Discrete chaotic calculus and covariance identities *

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Abstract

We show that for the binomial process (or Bernoulli random walk) the orthogonal functionals constructed in Kroeker [14] for Markov chains can be expressed using the Krawtchouk polynomials, and by iterated stochastic integrals. This allows to construct a chaotic calculus based on gradient and divergence operators and structure equations, and to establish a Clark representation formula. As an application we obtain simple infinite dimensional proofs of covariance identities on the discrete cube.

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1 Introduction

In the classical deterministic integration theory the polynomials $\{p_n(x) = x^n, n \geq 0\}$, and the exponential function $\exp(x) = \sum_{n=0}^{\infty} p_n(x)/n!$ play a special role because they satisfy $\int_0^t p_n(x)dx = \frac{1}{n+1}p_{n+1}(t)$ and $\int_0^t \exp(x)dx = \exp(t) - \exp(0)$. In some stochastic cases, it turns out that the role of p_n is taken up by orthogonal polynomials related to the distribution of the integrator. The most studied stochastic case is integration with respect to Brownian motion $\{W_t, t \geq 0\}$, where W_t has a normal distribution $\mathcal{N}(0,t)$, i.e. with mean zero and variance $t \geq 0$. The notion of multiple stochastic integration for this process was first introduced by Wiener. As is well known, in stochastic Itô integration theory with respect to standard Brownian motion, the Hermite polynomials play the role of the p_n:

$$\int_0^t H_n(W_s; s)dW_s = \frac{H_{n+1}(W_t; t)}{n+1},$$

where $H_n(x;t)$ is the monic (with leading coefficient equal to one) Hermite polynomial with parameter t. The monic Hermite polynomials $H_n(x,t)$ are orthogonal with respect

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to the normal distribution $\mathcal{N}(0,t)$ of W_t and $(H_n(W_t;t))_{t\in\mathbb{R}_+}\}$ is a martingale. Using the generating function $\sum_{n=0}^{\infty} H_n(x;t)z^n/n! = \exp(-tz^2/2 + zx)$, one can easily see that the role of the exponential function is now taken by the function $\exp(-t/2 + W_t)$ because we have $\int_0^t \exp(-s/2+W_s)dW_s = \exp(-t/2+W_t)-1$. The transformation $\exp(W_t-t/2)$ of the Brownian motion is sometimes called geometric Brownian motion or the stochastic exponent of the Brownian motion. There is a similar result for the compensated Poisson process $M_t = N_t - t$: the monic orthogonal polynomials with respect to the Poisson distribution $\mathrm{P}(t)$ are the monic Charlier polynomials, $C_n(x;t)$, defined by the generating function (Koekoek and Swarttow, [13], 1998): $W(x,t,z) = \sum_{n=0}^{\infty} C_n(x;t)z^n/n! = \exp(-tz)(1+x)^x$. We have:

$$\int_0^t C_n(N_{s-};s)dM_s = \frac{C_{n+1}(N_t;t)}{n+1}. \tag{1}$$

In terms of the generating function, this is equivalent with

$$\int_0^t W(N_{s-},s,w)dM_s = \frac{W(N_t,t,w)-1}{w}.$$

This result goes back to Ogura [17] (1972) and Engel [6] (1982), it implies that the monic Charlier polynomials $\{C_n(N_t;t)\}$ are martingales, see also Schoutens and Teugels [21] (1998), and Schoutens [20] (2000). Chaos expansions for Markov chains have been constructed in Kroeker [14] (1980) via orthogonal functionals that are the analogs of multiple stochastic integrals with respect to martingales. A natural question for investigation is the determination of martingales whose multiple stochastic integrals can be expressed as polynomials. In continuous time, Privault, Solé and Vives [19] (1997) proved that the only normal martingales solutions of structure equations which have an associated family of polynomials are the Poisson process and the Brownian motion. In the i.i.d. case it is shown in Feinsilver [8] (1986) that such polynomials have to be Meixner polynomials. In this paper we will show that the Markov chain approach to multiple stochastic integrals coincides with the i.i.d. approach of [8] only for the binomial process, and that the binomial process is the only i.i.d. discrete time process for which the multiple stochastic integrals of [8] can be expressed with polynomials, namely the Krawtchouk polynomials. Moreover, in this case these functionals can be also expressed as discrete iterated integrals with respect to the compensated binomial process. From this we deduce a chaotic calculus and Clark formula which are applied to the statement and proof of covariance identities in infinite dimensions, extending results obtained by Houdré and Pérez-Abreu [11] (1995) in the continuous time case, and by Bobkov, Götze and Houdré [3] (2000) on the finite discrete cube.

This paper is organized as follows. In Section 3.1 we reformulate the construction of [14] in the language of tensor products. We give a particular attention to this construction because it is valid for processes with non-independent increments, and the non-independence of increments is always a non-trivial problem in chaotic representation, cf. Emery [4] (1990) and Biane [2] (1989) in continuous time. Section 3.2 deals with the i.i.d. case. Section 3.3 is devoted to the representation of orthogonal functionals of the binomial process as Krawtchouk polynomials, and Section 4 presents the iterated stochastic integrals and the relation between the Krawtchouk polynomials and the binomial process. In Section 5 we obtain a Clark representation formula for functionals of the binomial process, using gradient and divergence operators. Section 6 is devoted to the statement and proof of covariance identities.

Other approaches to discrete time stochastic analysis can be found in Holden et al. [9] (1992), [10] (1993), Leitz-Martini [15] (2000), and also in Attal [1] (2000) in the framework of quantum stochastic calculus. In this paper our focus is on multiple stochastic integration and associated polynomials.

2 Notation

We denote by $l^2(\mathbb{N}^*)$ with $\mathbb{N}^* = \mathbb{N}\backslash\{0\} = \{1, 2, \ldots\}$, the set of all square-summable functions on the strictly positive integers and by $(e_k)_{k\in\mathbb{N}^*} = (1_{\{k\}})_{k\in\mathbb{N}^*}$ the canonical basis of $l^2(\mathbb{N}^*)$. The tensor product of functions in $l^2(\mathbb{N}^*)$ will be denoted as "$\otimes$". On the space $l^2(\mathbb{N}^*)^{\otimes n}$ of square summable functions in n strictly positive integer variables, the canonical inner product is defined by

$$\langle e_{i_1}^{\otimes n_1} \otimes \cdots \otimes e_{i_p}^{\otimes n_p}, e_{j_1}^{\otimes m_1} \otimes \cdots \otimes e_{j_q}^{\otimes m_q} \rangle_{l^2(\mathbb{N}^*)^{\otimes n}} = 1_{\{p=q\}} 1_{\{i_k=j_k, 1\leq k\leq p\}} 1_{\{n_k=m_k, 1\leq k\leq p\}},$$

$1 \leq i_1 < i_2 < \cdots < i_p$, $1 \leq j_1 < j_2 < \cdots < j_q$. The symmetric tensor product $l^2(\mathbb{N}^*)^{\circ n}$ is by definition the set of all square-summable symmetric functions in n strictly positive integer variables. Given $f_n \in l^2(\mathbb{N}^*)^{\otimes n}$, the symmetrization in n variables of f_n is the function $\tilde{f}_n \in l^2(\mathbb{N}^*)^{\circ n}$ defined as

$$\tilde{f}_n(k_1, \ldots, k_n) = \frac{1}{n!} \sum_{\sigma\in\Sigma_n} f_n(k_{\sigma(1)}, \ldots, k_{\sigma(n)}), \quad k_1, \ldots, k_n \geq 1, \tag{2}$$

where Σ_n is the set of all permutations of $\{1, \ldots, n\}$. The symmetric tensor product of functions in $l^2(\mathbb{N}^*)$ will be denoted by "$\circ$", i.e. $e_{i_1}^{\circ n_1} \circ \cdots \circ e_{i_d}^{\circ n_d}$ is the symmetrization in $n_1 + \cdots + n_d = n$ variables of $e_{i_1}^{\otimes n_1} \otimes \cdots \otimes e_{i_d}^{\otimes n_d}$. We have

$$\langle e_{i_1}^{\circ n_1} \circ \cdots \circ e_{i_d}^{\circ n_d}, e_{i_1}^{\circ n_1} \circ \cdots \circ e_{i_d}^{\circ n_d} \rangle_{l^2(\mathbb{N}^*)^{\otimes n}} = \frac{n_1! \cdots n_d!}{n!}. \tag{3}$$

Given $f_1, \ldots, f_n \in l^2(\mathbb{N}^*)$, we denote by $f_1 \odot \cdots \odot f_n$ the symmetrization in n variables of

$$(k_1, \ldots, k_n) \mapsto 1_{\{k_i \neq k_j, 1 \leq i < j \leq n\}} f_1(k_1) \cdots f_n(k_n),$$

with

$$\langle e_{i_1}^{\odot n_1} \odot \cdots \odot e_{i_p}^{\odot n_p}, e_{j_1}^{\odot m_1} \odot \cdots \odot e_{j_q}^{\odot m_q} \rangle_{l^2(\mathbb{N}^*)^{\otimes n}} = \frac{1}{n!} 1_{\{p=q,\ n_1=\cdots=n_p=m_1=\cdots=m_q=1,\ i_k=j_k, 1 \leq k \leq p\}}, \quad (4)$$

$1 \leq i_1 < \cdots < i_p$, $1 \leq j_1 < \cdots < j_q$. We call $l^2(\mathbb{N}^*)^{\odot n} \subset l^2(\mathbb{N}^*)^{\otimes n}$ the completed linear span generated by $\{f_1 \odot f_2 \odot \cdots \odot f_n \ : \ f_1, \ldots, f_n \in l^2(\mathbb{N}^*)$. The space $l^2(\mathbb{N}^*)^{\odot n}$ consists in fact of all the symmetric functions in $l^2(\mathbb{N}^*)^{\circ n}$ that vanish on every diagonal in $(\mathbb{N}^*)^n$, in fact $l^2(\mathbb{N}^*)^{\odot n}$ is also the n-th chaos of the toy Fock space, cf. Meyer [16] (1993), p. 14. The monic Krawtchouk polynomials are determined by the generating function Koekoek and Swarttouw [13] (1998):

$$Y(x, N, z) = \sum_{n=0}^{N} K_n(x; N, p) \frac{z^n}{n!} = (1 + qz)^x (1 - pz)^{N-x},$$

where $N \in \mathbb{N}$, $0 < p < 1$ and $p + q = 1$, with $K_n(x; N, p) = 0$ for all $x \in \mathbb{N}$, $n > N$. Explicitly, this implies

$$K_n(x; N, p) = p^n (-N)_n \sum_{i=0}^{i=n} \frac{(-n)_i (-x)_i}{(-N)_i} \frac{(1/p)^i}{i!}, \qquad x, n \in \mathbb{N}, \quad (5)$$

and in particular $K_1(x; N, p) = x - Np$, where $(a)_k = a(a+1)\ldots(a+k-1)$ denotes the Pochhammer symbol, with $(a)_0 = 1$ for all $a \in \mathbb{R}$. As mentioned above, the Krawtchouk polynomials are orthogonal with respect to the binomial distribution $\text{Bin}(N, p)$:

$$\begin{aligned} \sum_{x=0}^{x=N} \binom{N}{x} p^x q^{N-x} K_n(x; N, p) K_m(x; N, p) &= (-1)^n n! (-N)_n (pq)^n 1_{\{n=m\}} \\ &= \binom{N}{n} (pq)^n 1_{\{n=m\}}, \quad 0 \leq n, m \leq N, \end{aligned}$$

see [13].

The binomial process with parameter $0 < p < 1$, denoted by $(B_n)_{n \in \mathbb{N}}$, is a stochastic process such that, $B_0 = 0$ and $(X_i)_{i \geq 1} = (B_i - B_{i-1})_{i \geq 1}$ is a family of i.i.d. Bernoulli random variables with parameter $0 < p = P(X_i = 1) < 1$; B_N has the binomial distribution $\text{Bin}(N, p)$, given by the probabilities $\binom{N}{i} p^i q^{N-i}$, $i = 0, 1, \ldots, N$.

3 Orthogonal Expansions

In this section we formulate, in the language of tensor products, the construction of orthogonal functionals of Markov chains due to [14]. This construction does not seem to be related to the chaos expansions defined in [2] for finite Markov chains in continuous time. The notion of tensor product makes the exposition significantly different, but leads to the same objects. First we consider a general Markov chain. Next, we turn to the i.i.d. case and finally we consider the special case of a binomial process.

3.1 Orthogonal expansions for Markov chains

Let $(S_n)_{n\in\mathbb{N}}$ be a Markov chain with state space $\mathbb{N}$ and transition matrix $(P(x,y))_{x,y\in\mathbb{N}}$, starting from 0, on a probability space Ω. Let $\mu(k)\in\mathbb{N}\cup\{\infty\}$, denote the dimension of $l^2(\mathbb{N};P(k,\cdot))$, and let $(\phi^n(\cdot\mid k))_{0\leq n\leq\mu(k)}$ be a complete orthogonal set of polynomials in $l^2(\mathbb{N};P(k,\cdot))$, with $\phi^n(\cdot\mid k)$ of degree n, $\|\phi^n(\cdot\mid k)\|^2_{l^2(\mathbb{N},P(k,\cdot))}=n!$, $0\leq n\leq\mu(k)$, and $\phi^n(x\mid k)=0$, $n>\mu(k)$ for all $x,k\in\mathbb{N}$.

Remark 1 *In the construction of [14], the functional $\phi^n(x\mid y)$ is not constrained to be a polynomial in $x\in\mathbb{N}$, in this case and the choice of the family $(\phi^n(\cdot\mid y))_{n\in\mathbb{N}}$ is not unique.*

In [14], the data of the initial distribution π of $(S_n)_{n\in\mathbb{N}}$ is also considered. In our notation this can be easily taken into account by letting $P(0,\cdot)=\pi(\cdot)$. Next, we define the map J_n which will be the analog of the multiple stochastic integral in the case of continuous time martingales and maps a symmetric function to a random variable in $L^2(\Omega)$.

Definition 1 *We densely define the linear map $J_n: l^2(\mathbb{N}^*)^{\circ n}\longrightarrow L^2(\Omega)$ as:*

$$J_n(e_{i_1}^{\circ n_1}\circ\cdots\circ e_{i_d}^{\circ n_d})=\prod_{k=1}^{k=d}\phi^{n_k}(S_{i_k}\mid S_{i_k-1}),$$

with $1\leq i_1<\cdots<i_n$, and $n_1+\cdots+n_d=n\geq 1$.

We let $J_0(f_0)=1$, $f_0\in\mathbb{R}$, i.e. $l^2(\mathbb{N})^{\circ 0}$ is identified to $\mathbb{R}$. (By induction on $d\in\mathbb{N}$, the representation of $J_n(e_{i_1}^{\circ n_1}\circ\cdots\circ e_{i_d}^{\circ n_d})$ as $\prod_{k=1}^{k=d}\phi^{n_k}(S_{i_k}\mid S_{i_k-1})$ is unique). For all symmetric function $f_n\in l^2(\mathbb{N}^*)^{\circ n}$ of n variables and finite support, written as

$$f_n=\sum_{d=1}^{d=n}\sum_{\substack{1\leq i_1<\cdots<i_d\\ n_1+\cdots+n_d=n}}a_{i_1,\ldots,i_d}^{n_1,\ldots,n_d}e_{i_1}^{\circ n_1}\circ\cdots\circ e_{i_d}^{\circ n_d}, \tag{6}$$

we let

$$J_n(f_n)=\sum_{d=1}^{d=n}\sum_{\substack{1\leq i_1<\cdots<i_d\\ n_1+\cdots+n_d=n}}a_{i_1,\ldots,i_d}^{n_1,\ldots,n_d}J_n(e_{i_1}^{\circ n_1}\circ\cdots\circ e_{i_d}^{\circ n_d}).$$

Proposition 1 *The functional $J_n(f_n)$ is orthogonal to $J_m(g_m)$ in $L^2(\Omega)$ if $n \neq m$, and $J_n : l^2(\mathbb{N}^*)^{\circ n} \longrightarrow L^2(\Omega)$ extends as a linear continuous operator with*

$$E[J_n(f_n)^2] \leq n! \|f_n\|^2_{l^2(\mathbb{N}^*)^{\otimes n}}, \quad f_n \in l^2(\mathbb{N}^*)^{\circ n}, \ n \in \mathbb{N}. \tag{7}$$

The equality $E[J_n(f_n)^2] = n!\|f_n\|^2_{l^2(\mathbb{N}^*)^{\otimes n}}$, for $f_n \in l^2(\mathbb{N}^*)^{\circ n}$ and $n \in \mathbb{N}$, holds if $\mu(k) = \infty$ for all $k \in \mathbb{N}$.

Proof. By construction we have

$$E[\phi^{n_k}(S_{i_k} \mid S_{i_k-1})\phi^{m_k}(S_j \mid S_{j-1}) \mid S_{i_0}, \ldots, S_{i_k-1}] = n_k! 1_{\{i_k=j\}} 1_{\{n_k=m_k\}} 1_{\{n_k \leq \mu(S_{i_k-1})\}},$$

and for $n_k \geq 1$:

$$E[\phi^{n_k}(S_{i_k} \mid S_{i_k-1}) \mid S_{i_0}, \ldots, S_{i_k-1}] = 0,$$

hence by induction on $k = 1, \ldots, d$,

$$E[J_n(e_{i_1}^{\circ n_1} \circ \cdots \circ e_{i_d}^{\circ n_d}) J_m(e_{j_1}^{\circ m_1} \circ \cdots \circ e_{j_l}^{\circ m_l})] = 0$$

if $\{i_1, \ldots, i_d\} \neq \{j_1, \ldots, j_d\}$ or $n \neq m$, and

$$0 \leq E[J_n(e_{i_1}^{\circ n_1} \circ \cdots \circ e_{i_d}^{\circ n_d})^2] \leq n_1! \cdots n_d!.$$

With $f_n \in l^2(\mathbb{N}^*)^{\circ n}$ and $g_m \in l^2(\mathbb{N}^*)^{\circ m}$ as in (6) we have $E[J_n(f_n)J_m(g_m)] = 0$ if $n \neq m$, and

$$E[J_n(f_n)^2] \leq n! \sum_{d=1}^{d=n} \sum_{\substack{1 \leq i_1 < \cdots < i_d \\ n_1 + \cdots + n_d = n}} \frac{n_1! \cdots n_d!}{n!} a^2_{i_1,\ldots,i_d} = n! \langle f_n, f_n \rangle_{l^2(\mathbb{N}^*)^{\otimes n}},$$

from (3). □

Remark 2 *The isometry formula $E[J_n(f_n)J_m(g_m)] = 1_{\{n=m\}} n! \langle f_n, g_m \rangle_{l^2(\mathbb{N}^*)^{\otimes n}}$ (see Relation (13) in [14]) does not hold in general, e.g. for the binomial process we have*

$$E[J_n(1_{[1,N]}^{\circ n})^2] = (n!)^2 \binom{N}{n} < n! \|1_{[1,N]}^{\circ n}\|^2_{l^2(\mathbb{N}^*)^{\otimes n}},$$

cf. Section 3.3.

From the expression

$$f^{\circ n} = \sum_{d=1}^{d=n} \sum_{\substack{1 \leq i_1 < \cdots < i_d \\ n_1 + \cdots + n_d = n}} \frac{n!}{n_1! \cdots n_d!} f^{n_1}(i_1) \cdots f^{n_d}(i_d) e_{i_1}^{\circ n_1} \circ \cdots \circ e_{i_d}^{\circ n_d}, \quad f \in l^2(\mathbb{N}^*). \tag{8}$$

We have

$$J_n(1_{[1,N]}^{\circ n}) = \sum_{d=1}^{d=n} \sum_{\substack{1\leq i_1<\cdots<i_d\leq N\\ n_1+\cdots+n_d=n}} \frac{n!}{n_1!\cdots n_d!} \prod_{k=1}^{k=d} \phi^{n_k}(S_{i_k} \mid S_{i_k-1}),$$

and a stochastic exponential $\mathcal{E}_N^\circ(z)$ can be constructed as

$$\mathcal{E}_N^\circ(z) = \sum_{n=0}^{n=N} z^n J_n(1_{[1,N]}^{\circ n}) = \sum_{n=0}^{n=N} z^n \sum_{d=1}^{d=n} \sum_{\substack{1\leq i_1<\cdots<i_d\leq N\\ n_1+\cdots+n_d=n}} \frac{n!}{n_1!\cdots n_d!} \prod_{k=1}^{k=d} \phi^{n_k}(S_{i_k} \mid S_{i_k-1}), \quad z \in \mathbb{R}. \tag{9}$$

A Wick type product $\diamond$ of random variables may also be defined as

$$\left(\prod_{i=1}^{i=d} \phi^{n_i}(S_{k_i} \mid S_{k_i-1})\right) \diamond \left(\prod_{i=1}^{i=d} \phi^{m_i}(S_{k_i} \mid S_{k_i-1})\right) = \prod_{i=1}^{i=d} \phi^{n_i+m_i}(S_{k_i} \mid S_{k_i-1}).$$

(By induction on $d \in \mathbb{N}$, the representation of $J_n(e_{i_1}^{\circ n_1} \circ \cdots \circ e_{i_d}^{\circ n_d})$ as $\prod_{i=1}^{i=d} \phi^{n_i}(S_{k_i} \mid S_{k_i-1})$ is unique). By linearity we have

$$J_n(f_n) \diamond J_m(g_m) = J_{n+m}(f_n \circ g_m).$$

In the i.i.d. case (see below) this product coincides with the one studied in [9] and [10].

3.2 Orthogonal expansions for i.i.d. processes

From now on we consider processes with i.i.d. increments, i.e. $(S_n - S_{n-1})_{n\geq 1} = (X_n)_{n\geq 1}$ is a family of i.i.d. random variables. In this case the function $\phi^n(x \mid y)$ depends only on the difference $x - y$, so that we write $\phi^n(x \mid y) = \phi^n(x - y)$. We have

$$J_n(1_{[1,N]}^{\circ n}) = \sum_{d=1}^{d=n} \sum_{\substack{1\leq i_1<\cdots<i_d\leq N\\ n_1+\cdots+n_d=n}} \frac{n!}{n_1!\cdots n_d!} \phi^{n_1}(X_{i_1}) \cdots \phi^{n_d}(X_{i_d}).$$

For all symmetric function $f_n \in l^2(\mathbb{N}^*)^{\odot n}$ we have

$$J_n(f_n) = \sum_{1\leq i_1<\cdots<i_n} f_n(i_1,\ldots,i_n)\phi^1(X_{i_1})\cdots\phi^1(X_{i_n}),$$

with

$$f_n = \sum_{1\leq i_1<\cdots<i_n} f_n(i_1,\ldots,i_n) e_{i_1} \circ \cdots \circ e_{i_n}.$$

Proposition 2 *The functional $J_n(f_n)$ is orthogonal to $J_m(g_m)$ in $L^2(\Omega)$ if $n \neq m$, and $J_n : l^2(\mathbb{N}^*)^{\odot n} \longrightarrow L^2(\Omega)$ extends as a linear continuous operator with*

$$E[J_n(f_n)^2] = n! \|f_n\|^2_{l^2(\mathbb{N}^*)^{\otimes n}}, \quad f_n \in l^2(\mathbb{N}^*)^{\odot n}, \ n \in \mathbb{N}. \tag{10}$$

Proof. The orthogonality property follows from Prop. 1. By independence we have if $\{i_1,\dots,i_n\} \neq \{j_1,\dots j_n\}$,

$$E[J_n(e_{i_1}\odot\cdots\odot e_{i_n})J_n(e_{j_1}\odot\cdots\odot e_{j_n})] = 0, \quad n \geq 0,$$

and

$$E[(J_n(e_{i_1}\odot\cdots\odot e_{i_n}))^2] = 1_{\{i_k\neq i_l,\ 1\leq k<l\leq n\}}.$$

With $f_n \in l^2(\mathbb{N}^*)^{\odot n}$ we have

$$\langle f_n, f_n\rangle_{l^2(\mathbb{N}^*)^{\otimes n}} = \frac{1}{n!}\sum_{1\leq i_1<\cdots<i_n} f_n(i_1,\dots,i_n)^2,$$

hence

$$E[J_n(f_n)^2] = \sum_{1\leq i_1<\cdots<i_n} f_n(i_1,\dots,i_n)^2 = n!\langle f_n, f_n\rangle_{l^2(\mathbb{N}^*)^{\otimes n}}.$$

□

The corresponding exponential martingale $\mathcal{E}_N^{\odot}(z)$ is constructed as

$$\mathcal{E}_N^{\odot}(z) = \sum_{n=0}^{n=N} z^n J_n(1_{[1,N]}^{\odot n}) = \prod_{i=1}^{i=N}(1+z\phi^1(X_i)) = \prod_{i=1}^{i=N}(1+z(\alpha+\beta X_i)), \quad z\in\mathbb{R}. \tag{11}$$

We now complete the result of [8] by showing that the integral $J_n(1_{[1,N]}^{\odot n})$ is a polynomial in S_N if and only if $(S_n)_{n\in\mathbb{N}}$ is a binomial process $(B_n)_{n\in\mathbb{N}}$. We have

$$J_1(1_{[1,N]}) = \sum_{i=1}^{i=N} J_1(e_i) = \sum_{i=1}^{i=N}\phi^1(X_i) = \alpha N + \beta S_N.$$

Proposition 3 *Each of the following statements holds if and only if the law of X_n is supported by two points, $n \geq 1$:*

i) The exponentials $\mathcal{E}_N^{\circ}(z)$ and $\mathcal{E}_N^{\odot}(z)$ coincide, i.e.

$$\mathcal{E}_N^{\circ}(z) = \sum_{n=0}^{n=N} z^n J_n(1_{[1,N]}^{\circ n}) = \sum_{n=0}^{n=N} z^n J_n(1_{[1,N]}^{\odot n}) = \mathcal{E}_N^{\odot}(z), \quad z\in\mathbb{R}.$$

ii) The integrals $J_n(f^{\odot n})$ and $J_n(f^{\circ n})$ coincide for all $f\in l^2(\mathbb{N}^*)$ and $n\in\mathbb{N}$.

iii) The integral $J_2(1_{[1,N]}^{\odot 2})$ can be expressed as a second degree polynomial in S_N for all $N \geq 1$.

Proof. We note that (i) is equivalent to (ii). Taking $n = N = 2$, (ii) implies $\phi^2 = 0$, hence the distribution of X_k is supported by two points only. If (iii) is satisfied we have $1_{[1,N]}^{\odot 2} = \sum_{1\leq i\neq j\leq N} e_i \circ e_j$, and

$$J_2(1_{[1,N]}^{\odot 2}) = \sum_{1\leq i\neq j\leq N} \phi^1(X_i)\phi^1(X_j) = (\alpha N + \beta S_N)^2 - \sum_{1\leq i\leq N} \phi^1(X_i)^2.$$

If $J_2(1_{[1,N]}^{\odot 2})$ is a second degree polynomial in S_N, then $\sum_{i=1}^{i=N} \phi^1(X_i)^2 = c_N S_N^2 + d_N S_N + e_N$ is polynomial of degree at most two in S_N, hence

$$\phi^1(X_N)^2 = c_N S_N^2 + d_N S_N + e_N - c_{N-1}S_{N-1}^2 - d_{N-1}S_{N-1} - e_{N-1}$$

Hence, i.e. with $S_N = X_N + S_{N-1}$, we have for $N \geq 1$:

$$(\alpha + \beta X_N)^2 = c_N(X_N + S_{N-1})^2 + d_N(X_N + S_{N-1}) + e_N - c_{N-1}S_{N-1}^2 - d_{N-1}S_{N-1} - e_{N-1}$$

or

$$\begin{aligned} &X_N^2(\beta^2 - c_N) + X_N(2\alpha\beta - 2c_N S_{N-1} - d_N) \\ &\quad -c_N S_{N-1}^2 - d_N S_{N-1} - e_N + c_{N-1}S_{N-1}^2 + d_{N-1}S_{N-1} + e_{N-1} + \alpha^2 = 0. \end{aligned}$$

If X_N is allowed to take at least three distinct values, then $c_N = \beta^2$ and $2\alpha\beta - 2c_N S_{N-1} - d_N = 0$, $N \geq 1$, which is impossible. Hence X_N can only attain 2 values. The fact that $J_n(1_{[1,N]}^{\circ n})$ is a polynomial in S_N if the law of X_n is supported by two points, $n \geq 1$, will be proved in Section 3.3. □

If the conditions of Prop. 3 hold then the Wick product satisfies

$$J_n(f_n) \diamond J_m(g_m) = J_{n+m}(f_n \circ g_m) = J_{n+m}(f_n \odot g_m),$$

and the isometry formula can be written as

$$E[J_n(f_n)J_m(g_m)] = n! 1_{\{n=m\}} \langle 1_{\Delta_n} f_n, g_m \rangle_{l^2(\mathbb{N}^*)^{\otimes n}},$$

for all $f_n \in l^2(\mathbb{N}^*)^{\circ n}$ and $g_m \in l^2(\mathbb{N}^*)^{\circ m}$, $n, m \in \mathbb{N}^*$, where

$$\Delta_n = \{(k_1, \ldots, k_n) \in (\mathbb{N}^*)^n \ : \ k_i \neq k_j, \quad 1 \leq i < j \leq n\}.$$

Remark: We know from page 67 of [4] or Prop. 4 of [5] that in the i.i.d. case, the above conditions are equivalent to the chaotic representation property for $(S_n)_{n\in\mathbb{N}}$. The chaotic representation property will be explicitly studied in Section 3.3.

3.3 Orthogonal expansions for the binomial process

From now on we assume that $(X_n)_{n\geq 1}$ is an i.i.d. sequence of random variables whose laws are supported by two points a and b, $a < b$. Then $(B_n)_{n\in\mathbb{N}} = ((S_n - na)/(b-a))_{n\in\mathbb{N}}$ is the binomial process and ϕ^1 is of the form $\phi^1(x \mid y) = \alpha(x-y)+\beta$. Up to a rescaling we can assume that $a = 0$ and $b = 1$, i.e. $(S_n)_{n\in\mathbb{N}}$ is itself the binomial process $(B_n)_{n\in\mathbb{N}}$. In this case we have $P(x,y) = q1_{\{y=x\}} + p1_{\{y=x+1\}}$ for all $x, y \in \mathbb{N}$, which leads to $\phi^1(x \mid y) = (pq)^{-1/2}K_1(x-y;1,p) = (pq)^{-1/2}(x-y-p)$ and $\phi^n(x \mid y) = 0$ for all $x, y \in \mathbb{R}$ and $n > 1$.

Proposition 4 *With $(B_n)_{n\in\mathbb{N}}$ the binomial process we have*

$$J_n(1^{\circ n_1}_{[M_1+1,N_1]} \circ \cdots \circ 1^{\circ n_d}_{[M_d+1,N_d]}) = (pq)^{-n/2} \prod_{k=1}^{k=d} K_{n_k}(B_{N_k} - B_{M_k}; N_k - M_k, p),$$

$n_1 + \cdots + n_d = n$, $0 \leq M_i < N_i \leq M_{i+1}$, $i = 1, \ldots, d-1$, and $M_d < N_d$.

Proof. From (8) we have

$$1^{\circ n}_{[M+1,N]} = \sum_{d=1}^{d=n} \sum_{\substack{M<i_1<\cdots<i_d\leq N \\ n_1+\cdots+n_d=n}} \frac{n!}{n_1!\cdots n_d!} e^{\circ n_1}_{i_1} \circ \cdots \circ e^{\circ n_d}_{i_d},$$

hence

$$\begin{aligned} J_n(1^{\circ n}_{[M+1,N]}) &= 1_{\{0\leq n\leq N-M\}} n! \sum_{M<i_1<\cdots<i_n\leq N} \prod_{k=1}^{k=n} \phi^1(X_{i_k}) \\ &= 1_{\{0\leq n\leq N-M\}} (pq)^{-n/2} n! \sum_{M<i_1<\cdots<i_n\leq N} \prod_{k=1}^{k=n} (X_{i_k} - p). \end{aligned}$$

This also shows that $J_n(1^{\circ n}_{[M+1,N]})$ is a (polynomial) functional of $B_N - B_M$ since it depends only on the number of jumps of $(B_n)_{n\geq 1}$ on $\{M+1, \ldots, N\}$, and not on jump times. Moreover, $J_n(1^{\circ n}_{[M+1,N]})$ satisfies the same orthogonality property as the Krawtchouk polynomials. Since

$$E[J_n(1^{\circ n}_{[M+1,N]})^2] = (n!)^2 \binom{N-M}{n} = n!(-1)^n(M-N)_n$$

and from the orthogonality relation (6): $E[(K_n(B_N - B_M; N-M, p))^2] = n!(-1)^n(M-N)_n(pq)^n$, we obtain

$$J_n(1^{\circ n}_{[M+1,N]}) = (pq)^{-n/2} K_n(B_N - B_M; N-M, p).$$

Finally, since $\phi^1(X_{i_k})$ is independent of B_{i_k-1} and from the definition of J_n we have :

$$J_n(1^{\circ n_1}_{[M_1+1,N_1]} \circ \cdots \circ 1^{\circ n_d}_{[M_d+1,N_d]}) = \prod_{l=1}^{l=d} n_l! \sum_{M_l<i_1<\cdots<i_{n_l}\leq N_l} \prod_{k=1}^{k=n_l} \phi^1(X_{i_k})$$

$$= \prod_{l=1}^{l=d} J_{n_l}(1^{\circ n_l}_{[M_l+1,N_l]})$$

$$= (pq)^{-n/2} \prod_{l=1}^{l=d} K_{n_l}(B_{N_l} - B_{M_l}; N_l - M_l, p).$$

□

Note that as N goes to infinity, $N^{-n} E[J_n(1^{\circ n}_{[1,N]})^2]$ converges to $n!$, which is the usual value of the square norm of the multiple stochastic integral over $[0,1]^n$ with respect to a continuous time normal martingale. We also obtained the relation

$$(pq)^{n/2} J_n(1^{\circ n}_{[1,N]}) = K_n(B_N; N, p) = n! \sum_{1\leq i_1<\cdots<i_n\leq N} \prod_{k=1}^{k=n} (X_{i_k} - p) = \sum_{\substack{i_1,\ldots,i_n \\ 1\leq i_k\neq i_l\leq N}} \prod_{k=1}^{k=n} (X_{i_k} - p),$$

see §V-9-3 of [7] for the symmetric case $p = q = 1/2$.
The Wick product can be expressed in terms of Krawtchouk polynomials as

$$K_n(B_N - B_M) \diamond K_m(B_N - B_M) = K_{n+m}(B_N - B_M).$$

4 Iterated stochastic summation with respect to the binomial process

In the usual continuous time stochastic integration with respect to a normal martingale $(M_t)_{t\in\mathbb{R}_+}$, the multiple stochastic integral $I_n(f_n)$ of a symmetric function of n real variables is $n!$ times the iterated integral of f_n over the simplex $\{0 \leq t_1 < \cdots < t_n\}$:

$$\frac{1}{n!} I_n(f_n) = \int_0^\infty \int_0^{t_n^-} \cdots \int_0^{t_2^-} f_n(t_1, \ldots, t_n) dM_{t_1} \cdots dM_{t_n}.$$

Given $f_n \in L^2(\mathbb{R}_+^n)$ one lets $I_n(f_n) = I_n(\tilde{f}_n)$, where $\tilde{f}_n$ denotes the symmetrization (2) of f_n in n variables. Given $f_{n+1} \in L^2(\mathbb{R}_+)^{\circ n} \otimes L^2(\mathbb{R}_+)$ this implies

$$\int_0^\infty I_n(f_{n+1}(\cdot, t) 1_{[0,t]^n}(\cdot) 1_{\Delta_n}(\cdot)) dM_t = I_{n+1}(f_{n+1} 1_{\Delta_{n+1}}),$$

where $f_{n+1}(\cdot, t) 1_{[0,t]^n}(\cdot) 1_{\Delta_n}(\cdot)$ is the function of n variables defined as

$$(t_1, \ldots, t_n) \mapsto f_{n+1}(t_1, \ldots, t_n, t) 1_{[0,t]}(t_1) \cdots 1_{[0,t]}(t_n) 1_{\Delta_n}(t_1, \ldots, t_n), \qquad (12)$$

and $I_n(f_n) = n! I_n(f_n 1_{\Delta_n})$ for all $f_n \in L^2(\mathbb{R}_+)^{\circ n}$. We will show that analogously, the functional $\frac{1}{n!} J_n(f_n)$ is an iterated multiple stochastic integral in discrete time with respect to the compensated binomial process $(B_n - np)_{n\in\mathbb{N}}$. We set $J_n(f_n) = J_n(\tilde{f}_n)$ if $f_n \in l^2(\mathbb{N}^n)$

is not symmetric, and let $\Delta_n = \{0 \leq k_1 < \cdots < k_n\}$ denote the simplex in $\mathbb{N}^n$, and let $Y_k = (X_k - p)/\sqrt{pq}$, $k \geq 1$, denote the normalized (centered with variance one) increment of $(B_n)_{n\in\mathbb{N}^*}$.

Theorem 1 *We have for* $f_{n+1} \in l^2(\mathbb{N}^*)^{\circ n} \otimes l^2(\mathbb{N}^*)$*:*

$$\sum_{k=1}^{k=\infty} Y_k J_n(f_{n+1}(\cdot,k)1_{[1,k-1]^n}(\cdot)1_{\Delta_n}(\cdot)) = J_{n+1}(f_{n+1}1_{\Delta_{n+1}}), \qquad (13)$$

where $f_{n+1}(\cdot,k)1_{[1,k-1]^n}(\cdot)1_{\Delta_n}(\cdot)$*,* $k \geq n+1$*, is defined as in (12).*

Proof. First we note that $J_n(f_{n+1}(\cdot,k)1_{[1,k-1]^n}(\cdot)) = 0$ if $n > k-1$, so that the summation (13) actually starts at $k = n+1$. We start by proving that

$$\sum_{k=M+1}^{k=N} (X_k - p) K_n(B_{k-1} - B_M; k-1-M, p) = \frac{K_{n+1}(B_N - B_M; N-M, p)}{n+1}, \qquad (14)$$

with $K_n(x; N, p)$ the monic Krawtchouk polynomial of degree n. Using the generating function

$$Y(x, N, z) = \sum_{n=0}^{\infty} K_n(x; N, p)\frac{z^n}{n!} \sum_{n=0}^{N} K_n(x; N, p)\frac{z^n}{n!} = (1+qz)^x(1-pz)^{N-x},$$

it is sufficient to prove

$$\sum_{k=M+1}^{k=N} (X_k - p) Y(B_{k-1} - B_M, k-1-M, z) = \frac{Y(B_N - B_M, N-M, z) - 1}{z},$$

This follows immediately from the fact that

$$\begin{aligned} &\frac{Y(B_k - B_M, k-M, z) - Y(B_{k-1} - B_M, k-1-M, z)}{z} \\ &= \frac{Y(B_{k-1} - B_M, k-1-M, z)}{z}\left(\frac{(1+qz)^{X_k}}{(1-pz)^{X_k - 1}} - 1\right) \\ &= Y(B_{k-1} - B_M, k-1-M, z)(X_k - p). \end{aligned}$$

Another way of proving (14) is to directly use the representation formula (5). From the relation

$$J_n(1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ n_d}) = \prod_{k=1}^{k=d} J_{n_k}(1_{[M_k+1,N_k]}^{\circ n_k}),$$

we deduce that (13) holds for

$$f_{n+1} = 1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ n_d} \otimes 1_{[M_l+1,N_l]}.$$

For this it suffices to consider $l = d$ and to note that

$$\begin{aligned}
&\sum_{k=1}^{k=\infty} \frac{X_k - p}{\sqrt{pq}} J_n(f_{n+1}(\cdot, k) 1_{[1,k-1]^n}(\cdot) 1_{\Delta_n}(\cdot)) \\
&= \sum_{k=1}^{k=\infty} \frac{1}{n!} \frac{X_k - p}{\sqrt{pq}} J_n(f_{n+1}(\cdot, k) 1_{[1,k-1]^n}(\cdot)) \\
&= \frac{1}{n!} \sum_{k=M_d+1}^{k=N_d} \frac{X_k - p}{\sqrt{pq}} J_n(1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,k-1]}^{\circ n_d}) \\
&= \frac{1}{n!} (pq)^{-(n+1)/2} \prod_{k=1}^{k=d-1} K_{n_k}(B_{N_k} - B_{M_k}; N_k - M_k, p) \\
&\quad \times \sum_{k=M_d+1}^{k=N_d} K_{n_d}(B_{k-1} - B_{M_d}; k - 1 - M_d, p)(X_k - p) \\
&= \frac{(pq)^{-(n+1)/2}}{n!(n_d+1)} K_{n_d+1}(B_{N_d} - B_{M_d}; N_d - M_d, p) \prod_{k=1}^{k=d-1} K_{n_k}(B_{N_k} - B_{M_k}; N_k - M_k, p) \\
&= \frac{1}{n!(n_d+1)} J_{n+1}(1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ (n_d+1)}).
\end{aligned}$$

Now, for $0 \leq k_1 < \cdots < k_{n+1}$ we have

$$\begin{aligned}
&\left(1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ (n_d+1)}\right)(k_1, \ldots, k_{n+1}) \\
&= \frac{n!(n_d+1)}{(n+1)!} \left(1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ n_d} \otimes 1_{[M_d+1,N_d]}\right)(k_1, \ldots, k_{n+1}) \\
&= \frac{n!(n_d+1)}{(n+1)!} f_{n+1}(k_1, \ldots, k_{n+1}) \\
&= \frac{n!(n_d+1)}{(n+1)!} f_{n+1}(k_1, \ldots, k_{n+1}) 1_{\Delta_{n+1}}(k_1, \ldots, k_{n+1}).
\end{aligned}$$

This shows that $1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ (n_d+1)}$ is $n!(n_d+1)$ times the symmetrization of $f_{n+1} 1_{\Delta_{n+1}}$. Hence

$$\sum_{k=1}^{k=\infty} \frac{1}{n!} \frac{X_k - p}{\sqrt{pq}} J_n(f_{n+1}(\cdot, k) 1_{[1,k-1]^n}(\cdot)) = J_{n+1}(f_{n+1} 1_{\Delta_{n+1}}).$$

Finally from (7), by linearity and density, Relation (13) holds for all $f_{n+1} \in l^2(\mathbb{N}^*)^{\circ n} \otimes l^2(\mathbb{N}^*)$.

□

The interpretation of this result is that the Krawtchouk polynomials are the stochastic counterparts of the usual powers $(B_N - Np)^n = (K_1(B_N; N, p))^n$, $n \geq 0$ for the compensated binomial process $\{B_n - np, n \in \mathbb{N}\}$. Also we found that the role of the classical exponential function, now is taken by $Y(B_n, n, 1) = (1+q)^{B_n} q^{n-B_n}$ because of the relation

$$\sum_{i=1}^{i=n} \left(\frac{1+q}{q}\right)^{B_{i-1}} q^{i-1}(X_i - p) = \left(\frac{1+q}{q}\right)^{B_n} q^n - 1.$$

Furthermore there is a striking similarity with integration with respect to Brownian motion and Hermite polynomials on the one hand and the Poisson process and the Charlier polynomials on the other hand.

5 Gradient, divergence and Clark formula

In this section we introduce gradient and divergence operators, and obtain a Clark formula for the functionals of $(B_n)_{n\geq 1}$. We use the convention $1_{[N,M]} = 0$ if $M < N$. Let $\mathcal{P}$ denote the set of polynomials in $X_1, X_2, X_3, \ldots$, i.e. $\mathcal{P}$ is the linear space spanned by

$$J_n(1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ n_d}),$$

$0 \leq M_1 \leq N_1 < \cdots < M_d \leq N_d$, $n_1, \ldots, n_d \in \mathbb{N}$. Let $\mathcal{U}$ denote the space of discrete-time processes $(u(k))_{k\geq 1}$, with finite support in $k \geq 1$ and such that $u(k) \in \mathcal{P}$, $k \geq 1$. The space $\mathcal{P}$ is clearly dense in $L^2(\Omega, P)$, hence the process $(B_n)_{n\geq 1}$ has the chaos representation property, i.e. any $F \in L^2(\Omega, P)$ can be represented as a series of multiple stochastic integrals:

$$F = \sum_{n=0}^{\infty} J_n(f_n), \quad f_k \in l^2(\mathbb{N}^*)^{\circ k},\ k \in \mathbb{N}^*,$$

with $J_0(f_0) = E[F]$.

Definition 2 *We densely define the linear gradient and divergence operators $D : L^2(\Omega) \longrightarrow L^2(\Omega \times \mathbb{N}^*)$ and $\delta : L^2(\Omega \times \mathbb{N}^*) \longrightarrow L^2(\Omega)$ as*

$$D_k J_n(f_n) = n J_{n-1}(f_n(*, k) 1_{\Delta_n}(*, k)), \qquad f_n \in l^2(\mathbb{N}^*)^{\circ n}, \quad n \in \mathbb{N},$$

and

$$\delta(J_n(f_{n+1}(*, \cdot))) = J_{n+1}(\tilde{f}_{n+1}) = J_{n+1}(f_{n+1}), \quad f_{n+1} \in l^2(\mathbb{N}^*)^n \otimes l^2(\mathbb{N}^*),$$

where $\tilde{f}_{n+1}$ denotes the symmetrization of f_{n+1}.

Let $\mathcal{F}_k$ denote the σ-algebra generated by $X_1, \ldots, X_k$.

Proposition 5 *Let $(u(k))_{k\geq 1}$ be a predictable square-integrable process, i.e. $u(k)$ is $\mathcal{F}_{k-1}$-measurable, $k \geq 1$, and $E[\|u\|^2_{l^2(\mathbb{N}^*)}] < \infty$. Then $\delta(u)$ coincides with the discrete time stochastic integral with respect to $(B_n)_{n\geq 1}$:*

$$\delta(u) = \sum_{k=1}^{\infty} Y_k u(k),$$

with $E[\delta(u)^2] = E[\|u\|^2_{l^2(\mathbb{N}^*)}]$.

Proof. Given $f_{n+1} \in l^2(\mathbb{N})^{\circ n} \otimes l^2(\mathbb{N}^*)$ and $u(k) = J_n(f_{n+1}(\cdot, k))$, $k \geq 1$, the predictability condition means that $f_{n+1}(\cdot, k) = f_{n+1}(\cdot, k) 1_{[1,k-1]^n}(\cdot)$, hence the symmetrization of f_{n+1} is $n!$ times the symmetrization of $f_{n+1} 1_{\Delta_{n+1}}$. Thus from Th. 1 we have

$$\begin{aligned}
\delta(J_n(f_{n+1}(*, \cdot))) &= J_{n+1}(f_{n+1}) = n! J_{n+1}(f_{n+1} 1_{\Delta_{n+1}}) \\
&= n! \sum_{k=1}^{\infty} Y_k J_n(f_{n+1}(\cdot) 1_{[1,k-1]^n}(\cdot) 1_{\Delta_n}(\cdot)) \\
&= \sum_{k=1}^{\infty} Y_k J_n(f_{n+1}(\cdot) 1_{[1,k-1]^n}(\cdot)) = \sum_{k=1}^{\infty} Y_k u(k).
\end{aligned}$$

A density argument completes the proof. □

The truncation by the function 1_{Δ_n} in

$$D_k J_n(1_{[1,N]}^{\circ n}) = n J_{n-1}(1_{[1,N]}^{\circ(n-1)}(*) 1_{\Delta_n}(*, k)),$$

is not present in continuous time. It disappears after taking the predictable projection of the gradient process:

$$\begin{aligned}
E[D_k J_n(f_n) \mid \mathcal{F}_{k-1}] &= n E[J_{n-1}(f_n(*, k) 1_{\Delta_n}(*, k)) \mid \mathcal{F}_{k-1}] \\
&= n J_{n-1}(f_n(*, k) 1_{\Delta_n}(*, k) 1_{[1,k-1]^{n-1}}(*)) \\
&= n J_{n-1}(f_n(*, k) 1_{[1,k-1]^{n-1}}(*)),
\end{aligned}$$

hence

$$E[D_k J_n(f_n) \mid \mathcal{F}_{k-1}] = n J_{n-1}(f_n(\cdot, k) 1_{[1,k-1]^{n-1}}(\cdot)), \quad k \in \mathbb{N}^*.$$

In fact, under the conditional expectation, D coincides with the annihilation operator on Krawtchouk polynomials:

$$E[D_k K_n(B_N - B_M; N-M, p) \mid \mathcal{F}_{k-1}] = \frac{n}{\sqrt{pq}} 1_{[M,N]}(k) E[K_{n-1}(B_N - B_M; N-M, p) \mid \mathcal{F}_{k-1}].$$

The following proposition shows that $D : L^2(\Omega) \longrightarrow L^2(\Omega \times \mathbb{N}^*)$ and $\delta : L^2(\Omega \times \mathbb{N}^*) \longrightarrow L^2(\Omega)$ are mutually adjoint.

Proposition 6 *We have*

$$E[\langle DF, u \rangle_{l^2(\mathbb{N}^*)}] = E[F \delta(u)], \quad u \in \mathcal{U}, \ F \in \mathcal{P},$$

and $D : L^2(\Omega) \longrightarrow L^2(\Omega \times \mathbb{N}^*)$, $\delta : L^2(\Omega \times \mathbb{N}^*) \longrightarrow L^2(\Omega)$ *are closable.*

Proof. It suffices to consider $F = J_n(f_n)$ and $u(k) = J_m(g_m(*,k))$, with $f_n \in l^2(\mathbb{N}^*)^{\circ n}$, $g_m \in l^2(\mathbb{N}^*)^{\circ m} \otimes l^2(\mathbb{N}^*)$, and $n = m+1$:

$$\begin{aligned} E[\langle DF, u\rangle_{l^2(\mathbb{N}^*)}] &= n\sum_{k=1}^{\infty} E[J_{n-1}(f_n(*,k)1_{\Delta_n}(*,k))J_m(g_m(*,k))] \\ &= nm!\sum_{k=1}^{\infty}\langle f_n(*,k)1_{\Delta_n}(*,k), g_m(*,k)\rangle_{l^2(\mathbb{N}^*)^{\odot m}} \\ &= n!\langle f_n, g_m\rangle_{l^2(\mathbb{N}^*)^{\odot n}} = E[F\delta(u)]. \end{aligned}$$

The closability of D and δ is a consequence of the duality formula and of the density of $\mathcal{P}$ and $\mathcal{U}$ in $L^2(\Omega)$ and $L^2(\Omega \times \mathbb{N}^*)$ respectively. □

In fact D_k also coincides with the operator $(pq)^{-1/2}a_k^-$ of §II-2-2 in [16], since we have for $l_i \neq l_j$, $i \neq j$:

$$D_k\prod_{l=1}^{l=n}(X_{i_l} - p) = \frac{1}{\sqrt{pq}}1_{\{l\in\{i_1,\dots,i_n\}\}}\prod_{l=1,\ i_l\neq k}^{l=n}(X_{i_l} - p),$$

and the probabilistic interpretation of D_k

$$D_kF(S_\cdot) = \sqrt{pq}F(S_\cdot + 1_{\{X_k=0\}}1_{\{k\leq\cdot\}}) - F(S_\cdot - 1_{\{X_k=1\}}1_{\{k\leq\cdot\}}).$$

cf. also [9], [10], and also [15].

The normalized increment Y_i of $(B_n)_{n\in\mathbb{N}^*}$ satisfies the structure equation

$$Y_i^2 = 1 + \varphi Y_i, \quad i \geq 1, \quad \text{with } \varphi = \frac{q-p}{\sqrt{pq}},$$

see §II-2-1 of [16], and [4]. This implies in particular the following elementary product formula for single stochastic integrals:

$$J_1(f)J_1(g) = J_2(f\circ g) + \langle f, g\rangle_{l^2(\mathbb{N}^*)} + \varphi J_1(fg), \tag{15}$$

for $f, g \in l^2(\mathbb{N}^*)$ such that $fg \in l^2(\mathbb{N}^*)$. We also have

$$\begin{aligned} &J_n(1_{[M+1,N]}^{\circ n})J_1(1_{[M+1,N]}) \\ &\quad = J_{n+1}(1_{[M+1,N]}^{\circ(n+1)}) + n(N-M-n+1)J_{n-1}(1_{[M+1,N]}^{\circ(n-1)}) + \varphi nJ_n(1_{[M+1,N]}^{\circ n}), \end{aligned}$$

from the three term recurrence relation for Krawtchouk polynomials, see e.g. [13]:

$$\begin{aligned} &K_1(B_N - B_M; N-M, p)K_n(B_N - B_M; N-M, p) = K_{n+1}(B_N - B_M; N-M, p) \\ &\quad + npq(N-M-n+1)K_{n-1}(B_N - B_M; N-M, p) + n(q-p)K_n(B_N - B_M; N-M, p). \end{aligned}$$

This implies in particular

$$F\delta(u) = \delta(Fu) + \langle DF, u\rangle_{l^2(\mathbb{N}^*)} + \delta(\varphi u DF), \quad F \in \mathcal{P},\ u \in l^2(\mathbb{N}^*).$$

The operator D and δ do not satisfy the same product rules as in the continuous time case (cf. Prop. 1.3 of [18]), since we have if $X_k = 0$:

$$D_k(FG) = FD_kG + GD_kF + \frac{1}{\sqrt{pq}} D_kFD_kG,$$

and if $X_k = 1$:

$$D_k(FG) = FD_kG + GD_kF - \frac{1}{\sqrt{pq}} D_kFD_kG.$$

Thus in general:

$$D_k(FG) \neq FD_kG + GD_kF + \varphi D_kFD_kG, \quad k \in \mathbb{N}^*, \tag{16}$$

and

$$F\delta(u) \neq \delta(Fu) + \langle DF, u\rangle_{l^2(\mathbb{N}^*)} + \delta(\varphi u DF), \quad F \in \mathcal{P},\ u \in \mathcal{U}, \tag{17}$$

except if u is deterministic. The latter inequality expresses the fact that there is no explicit formula for the product $K_n(x; N, p)K_m(x; N, p)$, with $n, m > 1$. The next result is the predictable representation of the functionals of $(B_n)_{n\geq 1}$.

Proposition 7 *We have the Clark formula*

$$F = E[F] + \sum_{k=1}^{\infty} E[D_kF \mid \mathcal{F}_{k-1}]Y_k = E[F] + \delta(E[D_\cdot F \mid \mathcal{F}_{\cdot-1}]), \quad F \in L^2(\Omega).$$

Proof. For $F = J_n(f_n)$ we have

$$E[D_kJ_n(f_n) \mid \mathcal{F}_{k-1}] = nJ_{n-1}(f_n(\cdot, k)1_{[1,k-1]^{n-1}}) = n!J_{n-1}(f_n(\cdot, k)1_{[1,k-1]^{n-1}}(\cdot)1_{\Delta_{n-1}}(\cdot)).$$

We apply Th. 1:

$$\begin{aligned} F &= J_n(f_n) = n!J_n(f_n1_{\Delta_n}) = n!\sum_{k=1}^{\infty} J_{n-1}(f_n(\cdot, k)1_{[1,k-1]^{n-1}}(\cdot)1_{\Delta_{n-1}}(\cdot))Y_k \\ &= n\sum_{k=1}^{\infty} J_{n-1}(f_n(\cdot, k)1_{[1,k-1]^{n-1}}(\cdot))Y_k = \sum_{k=1}^{\infty} E[D_kJ_n(f_n) \mid \mathcal{F}_{k-1}]Y_k. \end{aligned}$$

Next we apply Prop. 5 to the predictable process $u = (E[D_kF \mid \mathcal{F}_{k-1}])_{k\geq 1}$:

$$F = \sum_{k=1}^{\infty} E[D_kJ_n(f_n) \mid \mathcal{F}_{k-1}]Y_k = \delta(E[D_\cdot F \mid \mathcal{F}_{\cdot-1}]), \quad F \in \mathcal{P}.$$

This identity also shows that $F \mapsto E[D_\cdot F \mid \mathcal{F}_{\cdot-1}]$ has norm bounded by one as an operator from $L^2(\Omega)$ into $L^2(\Omega \times \mathbb{N}^*)$:

$$\|E[D_\cdot F \mid \mathcal{F}_{\cdot-1}]\|^2_{L^2(\Omega\times\mathbb{N}^*)} = \|F - E[F]\|^2_{L^2(\Omega)} \leq \|F - E[F]\|^2_{L^2(\Omega)} + E[F]^2 \leq \|F\|^2_{L^2(\Omega)},$$

hence the Clark formula extends to $F \in L^2(\Omega)$. □

A generalization consists in replacing the constant φ in the structure equation $Y_k^2 = 1+\varphi Y_k$ by a deterministic function $\varphi : \mathbb{N}^* \longrightarrow \mathbb{R}$ and considering the solution of the structure equation

$$Z_i^2 = 1 + \varphi_i Z_i, \qquad i \in \mathbb{N}^*,$$

i.e.

$$Z_i = \sqrt{\varphi_i^2 + 4}\left(X_i + \frac{-1 + \varphi_i/\sqrt{\varphi_i^2+4}}{2} \right), \qquad i \geq 1.$$

The process $(Z_1 + \cdots + Z_n)_{n\geq 1}$ will be a martingale under P if and only if

$$\frac{-1 + \varphi_i/\sqrt{\varphi_i^2+4}}{2} = -p,$$

i.e. $\varphi = (q-p)/\sqrt{qp}$. In fact it is a Markov chain with transition matrix

$$P(i,i) = \frac{1}{2} + \frac{\varphi_i}{2\sqrt{\varphi_i^2+4}} = q_i, \qquad P(i,i+1) = \frac{1}{2} - \frac{\varphi_i}{2\sqrt{\varphi_i^2+4}} = p_i, \quad i \geq 1.$$

Remark: The results of Sects. 4 and 5 can be generalized by replacing $(Y_1 + \cdots + Y_n)_{n\geq 1}$ by the process $(Z_1 + \cdots + Z_n)_{n\geq 1}$ solution of $Z_i^2 = 1 + \varphi_i Z_i$ for $i \in \mathbb{N}^*$, except for the fact that $J_n(1_{[1,N]}^{\circ n})$ is not a (polynomial) functional of B_N if φ_i is dependent on i since $J_n(1_{[1,N]}^{\circ n})$ will not only depend on the number of jumps from 1 to N but also on their respective positions.

6 Covariance identities

In this section we apply the above construction to the derivation of covariance identities on the infinite cube $\{-1,1\}^\infty$, and recover with simple chaotic proofs some results of [3] on the finite discrete cube, cf. also [11] in continuous time. These identities have been recently applied in [12] to the proof of deviation inequalities in discrete settings. Let $\mathbb{D}([a,\infty[)$, $a \in \mathbb{N}^*$, denote the completion of $\mathcal{P}$ under the norm

$$\|F\|^2_{\mathbb{D}([a,\infty[)} = E[F^2] + \sum_{k=a}^{\infty}(D_k F)^2,$$

i.e. $(D_k F)_{k\geq a}$ is defined in $L^2(\Omega, l^2([a,\infty[))$ for $F \in \mathbb{D}([a,\infty[)$. The following Lemma is a consequence of the Clark formula:

Lemma 1 *Let $a \in \mathbb{N}$ and $F \in \mathbb{D}([a,\infty[)$. We have*

$$F = E[F \mid \mathcal{F}_a] + \sum_{k=a+1}^{\infty} E[D_k F \mid \mathcal{F}_{k-1}] Y_k, \tag{18}$$

and

$$E[(E[F \mid \mathcal{F}_a])^2] = E[F^2] - E\left[\sum_{k=a+1}^{\infty} (E[D_k F \mid \mathcal{F}_{k-1}])^2\right]. \tag{19}$$

Proof. Relation (18) holds because

$$F - \sum_{k=a+1}^{\infty} E[D_k F \mid \mathcal{F}_{k-1} Y_k] = E[F] + \sum_{k=1}^{k=a} E[D_k F \mid \mathcal{F}_{k-1}] Y_k$$

is $\mathcal{F}_a$-measurable, and

$$\sum_{k=a+1}^{\infty} E[D_k F \mid \mathcal{F}_{k-1}] Y_k$$

is orthogonal to $L^2(\Omega, \mathcal{F}_a)$ in $L^2(\Omega)$. Relation (19) is an immediate consequence of (18).

□

Next we prove a covariance identity in discrete time, cf. [11] for the Wiener and Poisson processes. Let $\mathbb{D}(\Delta_n)$ be the completion of $\mathcal{P}$ under the norm

$$\|F\|^2_{\mathbb{D}(\tilde{\Delta}_n)} = E[F^2] + E\left[\sum_{1 \leq k_1 < \cdots < k_n} (D_{k_n} \cdots D_{k_1} F)^2\right],$$

where

$$\tilde{\Delta}_n = \{(k_1, \ldots, k_n) \in (\mathbb{N}^*)^n \ : \ k_i \neq k_j, \quad 1 \leq i < j \leq n\}.$$

Theorem 2 *Let $n \in \mathbb{N}$ and $F, G \in \bigcap_{k=1}^{k=n+1} \mathbb{D}(\tilde{\Delta}_k)$. We have*

$$\begin{aligned} \mathrm{Cov}(F, G) &= \sum_{d=1}^{d=n} (-1)^{d+1} E\left[\sum_{\{1 \leq k_1 < \cdots < k_d\}} (D_{k_d} \cdots D_{k_1} F)(D_{k_d} \cdots D_{k_1} G)\right] \qquad (20) \\ &+ (-1)^n E\left[\sum_{\{1 \leq k_1 < \cdots < k_{n+1}\}} E\left[D_{k_{n+1}} \cdots D_{k_1} F \mid \mathcal{F}_{k_{n+1}-1}\right] E\left[D_{k_{n+1}} \cdots D_{k_1} G \mid \mathcal{F}_{k_{n+1}-1}\right]\right]. \end{aligned}$$

Proof. We take $F = G$. For $n = 0$, (20) is a consequence of the Clark formula. Let $n \geq 1$. Applying Lemma 1 to $D_{k_n} \cdots D_{k_1} F$ with $a = k_n$ and $b = k_{n+1}$, and summing on $(k_1, \cdots, k_n) \in \tilde{\Delta}_n$, we obtain

$$\begin{aligned} E\left[\sum_{\{1 \leq k_1 < \cdots < k_n\}} (E[D_{k_n} \cdots D_{k_1} F \mid \mathcal{F}_{k_n - 1}])^2\right] &= E\left[\sum_{\{1 \leq k_1 < \cdots < k_n\}} (D_{k_n} \cdots D_{k_1} F)^2\right] \\ &\quad - E\left[\sum_{\{1 \leq k_1 < \cdots < k_{n+1}\}} \left(E\left[D_{k_{n+1}} \cdots D_{k_1} F \mid \mathcal{F}_{k_{n+1}-1}\right]\right)^2\right], \end{aligned}$$

which concludes the proof by induction. □

The variance inequality

$$\sum_{k=1}^{k=2n}(-1)^{k+1}\|D^kF\|^2_{l^2(\tilde{\Delta}_k)} \leq \mathrm{Var}(F) \leq \sum_{k=1}^{k=2n-1}(-1)^{k+1}\|D^kF\|^2_{l^2(\tilde{\Delta}_k)},$$

$F \in L^2(\Omega)$, is a consequence of Th. 2, see (2.15) in [11] in continuous time. We define an Ornstein-Uhlenbeck type operator Q_t on $L^2(\Omega)$, for all $t \in \mathbb{R}_+$, as

$$Q_tF = \sum_{n=0}^{\infty} e^{-nt}J_n(f_n),$$

if $F = \sum_{n=0}^{\infty} J_n(f_n)$. Next, in Prop. 8 and Prop. 9 we provide a simple proof of Th. 3.1 in [3] via chaos expansions.

Proposition 8 *Let $F, G \in \mathbb{D}([1,\infty[)$. We have*

$$\mathrm{Cov}(F,G) = \int_0^{\infty} E[\langle DF, DQ_tG\rangle_{l^2(\mathbb{N}^*)}]dt.$$

Proof. We consider $F = J_n(f_n)$ and $G = J_n(g_n)$. We have

$$\begin{aligned}
\int_0^{\infty} E[\langle DJ_n(f_n), DQ_tJ_n(g_n)\rangle_{l^2(\mathbb{N}^*)}]dt &= \int_0^{\infty}\sum_{k=1}^{\infty} E[D_kJ_n(f_n)D_kJ_n(f_n)]e^{-nt}dt \\
&= \frac{1}{n}\sum_{k=1}^{\infty} E[D_kJ_n(f_n)D_kJ_n(f_n)] \\
&= n!\sum_{k=1}^{\infty}\langle 1_{\Delta_n}(*,k)f_n(*,k), g_n(*,k)\rangle_{l^2(\mathbb{N}^*)^{\otimes(n-1)}} \\
&= n!\langle 1_{\Delta_n}f_n, g_n\rangle_{l^2(\mathbb{N}^*)^{\otimes n}} \\
&= E[J_n(f_n)J_n(g_n)] = \mathrm{Cov}(F,G).
\end{aligned}$$

The extension of the identity to $\mathbb{D}([1,\infty[)$ is obtained by orthogonality of multiple stochastic integrals of different orders, by closability of the operator D, and by continuity of the operator Q_t on $\mathbb{D}([1,\infty[)$. □

We now assume that $p = q = 1/2$.

Proposition 9 *We have for $F \in L^2(\Omega, \mathcal{F}_N)$:*

$$Q_tF(\omega') = \int_{\Omega} F(\omega)q_t^N(\omega,\omega')dP(\omega), \quad \omega,\omega' \in \Omega,$$

where $q_t^N(\omega,\omega')$ is the kernel

$$q_t^N(\omega,\omega') = \prod_{i=1}^{i=N}(1 + e^{-t}Y_i(\omega)Y_i(\omega')), \quad \omega,\omega' \in \Omega.$$

Proof. Since $L^2(\Omega, \mathcal{F}_N)$ is finite dimensional it suffices to consider the exponential function

$$Y(B_N, N, z) = \sum_{n=0}^{n=N} K_n(B_N; N, p)\frac{z^n}{n!} = (1+qz)^{B_N}(1-pz)^{N-B_N}.$$

We have

$$\begin{aligned} E\left[\left(1+\frac{1}{2}z\right)^{X_i}\left(1-\frac{1}{2}z\right)^{1-X_i}(1+e^{-t}Y_iY_i(\omega'))\right] &= \frac{1}{2}\left(1+\frac{1}{2}z\right)(1+e^{-t}Y_i(\omega')) \\ &\quad +\frac{1}{2}\left(1-\frac{1}{2}z\right)(1-e^{-t}Y_i(\omega')) \\ &= 1+\frac{1}{2}e^{-t}Y_i(\omega'), \quad \omega' \in \Omega, \end{aligned}$$

which allows to conclude by independence of the sequence $(X_i)_{i\in\mathbb{N}^*}$. □

In particular for $n = 0$, Relation (20) can be written as

$$\mathrm{Cov}(F, G) = E\left[\sum_{k=1}^{\infty} E\left[D_kF \mid \mathcal{F}_{k-1}\right] E\left[D_kG \mid \mathcal{F}_{k-1}\right]\right] = \int_0^\infty E[\langle DF, DQ_tG\rangle_{l^2(\mathbb{N}^*)}]dt,$$

for all $F, G \in I\!\!D([1,\infty[)$. The following result is then a consequence of Th. 2, Prop. 8 and Prop. 9.

Corollary 1 *Let $n \in \mathbb{N}$ and $F, G \in L^2(\Omega, \mathcal{F}_N)$. We have*

$$\mathrm{Cov}(F,G) = \sum_{d=1}^{d=n}(-1)^{d+1}E\left[\sum_{\{1\le k_1<\cdots<k_d\le N\}}(D_{k_d}\cdots D_{k_1}F)(D_{k_d}\cdots D_{k_1}G)\right] \tag{21}$$

$$+(-1)^n\int_\Omega\int_\Omega \sum_{\{1\le k_1<\cdots<k_{n+1}\le N\}} D_{k_{n+1}}\cdots D_{k_1}F(\omega)D_{k_{n+1}}\cdots D_{k_1}G(\omega')q_t^N(\omega,\omega')P(d\omega)P(d\omega').$$

In particular, for $n = 0$ we obtain

$$\mathrm{Cov}(F,G) = \int_\Omega\int_\Omega \sum_{k=1}^{k=N} D_kF(\omega)D_kG(\omega')q_t^N(\omega,\omega')P(d\omega)P(d\omega').$$

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