Stein approximation for Itô and Skorohod integrals by Edgeworth type expansions

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Abstract

We derive Edgeworth-type expansions for Skorohod and Itô integrals with respect to Brownian motion, based on cumulant operators defined by the Malliavin calculus. As a consequence we obtain Stein approximation bounds for stochastic integrals, which apply to SDE solutions and to multiple stochastic integrals.

Key words: Stein method; cumulants; Malliavin calculus; Wiener space; Edgeworth expansions; Itô integral; Skorohod integral.
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1 Introduction

Classical Edgeworth series around the Gaussian cumulative distribution function $\Phi(x)$ take the form

$$\Phi(x) + c_1\phi(x)H_1(x) + \cdots + c_m\phi(x)H_m(x) + \cdots ,$$

where $\phi(x)$, $x \in \mathbb{R}$, is the standard Gaussian density, $H_k(x)$ is the Hermite polynomial of degree $k \geq 1$, and c_k is a coefficient depending on the sequence of cumulants

$(\kappa_n)_{n\geq 1}$ of a random variable F, cf. Chapter 5 of [6] and § A.4 of [11]. Edgeworth expansions are used in particular as asymptotic expansions for the cumulative distribution function $P(F \leq x)$ (or, in more general forms, as asymptotic expansions for expectations of the type $E[h(F)]$, where h is some test function - see [6]), when F is centered with unit variance $E[F^2] = 1$, for example when F is a renormalized sum of independent random variables that can be approximated by the central limit theorem, cf. Chapter 2 of [5].

In [1], Edgeworth type expansions of the form

$$E[Ff(F) - f'(F)] = \sum_{l=2}^{\infty} \frac{\kappa_{l+1}}{l!} E[f^{(l)}(F)], \qquad f \in \mathcal{C}^\infty(\mathbb{R}),$$

have been derived and connected to classical Edgeworth series for $E[h(F)]$ by the Stein equation

$$h(F) = E[h(\mathcal{N})] + Ff(F) - f'(F),$$

where h is some adequate test function and $\mathcal{N} \simeq \mathcal{N}(0,1)$ is a standard Gaussian random variable. Recently, Edgeworth type expansions with exact remainder term of the form

$$E[Ff(F) - f'(F)] = \sum_{l=2}^{n} \frac{\kappa_{l+1}}{l!} E[f^{(l)}(F)] + E[f^{(n+1)}(F)\Gamma_{n+1}F], \tag{1.1}$$

have been obtained by the Malliavin calculus in [8], [2], [3], written here for F a centered random variable with unit variance, where Γ_{n+1} is a cumulant type operator on the Wiener space satisfying the relation $n!E[\Gamma_n F] = \kappa_{n+1}$, $n \in \mathbb{N}$, cf. [10]. This approach refines and extends the application of the Malliavin calculus to Stein approximation, Berry-Esseen bounds and the fourth moment theorem initiated in [9], see also [12], and [7] for a review.

The approaches of [2], [8], [9] and the cumulant operators of [10] rely on covariance identities based on the number (or Ornstein-Uhlenbeck) operator L and its inverse on the Wiener space, and they are particularly well suited to the study of multiple stochastic integrals.

In this paper we derive a Edgeworth type expansions for random variables represented as the Itô or Skorohod integral $F = \delta(u)$ of a process u on the Wiener space. Our expansions rely on properties of the operator δ, which coincides with the Itô stochastic integral with respect to d-dimensional Brownian motion on the square-integrable adapted processes, and are applied to Stein approximation bounds. Although this approach does not rely on the operator L, it nevertheless also covers the case of multiple stochastic integrals.

In Section 2 we derive expansions of the form (1.1) for $E[\delta(u)f(\delta(u)) - f'(\delta(u))]$, based on a family of cumulant operators that are associated to the process u and specially defined for the Skorohod integral operator δ. In Section 3 we derive Stein type approximation bounds for stochastic integrals, and we apply them to the solutions of stochastic differential equations. In Section 4 we also provide an alternative approach to the results of [9] on multiple stochastic integrals.

Notation and cumulant operators for the Skorohod integral

Consider a standard d-dimensional Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ on the Wiener space Ω. Letting $H = L^2(\mathbb{R}_+;\mathbb{R}^d)$, we consider the standard Sobolev spaces of real-valued, resp. H-valued, functionals $\mathbb{D}_{p,k}$, resp. $\mathbb{D}_{p,k}(H)$, $p, k \geq 1$, for the Malliavin gradient D on the Wiener space, cf. [13] for definitions. Recall that the Skorohod operator δ is the adjoint of the gradient D through the duality relation

$$E[F\delta(v)] = E[\langle DF, v\rangle_H], \quad F \in \mathrm{Dom}(D), \quad v \in \mathrm{Dom}(\delta), \tag{1.2}$$

and we have the commutation relation

$$D_t\delta(u) = u(t) + \delta(D_t u), \qquad t \in \mathbb{R}_+, \tag{1.3}$$

provided $u \in \mathbb{D}_{2,1}(H)$ and $D_t u \in \mathrm{Dom}(\delta)$, dt-*a.e.*, cf. Proposition 1.3.2 of [13].

Next we define an operator composition $(Du)^k$ and its adjoint D^* in the sense of matrix powers with continuous indices. Namely, given $u \in \mathbb{D}_{2,1}(H)$ and $k \geq 1$, we let $(Du)^k$ denote the random operator on H almost surely defined by

$$(Du)^k h_s = \int_0^\infty \cdots \int_0^\infty (D_{t_k} u_s D_{t_{k-1}} u_{t_k} \cdots D_{t_1} u_{t_2}) h_{t_1} dt_1 \cdots dt_k, \quad s \in \mathbb{R}_+, \quad h \in H, \tag{1.4}$$

cf. e.g. § 7 of [17], [16], [15] for details. In the sequel we will simply denote $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_H$. The adjoint $D^* u$ of Du on H satisfies

$$\langle (Du)v, h \rangle = \langle v, (D^*u)h \rangle, \qquad h, v \in H,$$

and is given by

$$(D^*u)v_s = \int_0^\infty (D_s u_t) v_t dt, \qquad s \in \mathbb{R}_+, \quad v \in L^2(W; H).$$

Given $u \in \mathbb{D}_{k,2}(H)$, our results will be based on a family of cumulant operators

$$\Gamma_k^u : \mathbb{D}_{2,1} \longrightarrow L^2(\Omega), \qquad k \geq 1,$$

defined by $\Gamma_1^u F := \langle u, DF \rangle$ and

$$\Gamma_k^u F := F \langle (Du)^{k-2} u, u \rangle + F \langle D^* u, D((Du)^{k-2} u) \rangle + \langle (Du)^{k-1} u, DF \rangle, \qquad k \geq 2.$$

Note that the operator Γ^u is directly relevant to the integrand u in the stochastic integral representation $\delta(u)$ and as such it differs from the Γ operator of [10] appearing in (1.1), in addition, those operators are not directly related to the Bakry-Émery-Ledoux Γ and Γ_2 operators.

Recall that by the proof of Lemma 3.1 in [14] we have

$$\langle D^* u, D((Du)^k v) \rangle_{H \otimes H} = \mathrm{trace}\,((Du)^{k+1} Dv) + \sum_{i=2}^{k+1} \frac{1}{i} \langle (Du)^{k+1-i} v, D\,\mathrm{trace}\,(Du)^i \rangle, \tag{1.5}$$

$u \in \mathbb{D}_{2,2}(H)$, $v \in \mathbb{D}_{2,1}(H)$, $k \in \mathbb{N}$, hence by the relation

$$\langle (Du)^k h, u \rangle = \langle (D^*u)^k u, h \rangle = \frac{1}{2} \langle (D^*u)^{k-1} D \langle u, u \rangle, h \rangle = \frac{1}{2} \langle (Du)^{k-1} h, D \langle u, u \rangle \rangle, \tag{1.6}$$

$h \in H$, $k \geq 1$, $u \in \mathbb{D}_{2,1}(H)$, which follows from $D\langle u, u\rangle = 2(D^*u)u$, for any $u \in \mathbb{D}_{2,2}(H)$ we have

$$\Gamma_k^u \mathbf{1} = \frac{1}{2}\langle (Du)^{k-3}u, D\langle u, u\rangle\rangle + \mathrm{trace}\,(Du)^k + \sum_{i=2}^{k-1} \frac{1}{i}\langle (Du)^{k-1-i}u, D\,\mathrm{trace}\,(Du)^i\rangle, \tag{1.7}$$

for all $k \geq 3$.

2 Edgeworth type expansions

The duality (1.2) and the commutation relation (1.3) show that

$$E\big[f'(\delta(u))\langle u, u\rangle - \delta(u)f(\delta(u))\big] = -E\left[f'(\delta(u))\langle u, \delta(Du)\rangle\right], \tag{2.1}$$

for $u \in \mathbb{D}_{1,2}(H)$, $F \in \mathbb{D}_{2,1}$ and $f \in \mathcal{C}_b^1(\mathbb{R})$. Applying the above relation (2.1) with $d = 1$ to the solution f_x of the Stein equation

$$\mathbf{1}_{(-\infty,x]}(z) - \Phi(x) = f_x'(z) - zf_x(z), \qquad z \in \mathbb{R}, \tag{2.2}$$

satisfying the bounds $\|f_x\|_\infty \leq \sqrt{2\pi}/4$ and $\|f_x'\|_\infty \leq 1$, cf. Lemma 2.2-(v) of [4], yields the expansion

$$P(\delta(u) \leq x) - \Phi(x) = E\left[(1 - \langle u, u\rangle)f_x'(\delta(u))\right] - E\left[\langle u, \delta(Du)\rangle f_x'(\delta(u))\right], \qquad x \in \mathbb{R},$$

around the Gaussian cumulative distribution function $\Phi(x)$, with $u \in \mathbb{D}_{1,2}(H)$. In the next proposition we extend (2.1) into an expansion of all orders that will be applied to Stein approximation in the next section. By comparison with Proposition 3.11 of [2], the last term in the expansion (2.3) below is not given by a cumulant operator.

Proposition 2.1 *Let $n \geq 1$ and assume that $u \in \mathbb{D}_{k,2}(H)$ for all $k = 1, \ldots, n+2$. Then for all $f \in \mathcal{C}_b^{n+1}(\mathbb{R})$ and $F \in \mathbb{D}_{2,1}$ we have*

$$E\left[F\delta(u)f(\delta(u))\right] = \sum_{k=0}^{n} E\left[f^{(k)}(\delta(u))\Gamma_{k+1}^u F\right] \tag{2.3}$$
$$+\frac{1}{2}E\left[Ff^{(n+1)}(\delta(u))\langle (Du)^{n-1}u, D\langle u, u\rangle\rangle\right] + E\left[Ff^{(n+1)}(\delta(u))\langle (Du)^n u, \delta(Du)\rangle\right].$$

Proof. By the duality (1.2) between D and δ, the chain rule of derivation for D and the commutation relation (1.3), for $F \in \mathbb{D}_{2,1}$, $u \in \mathbb{D}_{n+1,2}(H)$, and all $k \in \mathbb{N}$ we have

$$\begin{aligned}
& E[Ff(\delta(u))\langle (Du)^k u, \delta(D^*u)\rangle] - E[Ff'(\delta(u))\langle (D^*u)^{k+1}u, \delta(D^*u)\rangle] \\
&= E[\langle D^*u, D(Ff(\delta(u))(Du)^k u)\rangle] - E[Ff'(\delta(u))\langle (D^*u)^{k+1}u, \delta(D^*u)\rangle] \\
&= E[Ff'(\delta(u))\langle (Du)^{k+1}u, D\delta(u)\rangle] \\
&\quad -E[Ff'(\delta(u))\langle (D^*u)^{k+1}u, \delta(D^*u)\rangle] + E[f(\delta(u))\langle D^*u, D(F(Du)^k u)\rangle] \\
&= E[Ff'(\delta(u))\langle (Du)^{k+1}u, u\rangle] + E[Ff'(\delta(u))\langle (Du)^{k+1}u, \delta(D^*u)\rangle] \\
&\quad -E[Ff'(\delta(u))\langle (D^*u)^{k+1}u, \delta(D^*u)\rangle] + E[f(\delta(u))\langle D^*u, D(F(Du)^k u)\rangle] \\
&= E[Ff'(\delta(u))\langle (Du)^{k+1}u, u\rangle] + E[f(\delta(u))\langle D^*u, D(F(Du)^k u)\rangle] \\
&= E[Ff'(\delta(u))\langle (Du)^{k+1}u, u\rangle] + E[f(\delta(u))\langle (Du)^{k+1}u, DF\rangle] \\
&\quad +E[Ff(\delta(u))\langle D^*u, D((Du)^k u)\rangle],
\end{aligned}$$

which shows that

$$\begin{aligned}
& E[Ff(\delta(u))\langle (Du)^k u, \delta(D^*u)\rangle] - E[Ff'(\delta(u))\langle (D^*u)^{k+1}u, \delta(D^*u)\rangle] \\
&= E[Ff'(\delta(u))\langle (Du)^{k+1}u, u\rangle] + E[Ff(\delta(u))\langle D^*u, D((Du)^k u)\rangle] \\
&\quad +E[f(\delta(u))\langle (Du)^{k+1}u, DF\rangle].
\end{aligned} \tag{2.4}$$

Consequently, since $(Du)^{k-1}u \in \mathbb{D}_{(n+1)/k,1}(H)$ we have $\delta(u) \in \mathbb{D}_{(n+1)/(n-k+1),1}$, and by (2.4) we get

$$\begin{aligned}
& E\left[Ff(\delta(u))\langle (Du)^k u, D\delta(u)\rangle\right] - E\left[Ff'(\delta(u))\langle (Du)^{k+1}u, D\delta(u)\rangle\right] \\
&= E\left[Ff(\delta(u))\langle (Du)^k u, u\rangle\right] + E\left[Ff(\delta(u))\langle (Du)^k u, \delta(D^*u)\rangle\right] \\
&\quad -E\left[Ff'(\delta(u))\langle (Du)^{k+1}u, u\rangle\right] - E\left[Ff'(\delta(u))\langle (Du)^{k+1}u, \delta(D^*u)\rangle\right] \\
&= E\left[Ff(\delta(u))\langle (Du)^k u, u\rangle\right] + E[Ff(\delta(u))\langle D^*u, D((Du)^k u)\rangle] + E[f(\delta(u))\langle (Du)^{k+1}u, DF\rangle] \\
&= E\left[f(\delta(u))\Gamma^u_{k+2}F\right],
\end{aligned}$$

and therefore

$$\begin{aligned}
& E\left[F\delta(u)f(\delta(u))\right] = E\left[Ff'(\delta(u))\langle u, D\delta(u)\rangle\right] + E\left[f(\delta(u))\langle u, DF\rangle\right] \\
&= E\left[f(\delta(u))\langle u, DF\rangle\right] + E\left[Ff^{(n+1)}(\delta(u))\langle (Du)^n u, D\delta(u)\rangle\right]
\end{aligned}$$

$$+\sum_{k=0}^{n-1}\Big(E\left[Ff^{(k+1)}(\delta(u))\langle (Du)^k u, D\delta(u)\rangle\right] - E\left[Ff^{(k+2)}(\delta(u))\langle (Du)^{k+1}u, D\delta(u)\rangle\right]\Big)$$

$$\begin{aligned}
&= E\left[f(\delta(u))\langle u, DF\rangle\right] + \sum_{k=1}^{n} E\left[f^{(k)}(\delta(u))\Gamma_{k+1}^u F\right] + E\left[Ff^{(n+1)}(\delta(u))\langle (Du)^n u, D\delta(u)\rangle\right]\\
&= \sum_{k=0}^{n} E\left[f^{(k)}(\delta(u))\Gamma_{k+1}^u F\right] + E\left[Ff^{(n+1)}(\delta(u))\langle (Du)^n u, D\delta(u)\rangle\right]\\
&= E\left[f(\delta(u))\langle u, DF\rangle\right] + \sum_{k=1}^{n} E\left[f^{(k)}(\delta(u))\Gamma_{k+1}^u F\right] + E\left[Ff^{(n+1)}(\delta(u))\langle (Du)^n u, D\delta(u)\rangle\right]\\
&= \sum_{k=0}^{n} E\left[f^{(k)}(\delta(u))\Gamma_{k+1}^u F\right] + E\left[Ff^{(n+1)}(\delta(u))\langle (Du)^n u, u\rangle\right]\\
&\quad + E\left[Ff^{(n+1)}(\delta(u))\langle (Du)^n u, \delta(Du)\rangle\right]\\
&= \sum_{k=0}^{n} E\left[f^{(k)}(\delta(u))\Gamma_{k+1}^u F\right] + \frac{1}{2}E\left[Ff^{(n+1)}(\delta(u))\langle (Du)^{n-1} u, D\langle u, u\rangle\rangle\right]\\
&\quad + E\left[Ff^{(n+1)}(\delta(u))\langle (Du)^n u, \delta(Du)\rangle\right],
\end{aligned}$$

where we used the relation (1.6). □

Based on Proposition 2.1 we make the following remarks for random isometries and quasi-nilpotent processes satisfying $\text{trace}\,(Du)^k = 0$, $k \geq 2$. Recall that the setting of quasi-nilpotent processes includes the particular case where $(u_t)_{t\in\mathbb{R}_+}$ is an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted process, cf. e.g. Lemma 3.5 of [15] and references therein, in which case $\delta(u)$ coincides with the Itô integral of u, cf. Proposition 1.3.11 of [13].

(i) Quasi-nilpotent processes. When $\text{trace}\,(Du)^k = 0$ for all $k = 2, \ldots, n+1$ we have

$$\begin{aligned}
E\left[\delta(u)f(\delta(u))\right] &= E\left[\langle u, u\rangle f'(\delta(u))\right] + \frac{1}{2}\sum_{k=2}^{n+1} E\left[\langle (Du)^{k-2}u, D\langle u, u\rangle\rangle f^{(k)}(\delta(u))\right]\\
&\quad + E\left[f^{(n+1)}(\delta(u))\langle (Du)^n u, \delta(Du)\rangle\right], \qquad n \geq 0.
\end{aligned}$$

(ii) Random isometries. When $\langle u, u\rangle$ is deterministic we find

$$\begin{aligned}
E\left[\delta(u)f(\delta(u))\right] &= \langle u, u\rangle E\left[f'(\delta(u))\right] + \sum_{k=1}^{n} E\left[\langle D^* u, D((Du)^{k-1}u)\rangle_{H\otimes H} f^{(k)}(\delta(u))\right]\\
&\quad + E\left[f^{(n+1)}(\delta(u))\langle (Du)^n u, \delta(Du)\rangle\right], \qquad n \geq 0.
\end{aligned}$$

(iii) Multiple stochastic integral processes. Taking $u_t = I_n(f_{n+1}(*,t))$ where $n \in \mathbb{N}$ and f_{n+1} is a symmetric square-integrable function on $\mathbb{R}_+^{n+1}$, we have $\delta(u) = I_{n+1}(f_{n+1})$ and

$$\delta(D_t u) = nI_n(f_{n+1}(*,t)) = nu_t, \qquad t \in \mathbb{R}_+. \tag{2.5}$$

Hence, applying again Proposition 2.1 and (1.6) to $u_t = I_{n-1}(f_n(*,t))$, $n \geq 1$, we get

$$\begin{aligned} & E\left[F I_n(f_n) f(I_n(f_n))\right] \\ &= \sum_{k=0}^{n} E\left[f^{(k)}(I_n(f_n))\Gamma^u_{k+1}F\right] + \frac{n}{2} E\left[F f^{(n+1)}(I_n(f_n))\langle (Du)^{n-1}u, D\langle u,u\rangle\rangle\right]. \end{aligned}$$

In the case of random and quasi-nilpotent isometries we get

$$E\left[\delta(u)f(\delta(u))\right] = \langle u,u\rangle E\left[f'(\delta(u))\right] + E\left[f^{(n+1)}(\delta(u))\langle (Du)^n u, \delta(Du)\rangle\right],$$

which shows that $E\left[f^{(n+1)}(\delta(u))\langle (Du)^n u, \delta(Du)\rangle\right] = 0$, $n \in \mathbb{N}$, and recovers the standard Gaussian integration by parts $E\left[\delta(u)f(\delta(u))\right] = \langle u,u\rangle E\left[f'(\delta(u))\right]$, cf. [18].

3 Stein approximation

From now on we work with $d = 1$ and a one-dimensional Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, and we let $\mathcal{N} \simeq \mathcal{N}(0,1)$ denote a standard Gaussian random variable. In comparison with the results of [2], our bounds apply to a different stochastic integral representation.

Given $h : \mathbb{R} \to \mathbb{R}$ an absolutely continuous function with bounded derivative, the functional equation

$$h(z) - \mathrm{E}[h(\mathcal{N})] = f'(z) - zf(z), \qquad z \in \mathbb{R}, \tag{3.1}$$

has a solution $f_h \in \mathcal{C}_b^1(\mathbb{R})$ which is twice differentiable and satisfies the bounds

$$\|f_h'\|_\infty \leq \|h'\|_\infty \quad \text{and} \quad \|f_h''\|_\infty \leq 2\|h'\|_\infty, \qquad x \in \mathbb{R},$$

cf. Lemma 1.2-(v) of [9] and references therein. Let

$$d(F,G)=\sup_{h\in\mathcal{L}}|\mathrm{E}[h(F)]-\mathrm{E}[h(G)]|$$

denote the Wasserstein distance between the laws of F and G, where $\mathcal{L}$ denotes the class of 1-Lipschitz functions. In the sequel we let $\|u\|_2=\|u\|_{L^2(\Omega\times\mathbb{R}_+)}$.

Proposition 3.1 *Let $u\in\bigcap_{k=1}^{3}\mathbb{D}_{k,2}(H)$. We have*

$$d(\delta(u),\mathcal{N})\le E\left[|1-\langle u,u\rangle-\mathrm{trace}\,(Du)^2|\right]+\|u\|_2\|D\langle u,u\rangle\|_2+2E\left[|\langle(Du)u,\delta(Du)\rangle|\right]. \tag{3.2}$$

Proof. For $n=1$ and $F=1$, Proposition 2.1 shows that

$$\begin{aligned}E[\delta(u)f(\delta(u))] &= E[f'(\delta(u))\Gamma_2^u\mathbf{1}]\\ &\quad+\frac{1}{2}E[f''(\delta(u))\langle u,D\langle u,u\rangle\rangle]+E[f''(\delta(u))\langle(Du)u,\delta(Du)\rangle],\end{aligned}$$

hence for any continuous function $h:\mathbb{R}\to[0,1]$, denoting by f_h the solution to (3.1) we have

$$\begin{aligned}&E[h(\delta(u))]-E[h(\mathcal{N})]=E[\delta(u)f_h(\delta(u))-f_h'(\delta(u))]\\ &\quad= E[f_h'(\delta(u))(\Gamma_2^u\mathbf{1}-1)]+\frac{1}{2}E[f_h''(\delta(u))\langle u,D\langle u,u\rangle\rangle]+2E[f_h''(\delta(u))\langle(Du)u,\delta(Du)\rangle],\end{aligned}$$

hence

$$\begin{aligned}&|E[h(\delta(u))]-E[h(\mathcal{N})]|\\ &\quad\le \|h'\|_\infty E\left[|1-\Gamma_2^u\mathbf{1}|\right]+\|h'\|_\infty E\left[|\langle u,D\langle u,u\rangle\rangle|\right]+2\|h'\|_\infty E\left[|\langle(Du)u,\delta(Du)\rangle|\right],\end{aligned}$$

which yields (3.2) by the relation

$$\Gamma_2^u\mathbf{1}=\langle u,u\rangle+\langle D^*u,Du\rangle_{H\otimes H}=\langle u,u\rangle+\mathrm{trace}\,(Du)^2. \tag{3.3}$$

□

By (3.2) and (3.3) we find

$$d(\delta(u),\mathcal{N})\le\|1-\langle u,u\rangle\|_2+\|\mathrm{trace}\,(Du)^2\|_2+\|u\|_2\|D\langle u,u\rangle\|_2+2E\left[|\langle(Du)u,\delta(Du)\rangle|\right],$$

which, as in Section 2, yields the following remarks.

(i) Quasi-nilpotent processes. When $\text{trace}(Du)^2 = 0$, and in particular when $(u_t)_{t\in\mathbb{R}_+}$ is an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted process, we have

$$d(\delta(u),\mathcal{N}) \le E\left[|1-\langle u,u\rangle|\right] + \|u\|_2\|D\langle u,u\rangle\|_2 + 2E\left[|\langle (Du)u, \delta(Du)\rangle|\right]. \quad (3.4)$$

(ii) Random isometries. When $\langle u,u\rangle$ is deterministic we find

$$d(\delta(u),\mathcal{N}) \le |1-\langle u,u\rangle| + \|\text{trace}(Du)^2\|_2 + 2E\left[|\langle (Du)u, \delta(Du)\rangle|\right].$$

As another consequence of Proposition 3.1 and of the Skorohod isometry

$$\text{Var}[\delta(u)] = E[\delta(u)^2] = E[\langle u,u\rangle] + E[\text{trace}(Du)^2],$$

we also find the bound

$$\begin{aligned} d(\delta(u),\mathcal{N}) \;\le\; & |1-\text{Var}[\delta(u)]| + \sqrt{\text{Var}\left[\|u\|_H^2 + \text{trace}(Du)^2\right]} \\ & + \|u\|_2\|D\langle u,u\rangle\|_2 + 2E\left[|\langle (Du)u, \delta(Du)\rangle|\right], \end{aligned} \quad (3.5)$$

for $u \in \bigcap_{k=1}^3 I\!D_{k,2}(H)$, which will be applied below to multiple stochastic integrals. In particular we have the following.

(i) Quasi-nilpotent processes. When $\text{trace}(Du)^2 = 0$ the bound (3.5) yields

$$d(\delta(u),\mathcal{N}) \le |1-\text{Var}[\delta(u)]| + \sqrt{\text{Var}\left[\|u\|_H^2\right]} + \|u\|_2\|D\langle u,u\rangle\|_2 + 2E\left[|\langle (Du)u, \delta(Du)\rangle|\right].$$

(ii) Unit variance. In case $\text{Var}[\delta(u)] = 1$, (3.5) shows that

$$d(\delta(u),\mathcal{N}) \le \sqrt{\mathbb{E}\left[\left(\|u\|_H^2 + \text{trace}(Du)^2\right)^2\right] - 1} + \|u\|_2\|D\langle u,u\rangle\|_2 + 2E\left[|\langle (Du)u, \delta(Du)\rangle|\right].$$

(iii) Multiple stochastic integral processes. Taking $u_t = I_{n-1}(f_n(*,t))$ where f_n is a symmetric square-integrable function on $\mathbb{R}_+^n$, and applying again (1.6) and (2.5), by (3.5) we get

$$d(I_n(f_n),\mathcal{N}) \le |1-n!\|f_n\|_2^2| + \sqrt{\text{Var}\left[\|u\|_H^2 + \text{trace}(Du)^2\right]} + n\|u\|_2\|D\langle u,u\rangle\|_2. \quad (3.6)$$

The above bound (3.6) will be computed in terms of the kernel function f_n in the next section.

When $\delta(u)$ has unit variance and in addition trace$(Du)^2 = 0$ or $(u_t)_{t\in\mathbb{R}_+}$ is an adapted process, we find

$$d(\delta(u),\mathcal{N}) \leq \sqrt{\mathbb{E}\left[\|u\|_H^4\right] - 1} + \|u\|_2\|D\langle u,u\rangle\|_2 + 2E\left[|\langle (Du)u, \delta(Du)\rangle|\right].$$

4 Applications

a) Stochastic differential equations

Consider the stochastic differential equation

$$dX_t = \sigma(X_t)dW_t, \qquad X_0 = x_0,$$

where $\sigma \in \mathcal{C}_b^1(\mathbb{R})$. From Theorem 2.2.1 and Exercise 2.2.1 of [13], we have $X_t \in \text{Dom}(D)$, $t \in [0,T]$, and

$$D_s X_r = \mathbf{1}_{[0,r]}(s)\sigma(X_s)e^{\int_s^r \sigma'(X_u)dW_u - \int_s^r |\sigma'(X_u)|^2 du/2}, \qquad 0 \leq s \leq r. \tag{4.1}$$

Since $X_T = \delta(\mathbf{1}_{[0,T]}\sigma(X))$, and taking $H = L^2([0,T])$, from (3.4) we get

$$\begin{aligned} d(X_T,\mathcal{N}) \quad &\leq \quad E\left[|1 - \langle\sigma(X),\sigma(X)\rangle|\right] + \|\sigma(X)\|_2\|D\langle\sigma(X),\sigma(X)\rangle\|_2 \\ &\quad\quad +2E\left[|\langle(D\sigma(X))\sigma(X),\delta(D\sigma(X))\rangle|\right], \end{aligned} \tag{4.2}$$

where $\langle(D\sigma(X))\sigma(X),\delta(D\sigma(X))\rangle$ is given by

$$D_r\langle\sigma(X),\sigma(X)\rangle = 2\sigma(X_r)\int_r^T \sigma(X_s)\sigma'(X_s)e^{\int_r^s \sigma'(X_u)dW_u - \int_r^s |\sigma'(X_u)|^2 du/2}ds, \quad r \in \mathbb{R}_+.$$

In order to bound the last term in (4.2) we note that

$$\delta(D_r\sigma(X_.)) = \sigma(X_r)\int_r^T \sigma'(X_t)e^{\int_r^t \sigma'(X_u)dW_u - \int_r^t |\sigma'(X_u)|^2 du/2}dW_t, \qquad 0 \leq r \leq T,$$

and by (4.1) we have

$$\langle(D\sigma(X))\sigma(X),\delta(D\sigma(X))\rangle$$

$$
\begin{aligned}
&= \int_0^T \int_0^T \sigma(X_s) D_s\sigma(X_r) ds \delta(D_r\sigma(X_\cdot)) dr \\
&= \int_0^T \sigma'(X_t) \int_0^t |\sigma(X_s)|^2 \int_s^t \sigma(X_r)\sigma'(X_r) e^{\int_r^t \sigma'(X_u)dW_u - \int_r^t |\sigma'(X_u)|^2 du/2} dr ds dW_t,
\end{aligned}
$$

hence the last term in (4.2) can be bounded as

$$
\begin{aligned}
&E[|\langle (D\sigma(X))\sigma(X), \delta(D\sigma(X))\rangle|] \\
&\leq \sqrt{E\left[\int_0^T |\sigma'(X_t)|^2 \left(\int_0^t |\sigma(X_s)|^2 \int_s^t \sigma(X_r)\sigma'(X_r) e^{\int_r^t \sigma'(X_u)dW_u - \int_r^t |\sigma'(X_u)|^2 du/2} dr ds\right)^2 dt\right]} \\
&\leq \sqrt{E\left[\int_0^s t|\sigma'(X_t)|^2 \int_0^t |\sigma(X_s)|^4 (t-s) \int_s^t |\sigma(X_r)\sigma'(X_r)|^2 e^{2\int_r^t \sigma'(X_u)dW_u - \int_r^t |\sigma'(X_u)|^2 du} dr ds dt\right]} \\
&\leq \frac{T^{5/2}}{\sqrt{15}} \|\sigma\|_\infty^3 \|\sigma'\|_\infty^2 e^{T\|\sigma'\|_\infty/2},
\end{aligned}
$$

hence (4.2) provides an asymptotic bound on the distance $d(X_T, \mathcal{N})$ as $\|\sigma'\|_\infty$ tends to 0.

b) Multiple stochastic integrals

We now show that (3.5) can be used to recover the results [9] on multiple stochastic integrals. The bound (3.6) reads

$$
d(I_n(f_n), \mathcal{N}) \leq |1 - n!\|f_n\|_2^2| + \sqrt{\mathrm{Var}\left[\|u\|_H^2 + \mathrm{trace}\,(Du)^2\right]} + n\|u\|_2 \|D\langle u, u\rangle\|_2.
$$

By the multiplication formula for multiple stochastic integrals, cf. e.g. Relation (2.29) in [9] we have

$$
\langle u, u\rangle = \int_0^\infty (I_{n-1}(f_n(*, t)))^2 dt = \sum_{k=1}^n (k-1)! \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}^2 I_{2n-2k}(f_n \otimes_k f_n),
$$

and, since $D_s u(t) = (n-1) I_{n-2}(f_n(*, s, t))$,

$$
\begin{aligned}
\mathrm{trace}\,(Du)^2 &= (n-1)^2 \int_0^\infty \int_0^\infty I_{n-2}(f_n(*, s, t)) I_{n-2}(f_n(*, t, s)) ds dt \\
&= (n-1)^2 \sum_{k=0}^{n-2} k! \begin{pmatrix} n-2 \\ k \end{pmatrix}^2 \int_0^\infty \int_0^\infty I_{2n-4-2k}(f_n(*, s, t) \otimes_k f_n(*, s, t)) ds dt \\
&= (n-1)^2 \sum_{k=2}^{n} (k-2)! \begin{pmatrix} n-2 \\ k-2 \end{pmatrix}^2 I_{2n-2k}(f_n \otimes_k f_n),
\end{aligned}
$$

hence

$$
\begin{aligned}
\Gamma_2^u \mathbf{1} &= I_{2n-2}(f_n \otimes_1 f_n) \\
&\quad + \sum_{k=2}^{n} \left((n-1)^2 (k-2)! \begin{pmatrix} n-2 \\ k-2 \end{pmatrix}^2 + (k-1)! \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}^2 \right) I_{2n-2k}(f_n \otimes_k f_n) \\
&= \sum_{k=1}^{n} k! \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}^2 I_{2n-2k}(f_n \otimes_k f_n),
\end{aligned}
$$

and

$$
\mathrm{Var}\left[\Gamma_2^u \mathbf{1}\right] = \sum_{k=1}^{n-1} k!^2 \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}^4 \|f_n \otimes_k f_n\|_2^2.
$$

We also have

$$
D_r \langle u, u \rangle = 2 \sum_{k=1}^{n-1} (n-k)(k-1)! \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}^2 I_{2n-2k-1}((f_n \otimes_k f_n)(*, r)),
$$

hence

$$
\begin{aligned}
E\left[\int_0^\infty |D_r \langle u, u \rangle|^2 dr\right] &= 4 \sum_{k=1}^{n-1} ((n-k)(k-1)!)^2 \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}^4 \int_0^\infty \|(f_n \otimes_k f_n)(*, r)\|_2^2 dr \\
&= 4 \sum_{k=1}^{n-1} ((n-k)(k-1)!)^2 \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}^4 \|(f_n \otimes_k f_n)\|_2^2.
\end{aligned}
$$

Finally we get

$$
\begin{aligned}
d(I_n(f_n), \mathcal{N}) &\leq |1 - n! \|f_n\|_2^2| + \sqrt{\sum_{k=1}^{n-1} k!^2 \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}^4 \|f_n \otimes_k f_n\|_2^2} \\
&\quad + 2(n-1)\sqrt{(n-1)!} \|f_n\|_2 \sqrt{\sum_{k=1}^{n-1} ((n-k)(k-1)!)^2 \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}^4 \|f_n \otimes_k f_n\|_2^2},
\end{aligned}
$$

which recovers Proposition 3.2 of [9], with different constants.

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