The Dothan pricing model revisited

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Abstract

We compute zero coupon bond prices in the Dothan model by solving the associated PDE using integral representations of heat kernels and Hartman-Watson distributions. We obtain several integral formulas for the price $P(t,T)$ at time $t > 0$ of a bond with maturity $T > 0$ that complete those of the original paper [7], which are shown not to always satisfy the boundary condition $P(T,T) = 1$.

Key words: Interest rate models, Dothan model, PDE, heat kernel, option pricing, Hartman-Watson distribution, Bessel functions.
MSC 2000: 91B24, 60J65, 60H30, 81S40, 33C10, 35A20.

1 Introduction

In the Dothan [7] model, the short term interest rate process $(r_t)_{t\in\mathbb{R}_+}$ is modeled according to a geometric Brownian motion

$$dr_t = \lambda r_t dt + \sigma r_t dB_t, \tag{1.1}$$

where the volatility $\sigma > 0$ and the drift $\lambda \in \mathbb{R}$ are constant parameters and $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. In the Dothan model, the short term interest rate r_t remains always positive, while the proportional volatility term σr_t accounts for the

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sensitivity of the volatility of interest rate changes to the level of the rate r_t.

On the other hand, the Dothan model is the only lognormal short rate model that allows for an analytical formula for the zero coupon bond price

$$P(t,T) = \mathbb{E}\left[e^{-\int_t^T r_s ds} \Big| \mathcal{F}_t \right], \qquad 0 \le t \le T,$$

cf. [7], and it is commonly referenced in the bond pricing literature, cf. e.g. [5]. Other lognormal interest rate models include the BGM [4] model. For convenience of notation we let $p = 1 - 2\lambda/\sigma^2$ and write the solution of (1.1) as

$$r_t = r_0 \exp\left(\sigma B_t - p\sigma^2 t/2\right), \qquad t \in \mathbb{R}_+,$$

where $p\sigma/2$ identifies to the market price of risk, cf. e.g. [13], Section 4.2. By the Markov property of $(r_t)_{t\in\mathbb{R}_+}$, the bond price $P(t,T)$ is a function $F(\tau, r_t)$ of r_t and of the time to maturity $\tau = T - t$:

$$P(t,T) = F(\tau, r_t) = \mathbb{E}\left[e^{-\int_t^T r_s ds} \Big| r_t \right], \qquad 0 \le t \le T. \tag{1.2}$$

In addition, by a standard arbitrage argument, $F(\tau, r)$ satisfies the PDE

$$\begin{cases} \dfrac{\partial F}{\partial \tau}(\tau, r) = \dfrac{1}{2}\sigma^2 r^2 \dfrac{\partial^2 F}{\partial r^2}(\tau, r) + \lambda r \dfrac{\partial F}{\partial r}(\tau, r) - rF(\tau, r) \\ F(0, r) = 1, \qquad r \in \mathbb{R}_+. \end{cases} \tag{1.3}$$

The zero coupon bond price given in [7], page 64, cf. also [5] page 63, is

$$\begin{aligned} &F(\tau, r) = \\ &\frac{x^{p/2}}{\pi^2} e^{-\sigma^2\tau p^2/8} \int_0^\infty \sin(2\sqrt{x}\sinh a) \int_0^\infty u \sin(ua) e^{-u^2\sigma^2\tau/8} \cosh\left(\frac{\pi u}{2}\right) \left| \Gamma\left(-\frac{p}{2} + i\frac{u}{2}\right)\right|^2 du da \\ &+ \frac{2x^{p/2}}{\Gamma(p)} K_p(2\sqrt{x}), \end{aligned} \tag{1.4}$$

with $x = 2r/\sigma^2$, where $\Gamma(z) = \displaystyle\int_0^{+\infty} t^{z-1} e^{-t}\, dt$, $z \in \mathbb{C}$, $\mathcal{R}(z) > 0$, is the Gamma function and

$$K_w(x) = \int_0^\infty e^{-x\cosh z} \cosh(wz) dz = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-x\cosh z + wz} dz, \qquad x \in \mathbb{R}, \tag{1.5}$$

is the modified Bessel function of the second kind of order $w \in \mathbb{C}$, cf. page 376 of [1] or page 181 of [14]. A proof of (1.4) is given in [7] in case $p = 1$, while the argument given therein is not complete in the general case $p \in \mathbb{R}$. We will show in particular that (1.4) does not satisfy the correct initial condition $F(0, r) = 1$, $r > 0$, for all values of the parameter p, although it satisfies $F(\tau, 0) = 1$ for all $p \geq 0$ and $\tau \geq 0$.

In Section 2 we obtain a bond pricing formula based on the joint probability density of [15]. As an example, Figure 1 provides a numerical comparison between the result of Corollary 2.3 below and Relation (1.4) as functions of $\tau > 0$ with $r = 0.06$, $\sigma = 0.5$ and $p = -0.8441$, in which the bond price given by (1.4) appears to be an underestimate that can become negative and does not match the terminal condition $P(T, T) = 1$.

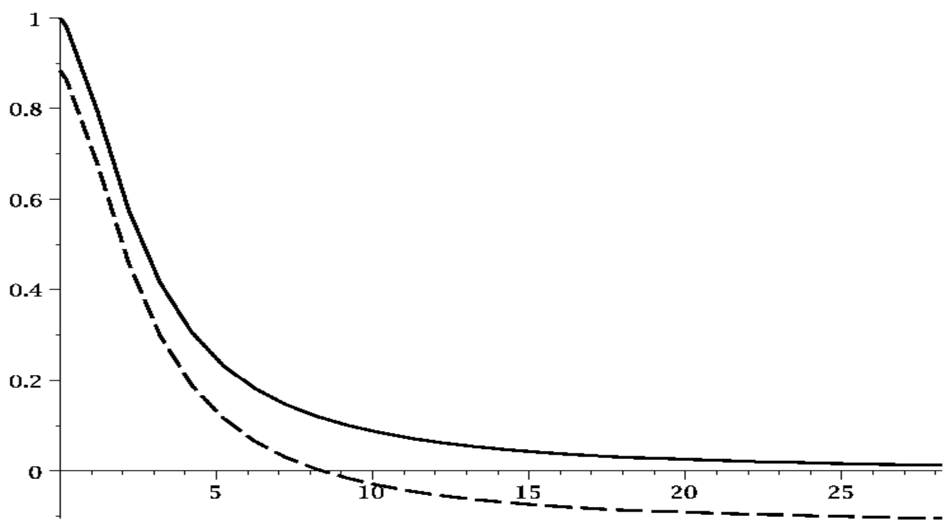

Figure 1: Comparison between Relations (2.9) (straight line) and (1.4) (dotted line).

As can be expected from (1.2), tractable expressions for the bond price $P(t, T)$ are more difficult to obtain for large values of $p \in \mathbb{R}$. We also derive an analytical formula for the pricing of bond options in Proposition 2.4. Note that related techniques have been applied to the pricing of Asian options, cf. e.g. [2], [6], [8], and references therein.

In Section 3 we present other expressions for $P(t, T)$, which are closer to the original formula (1.4), by solving the PDE (1.3) using a heat kernel representation and Gamma functions, and in Corollary 3.3 we obtain another probabilistic interpretation for $P(t,T)$ using hyperbolic cosine random variables.

2 Probabilistic approach

In [15], Proposition 2, the joint probability density of

$$\left(\int_0^\tau e^{\sigma B_s - p\sigma^2 s/2} ds, B_\tau\right), \qquad \tau > 0,$$

has been computed in the case $\sigma = 2$, cf. also [10]. Applying Brownian rescaling, this density can be written for an arbitrary variance parameter σ as

$$\begin{aligned} &\mathbb{P}\left(\int_0^\tau e^{\sigma B_s - p\sigma^2 s/2} ds \in du, B_\tau - p\sigma\tau/2 \in dy\right) \qquad (2.1) \\ &\quad = \frac{\sigma}{2} e^{-p\sigma y/2 - p^2\sigma^2\tau/8} \exp\left(-2\frac{1+e^{\sigma y}}{\sigma^2 u}\right) \theta\left(\frac{4e^{\sigma y/2}}{\sigma^2 u}, \frac{\sigma^2\tau}{4}\right) \frac{du}{u} dy, \end{aligned}$$

$u > 0$, $y \in \mathbb{R}$, $\tau > 0$, where

$$\theta(v,t) = \frac{v e^{\pi^2/(2t)}}{\sqrt{2\pi^3 t}} \int_0^\infty e^{-\xi^2/(2t)} e^{-v\cosh\xi} \sinh(\xi) \sin(\pi\xi/t)\, d\xi, \qquad v, t > 0. \qquad (2.2)$$

The following result is obtained by applying (2.1) to the computation of the conditional expectation (1.2).

Proposition 2.1 *The zero-coupon bond price $P(t,T) = F(T-t, r_t)$ is given for all $p \in \mathbb{R}$ by*

$$F(\tau, r) = e^{-\sigma^2 p^2 \tau/8} \int_0^\infty \int_0^\infty e^{-ur} \exp\left(-\frac{2(1+z^2)}{\sigma^2 u}\right) \theta\left(\frac{4z}{\sigma^2 u}, \frac{\sigma^2\tau}{4}\right) \frac{du}{u} \frac{dz}{z^{p+1}}. \qquad (2.3)$$

Proof. We have

$$\begin{aligned} F(T-t, r_t) &= P(t,T) \\ &= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) \Big| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\exp\left(-r_t \int_t^T e^{\sigma(B_s - B_t) - \sigma^2 p(s-t)/2} ds\right) \Big| \mathcal{F}_t\right] \qquad (2.4) \\ &= \mathbb{E}\left[\exp\left(-r \int_t^T e^{\sigma(B_s - B_t) - \sigma^2 p(s-t)/2} ds\right)\right]_{r=r_t} \\ &= \mathbb{E}\left[\exp\left(-r \int_0^{T-t} e^{\sigma(B_{t+s} - B_t) - \sigma^2 ps/2} ds\right)\right]_{r=r_t} \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\exp\left(-r\int_0^{T-t} e^{\sigma B_s-\sigma^2 ps/2}ds\right)\right]_{r=r_t} \\
&= \int_0^\infty \int_{-\infty}^\infty e^{-r_t u}\mathbb{P}\left(\int_0^\tau e^{\sigma B_s - p\sigma^2 s/2}ds \in du, B_\tau \in dy\right),
\end{aligned}
$$

with $\tau := T-t$, and the conclusion follows from the change of variable $z = e^{\sigma y/2}$, using (2.1). □

The above formula involves a triple integral which can be difficult to evaluate in practice. Next we present alternative representation formulas under some integrability conditions that involve only double integrals and special functions as in [7], and are more appropriate for numerical computation.

Corollary 2.2 *The zero-coupon bond price $P(t,T) = F(T-t, r_t)$ is given for all $p \in \mathbb{R}$ by*

$$
F(\tau, r) = \frac{8\sqrt{r}}{\sigma^2\sqrt{\pi^3\tau}} e^{-\sigma^2p^2\tau/8 + 2\pi^2/(\sigma^2\tau)} \tag{2.5}
$$
$$
\int_0^\infty \int_0^\infty e^{-2\xi^2/(\sigma^2\tau)} \frac{\sinh(\xi)\sin\left(4\pi\xi/(\sigma^2\tau)\right)}{\sqrt{1+2z\cosh\xi + z^2}} K_1\left(\sqrt{8r}\sqrt{1+2z\cosh\xi+z^2}/\sigma\right) d\xi \frac{dz}{z^p}.
$$

Proof. Relation (2.3) can be rewritten as

$$
F(\tau, r) = e^{-\sigma^2p^2\tau/8}\int_{-\infty}^\infty \int_0^\infty \exp\left(-\frac{4zr}{v\sigma^2} - v\frac{1+z^2}{2z}\right)\theta\left(v, \frac{\sigma^2\tau}{4}\right)\frac{dv}{v}\frac{dz}{z^{p+1}}, \tag{2.6}
$$

after the change of variable $v = 4z/(\sigma^2 u)$. Now, applying the Fubini theorem we have

$$
\int_0^\infty \exp\left(-\frac{4zr}{v\sigma^2} - v\frac{1+z^2}{2z}\right)\theta\left(v, \frac{\sigma^2\tau}{4}\right)\frac{dv}{v} \tag{2.7}
$$
$$
= \frac{e^{2\pi^2/(\sigma^2\tau)}}{\sqrt{\pi^3\sigma^2\tau/2}}\int_0^\infty e^{-2\xi^2/(\sigma^2\tau)}\sin\left(\frac{4\pi\xi}{\sigma^2\tau}\right)\sinh(\xi)\int_0^\infty \exp\left(-\frac{4rz}{v\sigma^2} - v\frac{1+2z\cosh\xi+z^2}{2z}\right)dvd\xi,
$$

since the integrand in (2.7) belongs to $L^1(\mathbb{R}_+^2)$ as it is bounded by

$$
(\xi, v) \mapsto \frac{1}{\sqrt{\pi^3\sigma^2\tau/2}} e^{2(\pi^2-\xi^2)/(\sigma^2\tau)}\sinh(\xi)\exp\left(-v\frac{1+z^2}{z} - \frac{4zr}{v\sigma^2}\right).
$$

Next we have

$$
\int_0^\infty \exp\left(-\frac{4rz}{v\sigma^2} - v\frac{1+2z\cosh\xi+z^2}{2z}\right)dv = \frac{4rz}{\sigma^2}\int_0^\infty \exp\left(-u - 2r\frac{1+2z\cosh\xi+z^2}{u\sigma^2}\right)\frac{du}{u^2}
$$

$$= \quad \frac{4z\sqrt{2r}}{\sigma}\frac{K_1\left(\sqrt{8r}\sqrt{1+2z\cosh\xi+z^2}/\sigma\right)}{\sqrt{1+2z\cosh\xi+z^2}},$$

where we used the identity

$$K_\nu(z) = \frac{z^\nu}{2^{\nu+1}}\int_0^\infty \exp\left(-u-\frac{z^2}{4u}\right)\frac{du}{u^{\nu+1}}, \quad \nu\in\mathbb{R}, \tag{2.8}$$

cf. [14] page 183, provided $\mathcal{R}(z^2) > 0$. □

The next corollary provides an alternative expression for the bond price using a double integral, which is however valid only for $p < 1$.

Corollary 2.3 *The zero-coupon bond price $P(t,T) = F(T-t, r_t)$ is given for all $p < 1$ by*

$$F(\tau,r) = 2e^{-\sigma^2p^2\tau/8}\int_0^\infty (v^2+8r/\sigma^2)^{p/2}\,\theta\left(v,\frac{\sigma^2\tau}{4}\right)K_p\left(\sqrt{v^2+8r/\sigma^2}\right)\frac{dv}{v^{p+1}}. \tag{2.9}$$

Proof. From Relations (2.6) and (2.8) we get

$$\begin{aligned} F(\tau,r) &= e^{-\sigma^2p^2\tau/8}\int_0^\infty \theta\left(v,\frac{\sigma^2\tau}{4}\right)\int_0^\infty \exp\left(-\frac{v}{2z}-z\left(\frac{v}{2}+\frac{4r}{\sigma^2 v}\right)\right)\frac{dz}{z^{p+1}}\frac{dv}{v} \\ &= 2e^{-\sigma^2p^2\tau/8}\int_0^\infty \theta\left(v,\frac{\sigma^2\tau}{4}\right)\left(v^2+\frac{8r}{\sigma^2}\right)^{p/2}K_p\left(\sqrt{v^2+\frac{8r}{\sigma^2}}\right)\frac{dv}{v^{p+1}}, \end{aligned}$$

where, letting

$$C(t) := \frac{e^{\pi^2/(2t)}}{\sqrt{2\pi^3 t}}\int_0^\infty e^{-\xi^2/(2t)}\sinh(\xi)d\xi = \frac{1}{\sqrt{\pi^3}}e^{t/2+\pi^2/(2t)}\int_0^{\sqrt{t/2}} e^{-x^2}\,dx < \infty,$$

$t > 0$, we have applied the Fubini theorem as

$$\begin{aligned} &\int_0^\infty\int_0^\infty \left|\theta\left(v,\frac{\sigma^2\tau}{4}\right)\exp\left(-\frac{v}{2z}(1+z^2)-\frac{4rz}{\sigma^2 v}\right)\right|\frac{dv}{v}\frac{dz}{z^{p+1}} \\ &\leq C(\sigma^2\tau/4)\int_0^\infty\int_0^\infty \exp\left(-\frac{v}{2z}(1+z^2)-\frac{4rz}{\sigma^2 v}\right)dv\frac{dz}{z^{p+1}} \\ &= \frac{4\sqrt{2r}}{\sigma}C(\sigma^2\tau/4)\int_0^\infty K_1\left(\sqrt{8r}\sqrt{1+z^2}/\sigma\right)\frac{z}{z+1}\frac{dz}{z^{p+1}} \\ &< \infty, \end{aligned}$$

for all $p < 1$, since from [1] page 378 we have $K_1(y) \underset{y\to\infty}{\sim} \sqrt{\frac{\pi}{2y}}e^{-y}$. □

Figure 2 provides a numerical comparison between the result (2.9) of Corollary 2.3 and (1.4) as functions of $r > 0$ with $T - t = 10.8$, $\sigma = 0.6$, and $p = -0.48$. Here the bond price given by (1.4) may also become negative and does not match the terminal condition $F(0, r) = 1$. In addition it is numerically less stable than (2.9), given that the same numerical algorithm has been used for the discretization of integrals.

We close this section with an analytical formula for the price of a bond option, obtained from the probability density function (2.1) and the same argument as in Corollary 2.2.

Proposition 2.4 *The price of a bond option with payoff function $h(x)$ is given by*

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) h(F(S-T, r_T))\Big|\mathcal{F}_t\right] \\
&= \frac{8\sqrt{r_t}}{\sigma^2\sqrt{\pi^3\tau}} e^{2\pi^2/(\sigma^2\tau) - p^2\sigma^2\tau/8} \int_0^\infty z^{-p-1} h(F(S-T, r_t e^{-p\sigma^2\tau/2} z^2)) \\
&\quad \times \int_0^\infty e^{-2\xi^2/(\sigma^2\tau)} \frac{\sinh(\xi)\sin(4\pi\xi/\sigma^2\tau))}{\sqrt{(z+e^\xi)(z+e^{-\xi})}} K_1\left(\frac{\sqrt{8r_t}}{\sigma}\sqrt{(z+e^\xi)(z+e^{-\xi})}\right) d\xi dz.
\end{aligned}$$

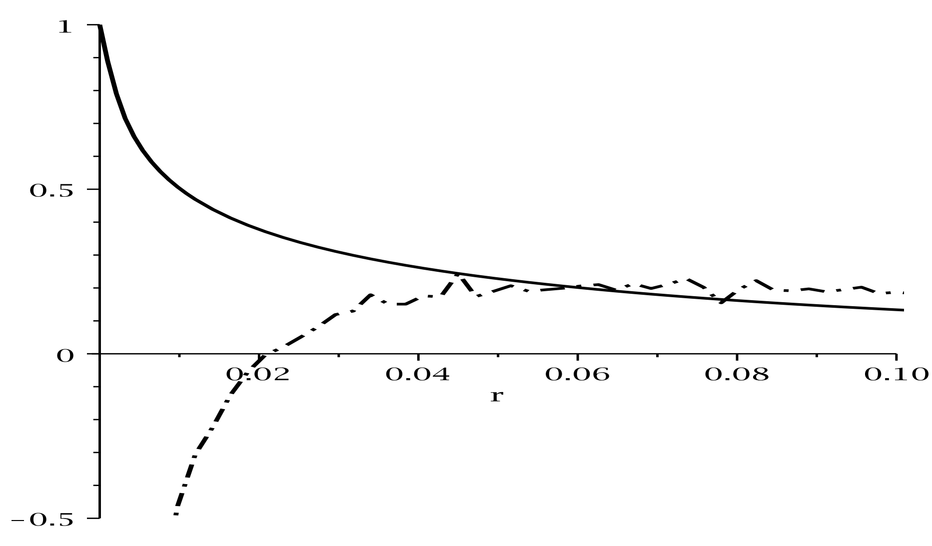

Figure 2: Comparison between Relations (2.9) (straight line) and (1.4) (dotted line).

3 PDE approach

In this section we derive another integral representation for the solution of the bond pricing PDE for $p \in (-\infty, 0)$, which is closer to Dothan's original formula (1.4). The

Dothan PDE (1.3) can be transformed into the simpler equation

$$\begin{cases} \dfrac{\partial U}{\partial s}(s,y) = -\left(H + \dfrac{p^2}{2}\right) U(s,y) \\ \\ U(0,y) = e^{-py}, \qquad y \in \mathbb{R}, \end{cases} \tag{3.1}$$

under the change of variable

$$F(\tau, r) := \left(\frac{2^{3/2}\sqrt{r}}{\sigma}\right)^p U\left(\frac{\sigma^2 \tau}{4}, \log\left(\frac{2^{3/2}\sqrt{r}}{\sigma}\right)\right),$$

where $H := -\dfrac{1}{2}\dfrac{\partial^2}{\partial y^2} + \dfrac{1}{2}e^{2y}$ is a Hamiltonian operator with Sturm-Liouville potential, cf. [9], hence the solution $U(s,y)$ of (3.1) is given by

$$U(s,y) = e^{-sp^2/2} \int_{-\infty}^{\infty} q_s(y,x) e^{-px} dx, \tag{3.2}$$

where the kernel $q_s(x,y)$ of $(e^{-sH})_{s \in \mathbb{R}_+}$ can be expressed as

$$q_t(x,y) = \frac{2}{\pi^2} \int_0^\infty u e^{-u^2 t/2} \sinh(\pi u) K_{iu}(e^y) K_{iu}(e^x) du, \tag{3.3}$$

$t > 0$, $x, y \in \mathbb{R}$, cf. [3], page 115, and [9] page 228. As a consequence the zero-coupon bond price $P(t,T) = F(T-t, r_t)$ is given for all $p \in \mathbb{R}$ by

$$\begin{aligned} F(\tau, r) &= \left(\frac{2^{3/2}\sqrt{r}}{\sigma}\right)^p U\left(\frac{\sigma^2 \tau}{4}, \log\left(\frac{2^{3/2}\sqrt{r}}{\sigma}\right)\right) \\ &= \frac{2^p (2r)^{p/2}}{\sigma^p} \exp\left(-\frac{\sigma^2 p^2 \tau}{8}\right) \int_{-\infty}^{\infty} e^{-py} q_{\sigma^2\tau/4}\left(\log\left(\frac{2\sqrt{2r}}{\sigma}\right), y\right) dy. \end{aligned}$$

Using the integral representation (3.3) of the kernel $q_t(x,y)$ we obtain the following result which clearly does not coincide with Dothan's formula (1.4) when $p < 0$, due to the absence of the Bessel function term $\dfrac{2x^{p/2}}{\Gamma(p)} K_p(2\sqrt{x})$ in (3.5) below.

Proposition 3.1 *The zero-coupon bond price $P(t,T) = F(T-t, r_t)$ is given for all $p \in \mathbb{R}$ by $F(\tau, r) =$*

$$\frac{2^{p+1}(\sqrt{2r})^p}{\pi^2 \sigma^p} e^{-p^2\sigma^2\tau/8} \int_0^\infty e^{-py} \int_0^\infty u \sinh(\pi u) e^{-u^2\sigma^2\tau/8} K_{iu}(\sqrt{8r}/\sigma) K_{iu}(e^y) du dy, \tag{3.4}$$

$r > 0$, $\tau > 0$.

From a computational point of view the above formula actually involves a triple integral of a Bessel function, which can be simplified to a double integral of a Gamma function under some additional conditions on p.

Corollary 3.2 *The zero-coupon bond price $P(t,T) = F(T-t, r_t)$ is given for all $p < 0$ by $F(\tau, r) =$*

$$\frac{x^{p/2}}{\pi^2} e^{-p^2\sigma^2\tau/8} \int_0^\infty \sin(2\sqrt{x}\sinh a) \int_0^\infty u e^{-u^2\sigma^2\tau/8} \cosh\left(\frac{\pi u}{2}\right) \left|\Gamma\left(-\frac{p}{2} + i\frac{u}{2}\right)\right|^2 \sin(ua) du da, \tag{3.5}$$

$r > 0$, $\tau > 0$, with $x = 2r/\sigma^2$.

Proof. First, from (3.4), after the change of variable $z = e^y$, we notice that

$$F(\tau, r) = r^{p/2} \frac{2^{1+3p/2}}{\pi^2 \sigma^p} e^{-p^2\sigma^2\tau/8} \int_0^\infty \int_0^\infty u e^{-\sigma^2 u^2 \tau/8} \sinh(\pi u) K_{iu}(z) K_{iu}\left(\sqrt{8r}/\sigma\right) du \frac{dz}{z^{p+1}},$$

and we note that

$$\begin{aligned} & \int_0^\infty \frac{1}{z^{p+1}} \int_0^\infty \left| u e^{-\sigma^2 u^2 \tau/8} \sinh(\pi u) K_{iu}(z) K_{iu}\left(\sqrt{8r}/\sigma\right) \right| du dz \\ \leq \; & C \int_0^\infty u e^{-\sigma^2 u^2 \tau/8} \sinh(\pi u) du \times \sup_{w \in \mathbb{R}_+} \int_0^\infty z^{-p-1} \left| K_{iw}(z) \right| dz < \infty, \end{aligned}$$

where $C > 0$ is a constant. From this bound we can apply the Fubini theorem, hence from Proposition 3.1 and the relation

$$\left|\Gamma\left(-\frac{p}{2} - i\frac{w}{2}\right)\right|^2 = \frac{1}{2^{-2-p}} \int_{-\infty}^\infty K_{iu}(e^y) e^{-py} dy du, \qquad \omega \in \mathbb{R},$$

we get

$$\begin{aligned} F(\tau, r) &= \frac{2^{p+1}(\sqrt{2r})^p}{\pi^2 \sigma^p} e^{-p^2\sigma^2\tau/8} \int_0^\infty u \sinh(\pi u) K_{iu}(\sqrt{8r}/\sigma) e^{-u^2\sigma^2\tau/8} \int_{-\infty}^\infty K_{iu}(e^y) e^{-py} dy du \\ &= \frac{(\sqrt{2r})^p}{2\pi^2\sigma^p} e^{-p^2\sigma^2\tau/8} \int_0^\infty u \sinh(\pi u) K_{iu}(\sqrt{8r}/\sigma) e^{-u^2\sigma^2\tau/8} \left|\Gamma\left(-\frac{p}{2} + i\frac{u}{2}\right)\right|^2 du, \qquad (3.6) \\ &= \frac{x^{p/2}}{\pi^2} e^{-\sigma^2 p^2 \tau/8} \int_0^\infty u e^{-\sigma^2 u^2 \tau/8} \cosh\left(\frac{\pi u}{2}\right) \int_0^\infty \sin(2\sqrt{x}\sinh a) \sin(ua) da \left|\Gamma\left(-\frac{p}{2} + i\frac{u}{2}\right)\right|^2 du, \end{aligned}$$

which yields (3.5) for all $p < 0$, where $x = \sqrt{2r}/\sigma$ and we used the integral representation $K_{i\mu}(z) = \dfrac{1}{\sinh(\pi\mu/2)} \displaystyle\int_0^{+\infty} \sin(z \sinh t) \sin(\mu t) dt$, $z \in \mathbb{R}_+$, $\mu \in \mathbb{R}$, that can be derived from the relations on pages 182-183 of [14]. □

Finally we derive a probabilistic representation of the bond price that can be useful for Monte Carlo estimation, using the hyperbolic cosine distribution with characteristic function $u \mapsto (\cosh u)^p$, $p < 0$, cf. [12].

Corollary 3.3 *For all $p < 0$ we have*

$$F(\tau, r) = \Gamma(-p) \frac{2^{3p/2} r^{p/2}}{\pi \sigma^p} e^{-\sigma^2 \tau p^2/8} \, \mathbb{E}\left[Z_p e^{-\sigma^2 \tau Z_p^2/8} \sinh(\pi Z_p) K_{iZ_p}(\sqrt{8r}/\sigma) \right],$$

$r > 0$, $\tau > 0$, where Z_p is an hyperbolic cosine random variable with parameter $-p$.

Proof. We use Relation (3.6) above and the fact that the density of Z_p is given by

$$u \mapsto \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iuy} (\cosh y)^p dy = \frac{2^{-p-2}}{\pi \Gamma(-p)} \left| \Gamma\left(-\frac{p}{2} - i\frac{u}{2} \right) \right|^2, \qquad u \in \mathbb{R},$$

cf. [12], page 300. □

Note added in proof

The result of Corollary 3.2 can be extended to all $p \in \mathbb{R}$ using spectral expansions for the Fokker-Planck equation, cf. [11] and the references therein.

References

[1] M. Abramowitz and I.A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* Dover Publications, New York, 1964.

[2] P. Barrieu, A. Rouault, and M. Yor. A study of the Hartman-Watson distribution motivated by numerical problems related to the pricing of Asian options. *J. Appl. Probab.*, 41(4):1049–1058, 2004.

[3] A. N. Borodin and P. Salminen. *Handbook of Brownian motion – Facts and formulae.* Probability and its Applications. Birkhäuser Verlag, Basel, 1996.

[4] A. Brace, D. Gatarek, and M. Musiela. The market model of interest rate dynamics. *Math. Finance*, 7(2):127–155, 1997.

[5] D. Brigo and F. Mercurio. *Interest rate models—theory and practice.* Springer Finance. Springer-Verlag, Berlin, second edition, 2006.

[6] P. Carr and M. Schröder. Bessel processes, the integral of geometric Brownian motion, and Asian options. *Theory Probab. Appl.*, 48(3):400–425, 2004.

[7] L.U. Dothan. On the term structure of interest rates. *Jour. of Fin. Ec.*, 6:59–69, 1978.

[8] D. Dufresne. Laguerre series for Asian and other options. *Math. Finance*, 10(4):407–428, 2000.

[9] C. Grosche and F. Steiner. *Handbook of Feynman path integrals*, volume 145 of *Springer Tracts in Modern Physics.* Springer-Verlag, Berlin, 1998.

[10] H. Matsumoto and M. Yor. Exponential functionals of Brownian motion. I. Probability laws at fixed time. *Probab. Surv.*, 2:312–347 (electronic), 2005.

[11] C. Pintoux and N. Privault. A direct solution to the Fokker-Planck equation for exponential Brownian functionals. *Analysis and Applications*, 8(3):287–304, 2010.

[12] J. Pitman and M. Yor. Infinitely divisible laws associated with hyperbolic functions. *Canad. J. Math.*, 55(2):292–330, 2003.

[13] N. Privault. *An Elementary Introduction to Stochastic Interest Rate Modeling.* Advanced Series on Statistical Science & Applied Probability, 12. World Scientific Publishing Co., Singapore, 2008.

[14] G. N. Watson. *A treatise on the theory of Bessel functions.* Cambridge University Press, Cambridge, 1995. Reprint of the second (1944) edition.

[15] M. Yor. On some exponential functionals of Brownian motion. *Adv. in Appl. Probab.*, 24(3):509–531, 1992.