respect to the filtrations $\mathscr{P}_t$ and $\mathscr{F}_t$. So we summarize the $\mathscr{F}_t$ proof only for completeness. We first show that

$$\min_{\hat{S}^*_t}\left(\mathcal{L}^*_{\exp(\hat{S}^*_t/\hbar)}A^*_t(p)+L(p,\hat{S}^*_t)\right)=-e^{A^*_t/\hbar}\hat{H}e^{-A^*_t/\hbar}, \tag{4.9}$$

and the minimum is attained for $\hat{S}^*_t = A^*_t$. Let

$$\begin{aligned}F^*(p,k) &= e^{(\hat{S}^*_t(p-\hbar k)-\hat{S}^*_t(p))/\hbar}\frac{A^*_t(p+\hbar k)-A^*_t(p)}{\hbar}+e^{(\hat{S}^*_t(p-\hbar k)-\hat{S}^*_t(p))/\hbar}\frac{f^*_t(p+\hbar k)-f^*_t(p)}{\hbar}\\ &\quad -e^{(\hat{S}^*_t(p-\hbar k)-\hat{S}^*_t(p))/\hbar}+e^{-(A^*_t(p-\hbar k)-A^*_t(p))/\hbar}.\end{aligned}$$

We have

$$\begin{aligned}&L(p,\hat{S}^*)+\mathcal{L}^*_{\exp(\hat{S}^*_t/\hbar)}A^*_t(p)-e^{A^*_t(p)/\hbar}\hat{H}e^{-A^*_t(p)/\hbar}\\ &= \hbar\int_{\mathrm{R}^d}F^*(p,k)\nu(dk)+\frac{1}{2}\|\nabla A^*_t(p)+\nabla\hat{S}^*_t(p)\|^2\\ &\geq \hbar\int_{\mathrm{R}^d}F^*(p,k)\nu(dk),\end{aligned}$$

Proceeding as in the forward case we obtain (4.9) and the dynamic programming equation (4.3) becomes

$$\frac{\partial A^*_t}{\partial t}-e^{A^*_t/\hbar}\hat{H}e^{-A^*_t/\hbar}=0,$$

with solution $A^*_t=-\hbar\log\hat{\eta}^*_t$. Finally we have

$$\begin{aligned}&e^{A^*_t(p)/\hbar}\hat{H}e^{-A^*_t(p)/\hbar}\\ &= \frac{1}{2}\|p\|^2-e^{A^*_t(p)/\hbar}\left(\frac{\hbar^2}{2}\Delta e^{-A^*_t(p)/\hbar}+\int_{\mathrm{R}^d}(e^{-A^*_t(p+\hbar k)/\hbar}-e^{-A^*_t(p)/\hbar})\nu(dk)\right)\\ &= \frac{1}{2}\|p\|^2+\frac{\hbar}{2}\Delta A^*_t(p)-\frac{1}{2}\|\nabla A^*_t(p)\|^2-\int_{\mathrm{R}^d}(e^{-\hbar^{-1}(A^*_t(p+\hbar k)-A^*_t(p))}-1)\nu(dk),\end{aligned}$$

which yields (4.8). □

5 Equations of motion

We now derive a.s. equations of motion associated with $(\hat{z}_t)_{t\in I}$. Here, $(\hat{z}_t)_{t\in[r,v]}$ represents the process associated to our system in momentum representation, and the expectations of the almost sure equations of motion can be interpreted as the probabilistic counterpart of the Ehrenfest Theorem in quantum dynamics. The forward and backward derivatives defined before as the generators $D_t = \frac{\partial}{\partial t} + \mathcal{L}_{\hat{\eta}_t}$ and $D_t^* = \frac{\partial}{\partial t} - \mathcal{L}_{\hat{\eta}_t^*}$ have natural probabilistic interpretations as the following limits of conditional expectations, for f regular enough

$$D_t f_t(\hat{z}_t) = \lim_{\Delta t \downarrow 0} E\left[\frac{f_{t+\Delta t}(\hat{z}_{t+\Delta t}) - f_t(\hat{z}_t)}{\Delta t} \Big| \mathscr{P}_t \right] = E\left[\frac{d}{dt^+} f_t(\hat{z}_t) \Big| \mathscr{P}_t \right], \tag{5.1}$$

and

$$D_t^* f_t(\hat{z}_t) = \lim_{\Delta t \downarrow 0} E\left[\frac{f_t(\hat{z}_t) - f_{t-\Delta t}(\hat{z}_{t-\Delta t})}{\Delta t} \Big| \mathscr{F}_t \right] = E\left[\frac{d}{dt^-} f_t(\hat{z}_t) \Big| \mathscr{F}_t \right], \tag{5.2}$$

where $\frac{d}{dt^+} f$, $\frac{d}{dt^-} f$ denote the right hand side and left hand side derivatives corresponding to the formal limits of (5.1) and (5.2) when Planck's constant $\hbar$ is equal to 0. Of course, the expectation E denotes, here, the one with respect to the process $\hat{z}_t$ solving Equations 3.4 and 3.5. The definitions (5.1) and (5.2) provide a probabilistic interpretation of M_t, M_t^* such that $D_t M_t = 0$ and $D_t^* M_t^* = 0$. Indeed, when this happens we have clearly for all $\Delta t > 0$,

$$E[M_{t+\Delta t}(\hat{z}_{t+\Delta t}) \mid \mathscr{P}_t] = M_t(\hat{z}_t), \qquad \text{and} \qquad M_t^*(\hat{z}_t) = E[M_{t-\Delta t}^*(\hat{z}_{t-\Delta t}) \mid \mathscr{F}_t].$$

As indicated after (2.6), the first condition means that $M_t(\hat{z}_t)$ is a $\mathscr{P}_t$-martingale and the second that $M_t^*(\hat{z}_t^*)$ is a $\mathscr{F}_t$-martingale. For instance, $f_t(\hat{z}_t) = \frac{\tilde{\eta}_t(\hat{z}_t)}{\hat{\eta}_t(\hat{z}_t)}$, as defined above, is a $\mathscr{P}_t$-martingale.

The relation with quantum dynamics is clearer when expressed in terms of (absolute, in contrast with conditional) expectations. For this purpose it is sufficient to observe that

Corollary 5.1 *Under (absolute) expectations and when f_t, $D_t f$, $D_t^* f$ are integrable we have*

$$\frac{d}{dt} E[f_t(\hat{z}_t)] = E[D_t f_t(\hat{z}_t)] = E[D_t^* f_t(\hat{z}_t)], \quad f \in \mathcal{S}(\mathbb{R}^{d+1}).$$

Proof. This follows from the Itô formula, written as

$$\begin{aligned} df_t(\hat{z}_t) &= D_t f_t(\hat{z}_t) dt + \langle \nabla f_t(\hat{z}_t), dW_t^\hbar \rangle \\ &\quad + \int_{\mathbb{R}^d} (f_t(\hat{z}_{t^-} + \hbar k) - f_t(\hat{z}_{t^-})) \left(\mu(dk, dt) - \frac{\hat{\eta}_t(\hat{z}_{t^-} + \hbar k)}{\hat{\eta}_t(\hat{z}_{t^-})} \nu(dk) dt \right), \end{aligned}$$

and

$$\begin{aligned} d_* f(\hat{z}_t) &= D_t^* f_t(\hat{z}_t)dt + \langle \nabla f(\hat{z}_t), d_* W_t^{\hbar *} \rangle \\ &\quad + \int_{\mathrm{R}^d} (f(\hat{z}_{t^+}) - f(\hat{z}_{t^+} - \hbar k)) \left(\mu_*(dk, dt) - \frac{\hat{\eta}_t^*(\hat{z}_{t^+} - \hbar k)}{\hat{\eta}_t^*(\hat{z}_{t^+})} \nu(dk)dt \right). \end{aligned}$$

□

In the next proposition we make the assumption:

$$\int_{\mathrm{R}^d} \|k\| \nu(dk) < \infty, \tag{5.3}$$

and let $D_t p_{|p=\hat{z}_t}$ be denoted by $D_t \hat{z}_t$.

Proposition 5.2 *The process $(\hat{z}_t)_{t\in[r,v]}$, critical point of the action functional introduced before Proposition 4.1, solves the almost sure equations of motion*

$$D_t \left(\hbar \frac{\nabla \hat{\eta}_t}{\hat{\eta}_t} \right) (\hat{z}_t) = \hat{z}_t, \tag{5.4}$$

where ∇ denotes ∇_p except if otherwise specified, and

$$D_t \hat{z}_t = \frac{1}{\hat{\eta}_t(\hat{z}_t)} (-i\nabla_q V)(i\hbar \nabla_p) \hat{\eta}_t(\hat{z}_t). \tag{5.5}$$

Proof. By (2.11),

$$-i\nabla_q V(q) = -iq + \frac{1}{\hbar} \int_{\mathrm{R}^d} k e^{-\frac{i}{\hbar}\langle q,k\rangle} \nu(dk).$$

Therefore

$$(-i\nabla_q V)(i\hbar\nabla)\hat{\eta}_t(p) = \hbar \nabla \hat{\eta}_t(p) + \int_{\mathrm{R}^d} k \hat{\eta}_t(p + \hbar k) \nu(dk).$$

On the other hand we have

$$D_t p \;=\; \mathcal{L}_{\hat{\eta}_t} p = \hbar \nabla \log \hat{\eta}_t(p) + \int_{\mathrm{R}^d} \hbar k \frac{\hat{\eta}_t(p + \hbar k)}{\hat{\eta}_t(\hbar k)} \nu(dk)$$

which proves the first relation (5.5). Concerning (5.4) we have:

$$D_t \left(\hbar \frac{\nabla \hat{\eta}_t}{\hat{\eta}_t} \right)(p) = \frac{1}{\hat{\eta}_t} \left(\frac{\partial}{\partial t} - \frac{1}{\hbar} \hat{H} \right) (\hbar \nabla \hat{\eta}_t) = p + \frac{1}{\hat{\eta}_t} \nabla \left(\hbar \frac{\partial}{\partial t} - \hat{H} \right) \hat{\eta}_t = p.$$

This relation can also be obtained by differentiation with respect to p of the heat equation for $\hat{\eta}$:

$$\frac{\partial \nabla \hat{\eta}_t}{\partial t}(p) = \nabla \frac{\partial \hat{\eta}_t}{\partial t}(p) = \frac{1}{\hbar} \nabla \hat{H} \hat{\eta}_t(p) = \frac{1}{\hbar} \hat{H} \nabla \hat{\eta}_t(p) + p.$$

□

In the backward case, similar calculations yield

$$D_t^* \hat{z}_t = \frac{1}{\hat{\eta}_t^*(\hat{z}_t)} (i\nabla_q \overline{V})(i\hbar\nabla_p)\hat{\eta}_t^*(\hat{z}_t), \qquad D_t^* \left(\hbar \frac{\nabla \hat{\eta}_t^*}{\hat{\eta}_t^*}\right)(\hat{z}_t) = -\hat{z}_t.$$

Since $D_t \mapsto -D_t^*$ and $p \mapsto -p$ under time reversal, it is clear that these equations are time reversed of (5.5). The (forward) analog of the Newton equation in momentum representation becomes

$$D_t D_t \left(\frac{\nabla \hat{\eta}_t}{\hat{\eta}_t}\right)(\hat{z}_t) = \frac{1}{\hat{\eta}_t(\hat{z}_t)}(-i\nabla_q V)(i\hbar\nabla)\hat{\eta}_t(\hat{z}_t).$$

If $V(q) = q^2/2$ we obtain $D_t \hat{z}_t = \hbar\nabla \log \hat{\eta}_t$ and $D_t D_t \hat{z}_t = \hat{z}_t$. This is the purely diffusive case, already known [6].

Let us come back on the interpretation of what we found and, in particular, on its associated classical limit. In position representation the classical action functional S should be regarded as the integral of the ("reduced") Poincaré-Cartan differential form:

$$\tilde{\omega}_q = p(q,t)dt - h(q, p(q,t))dt$$

where $p = p(q,t)$ denotes the momentum expressed as a function of the position and time and $h = h(q, p(q,t))$ the Hamiltonian observable. This means that the underlying flow is the one associated with the first equation of Hamilton, for our elementary class of $h(q,p) = \frac{\|p\|^2}{2} + V(q)$, reduced to the q-variable:

$$\frac{dq}{dt} = p(q,t). \tag{5.6}$$

The second Hamilton equation follows from the integrability condition of $\tilde{\omega}_q$. For Lipschitz p, let $s \mapsto q(s)$ be the solution of (5.6) such that $q(t) = q$. Let us denote by $\int^{q,t} \tilde{\omega}_q$ the associated line integral. When it is locally univocal, this integral defines the action functional, say $S(q,t)$. Then S solves the Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + h(q, \nabla_q S) = 0,$$

whose ∇_q coincides with the above second Hamilton equation. For the momentum representation, the construction is symmetric. Instead of $\tilde{\omega}_q$ we have to consider the reduced form

$$\tilde{\omega}_p = q(p,t)dp + h(q(p,t),p)dt$$

whose underlying flow is the one associated with the second equation of Hamilton reduced to its p-variable:

$$\frac{dp}{dt} = -\nabla V(q(p,t)).$$

When

$$\hat{S}(p,t) = \int^{p,t} \tilde{\omega}_p$$

makes sense, so does the Hamilton-Jacobi equation:

$$\frac{\partial \hat{S}}{\partial t} + h(-\nabla_p \hat{S}, p) = 0 \tag{5.7}$$

whose gradient ∇_p coincides with the integrability condition of $\tilde{\omega}_p$.

Let us specialize this to our class of elementary Hamiltonians which are classical limits of (2.13). Then the action function becomes

$$\begin{aligned}
\hat{S}(p,t) &= \int (q(p(s),s)\frac{dp}{ds} + h(q(p(s),s),p))ds \\
&= \int \left(\frac{1}{2}p^2 - \nabla_p \hat{S}.\dot{p}(s) + V(-\nabla_p \hat{S})\right) ds \\
&= \int L(p(s),\dot{p}(s))ds,
\end{aligned} \tag{5.8}$$

whose integrand defines the Lagrangian L of our system. So, for the elementary class of Hamiltonians H considered here, the equations of motion reduce to

$$\begin{aligned}
&\frac{d}{dt}(-\nabla_p \hat{S}) = p \\
&\frac{dp}{dt} = -\nabla V(-\nabla_p \hat{S}).
\end{aligned} \tag{5.9}$$

Now let us compare this with the Lagrangian and a.s. equation of motion obtained for our class of time-reversible processes $\hat{z}_t$ with jumps. First notice that, in the above classical summary, the parameter is the usual ("real") time. In order to obtain the Euclidean counterpart we have to introduce the "Wick rotation" $t \mapsto -it$ seen in Section 1. It is easy to check that (5.9) can be transformed into

$$\frac{d}{dt}(i\nabla_p \hat{\mathscr{S}}) = ip \tag{5.10}$$

$$i\frac{dp}{dt} = -\nabla_q V(i\nabla_p \hat{\mathscr{S}})$$

where $\hat{\mathscr{S}} = \hat{\mathscr{S}}(p,t)$ solves the Euclidean counterpart of the Hamilton-Jacobi equation (5.7). Comparing with (5.5), those equations can be regarded as the quantum deformation of (5.10) when the smooth trajectories $t \mapsto ip(t)$ are replaced by the very irregular ones of our diffusion process with jumps $\hat{z}_t$. Since the classical (strong) time derivative does not make sense anymore, it is natural to replace it by its probabilistic counterpart D_t (5.1) or, regarded as an operator, by (3.10). The role of the classical action function $\hat{S}(p,t)$ is manifestly played, in (5.5), by $A_t(p) = -\hbar \log \hat{\eta}_t(p)$ solving the Hamilton-Jacobi-Bellman equation (4.4). In particular, it is clear from our first equation (5.5) that the position observable is now proportional to $\hbar\nabla \log \hat{\eta}_t(p) = -\nabla A_t(p)$. So the Hamilton-Jacobi-Bellman equation (4.4) is a quantum deformation of the Euclidean version of the classical equation (5.7). The integrand of the action functional, i.e. our Lagrangian $L(p, D_t p)$ defined in (4.2) of Section 4, is also a (Euclidean) deformation of the classical integrand of (5.8). The main deformation term, of order $o(\hbar^2)$, involves an integral with respect to the Lévy measure $\nu(dk)$. This "small" term is, however, necessary to validate the variational characterization of the process $\hat{z}_t$ given in Section 4. Notice that, because of the relation between the action function and the positive solution $\hat{\eta}_t$ of (2.17) (cf. Proposition 4.1) $A_t(p) = -\hbar \log \hat{\eta}_t(p)$, and also the fact that the underlying Lévy measure $\nu(dk)$ does not depend on the Planck constant $\hbar$, the limit $\hbar \to 0$ of the probabilistic construction is not trivial. In particular, the first term under the integral of the Hamilton-Jacobi-Bellman equation (4.4), which coincides with the factor $\frac{\hat{\eta}_t(p+\hbar k)}{\hat{\eta}_t(p)}$ in the integral term of the forward generator $\mathcal{L}_{\hat{\eta}_t}$ of $\hat{z}_t$, reduces to

$$\int e^{-\hbar\left\langle \frac{\nabla \hat{\eta}_t}{\hat{\eta}_t}(p),k\right\rangle}\nu(dk) \longrightarrow \int e^{\left\langle \nabla \hat{S}(p),k\right\rangle}\nu(dk) \equiv \int e^{-\langle q,k\rangle}\nu(dk),$$

namely the Laplace transform of ν (or Fourier transform in imaginary time). Indeed, both quantically and classically, the position q is proportional to the gradient of the action. Using this and the representation (2.11) of the classical potential V, one can reinterpret our first order classical equation of motion (5.10) for p as the solution of a deterministic variational principle, limit of the one given in Section 4, whose "control"

$u(k)$ becomes optimal precisely when $u^*(k) = e^{-\langle q,k\rangle}$. Of course, then, the associated classical Lagrangian reduces to the Euclidean version of the one in (5.8) or, equivalently, to the classical limit of $L(p, D_t p)$ as given in Section 4, for the semiclassical state $\hat{\eta}_t(p) = \exp -\frac{1}{\hbar}\hat{\mathscr{S}}_t(p)$. However, since this deterministic variational principle does not seem to have an obvious physical interpretation it will not be given here (cf. [30]).

A number of the properties of these processes remains to be investigated. Many of those known to hold for pure diffusions should survive for the much richer class of diffusions with jumps considered here. In particular, a systematic study of their symmetries, in term of a Noether Theorem, on the model of [32], [33], is possible and should provide further informations on the general structure of the construction. A more geometrical approach to these symmetries [20] can probably be extended as well to this class. Moreover, the almost sure equations of motion could be more elegantly deduced from an appropriate generalization of the stochastic calculus of variations used in [7].

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