Integration by parts for point processes and Monte Carlo estimation

Nicolas PRIVAULT
Département de Mathématiques
Université de Poitiers
Téléport 2 - BP 30179
86962 Futuroscope Chasseneuil Cedex, France

Xiao WEI
School of Insurance
Central University of Finance and Economics
100081, Beijing, P.R. China

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Abstract

We develop an integration by parts technique for point processes, with application to the computation of sensitivities via Monte Carlo simulations in stochastic models with jumps. The method is applied to density estimation with respect to the Lebesgue measure via a modified kernel estimator which is less sensitive to variations of the bandwidth parameter than standard kernel estimators. This applies to random variables whose densities are not analytically known and requires the knowledge of the point process jump times.

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1 Introduction

Kernel estimators for the density ϕ_F of a random variable F from a random sample $\{F(k)\}_{k=1,\dots,N}$ of F have been introduced in [17], [14]. More precisely in [17], finite difference estimators of the form

$$\phi_F(y) \simeq \frac{1}{h} E[\mathbf{1}_{[-\frac{h}{2},\frac{h}{2}]}(F-y)] \simeq \frac{1}{2Nh} \sum_{k=1}^{N} \mathbf{1}_{[-h,h]}(F(k)-y), \qquad y \in \mathbb{R}_+, \tag{1.1}$$

have been constructed, and extended in [14] to estimators of the form

$$\phi_F(y) \simeq \frac{1}{Nh} \sum_{k=1}^{N} K\left(\frac{F(k) - y}{h}\right), \tag{1.2}$$

where $K : \mathbb{R} \to \mathbb{R}_+$ is a kernel satisfying

$$\int_{-\infty}^{\infty} K(x)dx = 1.$$

The performance of kernel estimators is dependent on the choice of the bandwidth parameter h, whose optimal value is function of the number N of samples, i.e. it should decrease as N increases. It is known since [17] that the optimal rate of decrease of h in the mean square sense is $N^{-1/4}$ for the finite difference estimator, while in [14] optimal values of h have been obtained for kernel estimators, in terms of N and K.

On the other hand, integration by parts and related Malliavin calculus techniques can be used to represent the density ϕ_F of F with respect to the Lebesgue measure as

$$\phi_F(y) = \frac{\partial}{\partial y} P(F \le y) = E[W \mathbf{1}_{\{F \le y\}}], \tag{1.3}$$

under certain technical assumptions, cf. e.g. § 2.1 of [13] on the Wiener space, where W is a random variable called a weight. This provides another way to estimate the density of F with respect to the Lebesgue measure by Monte Carlo methods: denoting by $\{F(k)\}_{k=1,\dots,N}$ a random sample distributed according to the law of F we have

$$\phi_F(y) \simeq \frac{1}{N} \sum_{k=1}^{N} W(k) \mathbf{1}_{\{F(k) \le y\}}, \tag{1.4}$$

where $\{W(k)\}_{k=1,\dots,N}$ denote independent random samples of W. The interest in (1.4), compared to kernel estimators, is that it is independent on the value of a bandwidth parameter. Note however that in addition to the samples of F, this estimator requires the knowledge of the random path of the underlying stochastic process in order to evaluate W. On the other hand, the integrability of the weight W in (1.3) entails the existence of the density of F with respect to the Lebesgue measure, thus excluding discrete random variables from this approach.

More generally, the Malliavin calculus has been applied to sensitivity analysis in continuous and discontinuous financial markets, cf. [10], [9], [11], [7], [6], [2], [1] and in insurance, cf. [16], to express derivatives of the form $\frac{\partial}{\partial \zeta} E[f(F_\zeta)]$ as:

$$\frac{\partial}{\partial \zeta} E[f(F_\zeta)] = E[W_\zeta f(F_\zeta)], \tag{1.5}$$

where (F_ζ) is a family of random variables in $\mathcal{S}_T$ depending on a parameter $\zeta \in \mathbb{R}$. Here, W_ζ is a weight independent of the function f, which need not be differentiable: in particular the estimation of density (1.4) corresponds to $f = \mathbf{1}_{(-\infty,0)}$ and $F_y = F - y$, with W independent of y. Note that in mathematical finance, each value of the bandwidth parameter h in the finite difference

$$\frac{1}{2h} E[f(F_{\zeta+h}) - f(F_{\zeta-h})]$$

yields a different estimate of the corresponding sensitivity (also called “Greek”), see e.g. [5], p. 40, whereas (1.5) is again independent of a bandwidth parameter.

In Proposition 3.3 below we derive a general integration by parts formula for point processes, extending the results obtained in the Poisson case in [3], [8], [15], [11], [16], with potential application to sensitivity analysis and density estimation for stochastic models in finance, insurance, and engineering. Using this integration by parts formula we obtain an expression of the form (1.3)-(1.4):

$$\phi_F(y) = E[W \mathbf{1}_{\{F \le y\}}] \simeq \frac{1}{N} \sum_{k=1}^{N} \mathbf{1}_{\{F(k) \le y\}} W(k), \tag{1.6}$$

for the density of a random functional F of a point process with respect to the Lebesgue measure. This expression requires the knowledge of the characteristics (the Janossy densities) of the underlying point process in order to compute the weight W, while the density of F may be unknown or not analytically computable and thus requiring a numerical estimation.

It turns out that the performance of the corresponding estimator (1.6) decreases when y is large, in which case the term $W \mathbf{1}_{\{F \le y\}}$ has a large variance. This problem is tackled by a localization procedure, mixing (1.6) with a standard kernel estimate:

$$\begin{aligned} \phi_F(y) &= \frac{1}{\eta} E\left[K\left(\frac{F-y}{\eta}\right)\right] - E\left[W \times f\left(\frac{F-y}{\eta}\right)\right] \qquad (1.7) \\ &\simeq \frac{1}{N\eta} \sum_{k=1}^{N} K\left(\frac{F(k)-y}{\eta}\right) - \frac{1}{N} \sum_{k=1}^{N} W(k) f\left(\frac{F(k)-y}{\eta}\right), \end{aligned}$$

where K is a kernel supported in $[0, \infty)$ and

$$f(x) = \mathbf{1}_{[0,\infty)}(x) \left(1 - \int_0^x K(y)dy\right), \qquad x \in \mathbb{R}.$$

As shown in Section 6, this estimator combines the advantages of Malliavin type estimators (1.6) and kernel estimators (1.2), in that it is little sensitive to values of the bandwidth parameter h, while at the same time it does not present the above mentioned variance problem. Actually, (1.7) recovers with a simple proof an analog of Theorem 2.1 proved in [12] on the Wiener space. The optimization results of [12] in terms of kernel K and bandwidth parameter h also apply here and are used in numerical simulations, cf. Figure 6.3.

We proceed as follows. In Section 2 we review some properties of point processes, and in Section 3 we establish the integration by parts formula (Proposition 3.3) which will be our main tool for density estimation. In Section 4 we present an application of the integration by parts formula to the computation of sensitivities, in particular for functionals of the form

$$F = \int_0^T h(t)dX_t, \tag{1.8}$$

where h is a $\mathcal{C}^1$ function and

$$X_t = \sum_{k=1}^{N_t} Y_k, \qquad t \in \mathbb{R}_+,$$

is a compound log-normal renewal process with random marks $(Y_k)_{k\geq 1}$ independent of $(N_t)_{t\in\mathbb{R}_+}$. These results are used in Section 5 to construct a modified kernel density estimator. Simulations and comparisons of different methods for density estimation are presented in Section 6 for functionals of the form (1.8) with $h(t) = e^{-rt}$, $t \in [0, T]$. Such functionals can be used to express risk reserve processes for insurance portfolios in which the accumulated amount of claims occurring in the time interval $(0, t]$ is given by X_t, cf. e.g. [16].

2 Point processes

Let

$$N_t = \sum_{k=1}^{\infty} \mathbf{1}_{[T_k,\infty)}(t), \qquad t \in \mathbb{R}_+, \tag{2.1}$$

be a point process with increasing sequence of jump times $(T_k)_{k\geq 1}$, generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ on a probability space $(\Omega, \mathcal{F}, P)$. Set $T_0 = 0$ and let the inter-jump times of $(N_t)_{t\in\mathbb{R}_+}$ be denoted by $\tau_k := T_k - T_{k-1}$, $k \geq 1$.

Definition 2.1. *Let $T > 0$. We denote by $\mathcal{S}_T$ the subspace of $L^2(\Omega, \mathcal{F}_T)$ made of functionals of the form*

$$F = f_0 \mathbf{1}_{\{N_T=0\}} + \sum_{n=1}^{\infty} \mathbf{1}_{\{N_T=n\}} f_n(T_1, \ldots, T_n), \tag{2.2}$$

where $f_0 \in \mathbb{R}$ and f_n is $\mathcal{C}^2$ and symmetric in n variables on $[0,T]^n$, $n \geq 1$, $T > 0$.

The set of $F \in \mathcal{S}_T$ for which the expansion (2.2) is finite is denoted by $\mathcal{S}_T^f$ and is dense in $L^p(\Omega, \mathcal{F}_T)$, $p \geq 1$. The expectation of F equals

$$E[F] = j_{T,0} f_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^T \cdots \int_0^T f_n(t_1, \ldots, t_n) j_{T,n}(t_1, \ldots, t_n) dt_1 \cdots dt_n, \tag{2.3}$$

where $j_{T,n} : \mathbb{R}_+^n \to \mathbb{R}_+$, $n \geq 1$, are nonnegative symmetric functions on $[0,T]^n$ called the Janossy densities, and $j_{T,0} \in \mathbb{R}_+$, cf. [18], §5.3 of [4], and references therein. In other terms we have

$$P(T_1 \in dt_1, \ldots, T_n \in dt_n, N_T = n) = j_{T,n}(t_1, \ldots, t_n) dt_1 \cdots dt_n,$$

$0 \leq t_1 < t_2 < \cdots < t_n \leq T$. We turn to some examples of point processes and their Janossy densities.

Poisson processes

In the case of Poisson processes with arbitrary deterministic intensity $\lambda(t)$ we have

$$j_{T,n}(t_1, \ldots, t_n) = \lambda(t_1) \cdots \lambda(t_n) \exp\left(- \int_0^T \lambda(t) dt \right),$$

i.e. for the standard Poisson process with intensity $\lambda > 0$ we have

$$j_{T,n}(t_1, \ldots, t_n) = \lambda^n e^{-\lambda T}, \qquad t_1, \ldots, t_n \in [0,T].$$

Renewal processes

A point process $(N_t)_{t\in\mathbb{R}_+}$ as in (2.1) is called a renewal process with inter-occurrence time distribution function $Z(x)$ and density $z(x)$ if the random variables $\tau_k = T_k - T_{k-1}$, $k \geq 1$, are independent and identically distributed with

$$Z(x) = P(\tau_k \leq x) = \int_0^x z(y) dy, \qquad x \in \mathbb{R}_+, \quad k \geq 1.$$

Since the sequence $(\tau_k)_{k\geq 1}$ is i.i.d., for $0 \leq t_1 < t_2 < \cdots < t_n \leq T$ we have

$$\begin{aligned} & P(T_1 \in dt_1, \ldots, T_n \in dt_n, N_T = n) \\ &= P(\tau_1 \in dt_1, t_1 + \tau_2 \in dt_2, \ldots, t_{n-1} + \tau_n \in dt_n, \tau_{n+1} > T - t_n) \\ &= z(t_1)z(t_2 - t_1) \cdots z(t_n - t_{n-1})(1 - Z(T - t_n))dt_1 \cdots dt_n, \end{aligned}$$

hence the Janossy densities $j_{T,n}(t_1, \ldots, t_n)$ are given by

$$j_{T,n}(t_1, \ldots, t_n) = z(t_1)z(t_2 - t_1) \cdots z(t_n - t_{n-1}) \int_{T-t_n}^{\infty} z(s)ds, \qquad (2.4)$$

for $0 \leq t_1 < \cdots < t_n \leq T$. The value of $j_{T,n}(t_1, \ldots, t_n)$ on $(t_1, \ldots, t_n) \in [0, T]^n$ is obtained by symmetrization:

$$j_{T,n}(t_1, \ldots, t_n) = j_{T,n}(t_{(1)}, \ldots, t_{(n)}), \qquad t_1, \ldots, t_n \in [0, T],$$

where $(t_{(1)}, \ldots, t_{(n)})$ denotes the sequence $(t_1, \ldots, t_n)$ in ascending order, see §5.3 of [4].

3 Integration by parts

Definition 3.1. *Given $w \in \mathcal{C}^1([0, T])$, let D_w denote the gradient operator defined on $F \in \mathcal{S}_T$ of the form (2.2) by*

$$D_w F = -\sum_{n=1}^{\infty} \mathbf{1}_{\{N_T = n\}} \sum_{k=1}^{n} w(T_k) \frac{\partial f_n}{\partial t_k}(T_1, \ldots, T_n).$$

Let $\mathcal{C}_0^1([0, T])$ denote the space of $w \in \mathcal{C}^1([0, T])$ such that $w(0) = w(T) = 0$. In the sequel we assume that $j_{T,n} \in \mathcal{C}^1([0, T]^n)$, $n \geq 1$. Next, we state the definition of the divergence operator.

Definition 3.2. *Given $w \in \mathcal{C}_0^1([0, T])$ and $G \in \mathcal{S}_T$, let*

$$D_w^* G = G \int_0^T w'(t)dN_t - GD_w \log |G j_{T,N_T}(T_1, \ldots, T_{N_T})|, \qquad (3.1)$$

with the convention $0/0 = 0$.

Fix $p, q > 1$ satisfying $1/p + 1/q = 1$ and let $\mathrm{Dom}_p(D_w)$, resp. $\mathrm{Dom}_q(D_w^*)$, be defined as the sets of functionals $F \in L^p(\Omega, \mathcal{F}_T)$, resp. $F \in L^q(\Omega, \mathcal{F}_T)$, for which there exists $(F_n)_{n\in N}$ in $\mathcal{S}_T^f$ converging to F in $L^p(\Omega, \mathcal{F}_T)$, resp. in $L^q(\Omega, \mathcal{F}_T)$, and such that $(D_w F_n)_{n\in N}$, resp. $(D_w^* F_n)_{n\in N}$, converges in $L^p(\Omega, \mathcal{F}_T)$, resp in $L^q(\Omega, \mathcal{F}_T)$. In the next proposition we extend the integration by parts formulas of [3], [16] to the setting of point processes.

Proposition 3.3. *Let $w \in \mathcal{C}_0^1([0,T])$. The operators D_w and D_w^* are closable and can be extended to their closed domains* $\mathrm{Dom}_p(D_w)$ *and* $\mathrm{Dom}_q(D_w^*)$ *with the duality relation*

$$E[GD_wF] = E[FD_w^*G], \qquad F \in \mathrm{Dom}_p(D_w), \quad G \in \mathrm{Dom}_q(D_w^*). \tag{3.2}$$

Proof. For any $F \in \mathcal{S}_T^f$ we have

$$\begin{aligned}
E[D_wF] &= -\sum_{n=1}^\infty \frac{1}{n!}\int_0^T \cdots \int_0^T \sum_{k=1}^n w(t_k)\frac{\partial f_n}{\partial t_k}(t_1,\ldots,t_n) j_{T,n}(t_1,\ldots,t_n)dt_1\cdots dt_n \\
&= \sum_{n=1}^\infty \frac{1}{n!}\int_0^T \cdots \int_0^T f_n(t_1,\ldots,t_n)\sum_{k=1}^n \frac{\partial}{\partial t_k}\left(w(t_k)j_{T,n}(t_1,\ldots,t_n)\right)dt_1\cdots dt_n \\
&= \sum_{n=1}^\infty \frac{1}{n!}\int_0^T \cdots \int_0^T f_n(t_1,\ldots,t_n) j_{T,n}(t_1,\ldots,t_n) \\
&\quad \times \left(\sum_{k=1}^n w'(t_k) + \sum_{k=1}^n w(t_k)\frac{\partial \log j_{T,n}}{\partial t_k}(t_1,\ldots,t_n)\right)dt_1\cdots dt_n \\
&= E\left[\left(\int_0^T w'(t)dN_t - D_w \log j_{T,N_T}(T_1,\ldots,T_{N_T})\right)F\right],
\end{aligned}$$

hence for all $F, G \in \mathcal{S}_T^f$ we get

$$\begin{aligned}
&E[GD_wF] = E[D_w(FG) - FD_wG] \\
&= E\left[F\left(G\int_0^T w'(t)dN_t - GD_w\log j_{T,N_T}(T_1,\ldots,T_{N_T}) - D_wG\right)\right] \\
&= E[FD_w^*G].
\end{aligned}$$

Let now $(F_n)_{n\in N}$, $(\tilde{F}_n)_{n\in N}$ be two sequences in $\mathcal{S}_T^f$ converging to a same F in $L^p(\Omega,\mathcal{F}_T)$, and such that both $(D_wF_n)_{n\in N}$ and $(D_w\tilde{F}_n)_{n\in N}$ have limits denoted by U and V in $L^p(\Omega,\mathcal{F}_T)$. For all $G \in \mathcal{S}_T^f$ we have

$$\begin{aligned}
|\langle U-V, G\rangle_{L^2}| &= \lim_{n\to\infty} |\langle D_wF_n - D_w\tilde{F}_n, G\rangle_{L^2}| \\
&= |\lim_{n\to\infty}\langle F_n - \tilde{F}_n, D_w^*G\rangle_{L^2}| \\
&\le \|D_w^*G\|_{L^q}\lim_{n\to\infty}\|F_n-\tilde{F}_n\|_{L^p} \\
&= 0,
\end{aligned}$$

hence $U = V$, P-a.s. This shows that D_w can be extended to $F \in \mathrm{Dom}_p(D_w)$ by letting

$$D_wF = \lim_{n\to\infty} D_wF_n$$

for any sequence $(F_n)_{n\in N}$ in $\mathrm{Dom}_p(D_w)$ converging to F in $L^p(\Omega,\mathcal{F}_T)$, and such that $(D_wF_n)_{n\in N}$ converges in $L^p(\Omega,\mathcal{F}_T)$. A similar argument applies to D_w^* and allows us to extend the duality relation (3.2) to all $F\in\mathrm{Dom}_p(D_w)$ and $G\in\mathrm{Dom}_q(D_w^*)$. $\square$

We note the following:

Remark 3.4. *Let $F\in\mathcal{S}_T$ such that $F\in L^p(\Omega,\mathcal{F}_T)$ and $D_wF\in L^p(\Omega,\mathcal{F}_T)$, resp. $D_w^*F\in L^q(\Omega,\mathcal{F}_T)$. Then $F\in\mathrm{Dom}_p(D_w)$, resp. $F\in\mathrm{Dom}_q(D_w^*)$.*

Proof. It suffices to approximate F written as in (2.2) by the truncated sequence

$$F_m = f_0\mathbf{1}_{\{N_T=0\}} + \sum_{n=1}^{m}\mathbf{1}_{\{N_T=n\}}f_n(T_1,\ldots,T_n), \qquad m\geq 1$$

and to note that $(D_wF_m)_{m\geq1}$, resp. $(D_w^*F_m)_{m\geq1}$, is convergent in $L^p(\Omega,\mathcal{F}_T)$, resp. in $L^q(\Omega,\mathcal{F}_T)$. $\square$

This remark allows us to prove the following lemma, whose hypotheses will apply in the sequel.

Lemma 3.5. *Let $p\geq1$ and assume that there exists $c_0>0$ such that*

$$j_{T,n}^{1-p}(t_1,\ldots,t_n)\left|\frac{\partial j_{T,n}}{\partial t_k}(t_1,\ldots,t_n)\right|\leq c_0^n, \tag{3.3}$$

$k=1,\ldots,n$, $t_1,\ldots,t_n\in[0,T]^n$, $n\geq1$. Then $D_w\log j_{T,N_T}(T_1,\ldots,T_{N_T})\in L^p(\Omega,\mathcal{F}_T)$.

Proof. From (2.3) we have

$$\begin{aligned}
&\|D_w\log j_{T,N_T}(T_1,\ldots,T_{N_T})\|_{L^p}^p\\
&= \sum_{n=1}^{\infty}\frac{1}{n!}\int_0^T\cdots\int_0^T\left|\sum_{k=1}^{n}w(t_k)\frac{\partial j_{T,n}}{\partial t_k}(t_1,\ldots,t_n)\right|^p|j_{T,n}(t_1,\ldots,t_n)|^{1-p}dt_1\cdots dt_n\\
&\leq \|w\|_\infty^p c_0Te^{c_0T},
\end{aligned}$$

hence $D_w\log j_{T,N_T}(T_1,\ldots,T_{N_T})\in L^p(\Omega,\mathcal{F}_T)$, and $\log j_{T,N_T}(T_1,\ldots,T_{N_T})\in L^p(\Omega,\mathcal{F}_T)$ follows in the same way. $\square$

We now turn to the calculation of $D_w\log j_{T,N_T}(T_1,\ldots,T_{N_T})$ for examples of point processes satisfying (3.3) for all $p\geq1$.

Poisson processes

In the case of a Poisson process with arbitrary deterministic intensity $\lambda\in\mathcal{C}_b^1(\mathbb{R}_+)$ we have

$$\log j_{T,N_T}(T_1,\ldots,T_{N_T}) = \int_0^T\log\lambda(t)dN_t - \int_0^T\lambda(t)dt,$$

and

$$D_w \log j_{T,N_T}(T_1,\ldots,T_{N_T}) = -\int_0^T w(t)\frac{\lambda'(t)}{\lambda(t)}dN_t.$$

Renewal processes

In this case, (2.4) yields:

$$\begin{aligned}
&D_w \log j_{T,N_T}(T_1,\ldots,T_{N_T})\\
&= -\frac{w(T_{N_T})z(T-T_{N_T})}{1-Z(T-T_{N_T})} - \sum_{k=1}^{N_T} w(T_k)\frac{z'(T_k-T_{k-1})}{z(T_k-T_{k-1})} + \sum_{k=1}^{N_T-1} w(T_k)\frac{z'(T_{k+1}-T_k)}{z(T_{k+1}-T_k)}\\
&= \int_0^T w(t)\left(\frac{z'(T_{N_t+1}-T_{N_t})}{z(T_{N_t+1}-T_{N_t})} - \frac{z'(T_{N_t}-T_{N_t-1})}{z(T_{N_t}-T_{N_t-1})}\right)dN_t\\
&\quad -w(T_{N_T})\frac{z'(T_{N_T+1}-T_{N_T})}{z(T_{N_T+1}-T_{N_T})} - \frac{w(T_{N_T})z(T-T_{N_T})}{1-Z(T-T_{N_T})}\\
&= \int_0^T (w(t-\tau_{N_t})-w(t))\frac{z'(T_{N_t}-T_{N_t-1})}{z(T_{N_t}-T_{N_t-1})}dN_t - \frac{w(T_{N_T})z(T-T_{N_T})}{1-Z(T-T_{N_T})}.
\end{aligned}$$

Log-normal renewal process

In this example the inter-arrival times are independent and identically distributed according to the log-normal distribution with parameter $\sigma > 0$, i.e.

$$z(x) = \frac{e^{-(\log x)^2/(2\sigma^2)}}{\sigma x\sqrt{2\pi}}, \qquad x > 0.$$

In other terms $T_k - T_{k-1} = e^{\sigma\xi_k}$, where $(\xi_k)_{k\ge 1}$ is an i.i.d. sequence of standard Gaussian random variables, and

$$\begin{aligned}
&D_w \log j_{T,N_T}(T_1,\ldots,T_{N_T}) = \sum_{k=1}^{N_T}\frac{w(T_k)}{T_k-T_{k-1}}\left(1+\frac{\log(T_k-T_{k-1})}{\sigma^2}\right)\\
&\quad -\sum_{k=1}^{N_T-1}\frac{w(T_k)}{T_{k+1}-T_k}\left(1+\frac{\log(T_{k+1}-T_k)}{\sigma^2}\right) - \frac{w(T_{N_T})e^{-\left(\log(T-T_{N_T})\right)^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}(T-T_{N_T})(1-Z(T-T_{N_T}))}\\
&= \sum_{k=1}^{N_T}\frac{w(T_k)}{T_k-T_{k-1}}(1+\sigma^{-1}\xi_k) - \sum_{k=1}^{N_T-1}\frac{w(T_k)}{T_{k+1}-T_k}(1+\sigma^{-1}\xi_{k+1})\\
&\quad -\frac{w(T_{N_T})e^{-\left(\log(T-T_{N_T})\right)^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}(T-T_{N_T})(1-Z(T-T_{N_T}))}\\
&= -\frac{w(T_{N_T})e^{-\left(\log(T-T_{N_T})\right)^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}(T-T_{N_T})(1-Z(T-T_{N_T}))} + \sum_{k=1}^{N_T}(w(T_k)-w(T_{k-1}))\frac{1+\sigma^{-1}\xi_k}{T_k-T_{k-1}}\\
&= -\frac{w(T_{N_T})e^{-\left(\log(T-T_{N_T})\right)^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}(T-T_{N_T})(1-Z(T-T_{N_T}))} + \int_0^{T_{N_T}} w'(s)\frac{1+\sigma^{-1}\xi_{1+N_s}}{\tau_{1+N_s}}ds.
\end{aligned}$$

In the simulations of Section 5 we will take $w(t) = t(T-t)$, $t \in [0,T]$. In this case we have

$$\begin{aligned}
&\int_0^T w'(t)dN_t - D_w \log j_{T,N_T}(T_1,\ldots,T_{N_T}) \\
&= \frac{T_{N_T} e^{-\left(\log(T-T_{N_T})\right)^2/(2\sigma^2)}}{(1-Z(T-T_{N_T}))\sigma\sqrt{2\pi}} + \sum_{k=1}^{N_T}(T-2T_k) - \sum_{k=1}^{N_T}(T-T_k-T_{k-1})(1+\sigma^{-1}\xi_k) \\
&= \left(\frac{e^{-\left(\log(T-T_{N_T})\right)^2/(2\sigma^2)}}{(1-Z(T-T_{N_T}))\sigma\sqrt{2\pi}} - 1\right) T_{N_T} - \sigma^{-1}\sum_{k=1}^{N_T}(T-T_k-T_{k-1})\xi_k.
\end{aligned}$$

4 Sensitivity analysis

Let $I = (a,b)$ be an open interval of $\mathbb{R}$ and consider the derivative

$$\frac{\partial}{\partial\zeta}E[f(F_\zeta)] = E\left[\frac{\partial F_\zeta}{\partial\zeta}f'(F_\zeta)\right], \qquad \zeta \in (a,b), \tag{4.1}$$

where $(F_\zeta)_{\zeta\in(a,b)}$ a family of random variables differentiable in a parameter ζ and f is a $\mathcal{C}^1$ function on $\mathbb{R}$. This expression can be approximated by finite differences as

$$\frac{1}{2h}E[f(F_{\zeta+h}) - f(F_{\zeta-h})], \tag{4.2}$$

while (4.1) fails when f is not differentiable, e.g. when $f = \mathbf{1}_{[0,\infty)}$.

Proposition 4.1 below provides an expression for this derivative without using finite differences or requiring the differentiability of f. This formula will be applied in Section 5 to numerical simulations which will be compared to the results given by kernel estimates.

In the sequel and in Propositions 4.1, 4.2 and 4.3 we consider a family $(F_\zeta)_{\zeta\in(a,b)}$ of random functionals, continuously differentiable in $\mathrm{Dom}_p(D_w)$ in the parameter $\zeta \in (a,b)$, such that for some $n_0 \in \mathbb{N}$,

$$D_w F_\zeta \neq 0, \qquad \text{a.s. on } \{N_T \geq n_0\},$$

where w is a given element of $\mathcal{C}_0^1([0,T])$ and the function $f : \mathbb{R} \to \mathbb{R}$ is assumed to satisfy $f(F_\zeta) \in L^p(\Omega, \mathcal{F}_T)$, for all $\zeta \in (a,b)$.

Proposition 4.1. *Assume that*

$$\mathbf{1}_{\{N_T\geq n_0\}}\frac{\partial_\zeta F_\zeta}{D_w F_\zeta} \in \mathrm{Dom}_q(D_w^*), \qquad \zeta \in (a,b). \tag{4.3}$$

Then we have

$$\frac{\partial}{\partial\zeta}E[f(F_\zeta) \mid N_T \geq n_0] = E[W_\zeta f(F_\zeta) \mid N_T \geq n_0], \qquad \zeta \in (a,b), \tag{4.4}$$

where the weight W_ζ is given by

$$W_\zeta = D_w^*\left(\mathbf{1}_{\{N_T\geq n_0\}}\frac{\partial_\zeta F_\zeta}{D_w F_\zeta}\right), \qquad \zeta \in (a,b).$$

Proof. Assuming first that $f \in \mathcal{C}_b^\infty(\mathbb{R})$ we have from Proposition 3.3:

$$\begin{aligned}\frac{\partial}{\partial\zeta}E[\mathbf{1}_{\{N_T\geq n_0\}}f(F_\zeta)] &= E\left[\mathbf{1}_{\{N_T\geq n_0\}}f'(F_\zeta)\frac{\partial F_\zeta}{\partial\zeta}\right]\\ &= E\left[\mathbf{1}_{\{N_T\geq n_0\}}\frac{\partial_\zeta F_\zeta}{D_w F_\zeta}D_w(f(F_\zeta))\right]\\ &= E\left[f(F_\zeta)D_w^*\left(\mathbf{1}_{\{N_T\geq n_0\}}\frac{\partial_\zeta F_\zeta}{D_w F_\zeta}\right)\right].\end{aligned}$$

The extension to the general case is obtained from the bound

$$\left|\frac{\partial}{\partial\zeta}E[f_n(F_\zeta)\mathbf{1}_{\{N_T\geq n_0\}}] - E[W_\zeta f(F_\zeta)]\right| \leq \|f(F_\zeta) - f_n(F_\zeta)\|_{L^p}\|W_\zeta \mathbf{1}_{\{N_T\geq n_0\}}\|_{L^q},$$

and an approximating sequence $(f_n)_{n\in\mathbb{N}}$ of smooth functions. □

In the next proposition we focus on a sufficient condition for (4.3) to hold. These conditions can be checked using (2.2).

Proposition 4.2. *Assume that $F_\zeta \in \mathcal{S}_T$, $\zeta \in (a,b)$, and let $1/q' + 1/p' = 1/q$, $p' < q'$, such that $\partial_\zeta F_\zeta \in \mathrm{Dom}_{2q'}(D_w)$, $D_w F_\zeta \in \mathrm{Dom}_{2q'}(D_w)$, and $(D_w F_\zeta)^{-1} \in L^{2q'}(\{N_T \geq n_0\})$. Then (4.3) holds and we have*

$$\frac{\partial}{\partial\zeta}E[f(F_\zeta) \mid N_T \geq n_0] = E[W_\zeta f(F_\zeta) \mid N_T \geq n_0], \qquad \zeta \in (a,b),$$

where the weight W_ζ is given by

$$\begin{aligned}W_\zeta &= \frac{\mathbf{1}_{\{N_T\geq n_0\}}}{D_w F_\zeta}\left(\partial_\zeta F_\zeta\left(\int_0^T w'(t)dN_t - D_w \log|\partial_\zeta F_\zeta j_{T,N_T}(T_1,\ldots,T_{N_T})| + \frac{D_w D_w F_\zeta}{D_w F_\zeta}\right)\right.\\ &\quad \left. - D_w\partial_\zeta F_\zeta\right),\end{aligned} \tag{4.5}$$

and belongs to $L^q(\Omega, \mathcal{F}_T)$.

Proof. Since $F_\zeta \in \mathcal{S}_T$ we have from (3.1):

$$D_w^*\left(\mathbf{1}_{\{N_T\geq n_0\}}\frac{\partial_\zeta F_\zeta}{D_w F_\zeta}\right)$$

$$
\begin{aligned}
&= \mathbf{1}_{\{N_T \geq n_0\}} \frac{\partial_\zeta F_\zeta}{D_w F_\zeta} \left(\int_0^T w'(t) dN_t - D_w \log j_{T,N_T}(T_1, \ldots, T_{N_T}) \right) - D_w \left(\mathbf{1}_{\{N_T \geq n_0\}} \frac{\partial_\zeta F_\zeta}{D_w F_\zeta} \right) \\
&= \frac{\mathbf{1}_{\{N_T \geq n_0\}}}{D_w F_\zeta} \left(\partial_\zeta F_\zeta \left(\int_0^T w'(t) dN_t - D_w \log |\partial_\zeta F_\zeta j_{T,N_T}(T_1, \ldots, T_{N_T})| + \frac{D_w D_w F_\zeta}{D_w F_\zeta} \right) \right. \\
&\quad \left. - D_w \partial_\zeta F_\zeta \right).
\end{aligned}
$$

In order to apply Proposition 4.1 we need to check the domain condition

$$
\mathbf{1}_{\{N_T \geq n_0\}} \frac{\partial_\zeta F_\zeta}{D_w F_\zeta} \in \mathrm{Dom}_q(D_w^*),
$$

which is satisfied from Remark 3.4, provided W_ζ as in (4.5) belongs to $L^q(\Omega, \mathcal{F}_T)$. By Hölder's inequality we have

$$
\begin{aligned}
&\|W_\zeta\|_{L^q} \leq \left\| (D_w F_\zeta)^{-1} \right\|^2_{L^{2q'}(\{N_T \geq n_0\})} \|\partial_\zeta F_\zeta D_w D_w F_\zeta\|_{L^{q'}} \\
&+ \left\| (D_w F_\zeta)^{-1} \right\|_{L^{q'}(\{N_T \geq n_0\})} \left\| \partial_\zeta F_\zeta \int_0^T w'(t) dN_t + \partial_\zeta F_\zeta D_w \log j_{T,N_T}(T_1, \ldots, T_{N_T}) + D_w \partial_\zeta F_\zeta \right\|_{L^{p'}} \\
&\quad \leq \left\| (D_w F_\zeta)^{-1} \right\|_{L^{2q'}(\{N_T \geq n_0\})} \|\partial_\zeta F_\zeta\|_{L^{2q'}} \times \\
&\left(\left\| \int_0^T w'(t) dN_t \right\|_{L^{2p'}} + \|D_w \log j_{T,N_T}(T_1, \ldots, T_{N_T})\|_{L^{2p'}} + \|D_w \partial_\zeta F_\zeta\|_{L^{2p'}} + \|D_w D_w F_\zeta\|_{L^{2q'}} \right),
\end{aligned}
\tag{4.6}
$$

which allows us to conclude by Lemma 3.5. □

In the case of a Poisson process with deterministic intensity $\lambda \in \mathcal{C}^1(\mathbb{R}_+)$ we have

$$
W_\zeta = \mathbf{1}_{\{N_T \geq n_0\}} \left(\frac{\partial_\zeta F_\zeta}{D_w F_\zeta} \left(\int_0^T w'(t) dN_t - \int_0^T w(t) \frac{\lambda'(t)}{\lambda(t)} dN_t + \frac{D_w D_w F_\zeta}{D_w F_\zeta} \right) - \frac{D_w \partial_\zeta F_\zeta}{D_w F_\zeta} \right).
$$

In general we assume that the Janossy densities $j_{T,n}$ are known in order to compute the weight W_ζ while the density of F may not be analytically computable, or unknown as in the following example.

Consider now a compound point process of the form

$$
X_t = \sum_{k=1}^{N_t} Y_k, \qquad t \in \mathbb{R}_+,
$$

where $(Y_k)_{k \geq 1}$ is a sequence of random marks independent of $(N_t)_{t \in \mathbb{R}_+}$ and such that there exists $c_2 > 0$ such that $Y_k \geq c_2 > 0$ a.s., $k \geq 1$. We make the additional assumption

$$
j_{T,n}(t_1, \ldots, t_n) \leq c_0^n, \qquad t_1, \ldots, t_n \in [0, T]^n,
$$

$k = 1, \ldots, n$, $n \geq 1$.

Proposition 4.3. *Consider $g : (a,b) \to \mathbb{R}$ and $h : [a,b] \times [0,T] \to \mathbb{R}$ two $\mathcal{C}^1$ functions such that $\frac{\partial h}{\partial t}$ does not vanish on $[a,b] \times [0,T]$, and let*

$$F_\zeta = g(\zeta) + \int_0^T h(\zeta,t)dX_t = g(\zeta) + \sum_{k=1}^{N_t} Y_k h(\zeta, T_k), \qquad \zeta \in (a,b).$$

Let $\alpha > 0$ and

$$w(t) = t^\alpha (T-t)^\alpha, \qquad t \in [0,T].$$

Then (4.3) holds whenever $n_0 \geq 2\alpha$ and we have

$$\frac{\partial}{\partial \zeta} E[f(F_\zeta) \mid N_T \geq n_0] = E[W_\zeta f(F_\zeta) \mid N_T \geq n_0], \qquad \zeta \in (a,b).$$

where the weight W_ζ belongs to $L^q(\Omega)$, $\zeta \in (a,b)$.

Proof. We have

$$\partial_\zeta F_\zeta = g'(\zeta) + \int_0^T \frac{\partial h}{\partial \zeta}(\zeta,t)dX_t,$$

which belongs to $L^p(\Omega)$ for all $p \geq 1$. Since the gradient D_w does not act on Y_k, $k \in \mathbb{N}$, these random variables can be considered as constants in the integration by parts formula (3.2) and we have

$$D_w F_\zeta = -\int_0^T w(t) \frac{\partial h}{\partial t}(\zeta,t)dX_t.$$

Moreover there exists $c_1 > 0$ such that

$$\left| \frac{\partial h}{\partial t}(\zeta,t) \right| \geq c_1 > 0, \qquad (\zeta,t) \in [a,b] \times [0,T],$$

hence for any p', q' such that $1/q' + 1/p' = 1/q$ we have

$$\begin{aligned}
\left\| (D_w F_\zeta)^{-1} \right\|_{L^{2q'}(\{N_T \geq n_0\})}^{2q'} &= E\left[\mathbf{1}_{\{N_T \geq n_0\}} \left| \int_0^T w(t) \frac{\partial h}{\partial t}(\zeta,t)dX_t \right|^{-2q'} \right] \\
&= E\left[\sum_{n=n_0}^\infty \frac{1}{n!} \int_0^T \cdots \int_0^T \frac{j_{T,n}(t_1,\ldots,t_n)}{|\sum_{k=1}^n Y_k t_k^\alpha (T-t_k)^\alpha \frac{\partial h}{\partial t}(\zeta,t_k)|^{2q'}} dt_1 \cdots dt_n \right] \\
&\leq \frac{2^{2\alpha q'}}{(c_1 c_2)^{2q'}} \sum_{n=n_0}^\infty \frac{2^n c_0^n}{n!} \int_0^{1/2} \cdots \int_0^{1/2} \left(\sum_{k=1}^n t_k^\alpha \right)^{-2q'} dt_1 \cdots dt_n \\
&\leq \frac{2^{2\alpha q'}}{(c_1 c_2)^{2q'}} \sum_{n=n_0}^\infty \frac{c_0^n}{n!} \left(\frac{1}{4^{\alpha q'}} + 2^n \int_{\{\sum_{k=1}^n t_k^2 \leq 1/4\}} \left(\sum_{k=1}^n t_k^2 \right)^{-\alpha q'} dt_1 \cdots dt_n \right) \\
&= \frac{2^{2\alpha q'}}{(c_1 c_2)^{2q'}} \sum_{n=n_0}^\infty \frac{c_0^n}{n!} \left(\frac{1}{4^{\alpha q'}} + \frac{2^{n+1}\pi^{n/2}}{\Gamma(n/2)} \int_0^{1/2} r^{n-1-2\alpha q'} dr \right)
\end{aligned}$$

$$= (c_1c_2)^{-2q'} \sum_{n=n_0}^{\infty} \frac{c_0^n}{n!} \left(1 + \frac{2^{1+4\alpha q'} \pi^{n/2}}{\Gamma(n/2)(n-2\alpha q')}\right),$$

which is finite whenever $n_0 > 2\alpha q' > 2\alpha$, hence we can apply Proposition 4.2. $\square$

In practice one can choose $n_0 = 1$ provided $\alpha \in (0, 1/2)$. Note that at least four jumps can be required in other situations, see e.g. Proposition 3.2 of [1] in the Poisson case.

For example, taking $h(\zeta, t) = e^{-\zeta t}$, the weight W_ζ corresponding to the sensitivity

$$\frac{\partial}{\partial \zeta} E[\mathbf{1}_{\{N_T \geq n_0\}} f(F_\zeta)] = E[\mathbf{1}_{\{N_T \geq n_0\}} W_\zeta f(F_\zeta)] \tag{4.7}$$

with respect to the parameter $\zeta > 0$ is given on $\{N_T \geq n_0\}$ by

$$\begin{aligned} W_\zeta &= -\frac{1}{\zeta} + \frac{\int_0^T w(t) t e^{-\zeta t} dX_t}{\int_0^T w(t) e^{-\zeta t} dX_t} - \frac{\int_0^T t e^{-\zeta t} dX_t}{\zeta \int_0^T w(t) e^{-\zeta t} dX_t} \left(\frac{\int_0^T w(t)(\zeta w(t) - w'(t)) e^{-\zeta t} dX_t}{\int_0^T w(t) e^{-\zeta t} dX_t} \right. \\ &\quad \left. - \int_0^T w'(t) dN_t + D_w \log j_{T,N_T}(T_1, \ldots, T_{N_T}) \right). \end{aligned}$$

5 Density estimation

In this section we apply the above results to the computation of the conditional density $\phi_F(\cdot \mid N_T \geq n_0)$ of a random variable F with respect to the Lebesgue measure, written as the derivative

$$\phi_F(y \mid N_T \geq n_0) = -\frac{d}{dy} E[f(F-y) \mid N_T \geq n_0], \qquad y \in \mathbb{R},$$

with $f = \mathbf{1}_{(0,\infty)}$, i.e. we take $F^\zeta = F - \zeta$, $\zeta \in \mathbb{R}$.

Kernel estimators

The standard kernel estimator of the density ϕ_F with respect to the Lebesgue measure is given by

$$\phi_F(y) \simeq \frac{1}{h} E\left[K\left(\frac{F-y}{h}\right)\right] \simeq \frac{1}{Nh} \sum_{k=1}^{N} K\left(\frac{F(k)-y}{h}\right), \tag{5.1}$$

where K is a continuous positive function such that

$$\int_{-\infty}^{\infty} K(x) dx = 1.$$

Malliavin estimators

Taking $F_y = F - y$, Proposition 4.2 yields the following corollary.

Corollary 5.1. *Assume that $F \in \mathcal{S}_T$ and let $1/q' + 1/p' = 1/q$, $p' < q'$, such that $D_w F \in \mathrm{Dom}_{2q'}(D_w)$, and $(D_w F)^{-1} \in L^{2q'}(\{N_T \geq n_0\})$. Then we have*

$$\frac{\partial}{\partial y} E[\mathbf{1}_{\{N_T \geq n_0\}} f(F - y)] = E\left[W \mathbf{1}_{\{N_T \geq n_0\}} f(F - y)\right],$$

for f bounded and measurable on $\mathbb{R}$, where

$$W = \frac{\mathbf{1}_{\{N_T \geq n_0\}}}{D_w F} \left(\int_0^T w'(t) dN_t - D_w \log j_{T,N_T}(T_1, \ldots, T_{N_T}) + \frac{D_w D_w F}{D_w F} \right) \tag{5.2}$$

belongs to $L^q(\Omega)$.

In particular, taking $f = -\mathbf{1}_{[0,\infty)}$ we get

$$\phi_F(y \mid N_T \geq n_0) = -\frac{d}{dy} E[\mathbf{1}_{[0,\infty)}(F - y) \mid N_T \geq n_0] = -E[W \mathbf{1}_{[0,\infty)}(F - y) \mid N_T \geq n_0], \tag{5.3}$$

$y \in \mathbb{R}$, where the weight W is independent of y and of any bandwidth parameter. Here the condition $\{F > y\}$ in (5.3) with $y > 0$ actually ensures the integrability of $W \mathbf{1}_{[0,\infty)}(F - y)$ on $\{N_T \geq 1\}$. This yields the estimate

$$\phi_F(y \mid N_T \geq n_0) \simeq -\frac{1_{\{N_T \geq n_0\}}}{N P(N_T \geq n_0)} \sum_{i=1}^{N} W(i) \mathbf{1}_{[0,\infty)}(F(i) - y). \tag{5.4}$$

In case $F = \int_0^T h(t) dX_t$ the relation

$$D_w D_w F = \int_0^T w(t) \left(\frac{\partial h}{\partial t}(\zeta, t) w'(t) + \frac{\partial^2 h}{\partial t^2}(\zeta, t) w(t) \right) dX_t$$

yields

$$\begin{aligned} W \;=\; & \frac{\mathbf{1}_{\{N_T \geq n_0\}}}{\int_0^T w(t) \frac{\partial h}{\partial t}(\zeta, t) dX_t} \left(\int_0^T w'(t) dN_t - D_w \log j_{T,N_T}(T_1, \ldots, T_{N_T}) \right. \\ & \left. - \frac{\int_0^T w(t) (\frac{\partial h}{\partial t}(\zeta, t) w'(t) + \frac{\partial^2 h}{\partial t^2}(\zeta, t) w(t)) dX_t}{\int_0^T w(t) \frac{\partial h}{\partial t}(\zeta, t) dX_t} \right). \end{aligned} \tag{5.5}$$

Modified kernel estimators

When $D_w F$ is close to 0, the value of W becomes large, due to the division by $D_w F$ in (5.5), hence when y is small the term $W \mathbf{1}_{[0,\infty)}(F - y)$ is allowed to be non-zero for small values of F, and it has a large variance. A variance reduction technique called localization had been introduced in [9] to deal with related problems on the Wiener space. Here we apply a similar procedure to construct a modified kernel estimator using Malliavin weights. For this we will consider a decomposition of the form

$$\mathbf{1}_{[0,\infty)} = f + g,$$

where g is a $\mathcal{C}^1$ function. In the following proposition we obtain an analog of Theorem 2.1 in [12], via a somewhat simpler argument, under the hypotheses of Proposition 4.1.

Proposition 5.2. *Assume that $F \in \mathcal{S}_T$ and let $1/q' + 1/p' = 1/q$, $p' < q'$, such that $D_w F \in \mathrm{Dom}_{2q'}(D_w)$, and $(D_w F)^{-1} \in L^{2q'}(\{N_T \geq n_0\})$ and let f a function on $\mathbb{R}$ such that $f(0) = 1$, $f(x) = 0$, $x < 0$, and $\mathbf{1}_{(0,\infty)} f' \in L^2((0,\infty))$. We have for all $\eta > 0$:*

$$\begin{aligned}
&\phi_F(y \mid N_T \geq n_0) \\
&= -E\left[W f\left(\frac{F-y}{\eta}\right) \Big| N_T \geq n_0\right] - \frac{1}{\eta} E\left[\mathbf{1}_{\{F>y\}} f'\left(\frac{F-y}{\eta}\right) \Big| N_T \geq n_0\right], \qquad y \in \mathbb{R},
\end{aligned} \tag{5.6}$$

where W is given by (5.2).

Proof. Letting $g = \mathbf{1}_{[0,\infty)} - f$ we have

$$\begin{aligned}
\phi_F(y \mid N_T \geq n_0) &= -\frac{d}{dy} E[\mathbf{1}_{[0,\infty)}(F-y) \mid N_T \geq n_0] \\
&= -\frac{d}{dy} E\left[f\left(\frac{F-y}{\eta}\right) \Big| N_T \geq n_0\right] - \frac{d}{dy} E\left[g\left(\frac{F-y}{\eta}\right) \Big| N_T \geq n_0\right] \\
&= -E\left[W f\left(\frac{F-y}{\eta}\right) \Big| N_T \geq n_0\right] - \frac{1}{\eta} E\left[\mathbf{1}_{\{F>y\}} f'\left(\frac{F-y}{\eta}\right) \Big| N_T \geq n_0\right], \qquad y \in \mathbb{R},
\end{aligned}$$

where W is given by (5.5). $\square$

Letting $K(x) = -\mathbf{1}_{(0,\infty)}(x) f'(x)$, this leads by Monte Carlo approximation to a family of corrected kernel estimators:

$$\phi_F(y \mid N_T \geq n_0) \simeq \frac{1_{\{N_T \geq n_0\}}}{N P(N_T \geq n_0)} \sum_{i=1}^{N} \left(\frac{1}{\eta} K\left(\frac{F(i)-y}{\eta}\right) - W(i) f\left(\frac{F(i)-y}{\eta}\right)\right), \tag{5.7}$$

depending on $\eta > 0$. Note that (5.6) is an equality, whereas the standard kernel estimate

$$\phi_F(y \mid N_T \geq n_0) \simeq \frac{1}{\eta} E\left[K\left(\frac{F-y}{\eta}\right) \Big| N_T \geq n_0\right], \qquad y \in \mathbb{R},$$

is only an approximation.

The method for the determination of an optimal kernel $f : \mathbb{R} \to \mathbb{R}$ and bandwidth parameter $\eta > 0$ by minimization of

$$E\left[\mathbf{1}_{\{N_T \geq n_0\}\cap\{F>y\}}\left(Wf\left(\frac{F-y}{\eta}\right) - \frac{1}{\eta}\mathbf{1}_{\{F>y\}}f'\left(\frac{F-y}{\eta}\right)\right)^2\right], \qquad y \in \mathbb{R},$$

of [12], page 446, also applies here and yields

$$f(x) = \mathbf{1}_{[0,\infty)}(x)e^{-\lambda x}, \qquad x \in \mathbb{R},$$

and $\eta_{\mathrm{opt}} = \|W\|^{-1}_{L^2(\{N_T \geq n_0\})}$, for any $\lambda > 0$. Note that the criterion of optimality for η is not linked to the number of samples N, as is the case for the optimal decrease in $N^{-1/4}$ of the kernel estimator bandwidth parameter h.

6 Numerical results

Our results are illustrated by Monte Carlo density estimations with 10000 samples for the random variable

$$F_r := \alpha(r)\int_0^T e^{-rt}dN_t,$$

where $(N_t)_{t\in\mathbb{R}_+}$ is a log-normal renewal process and $T = 5$, $\sigma = 0.3$, and $\alpha(r) = \exp((1+r)^2 - 1)$ is a parameter chosen to enhance the readability of the simulation graphs. Clearly the law of F_r has a Dirac mass at $y = 0$, and we are interested in the values of the density on $\mathbb{R} \setminus \{0\}$ with respect to the Lebesgue measure.

Kernel estimators

We start by comparing several kernel estimators in Figure 6.1, with

$$K(x) = \frac{\pi}{2}\mathbf{1}_{[-1/2,1/2]}(x)\cos(\pi x),$$

and $\eta = 1, 0.1, 0.01$.

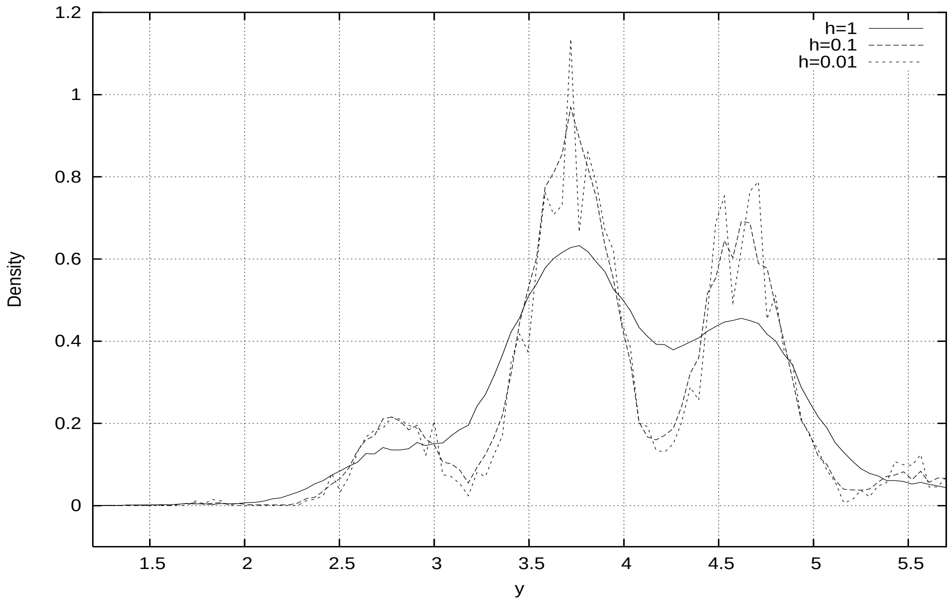

Figure 6.1: Kernel estimations of ϕ_{F_r} with 10000 samples and $r = 0.2$.

Malliavin estimator

For the Malliavin method we use the expression (5.3) where the weight W is given by

$$\begin{aligned} W &= -\frac{\mathbf{1}_{\{N_T \geq n_0\}}}{r\alpha(r) \int_0^T w(t) e^{-rt} dN_t} \left(\int_0^T w'(t) dN_t - D_w \log j_{T,N_T}(T_1, \ldots, T_{N_T}) \right. \\ & \left. + \frac{\int_0^T w(t)(rw(t) - w'(t)) e^{-rt} dN_t}{\int_0^T w(t) e^{-rt} dN_t} \right), \qquad (6.1) \end{aligned}$$

is independent of y and of any bandwidth parameter. The result of this estimation is shown in Figure 6.2 below.

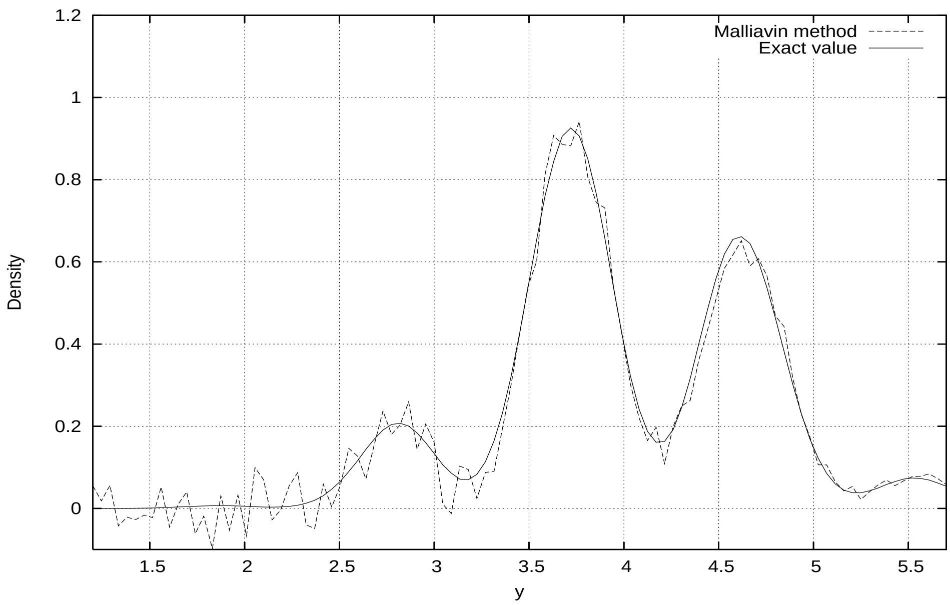

Figure 6.2: Probability density of F_r for $r = 0.2$ (Malliavin method with 10000 samples).

The graph labeled "exact value" has been obtained by finite differences with 10^7 samples. One can check in Figure 6.2 that although the Malliavin estimator (5.3) yields more precise values than the kernel estimator (5.1) when y is large, it behaves badly for small values of y due to a higher variance of $W\mathbf{1}_{[0,\infty)}(F-y)$ in this situation. This phenomenon is dealt with by the modified kernel estimator introduced in Section 5 by localization.

Modified kernel estimators

Figure 6.3 shows the result of this modified kernel estimation for $\eta = 1, 0.2, 0.01$, for comparison with the standard kernel estimate of Figure 6.1. The modified kernel estimator does depend on a parameter called η, but it appears more stable and less sensitive to variations of η than standard kernel estimators are sensitive to the value of the bandwidth parameter h. In our setting we found $\eta_{\text{opt}} = 0.1963$ by Monte Carlo simulation and we used the optimal kernel $K(x) = \mathbf{1}_{(0,\infty)}(x)e^{-x}$.

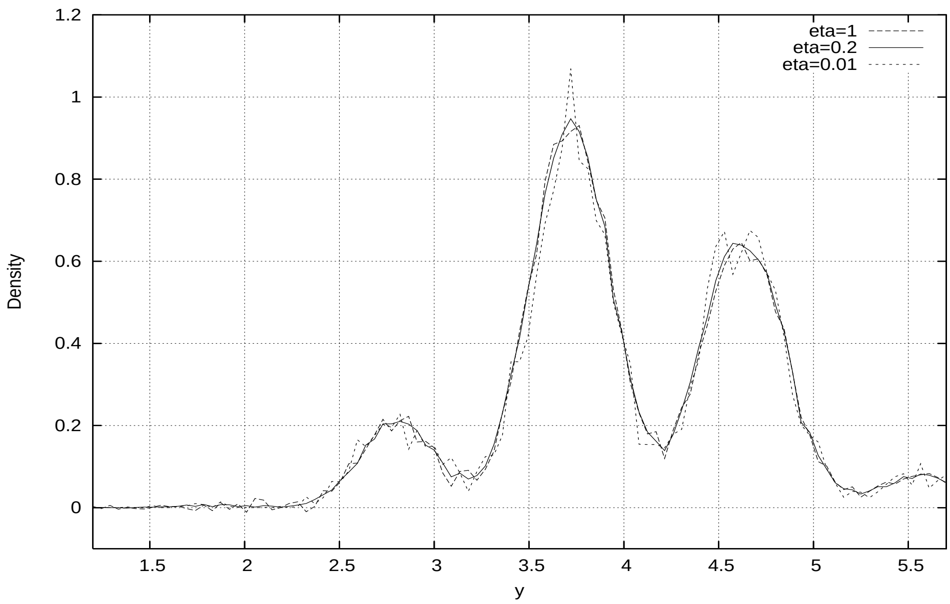

Figure 6.3: Modified kernel estimates of ϕ_{F_r} with 10000 samples and $r = 0.2$.

7 Conclusion

Both Malliavin and modified kernel estimators are consistent. The performances of kernel estimators are dependent on the choice of a bandwidth parameter η. The results of the Malliavin method are independent of η but may be degraded as the weight variance increases. In the examples considered in the paper, the latter performs better than the other estimators.

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