Weitzenböck and Clark-Ocone decompositions for differential forms on the space of normal martingales*

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Abstract

We present a framework for the construction of Weitzenböck and Clark-Ocone formulae for differential forms on the probability space of a normal martingale. This approach covers existing constructions based on Brownian motion, and extends them to other normal martingales such as compensated Poisson processes. It also applies to the path space of Brownian motion on a Lie group and to other geometries based on the Poisson process. Classical results such as the de Rham-Hodge-Kodaira decomposition and the vanishing of harmonic differential forms are extended in this way to finite difference operators by two distinct approaches based on the Weitzenböck and Clark-Ocone formulae.

Keywords: de Rham-Hodge-Kodaira decomposition, differential forms, Weitzenböck identity, Clark-Ocone formula, normal martingales, Malliavin calculus.

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1 Introduction

Vanishing theorems for harmonic forms on Riemannian manifolds can be proved by the Bochner method, which involves the Weitzenböck formula and relates the Hodge

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Laplacian on differential n-forms $\Delta_n = \mathrm{d}^{n-1}\mathrm{d}^{(n-1)*} + \mathrm{d}^{n*}\mathrm{d}^n$ to the Bochner Laplacian $L = \nabla^*\nabla$ through a zero order curvature term R_n, i.e.

$$\Delta_n = L + R_n. \tag{1.1}$$

Here, d^n is the exterior derivative on n-forms with adjoint d^{n*}, ∇ is the covariant derivative with adjoint ∇^*, and R_n is the Weitzenböck curvature which reduces to the usual Ricci tensor on one-forms. In particular, since both Laplacian operators Δ_n and L are non-negative, the identity (1.1) shows that there are no L^2 harmonic n-forms on a complete manifold when the curvature term R_n is positive.

The Bochner vanishing technique extends to infinite dimension, in particular in the linear case. On abstract Wiener spaces, the de Rham-Hodge decomposition and a Weitzenböck formula have been derived in [20] with $-L$ the Ornstein-Uhlenbeck operator and $R_n = n\mathrm{Id}$, and it has been shown therein that there exist no nontrivial harmonic n-forms for $n \geq 1$. Various other Weitzenböck-type formulae have been established on infinite-dimensional manifolds with curvature, for example, on submanifolds of the Wiener space in [12], on path spaces over Riemannian manifolds [6], and on loop spaces over Lie groups [10], with more complicated curvature terms R_n. On the path spaces over compact Lie groups, the Itô map has been in used in [11] to construct a diffeomorphism which transfers the Weitzenböck formula of [20], and thus the vanishing theorem, from the Wiener space to path groups. The vanishing of harmonic one-forms on loop groups has also been proved in [1] using the Weitzenböck formula [10].

Vanishing theorems on path spaces can also be proved using martingale representation and the Clark-Ocone formula

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F \mid \mathcal{F}_t] dB_t,$$

cf. [5], [15], which decomposes a square-integrable function into the sum of a constant and an Itô integral with respect to Brownian motion, where D_t denotes the Malliavin gradient, cf. (2.5) below. The Clark-Ocone formula has been extended to differential forms in [22] on the Wiener space and in [7] on the path space over a Riemannian manifold, in order to decompose an n-form into the sum of an exact form and a martingale. As the martingale component in the decomposition vanishes when the given n-form is closed, such a formula can be used to show that there exist no non-trivial harmonic n-forms. On the classical Wiener space, these generalised Clark-Ocone formulae [22] provide an alternative proof of the vanishing results of [20]; in addition, they give explicit expression for closed differential forms, while their dual versions apply to the representation of co-closed forms. The vanishing of harmonic one-forms on the path spaces over Riemannian manifolds, has been proved by the Clark-Ocone formula in [7].

Until now, those vanishing techniques have only been applied in the Brownian framework, where the underlying gradient operator satisfies the derivation property. The aim of this paper is to show that they also apply to a large family of stochastic processes, without requiring the derivation property of the gradient operator nor a Gaussian setting. In particular, our argument applies to gradient operators defined by chaos expansion methods with respect to normal martingales. In this way the Weitzenböck and Clark-Ocone decompositions are shown to apply not only on the Wiener space, but also to other normal continuous-time martingales such as the compensated Poisson process, for which the gradient operator can be defined by finite differences. Our approach relies on a direct proof inspired by the arguments of [9] and [10] for the Weitzenböck formula on path and loop groups.

In Section 3 we construct a Hodge Laplacian $\Delta_n = \mathrm{d}^{n-1}\mathrm{d}^{(n-1)*} + \mathrm{d}^{n*}\mathrm{d}^n$ on differential n-forms and we derive the de Rham-Hodge decomposition

$$L^2(\Omega; H^{\wedge n}) = \mathrm{Im}\, \mathrm{d}^{n-1} \oplus \mathrm{Im}\, \mathrm{d}^{n*} \oplus \mathrm{Ker}\, \Delta_n, \qquad n \geq 1,$$

cf. (3.7). Section 4 deals with examples to which our general framework applies, including chaos-based settings and a non-chaos based constructions such as the path space over a Lie group.

In Theorem 5.2 we prove the Weitzenböck identity

$$\Delta_n = n\mathrm{Id}_{H^{\wedge n}} + \nabla^* \nabla, \qquad n \geq 1,$$

and in Proposition 5.1 we show the vanishing of harmonic forms $\mathrm{Ker}\,\Delta_n = \{0\}$, from which the de Rham-Hodge decomposition

$$L^2(\Omega; H^{\wedge n}) = \mathrm{Im}\,\mathrm{d}^{n-1} \oplus \mathrm{Im}\,\mathrm{d}^{n*}, \qquad n \geq 1,$$

follows. This result is also derived in Corollary 6.2 from the Clark-Ocone formula of Theorem 6.1, showing the complementarity of the two approaches.

It can be shown in addition that this method goes beyond chaos expansions and encompasses other natural geometries in addition to the path space over a Lie group described in Section 4, for example on the Poisson space over the half line $\mathbb{R}_+$, cf. [19], in which case the gradient operator has the derivation property.

In [2], [3], n-differential forms on the configuration space over a Riemannian manifold under a Poisson random measure have been constructed in a different way by a lifting of the underlying differential structure on the manifold to the configuration space. We also refer the reader to [4] for a different approach to the construction of the Hodge decomposition on abstract metric spaces.

This paper is organized as follows. Sections 2 and 3 introduce the general framework of differential and divergence operators on functions and differential forms, including duality and commutation relations. Section 4 describes a number of examples to which this framework applies, while Sections 5 and 6 present the main results on the Weitzenböck identity and the generalised Clark-Ocone formulae, respectively, including the vanishing of harmonic forms. The examples of Section 4, which range from normal martingales to Lie-group valued Brownian motion, are revisited one by one in the frameworks of Sections 5 and 6. The appendix contains the proofs of the main results.

2 Differential forms and exterior derivative

In this section we introduce an abstract gradient and divergence framework based on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an algebra $\mathcal{S} \subset L^2(\Omega)$ for the pointwise product of random variables, dense in $L^2(\Omega)$.

Deterministic forms

We fix a measure space (X, σ) and we consider a linear space H of $\mathbb{R}^d$-valued functions, dense in $L^2(X, \mathbb{R}^d)$, $d \geq 1$, and endowed with the inner product induced from $L^2(X, \mathbb{R}^d)$. Denote by $H^{\otimes n}$ the n-th tensor power of H, and by $H^{\circ n}$, resp. $H^{\wedge n}$, its subspaces of symmetric, resp. skew-symmetric tensors, completed using the inherited Hilbert space cross norm. The exterior product $\wedge$ is defined as

$$h_1 \wedge \cdots \wedge h_n := \mathcal{A}_n(h_1 \otimes \cdots \otimes h_n), \quad h_1, \ldots, h_n \in H, \tag{2.1}$$

where $\mathcal{A}_n$ denotes the antisymmetrization map on n-tensors given by

$$\mathcal{A}_n(h_1 \otimes \cdots \otimes h_n) = \sum_{\sigma \in \Sigma_n} \mathrm{sign}(\sigma)(h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)}), \tag{2.2}$$

and the summation is over $n!$ elements of the symmetric group Σ_n consisting of all permutations of $\{1, \ldots, n\}$. We also equip $H^{\wedge n}$ with the inner product

$$\begin{aligned} \langle f_n, g_n \rangle_{H^{\wedge n}} &:= \frac{1}{n!} \int_X \cdots \int_X \langle f_n(x_1, \ldots, x_n), g_n(x_1, \ldots, x_n) \rangle_{(\mathbb{R}^d)^{\otimes n}} \sigma(dx_1) \cdots \sigma(dx_n) \\ &= \frac{1}{n!} \langle f_n, g_n \rangle_{H^{\otimes n}}, \qquad f_n, g_n \in H^{\wedge n}, \end{aligned}$$

so that we have in particular

$$\begin{aligned} &\langle h_1 \wedge \cdots \wedge h_n, k_1 \wedge \cdots \wedge k_n \rangle_{H^{\wedge n}} \\ &:= \frac{1}{n!} \int_X \cdots \int_X \langle \mathcal{A}_n(h_1 \otimes \cdots \otimes h_n), \mathcal{A}_n(k_1 \otimes \cdots \otimes k_n) \rangle_{(\mathbb{R}^d)^{\otimes n}} \sigma(dx_1) \cdots \sigma(dx_n) \\ &= \frac{1}{n!} \int_X \cdots \int_X \Big\langle \sum_{\sigma \in \Sigma_n} \mathrm{sign}(\sigma)(h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)}), \\ &\sum_{\eta \in \Sigma_n} \mathrm{sign}(\eta)(k_{\eta(1)} \otimes \cdots \otimes k_{\eta(n)}) \Big\rangle_{(\mathbb{R}^d)^{\otimes n}} \sigma(dx_1) \cdots \sigma(dx_n) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\sigma\in\Sigma_n} \mathrm{sign}(\sigma) \int_X \cdots \int_X \langle h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)}, k_1 \otimes \cdots \otimes k_n \rangle_{(\mathrm{R}^d)^{\otimes n}} \sigma(dx_1)\cdots\sigma(dx_n) \\
&= \sum_{\sigma\in\Sigma_n} \mathrm{sign}(\sigma) \int_X \cdots \int_X \langle h_{\sigma(1)}, k_1\rangle_{\mathrm{R}^d} \cdots \langle h_{\sigma(n)}, k_n\rangle_{\mathrm{R}^d} \sigma(dx_1)\cdots\sigma(dx_n) \\
&= \det((\langle h_i, k_j\rangle_H)_{1\le i,j\le n}).
\end{aligned}
$$

$h_1, \ldots, h_n,\, k_1, \ldots, k_n \in H$.

Covariant derivative

In the sequel we will use a covariant derivative operator

$$
\begin{aligned}
\nabla \;\; : H &\longrightarrow H \otimes H \\
h &\longmapsto \nabla h = (\nabla_x h)_{x\in X},
\end{aligned}
$$

where $\nabla_x h \in \mathbb{R}^d \otimes H$, $x \in X$, is defined from the relation

$$
\langle \nabla_x h, k\rangle_H := \langle \nabla h, k\rangle_H(x) \in \mathbb{R}^d, \qquad x \in X, \quad h, k \in H.
$$

We will extend the definition of ∇ to an operator

$$
\nabla : H^{\wedge n} \longrightarrow H \otimes H^{\wedge n}
$$

on differential forms in $H^{\wedge n}$, by the following steps.

(i) Let

$$
\nabla_x^{(l)}(h_1 \wedge \cdots \wedge h_n) \in \underbrace{H \wedge \cdots \wedge H}_{\text{l-1 times}} \wedge (\mathbb{R}^d \otimes H) \wedge \underbrace{H \wedge \cdots \wedge H}_{\text{n-l times}}
$$

as

$$
\nabla_x^{(l)}(h_1 \wedge \cdots \wedge h_n) := h_1 \wedge \cdots \wedge h_{l-1} \wedge \nabla_x h_l \wedge h_{l+1} \wedge \cdots \wedge h_n, \qquad x \in X,
$$

$l = 1, \ldots, n$.

(ii) We define $\nabla_x(h_1 \wedge \cdots \wedge h_n)$ in $\mathbb{R}^d \otimes H^{\wedge n}$, $x \in X$, as

$$
\nabla_x(h_1 \wedge \cdots \wedge h_n) \;\; := \;\; \sum_{j=1}^{n} \nabla_x^{(j)}(h_1 \wedge \cdots \wedge h_n) \tag{2.3}
$$

$$= \sum_{j=1}^{n} h_1 \wedge \cdots \wedge h_{j-1} \wedge \nabla_x h_j \wedge h_{j+1} \wedge \cdots \wedge h_n,$$

by canonically identifying the space

$$\underbrace{H \wedge \cdots \wedge H}_{\text{l-1 times}} \wedge (\mathbb{R}^d \otimes H) \wedge \underbrace{H \wedge \cdots \wedge H}_{\text{n-l times}}$$

to $\mathbb{R}^d \otimes H^{\wedge n}$, for $l = 1, \ldots, n$.

Given $g \in H$ we also define $\nabla_g(h_1 \wedge \cdots \wedge h_n) \in H^{\wedge n}$ by

$$\begin{aligned}
\nabla_g(h_1 \wedge \cdots \wedge h_n) &:= \int_X \langle g(x), \nabla_x(h_1 \wedge \cdots \wedge h_n)\rangle_{\mathrm{R}^d} \sigma(dx) \\
&= \sum_{j=1}^{n} \int_X \langle g(x), (h_1 \wedge \cdots \wedge h_{j-1} \wedge \nabla_x h_j \wedge h_{j+1} \wedge \cdots \wedge h_n)\rangle_{\mathrm{R}^d} \sigma(dx) \\
&= \sum_{j=1}^{n} (h_1 \wedge \cdots \wedge h_{j-1} \wedge \nabla_g h_j \wedge h_{j+1} \wedge \cdots \wedge h_n).
\end{aligned}$$

Exterior derivative - deterministic forms

We now define the exterior derivative on n-forms $u_n \in H^{\wedge n}$ by

$$\langle \mathrm{d}^n u_n, h_1 \wedge \cdots \wedge h_{n+1}\rangle_{H^{\wedge(n+1)}} := \sum_{k=1}^{n+1} (-1)^{k-1} \langle \nabla_{h_k} u_n, h_1 \wedge \cdots \wedge h_{k-1} \wedge h_{k+1} \wedge \cdots \wedge h_{n+1}\rangle_{H^{\wedge n}}, \tag{2.4}$$

where $h_1, \ldots, h_{n+1} \in H$, i.e. $\mathrm{d}^n = \dfrac{1}{n!}\mathcal{A}_{n+1}\nabla$, and the $(n+1)$-form $\mathrm{d}^n(h_1 \wedge \cdots \wedge h_n)$ is given by

$$\begin{aligned}
&\mathrm{d}^n_{x_{n+1}}((h_1 \wedge \cdots \wedge h_n)(x_1, \ldots, x_n)) = \mathrm{d}^n(h_1 \wedge \cdots \wedge h_n)(x_1, \ldots, x_{n+1}) \\
&= \sum_{j=1}^{n+1} (-1)^{j-1} \nabla_{x_j}(h_1 \wedge \cdots \wedge h_n)(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}) \\
&= \sum_{j=1}^{n+1} (-1)^{j-1} \sum_{i=1}^{n} \nabla^{(i)}_{x_j}(h_1 \wedge \cdots \wedge h_n)(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}) \\
&= \sum_{j=1}^{n+1} (-1)^{j-1} \sum_{i=1}^{n} (h_1 \wedge \cdots \wedge \nabla_{x_j} h_i \wedge \cdots \wedge h_n)(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}).
\end{aligned}$$

Random forms

In the sequel we will need a linear gradient operator

$$D \ : \mathcal{S} \longrightarrow L^2(\Omega; H) \qquad F \longmapsto DF = (D_x F)_{x \in X} \tag{2.5}$$

acting on random variables in $\mathcal{S}$.

We work on the space $\mathcal{S} \otimes H^{\wedge n}$ of elementary (random) n-forms that can be written as linear combinations of terms for the form

$$u_n = F \otimes h \in \mathcal{S} \otimes H^{\wedge n}, \qquad F \in \mathcal{S}, \quad h \in H^{\wedge n}. \tag{2.6}$$

The operator D is extended to $u_n \in \mathcal{S} \otimes H^{\wedge n}$ as in (2.6) by the pointwise equality

$$D_x u_n := (D_x F) \otimes (h_1 \wedge \cdots \wedge h_n), \qquad x \in X, \tag{2.7}$$

i.e. $Du_n \in \mathcal{S} \otimes H \otimes H^{\wedge n}$. We also extend ∇ to random forms $u_n = F \otimes f_n \in \mathcal{S} \otimes H^{\wedge n}$ by defining $\nabla u_n \in \mathcal{S} \otimes H \otimes H^{\wedge n}$ as

$$\begin{aligned} \nabla_y(u_n(x_1, \ldots, x_n)) &:= (D_y F) \otimes f_n(x_1, \ldots, x_n) + F \otimes \nabla_y f_n(x_1, \ldots, x_n) \\ &= (D_y F) \otimes f_n(x_1, \ldots, x_n) + F \otimes \sum_{l=1}^{n} \nabla_y^{(l)} f_n(x_1, \ldots, x_n), \end{aligned}$$

$x_1, \ldots, x_n, y \in X$. In particular for $n = 1$, ∇ extends to stochastic processes (or one-forms) as

$$\nabla_x(u_1(y)) = \nabla_x(F \otimes f_1(y)) = (D_x F) \otimes f_1(y) + F \otimes \nabla_x f_1(y),$$

with $u_1 = F \otimes f_1 \in \mathcal{S} \otimes H$.

Lie bracket and vanishing of torsion

The Lie bracket $\{f, g\}$ of $f, g \in H$, is defined to be the unique element w of H satisfying

$$(D_f D_g - D_g D_f)F = D_w F, \qquad F \in \mathcal{S}, \tag{2.8}$$

where

$$D_f F := \langle f, DF\rangle_H, \qquad f \in H, \quad F \in \mathrm{Dom}\,(D),$$

and is extended to $u, v \in \mathcal{S} \otimes H$ by

$$\{Fu, Gv\}(x) = FG\{u, v\}(x) + v(x)FD_uG - u(x)GD_vF, \qquad x \in X,$$

$u, v \in H$, $F, G \in \mathcal{S}$. In the sequel we will make the following assumption.

(A1) Vanishing of torsion. The connection defined by ∇ has a vanishing torsion, i.e. we have

$$\{u, v\} = \nabla_u v - \nabla_v u, \qquad u, v \in \mathcal{S} \otimes H. \tag{A1}$$

From (2.8) the vanishing of torsion Assumption (A1) can be written as

$$\begin{aligned}
&\int_X \int_X \langle (D_x D_y F - D_y D_x F), f(x) \otimes g(y)\rangle_{\mathrm{R}^d \otimes \mathrm{R}^d} \sigma(dx)\sigma(dy) \qquad (2.9)\\
&\quad = \int_X \int_X \langle \nabla_y g(x), D_x F \otimes f(y)\rangle_{\mathrm{R}^d \otimes \mathrm{R}^d} \sigma(dx)\sigma(dy)\\
&\qquad - \int_X \int_X \langle \nabla_y f(x), D_x F \otimes g(y)\rangle_{\mathrm{R}^d \otimes \mathrm{R}^d} \sigma(dx)\sigma(dy), \qquad F \in \mathcal{S}, \quad f, g \in H.
\end{aligned}$$

Exterior derivative - random forms

From Assumption (A1) we may now define the $(n+1)$-form $\mathrm{d}^n u_n$ as

$$\begin{aligned}
\mathrm{d}^n u_n(x_1, \ldots, x_{n+1}) &:= \mathrm{d}^n_{x_{n+1}}(u_n(x_1, \ldots, x_n))\\
&= \frac{1}{n!}\mathcal{A}_{n+1}(\nabla . u_n)(x_1, \ldots, x_{n+1})\\
&= (D.F \wedge f_n)(x_1, \ldots, x_{n+1}) + \frac{1}{n!} F \otimes \mathcal{A}_{n+1}(\nabla . f_n)(x_1, \ldots, x_{n+1}),
\end{aligned}$$

which is also equal to

$$\sum_{j=1}^{n+1} (-1)^{j-1} \nabla_{x_j} u_n(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1})$$

in $H^{\wedge(n+1)}$, for $u_n \in \mathcal{S} \otimes H^{\wedge n}$ of the form (2.6). In other words, on elementary forms we have

$$\mathrm{d}^n_{x_{n+1}}(F \otimes h_1 \wedge \cdots \wedge h_n(x_1, \ldots, x_n)) = \mathrm{d}^n(F \otimes h_1 \wedge \cdots \wedge h_n)(x_1, \ldots, x_{n+1}) \tag{2.10}$$

$$= \quad ((D.F) \wedge h_1 \wedge \cdots \wedge h_n)(x_1, \ldots, x_{n+1}) + F \otimes \mathrm{d}^n(h_1 \wedge \cdots \wedge h_n)(x_1, \ldots, x_{n+1}).$$

In particular, for $n = 0$ we have

$$\mathrm{d}_x^0 F = \nabla_x F = D_x F, \qquad F \in \mathcal{S} \otimes H^{\wedge 0} = \mathcal{S}.$$

and for $n = 1$,

$$\mathrm{d}^1(F \otimes h_1)(x_1, x_2) = \mathrm{d}_{x_2}^1(F \otimes h_1(x_1)) = F \otimes \mathrm{d}_{x_1}^1 h_1(x_2) + (D_{x_1} F) \otimes h(x_2) - D_{x_2} F \otimes h(x_1).$$

We also note that

$$\mathrm{d}^n(\mathcal{S} \otimes H^{\wedge n}) \subset \mathrm{Dom}\,(\mathrm{d}^{n+1}), \qquad n \in \mathbb{N}. \tag{2.11}$$

Assumption (A1) and the invariant formula for differential forms (see e.g. Prop. 3.11 page 36 of [13]) also show that we have

$$\mathrm{d}^{n+1}\mathrm{d}^n = 0, \qquad n \in \mathbb{N}, \tag{2.12}$$

which implies

$$\mathrm{Im}\,\mathrm{d}^n \subset \mathrm{Ker}\,\mathrm{d}^{n+1} \subset \mathrm{Dom}\,(\mathrm{d}^{n+1}), \qquad n \in \mathbb{N}. \tag{2.13}$$

3 Divergence of n-forms and duality

In this section we consider a divergence operator

$$\begin{aligned} \delta \ &: \mathcal{S} \otimes H \longrightarrow L^2(\Omega), \\ u = (u(x))_{x \in X} &\longmapsto \delta(u) \end{aligned}$$

acting on stochastic processes, and extended to elementary n-forms by letting

$$\begin{aligned} &\delta(h_1 \wedge \cdots \wedge h_n) \\ &\quad := \frac{1}{n} \sum_{j=1}^n (-1)^{j-1} \delta(h_j) \otimes (h_1 \wedge \cdots \wedge h_{j-1} \wedge h_{j+1} \wedge \cdots \wedge h_n) \in \mathcal{S} \otimes H^{\wedge(n-1)}, \end{aligned}$$

and to simple elements $u \in \mathcal{S} \otimes H^{\wedge n}$ of the form (2.6) by

$$\begin{aligned} &\delta(u_n)(x_1, \ldots, x_{n-1}) := \delta(u_n(\cdot, x_1, \ldots, x_{n-1})) \qquad (3.1) \\ &= \delta(F \otimes (h_1 \wedge \cdots \wedge h_n))(x_1, \ldots, x_{n-1}) \\ &= \frac{1}{n} \sum_{j=1}^n (-1)^{j-1} \delta(F \otimes h_j) \otimes (h_1 \wedge \cdots \wedge h_{j-1} \wedge h_{j+1} \wedge \cdots \wedge h_n)(x_1, \ldots, x_{n-1}). \end{aligned}$$

Divergence of random forms

The divergence operator d^{n*} on $(n+1)$-forms $u_{n+1} = F \otimes f_{n+1} \in \mathcal{S} \otimes H^{\wedge(n+1)}$ of the form (2.6) is defined by

$$\mathrm{d}^{n*} u_{n+1}(x_1, \ldots, x_n) := \delta(F \otimes f_{n+1}(\cdot, x_1, \ldots, x_n)) - F \,\mathrm{trace}\,(\nabla_\cdot f_{n+1}(\cdot, x_1, \ldots, x_n))$$

where

$$\mathrm{trace}\,(\nabla_\cdot f_{n+1}(\cdot, x_1, \ldots, x_n)) := \int_0^\infty \mathrm{Tr}\, \nabla_x f_{n+1}(x, x_1, \ldots, x_n) \sigma(dx)$$

and Tr denotes the trace on $\mathbb{R}^d \otimes \mathbb{R}^d$, i.e. we have

$$\begin{aligned} &\mathrm{d}^{n*} u_{n+1}(x_1, \ldots, x_n) \\ &\quad = \delta(F \otimes f_{n+1}(\cdot, x_1, \ldots, x_n)) - F \int_0^\infty \mathrm{Tr}\, \nabla_x f_{n+1}(x, x_1, \ldots, x_n) \sigma(dx), \end{aligned} \quad (3.2)$$

which belongs to $\mathcal{S} \otimes H^{\wedge n}$ from (3.1), $n \geq 1$, and

$$\mathrm{d}^{0*} u_1 = \delta(u_1), \qquad u_1 \in \mathcal{S} \otimes H, \quad (3.3)$$

when $n = 0$ since $\nabla_x f_1(x) = 0$.

Duality relations

We will make the following assumptions on δ, D and ∇:

(A2) The operators D and δ satisfy the duality relation

$$\mathbb{E}[\langle DF, u \rangle_H] = \mathbb{E}[F \delta(u)], \qquad F \in \mathrm{Dom}\,(D), \quad u \in \mathrm{Dom}\,(\delta). \quad \textbf{(A2)}$$

The above duality condition (A2) implies that the operators D and δ are closable, cf. Proposition 3.1.2 of [18], and the operators D and δ are extended to their respective closed domains $\mathrm{Dom}\,(D)$ and $\mathrm{Dom}\,(\delta)$.

(A3) The operator ∇ satisfies the condition

$$\int_{X^{n+1}} \langle g_{n+1}(x_1, \ldots, x_{n+1}), \nabla_{x_1} f_n(x_2, \ldots, x_{n+1}) \rangle_{(\mathbb{R}^d)^{\otimes(n+1)}} \sigma(dx_1) \cdots \sigma(dx_{n+1}) =$$

$$- \int_{X^{n+1}} \langle \operatorname{Tr} \nabla_x g_{n+1}(x, x_1, \ldots, x_n), f_n(x_1, \ldots, x_n) \rangle_{(\mathbb{R}^d)^{\otimes n}} \sigma(dx_1) \cdots \sigma(dx_n) \sigma(dx), \tag{A3}$$

$f_n \in H^{\otimes n}$, $g_{n+1} \in H^{\otimes(n+1)}$, $n \geq 1$.

The compatibility condition (A3) is weaker than the usual compatibility of ∇ with the metric $\langle \cdot, \cdot \rangle_H$, which reads

$$\langle \nabla_x f_n, g_n \rangle_{H^{\otimes n}} = -\langle f_n, \nabla_x g_n \rangle_{H^{\otimes n}}, \qquad x \in X, \tag{3.4}$$

$f_n, g_n \in H^{\otimes n}$. Indeed, when applying (3.4) to $f_{n+1}(x, \cdot) \in H^{\otimes n}$ and $g_n \in H^{\otimes n}$, $x \in X$, $n \geq 1$, we get

$$\begin{aligned}
&\int_{X^{n+1}} \langle \operatorname{Tr} \nabla_x f_{n+1}(x, x_1, \ldots, x_n), g_n(x_1, \ldots, x_n) \rangle_{(\mathbb{R}^d)^{\otimes n}} \sigma(dx_1) \cdots \sigma(dx_n) \sigma(dx) \\
&= \sum_{k=1}^{d} \int_{X^{n+1}} \langle (\nabla_x f_{n+1}^{(k)}(x, x_1, \ldots, x_n))^{(k)}, g_n(x_1, \ldots, x_n) \rangle_{(\mathbb{R}^d)^{\otimes n}} \sigma(dx_1) \cdot \sigma(dx_n) \sigma(dx) \\
&= -\sum_{k=1}^{d} \int_{X^{n+1}} \langle f_{n+1}^{(k)}(x, x_1, \ldots, x_n), (\nabla_x g_n(x_1, \ldots, x_n))^{(k)} \rangle_{(\mathbb{R}^d)^{\otimes n}} \sigma(dx_1) \cdot \sigma(dx_n) \sigma(dx) \\
&= -\int_{X^{n+1}} \langle f_{n+1}(x, x_1, \ldots, x_n), \nabla_x g_n(x_1, \ldots, x_n) \rangle_{(\mathbb{R}^d)^{\otimes(n+1)}} \sigma(dx_1) \cdot \sigma(dx_n) \sigma(dx),
\end{aligned}$$

where $f_{n+1}^{(k)}(x, x_1, \ldots, x_n)$ denotes the k-th component in $\mathbb{R}^d$ of the first component of $f_{n+1}(x, x_1, \ldots, x_n)$ in $(\mathbb{R}^d)^{\otimes(n+1)}$. In this sense, Assumption (A3) is automatically satisfied in all settings which incorporate the compatibility (3.4) with $\langle \cdot, \cdot \rangle$.

Proposition 3.1. (Duality). *Under Assumptions (A1)-(A3), for any $u_n \in \mathcal{S} \otimes H^{\wedge n}$ and $v_{n+1} \in \mathcal{S} \otimes H^{\wedge(n+1)}$ we have*

$$\langle \mathrm{d}^n u_n, v_{n+1} \rangle_{L^2(\Omega, H^{\wedge(n+1)})} = \langle u_n, \mathrm{d}^{n*} v_{n+1} \rangle_{L^2(\Omega, H^{\wedge n})}. \tag{3.5}$$

As above we note that the duality (3.5) implies the closability of both d^n and $\mathrm{d}^{n*} v_{n+1}$, which are extended to their closed domains $\operatorname{Dom}(\mathrm{d}^n)$ and $\operatorname{Dom}(\mathrm{d}^{n*})$, $n \in \mathbb{N}$, by the same argument as in Proposition 3.1.2 of [18]. When $n = 0$, the statement of Proposition 3.1 reduces to (A2).

The proof of Proposition 3.1 is postponed to the appendix. In the case of one-forms it reads

$$
\begin{aligned}
&\langle \mathrm{d}^1_{t_2}(Ff_1(t_1)), Gg_2(t_1,t_2)\rangle_{L^2(\Omega,H^{\wedge 2})} \\
&= \left\langle \nabla_{t_1}\left(\frac{F}{2}f_1(t_2)\right), Gg_2(t_1,t_2)\right\rangle_{L^2(\Omega,H^{\otimes 2})} - \left\langle \nabla_{t_2}\left(\frac{F}{2}f_1(t_1)\right), Gg_2(t_1,t_2)\right\rangle_{L^2(\Omega,H^{\otimes 2})} \\
&= \left\langle D_{t_1}\frac{F}{2}f_1(t_2), Gg_2(t_1,t_2)\right\rangle_{L^2(\Omega,H^{\otimes 2})} - \left\langle D_{t_2}\frac{F}{2}f_1(t_1), Gg_2(t_1,t_2)\right\rangle_{L^2(\Omega,H^{\otimes 2})} \\
&+ \left\langle \frac{F}{2}\nabla_{t_1}f_1(t_2), Gg_2(t_1,t_2)\right\rangle_{L^2(\Omega,H^{\otimes 2})} - \left\langle \frac{F}{2}\nabla_{t_2}f_1(t_1), Gg_2(t_1,t_2)\right\rangle_{L^2(\Omega,H^{\otimes 2})} \\
&= \left\langle \frac{F}{2}f_1(t_2), \delta(Gg_2(\cdot,t_2))\right\rangle_{L^2(\Omega,H)} - \left\langle \frac{F}{2}f_1(t_1), \delta(Gg_2(t_1,\cdot))\right\rangle_{L^2(\Omega,H)} \\
&- \left\langle \frac{F}{2}f_1(t_2), G\int_0^\infty \operatorname{Tr}\nabla_t g_2(t,t_2)dt\right\rangle_{L^2(\Omega,H)} - \left\langle \frac{F}{2}f_1(t_1), G\int_0^\infty \operatorname{Tr}\nabla_t g_2(t_1,t)dt\right\rangle_{L^2(\Omega,H)} \\
&= \left\langle \frac{F}{2}f_1(t_2), \delta(Gg_2(\cdot,t_2))\right\rangle_{L^2(\Omega,H)} + \left\langle \frac{F}{2}f_1(t_1), \delta(Gg_2(\cdot,t_1))\right\rangle_{L^2(\Omega,H)} \\
&- \left\langle \frac{F}{2}f_1(t_2), G\int_0^\infty \operatorname{Tr}\nabla_t g_2(t,t_2)dt\right\rangle_{L^2(\Omega,H)} + \left\langle \frac{F}{2}f_1(t_1), G\int_0^\infty \operatorname{Tr}\nabla_t g_2(t,t_1)dt\right\rangle_{L^2(\Omega,H)} \\
&= \langle Ff_1(t_1), \delta(Gg_2(\cdot,t_1))\rangle_{L^2(\Omega,H)} - \left\langle Ff_1(t_1), G\int_0^\infty \operatorname{Tr}\nabla_t g_2(t,t_1)dt\right\rangle_{L^2(\Omega,H)} \\
&= \langle Ff_1(t_1), \mathrm{d}^{1*}(Gg_2)(t_1)\rangle_{L^2(\Omega,H)}, \qquad F,G\in\mathcal{S}, \quad f_1\in H, \quad g_2\in H^{\wedge 2}.
\end{aligned}
$$

As in (A2) above, the duality (3.5) shows that d^n extends to a closed operator

$$\mathrm{d}^n : \mathrm{Dom}\,(\mathrm{d}^n) \longrightarrow L^2(\Omega; H^{\wedge(n+1)})$$

with domain $\mathrm{Dom}\,(\mathrm{d}^n) \subset L^2(\Omega; H^{\wedge n})$, and d^{n*}, $n\in\mathbb{N}$, extends to a closed operator

$$\mathrm{d}^{n*} : \mathrm{Dom}\,(\mathrm{d}^{n*}) \longrightarrow L^2(\Omega; H^{\wedge n})$$

with domain $\mathrm{Dom}\,(\mathrm{d}^{n*}) \subset L^2(\Omega; H^{\wedge(n+1)})$, by the same argument as in Proposition 3.1.2 of [18].

In addition, by the coboundary condition (2.12) and the duality (3.5) we find

$$\mathrm{d}^{n*}\mathrm{d}^{(n+1)*} = 0, \qquad n\in\mathbb{N}.$$

Based on (2.13) we define the Hodge Laplacian on differential n-forms as

$$\Delta_n = \mathrm{d}^{n-1}\mathrm{d}^{(n-1)*} + \mathrm{d}^{n*}\mathrm{d}^n, \tag{3.6}$$

and call harmonic n-forms the elements of the kernel $\mathrm{Ker}\,\Delta_n$ of Δ_n. By (2.12)-(2.13) and Proposition 3.1 we have the de Rham-Hodge decomposition

$$L^2(\Omega; H^{\wedge n}) = \mathrm{Im}\,\mathrm{d}^{n-1} \oplus \mathrm{Im}\,\mathrm{d}^{n*} \oplus \mathrm{Ker}\,\Delta_n, \qquad n \geq 1. \tag{3.7}$$

Indeed, the spaces of exact and co-exact forms $\mathrm{Im}\,\mathrm{d}^{n-1}$ and $\mathrm{Im}\,\mathrm{d}^{n*}$ are mutually orthogonal by (2.12) and the duality of Proposition 3.1. Moreover, the orthogonal complement $(\mathrm{Ker}\,\mathrm{d}^{(n-1)*}) \cap (\mathrm{Ker}\,\mathrm{d}^n)$ of $\mathrm{Im}\,\mathrm{d}^{n-1} \oplus \mathrm{Im}\,\mathrm{d}^{n*}$ in $L^2(\Omega; H^{\wedge n})$ is made of n-forms u_n that are both closed ($d^n u_n = 0$) and co-closed ($d^{(n-1)*}u_n = 0$), hence it is contained in (and equal to) $\mathrm{Ker}\,\Delta_n$ by (3.6).

Intertwinning relations

The statements and proofs of both the Weitzenböck identity and Clark-Ocone formula in the sequel will also require the following conditions. We assume that

(A4) the operator ∇ satisfies

$$\nabla_x f(y) \cdot \nabla_y f(x) = 0, \qquad \sigma(dx)\sigma(dy) - a.e., \qquad f \in H, \tag{A4}$$

i.e. $\sigma(dx)\sigma(dy) - a.e.$ we have $\nabla_x f(y) = 0$ or $\nabla_y f(x) = 0$.

(A5) Intertwinning relation. For all $u \in \mathcal{S} \otimes H$ of the form $u = F \otimes f$ we have

$$\langle g, D\delta(u)\rangle_H = \langle g, u\rangle_H + \delta(\nabla_g u) + \langle DF, \nabla_f g\rangle_H, \qquad g \in H. \tag{A5}$$

We make the following remarks.

Remark 3.1. *(i) When $\nabla = 0$ on H, Assumption (A5) reads*

$$D_x\delta(u) = u(x) + \delta(D_x u), \qquad x \in X,$$

for all $u = F \otimes f \in \mathcal{S} \otimes H$.

(ii) Under the torsion free Assumption (A1), Relation (2.9) *shows that (A5) reads*

$$D_x\delta(u) = u(x) + \delta(\nabla_x u) + \langle DD_xF, f\rangle_H - \langle D_xDF, f\rangle_H + \langle D.F, \nabla_x f(\cdot)\rangle_H, \quad (3.8)$$

$u = F \otimes f \in \mathcal{S} \otimes H$, $x \in X$.

(iii) As a consequence of (3.8)*, Assumption (A5) simplifies to*

$$D_x\delta(h) = h(x) + \delta(\nabla_x h), \qquad x \in X, \quad h \in H. \qquad \textbf{(A5')}$$

when D satisfies the Leibniz rule of derivation

$$D_x(FG) = FD_xG + GD_xF, \qquad F, G \in \mathcal{S}, \quad x \in X. \qquad (3.9)$$

This will be the case in examples where ∇ does not vanish on H.

(iv) When D has the derivation property (3.9)*, Relation* (3.2) *rewrites as the divergence formula*

$$\begin{aligned}&\mathrm{d}^{n*}u_{n+1}(x_1, \ldots, x_n)\\ &\quad = F\delta(f_{n+1}(\cdot, x_1, \ldots, x_n)) - \int_0^\infty \mathrm{Tr}\,\nabla_x(Ff_{n+1}(x, x_1, \ldots, x_n))\sigma(dx),\end{aligned}$$

$u_{n+1} = F \otimes f_{n+1} \in \mathcal{S} \otimes H^{\wedge(n+1)}$.

Proof. We only prove (iii) and (iv).

(iii) First, we note that the duality condition (A2) and the Leibniz rule (3.9) imply

$$\delta(Fh) = F\delta(h) - \langle DF, h\rangle_H, \qquad F \in \mathcal{S}, \quad h \in H. \qquad (3.10)$$

Hence by (A5') and (3.9) we have

$$\begin{aligned}D_x\delta(Fh) &= D_x(F\delta(h) - \langle DF, h\rangle_H)\\ &= \delta(h)D_xF + FD_x\delta(h) - D_x\langle DF, h\rangle_H\\ &= \delta(h)D_xF + Fh(x) + F\delta(\nabla_x h) - D_x\langle DF, h\rangle_H\\ &= Fh(x) + \delta(hD_xF) + \langle DD_xF, h\rangle_H + \delta(F\nabla_x h) + \langle D.F, \nabla_x h(\cdot)\rangle_H - \langle D_xDF, h\rangle_H\\ &= u_1(x) + \delta(\nabla_x u_1) + \langle DD_xF, h\rangle_H - \langle D_xDF, h\rangle_H + \langle D.F, \nabla_x h(\cdot)\rangle_H.\end{aligned}$$

The converse statement is immediate.

(iv) This is a consequence of (3.2) and the divergence formula (3.10). □

4 Examples

In this section we consider examples of frameworks satisfying Assumptions (A1)-(A5).

Commutative examples - chaos expansions

We start by considering a family of examples based on chaos expansions, in which we take $\nabla h = 0$ for all $h \in H$. Here (X, σ) is a measure space, we take $H = L^2(X, \sigma)$ and $d \geq 1$ and we assume that the chaos decomposition holds, i.e. every $F \in L^2(\Omega, \mathcal{F}, P)$ can be decomposed into a series

$$F = \sum_{n=0}^{\infty} I_n(f_n), \qquad f_n \in H^{\circ n},$$

of multiple stochastic integrals, where $I_0(f_0) = \mathbb{E}[F]$ and for all $n \geq 1$, the multiple stochastic integral $I_n : H^{\circ n} \to L^2(\Omega)$ satisfies the isometry condition

$$\langle I_n(f_n), I_m(g_m) \rangle_{L^2(\Omega)} = n! \mathbf{1}_{\{n=m\}} \langle f_n, g_m \rangle_{H^{\circ n}}, \qquad n, m \geq 1.$$

In this case the space $\mathcal{S}$ is made of the finite chaos expansions

$$\mathcal{S} = \left\{ C + \sum_{k=1}^{n} I_k(f_k), \qquad f_k \in L^2(X)^{\circ k}, \quad k = 1, \ldots, n, \quad n \geq 1, \ C \in \mathbb{R} \right\},$$

the operator

$$D : \mathrm{Dom}(D) \longrightarrow L^2(\Omega \times X, dP \times \sigma(dx))$$

is defined by

$$D_x I_n(f_n) := n I_{n-1}(f_n(*, x)), \qquad dP \times \sigma(dx) - a.e., \quad n \in \mathbb{N}. \tag{4.1}$$

On the other hand,

$$\delta : \mathrm{Dom}\,(\delta) \longrightarrow L^2(\Omega),$$

is defined on processes of the form $(I_n(f_{n+1}(*, t)))_{t \in \mathbb{R}_+}$ as

$$\delta(I_n(f_{n+1}(*, \cdot))) := I_{n+1}(\tilde{f}_{n+1}), \qquad n \in \mathbb{N}, \tag{4.2}$$

where $\tilde{f}_{n+1}$ denotes the symmetrization of $f_{n+1} \in H^{\circ n} \otimes H$ in $(n+1)$ variables. In this chaos expansion framework we have $\nabla = D$ and $\nabla f = 0$ for all $f \in H$, hence

Assumptions (A1) and (A3)-(A4) are obviously satisfied and the exterior derivative d is defined by the skew-symmetrisation of D, i.e. (2.10) becomes

$$\mathrm{d}^n_{x_{n+1}}(F \otimes h_1 \wedge \cdots \wedge h_n(x_1, \ldots, x_n)) = ((D.F) \wedge h_1 \wedge \cdots \wedge h_n)(x_1, \ldots, x_{n+1}), \quad (4.3)$$

$x_1, \ldots, x_{n+1} \in X$. As for Assumptions (A2) and (A5) we have the following:

- **(A2)** the duality relation holds in the general framework of chaos expansions; see, e.g., Proposition 4.1.3 of [18];
- **(A5)** given that $\nabla = D$, the commutation relation (A5) holds for $u \in \mathcal{S} \otimes H$, see, e.g., Proposition 4.1.4 of [18].

Note that here, Relations (2.12)-(2.13) hold by the definition (4.3) of the exterior derivative d and the symmetry of second derivative.

Next, we consider some specific examples based on chaos expansions.

Example 1.1 - Poisson random measures

On the probability space of a Poisson random measure $\omega(dx)$ with σ-finite intensity measure $\sigma(dx)$ on X, $I_n(f)$ is the multiple compensated Poisson stochastic integral

$$I_n(f_n) := \int_{\Delta_n} f_n(x_1, \ldots, x_n)(\omega(dx_1) - \sigma(dx_1)) \cdots (\omega(dx_n) - \sigma(dx_n))$$

of the symmetric function $f_n \in H^{\circ n}$ with respect to $\omega(dx)$, where

$$\Delta_n = \{(x_1, \ldots, x_n) \in X^n \ : \ x_i \neq x_j, \quad 1 \leq i < j \leq n\}.$$

Here the operator D_x defined in (4.1) acts by finite differences and addition of a configuration point at $x \in X$, i.e.

$$D_x F(\omega) = F(\omega \cup \{x\}) - F(\omega), \qquad x \in X,$$

where $\omega \cup \{x\}$ represents the addition of the point x to the point configuration ω, see e.g., Proposition 6.4.7 of [18]. Being a finite difference operator, D does not have the derivation property. We refer the reader to § 6.5 of [18] and references therein for the expression of δ defined in (4.2) in this setting.

Example 1.2 - Normal martingales

When $X = \mathbb{R}_+$, chaos-based examples satisfying the above conditions (A1)-(A5) include normal martingales having the chaos representation property (CRP). An $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-martingale $(M_t)_{t\in\mathbb{R}_+}$ on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ is called a normal martingale under P if

$$\mathbb{E}[(M_t - M_s)^2 \mid \mathcal{F}_s] = t - s, \qquad 0 \le s \le t,$$

see, e.g., [18] and references therein. Here we also assume that $(M_t)_{t\in\mathbb{R}_+}$ has the chaos representation property (CRP), and the multiple stochastic integral $I_n(f_n)$ of $f_n \in L^2(\mathbb{R}_+)^{\circ n}$ with respect to $(M_t)_{t\in\mathbb{R}_+}$ is given by

$$I_n(f_n) = n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \dots, t_n) dM_{t_1} \cdots dM_{t_n}, \qquad n \ge 1.$$

Examples of normal martingales satisfying the CRP include the Brownian motion and the compensated Poisson process, both of which we discuss in more details below, as well as certain processes with non-independent increments such as the Azéma martingales, for which the explicit expression of the gradient D is generally unknown; see [8] and § 2.10 of [18].

Example 1.2-a) - Brownian motion

When $X = \mathbb{R}_+$ and $(M_t)_{t\in\mathbb{R}_+}$ is the standard Brownian motion with respect to its own filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, it is usual to take $\mathcal{S}$ as the space of smooth cylindrical functionals of the form

$$F = f(I_1(h_1), \dots, I_1(h_n)), \qquad h_1, \dots, h_n \in H, \quad f \in C_b^\infty(\mathbb{R}^n),$$

on which the gradient D is defined by

$$DF = \sum_{i=1}^n h_i \partial_i f(I_1(h_1), \dots, I_1(h_n)), \tag{4.4}$$

e.g., see Definition 1.2.1 in [14]. Here, D is a derivation, whose adjoint δ is also called the Skorohod integral, the multiple integrals I_n are the well-known multiple Itô integrals, and $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is the standard Brownian filtration.

<u>Example 1.2-b)</u> - <u>Standard Poisson process</u>

In the special case $X = \mathbb{R}_+$ we can define a standard compensated Poisson process $(M_t)_{t\in\mathbb{R}_+}$ as $(M_t)_{t\in\mathbb{R}_+} := (N_t - t)_{t\in\mathbb{R}_+}$, which is a martingale with respect to its own filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, and D_t becomes a finite difference operator whose action is given by addition of a Poisson jump at time $t \in \mathbb{R}_+$, i.e.

$$D_t F(N.) = F(N. + \mathbf{1}_{[t,\infty)}(\cdot)) - F(N.), \qquad t \in \mathbb{R}_+, \tag{4.5}$$

which does not have the derivation property. The construction in [19] also applies to the standard Poisson process, via a different construction using differential operators on the Poisson space.

<u>Example 1.2-c)</u> - <u>Discrete-time chaos expansions</u>

Let $\Omega = \{-1,1\}^{\mathbb{N}}$ with $X = \mathbb{N}$, and consider the family $(Y_k)_{k\geq 1}$ of independent $\{-1,1\}$-valued Bernoulli random variables constructed from the canonical projections on Ω under P. That is, with $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(Y_0,\dots,Y_n)$ for $n \in \mathbb{N}$, the conditional probabilities $p_n := P(Y_n = 1 \mid \mathcal{F}_{n-1})$ and $q_n := P(Y_n = -1 \mid \mathcal{F}_{n-1})$ are given by

$$p_n = P(Y_n = 1) \quad \text{ and } \quad q_n = P(Y_n = -1),$$

respectively. We take $X = \mathbb{N}$ and $H = \ell^2(\mathbb{N}, \sigma)$, where σ is the counting measure on $\mathbb{N}$, and

$$\mathcal{S} = \left\{ F = f(Y_0,\dots,Y_n), \quad f : \mathbb{N}^{n+1} \longrightarrow \mathbb{R} \text{ bounded}, \ n \in \mathbb{N} \right\},$$

As in the continuous-time case, every $F \in L^2(\Omega, \mathcal{F}, P)$ can be decomposed into a series of discrete-time multiple stochastic integrals, which here take the form

$$I_n(f_n) = \sum_{k_1\neq\cdots\neq k_n\geq 0} f_n(k_1,\dots,k_n) Z_{k_1}\cdots Z_{k_n}, \tag{4.6}$$

where the sequence $Z_k := \mathbf{1}_{\{Y_k=1\}}\sqrt{\frac{q_k}{p_k}} - \mathbf{1}_{\{Y_k=-1\}}\sqrt{\frac{p_k}{q_k}}$, $k \in \mathbb{N}$, defines a normalized i.i.d. sequence of centered random variables with unit variance; see, e.g., Chapter 1 of [18]. The gradient D is given by

$$D_k I_n(f_n) = n I_{n-1}(f_n(*, k)\mathbf{1}_{\{k \notin *\}}), \qquad k \in \mathbb{N},$$

and more explicitly it satisfies, for $F \in \mathcal{S}$ and $k \in \mathbb{N}$,

$$D_k F(\omega) = \sqrt{p_k q_k}(F((\omega_i \mathbf{1}_{\{i \neq k\}} + \mathbf{1}_{\{i=k\}})_{i \in \mathbb{N}}) - F((\omega_i \mathbf{1}_{\{i \neq k\}} - \mathbf{1}_{\{i=k\}})_{i \in \mathbb{N}})). \qquad (4.7)$$

The divergence δ is defined as in (4.2), and again we have $\nabla = D$, hence Assumptions (A1) and (A3)-(A4) are automatically satisfied. Similarly, the duality relation (A2) is known to hold in the discrete-time case by e.g. Proposition 1.8.2 of [18].

Note however that here the operators D and δ do *not* satisfy the commutation relation (A5) in this discrete-time setting, due to the exclusion of diagonals in the construction (4.6) of multiple stochastic integrals. For this reason, the framework of Section 3 and the subsequent sections do not cover this discrete-time setting.

Noncommutative example

Here we consider an example which is not based on chaos (or multiple stochastic integral) expansions, and for which ∇ does not vanish on $H^{\wedge n}$, $n \geq 1$, with $X = \mathbb{R}_+$. In this case we need to show that Assumptions (A1)-(A5) are satisfied. A different noncommutative example, based on the standard Poisson process, is given in [19].

Example 1.3-) - The Lie-Wiener path space

Take $X = \mathbb{R}_+$ and let G be a compact connected m-dimensional Lie group, with identity e and whose Lie algebra $\mathcal{G}$, with orthonormal basis $(e_1, \ldots, e_m)$ and Lie bracket $[\cdot, \cdot]$, is identified to $\mathbb{R}^m$ and equipped with an Ad-invariant, left invariant metric $\langle \cdot, \cdot \rangle$.

Brownian motion $(\gamma(t))_{t\in\mathrm{R}_+}$ on G is constructed from a standard m-dimensional Brownian motion $(B_t)_{t\in\mathrm{R}_+}$ via the Stratonovich differential equation

$$\begin{cases} d\gamma(t) = \gamma(t) \circ dB_t \\ \\ \gamma(0) = \mathrm{e}, \end{cases} \tag{4.8}$$

with the image measure of the Wiener measure by the mapping $I : (B_t)_{t\in\mathrm{R}_+} \longmapsto (\gamma(t))_{t\in\mathrm{R}_+}$. Here we take $H = L^2(\mathbb{R}_+; \mathcal{G})$ with the inner product induced by $\mathcal{G}$, and let

$$\mathcal{S} = \{F = f(\gamma(t_1), \ldots, \gamma(t_n)) \ : \ f \in \mathcal{C}_b^\infty(\mathrm{G}^n)\}.$$

Next is the definition of the right derivative operator D, cf. [9].

Definition 4.1. *For F of the form*

$$F = f(\gamma(t_1), \ldots, \gamma(t_n)) \in \mathcal{S}, \qquad f \in \mathcal{C}_b^\infty(\mathrm{G}^n), \tag{4.9}$$

we let $DF \in L^2(\Omega \times \mathbb{R}_+; \mathcal{G})$ be defined as

$$\langle DF, h\rangle_H := \frac{d}{d\varepsilon} f\left(\gamma(t_1)e^{\varepsilon\int_0^{t_1} h_s ds}, \ldots, \gamma(t_n)e^{\varepsilon\int_0^{t_n} h_s ds}\right)_{|\varepsilon=0}, \quad h \in L^2(\mathbb{R}_+, \mathcal{G}).$$

Given F of the form (4.9) we also have

$$D_t F = \sum_{i=1}^n \partial_i f(\gamma(t_1), \ldots, \gamma(t_n)) \mathbf{1}_{[0,t_i]}(t), \qquad t \geq 0. \tag{4.10}$$

The covariant derivative operator $\nabla : \mathcal{S} \otimes H \to L^2(\Omega; H \otimes H)$ is defined as

$$\nabla_s u(t) = D_s u(t) + \mathbf{1}_{[0,t]}(s) \mathrm{ad} u(t) \in \mathcal{G} \otimes \mathcal{G}, \qquad s, t \in \mathbb{R}_+, \tag{4.11}$$

where $\mathrm{ad}(u)v = [u, v]$, $u, v \in \mathcal{G}$, $(\mathrm{ad}u)(t) := \mathrm{ad}(u(t))$, $t \in \mathbb{R}_+$, for $u \in \mathcal{S} \otimes H$, and $\mathrm{ad}u$ is the linear operator defined on $\mathcal{G}$ by

$$\langle e_i \otimes e_j, \mathrm{ad}u\rangle_{\mathcal{G}\otimes\mathcal{G}} = \langle e_j, \mathrm{ad}(e_i)u\rangle_{\mathcal{G}} = \langle e_j, [e_i, u]\rangle_{\mathcal{G}}, \quad i, j = 1, \ldots, m, \quad u \in \mathcal{G}.$$

We now check that all required assumptions are satisfied in the present setting.

(A1) The vanishing of torsion is satisfied from Theorem 2.3-(i) of [9].

(A2) The operator D admits an adjoint δ that satisfies the duality relation

$$E[F\delta(v)] = E[\langle DF, v\rangle_H], \quad F \in \mathcal{S},\ v \in L^2(\mathbb{R}_+;\mathcal{G}), \tag{4.12}$$

cf. e.g. [9], which shows that (A2) is satisfied.

(A3) We note that $\mathrm{ad}u$ is skew-adjoint as the inner product in $\mathcal{G}$ is chosen Ad-invariant, hence the connection from ∇ is torsion free and (3.4) is satisfied.

(A4) Assumption (A4) clearly holds by the definition (4.11) which shows that

$$\nabla_t f(s) = 0, \qquad 0 < s < t, \quad f \in H.$$

This also applies in the setting of loop groups [10].

(A5) By Theorem 2.4-(ii) of [9] it is known that D and ∇ satisfy the commutation relation (A5) for $f \in H$, hence by Remark 3.1, Assumption (A5) is satisfied for $u \in H \otimes \mathcal{S}$ since by (4.10) the operator D satisfies the chain rule of derivation (3.9). Here, (2.12)-(2.13) also hold from Corollary 2 of [11] which is proved using a mapping of ∇ on the path group to D on the Wiener space by the Itô map.

5 Weitzenböck identities for n-forms

In this section we will need the following additional assumption:

(B1) For all $n \geq 1$ the covariant derivative operator satisfies

$$\mathrm{d}_{x_n}^{n-1}\nabla_x f_n(x, x_1, \ldots, x_{n-1}) = \nabla_x \mathrm{d}_{x_n}^{n-1} f_n(x, x_1, \ldots, x_{n-1}), \tag{B1}$$

$x_1, \ldots, x_n \in X$, $x \in X$, $f_n \in H^{\wedge n}$.

Assumption (B1) is straightforwardly satisfied in all examples of Section 4, except for the discrete-time Example 1.2-c), however it requires a specific proof in the Poisson derivation case of [19]. Note that Assumption (B1) differs from the usual vanishing of curvature condition, which reads

$$\nabla_u \nabla_v - \nabla_v \nabla_u = \nabla_{\{u,v\}}$$

where $\{u, v\}$ is the Lie bracket of two vector fields u, v, cf. Theorem 2.3-(ii) of [9].

Lemma 5.1. *(Intertwinning relation) Under Assumptions (A1)-(A5) and (B1), for any $u_n \in \mathcal{S} \otimes H^{\wedge n}$ of the form $u_n = F \otimes f_n$, with $F \in \mathcal{S}$ and $f_n, g_n \in H^{\wedge n}$, we have*

$$\begin{aligned}
&\langle \mathrm{d}^{n-1}\mathrm{d}^{(n-1)*}u_n(x_1,\ldots,x_n), g_n(x_1,\ldots,x_n)\rangle_{H^{\wedge n}} = nF\langle f_n(x_1,\ldots,x_n), g_n(x_1,\ldots,x_n)\rangle_{H^{\wedge n}} \\
&+\sum_{j=1}^n \langle \delta(\nabla_{x_j}(F\otimes f_n(x_1,\ldots,x_{j-1},\cdot,x_{j+1},\ldots,x_n))), g_n(x_1,\ldots,x_n)\rangle_{H^{\wedge n}} \\
&-\left\langle \mathrm{d}_{x_n}^{n-1}(F\otimes \mathrm{trace}\nabla_\cdot f_n(\cdot,x_1,\ldots,x_{n-1})), g_n(x_1,\ldots,x_n)\right\rangle_{H^{\wedge n}} \\
&+\sum_{j=1}^n \int_X \langle D_xF\otimes f_n(x_1,\ldots,x_n), \nabla_{x_j}^{(j)} g_n(x_1,\ldots,x_{j-1},x,x_{j+1},\ldots,x_n)\rangle_{H^{\wedge n}}\sigma(dx).
\end{aligned} \tag{5.1}$$

The proof of Lemma 5.1 is deferred to the appendix. Here we verify that for $n = 1$, Lemma 5.1 coincides with Assumption (A5) since $\nabla_x f_1(x) = 0$ and

$$\begin{aligned}
\mathrm{d}_{x_1}^0 \mathrm{d}^{0*}u_1 &= D_{x_1}\delta(Ff_1) \\
&= \delta(f_1 D_{x_1}F) + Ff_1(x_1) + \delta(F\nabla_{x_1}f_1) \\
&\quad +\langle DD_{x_1}F, f_1\rangle_H - \langle D_{x_1}DF, f_1\rangle_H + \langle D_\cdot F, \nabla_{x_1}f_1(\cdot)\rangle_H \\
&= u_{x_1} + \delta(\nabla_{x_1}u_1) + \langle DD_{x_1}F, f_1\rangle_H - \langle D_{x_1}DF, f_1\rangle_H + \langle D_\cdot F, \nabla_{x_1}f_1(\cdot)\rangle_H,
\end{aligned}$$

for any $u_1, v_1 \in \mathcal{S}\otimes H$ of the form $u_1 = F\otimes f_1$ with $F \in \mathcal{S}$, and $f_1, g_1 \in H$. By the vanishing of torsion Assumption (A1), this yields

$$\begin{aligned}
&\langle \mathrm{d}_{x_1}^0 \mathrm{d}^{0*}u_1(x_1), g_1(x_1)\rangle_H = F\langle f(x_1), g_1(x_1)\rangle_H + \langle \delta(\nabla_{x_1}(Ff_1)), g_1(x_1)\rangle_H \\
&\qquad +\langle\langle DD_{x_1}F, f_1\rangle_H - \langle D_{x_1}DF, f_1\rangle_H, g_1(x_1)\rangle_H + \langle\langle D_\cdot F, \nabla_{x_1}f_1(\cdot)\rangle_H, g_1(x_1)\rangle_H \\
&\quad = F\langle f(x_1), g_1(x_1)\rangle_H + \langle \delta(\nabla_{x_1}(Ff_1)), g_1(x_1)\rangle_H \\
&\qquad + \int_X\int_X \langle \nabla_y g_1(x), D_xF\otimes f_1(y)\rangle_{\mathrm{R}^d\otimes\mathrm{R}^d}\sigma(dx)\sigma(dy) \\
&\qquad - \int_X\int_X \langle \nabla_y f_1(x), D_xF\otimes g_1(y)\rangle_{\mathrm{R}^d\otimes\mathrm{R}^d}\sigma(dx)\sigma(dy) \\
&\qquad +\langle\langle D_\cdot F, \nabla_{x_1}f_1(\cdot)\rangle_H, g_1(x_1)\rangle_H \\
&\quad = F\langle f(x_1), g_1(x_1)\rangle_H + \langle \delta(\nabla_{x_1}(F\otimes f_1)), g_1(x_1)\rangle_H + \langle\langle D_\cdot F, \nabla_{x_1}g_1(\cdot)\rangle_H, f_1(x_1)\rangle_H.
\end{aligned}$$

which coincides with Lemma 5.1 since $\nabla_x f_1(x) = 0$ when $n = 1$.

Weitzenböck identity

Recall that the de Rham-Hodge Laplacian (3.7) is given on n-forms by

$$\Delta_n = \mathrm{d}^{n-1}\mathrm{d}^{(n-1)*} + \mathrm{d}^{n*}\mathrm{d}^n, \qquad n \geq 1.$$

Theorem 5.2. *Under the Assumptions (A1)-(A5) and (B1) we have the Weitzenböck identity*

$$\Delta_n = n\mathrm{Id}_{H^{\wedge n}} + \nabla^*\nabla, \qquad u_n \in \mathcal{S} \otimes H^{\wedge n}, \quad n \geq 1. \tag{5.2}$$

By duality (5.2) shows that

$$\begin{aligned} n!\|\mathrm{d}^{(n-1)*}u_n\|^2_{L^2(\Omega,H^{\wedge(n-1)})} + n!\|\mathrm{d}^n u_n\|^2_{L^2(\Omega,H^{\wedge(n+1)})} \\ = nn!\|u_n\|^2_{L^2(\Omega,H^{\wedge n})} + \|\nabla u_n\|^2_{L^2(\Omega,H^{\otimes(n+1)})}, \end{aligned} \tag{5.3}$$

$u_n \in \mathcal{S} \otimes H^{\wedge n}$, $n \geq 1$. We first check that the Weitzenböck identity (5.3) holds for one-forms, i.e.

$$\|\mathrm{d}^{0*}u_1\|^2_{L^2(\Omega)} + \frac{1}{2}\|\mathrm{d}^1 u_1\|^2_{L^2(\Omega,H^{\otimes 2})} = \|u_1\|^2_{L^2(\Omega,H)} + \|\nabla u_1\|^2_{L^2(\Omega,H^{\otimes 2})}, \tag{5.4}$$

and we refer to the appendix for the proof in the case of n-forms. By Assumption (A5) and the commutation relation (2.9) we have, taking $u_1 = F \otimes f_1$ and following the argument of [9],

$$\begin{aligned}
\langle \mathrm{d}^0_{x_1}\mathrm{d}^{0*}u_1, u_1(x_1)\rangle_{L^2(\Omega,H)} &= \langle D_{x_1}\delta(F\otimes f_1), Ff_1(x_1)\rangle_{L^2(\Omega,H)} \\
&= \langle Ff_1(x_1) + \delta(\nabla_{x_1}(F\otimes f_1)) + \langle DD_{x_1}F, f_1\rangle_H - \langle D_{x_1}DF, f_1\rangle_H \\
&\quad + \langle D_{\cdot}F, \nabla_{x_1}f_1(\cdot)\rangle_H, Ff_1(x_1)\rangle_{L^2(\Omega,H)} \\
&= \langle u_1(x_1), u_1(x_1)\rangle_{L^2(\Omega,H)} + \langle \nabla_{x_1}u_1(\cdot), D_{\cdot}u_1(x_1)\rangle_{L^2(\Omega,H^{\otimes 2})} \\
&\quad + \langle F\langle DD_{x_1}F, f_1\rangle_H - F\langle D_{x_1}DF, f_1\rangle_H + F\langle D_{\cdot}F, \nabla_{x_1}f_1(\cdot)\rangle_H, f_1(x_1)\rangle_{L^2(\Omega,H)} \\
&= \langle u_1(x_1), u_1(x_1)\rangle_{L^2(\Omega,H)} + \langle \nabla_{x_1}u_1(\cdot), D_{\cdot}u_1(x_1)\rangle_{L^2(\Omega,H^{\otimes 2})} \\
&\quad + \langle\langle D_{x_1}F, F\nabla_{\cdot}f_1(x_1)\rangle_H, f_1(\cdot)\rangle_{L^2(\Omega,H)} \\
&= \langle u_1(x_1), u_1(x_1)\rangle_{L^2(\Omega,H)} + \langle \nabla_{x_1}u_1(x_2), \nabla_{x_2}u_1(x_1)\rangle_{L^2(\Omega,H^{\otimes 2})} \qquad (5.5) \\
&= \langle u_1(x_1), u_1(x_1)\rangle_{L^2(\Omega,H)} - \frac{1}{2}\langle \mathrm{d}^1 u_1, \mathrm{d}^1 u_1\rangle_{L^2(\Omega,H^{\otimes 2})} \\
&\quad + \langle \nabla_{x_2}u_1(x_1), \nabla_{x_2}u_1(x_1)\rangle_{L^2(\Omega,H^{\otimes 2})}, \qquad (5.6)
\end{aligned}$$

where we used Assumption (A4) to reach (5.5). This implies (5.4) and

$$\|\delta(u_1)\|^2_{L^2(\Omega,H)} + \frac{1}{2}\|\mathrm{d}^1 u_1\|^2_{L^2(\Omega,H^{\wedge 2})} = \|u_1\|^2_{L^2(\Omega,H)} + \|\nabla u_1\|^2_{L^2(\Omega,H^{\otimes 2})}, \quad u_1 \in \mathcal{S} \otimes H.$$

Theorem 5.2 shows that the Bochner Laplacian $L = -\nabla^*\nabla$ and the Hodge Laplacian Δ_n have same closed domain $\mathrm{Dom}(\Delta_n)$ on the random n-forms and that all eigenvalues λ_n of the Bochner Laplacian L satisfy $\lambda_n \geq n$. Indeed, if w_n is an eigenvector of Δ_n with eigenvalue λ_n, by rewriting (5.2) as

$$L = n\mathrm{Id}_{H^{\wedge n}} - \Delta_n,$$

we find that L and Δ_n share the same eigenvectors and that $\lambda_n \geq n \geq 1$ since

$$\begin{aligned} 0 &\leq -\langle L w_n, w_n\rangle_{L^2(\Omega,H^{\wedge n})} \\ &= \langle (\Delta_n - n\mathrm{Id}_{H^{\wedge n}}) w_n, w_n\rangle_{L^2(\Omega,H^{\wedge n})} \\ &= \langle \Delta_n w_n, w_n\rangle_{L^2(\Omega,H^{\wedge n})} - n\langle w_n, w_n\rangle_{L^2(\Omega,H^{\wedge n})} \\ &= (\lambda_n - n)\langle w_n, w_n\rangle_{L^2(\Omega,H^{\wedge n})}, \qquad n \geq 1. \end{aligned} \tag{5.7}$$

Proposition 5.1. *Under Assumptions (A1)-(A5) and (B1), the de Rham-Hodge-Kodaira decomposition* (3.7) *rewrites as*

$$L^2(\Omega; H^{\wedge n}) = \mathrm{Im}\,\mathrm{d}^{n-1} \oplus \mathrm{Im}\,\mathrm{d}^{n*}, \qquad n \geq 1. \tag{5.8}$$

Proof. By (5.7) the operator $\Delta_n = n\mathrm{Id}_{H^{\otimes n}} - L$ becomes invertible for all $n \geq 1$, and the space $\mathrm{Ker}\,\Delta_n$ of harmonic forms for the de Rham Laplacian Δ_n is equal to $\{0\}$. i.e. any harmonic form for the de Rham Laplacian Δ_n has to vanish, and we conclude by (3.7). □

Next we consider a number of examples to which the framework and results of this section can be applied.

Commutative examples - chaos expansions

All commutative examples of Section 3 satisfy Assumption (B1) since in this case, ∇ vanishes on $H^{\wedge n}$, $n \geq 1$, as in all chaos-based examples.

Example 2.1 - Poisson random measures

In the Poisson case the semi-group $(P_t)_{t\in\mathbb{R}_+}$ associated to the Bochner Laplacian $L := -\nabla^*\nabla$, cf. [21], admits an integral representation, cf. e.g. Lemma 6.8.1 of [18]. Proposition 5.1 shows here that any harmonic form for the de Rham Laplacian has to vanish.

Example 2.2 - Normal martingales

Example 2.2-a) - Brownian and Poisson cases

In the Brownian case, Theorem 5.2 covers Proposition 3.1 of [20] on the Weitzenböck decomposition, and Proposition 5.1 is known to hold also from [20].

Example 2.2-b) - Discrete-time case

Proposition 5.1 holds in this discrete-time setting as the semi-group $(P_t)_{t\in\mathbb{R}_+}$ is contractive, cf. Proposition 1.9.3 and Lemma 1.9.4 of [18]. However, Theorem 5.2 does not hold here as (A5) is not satisfied.

Noncommutative example

Example 2.3) - Lie-Wiener path space

We need to check the following condition, which immediately holds because the operation ad in (4.11) commutes with itself.

(B1) In other words, we can write $\mathrm{ad}u$ as

$$\begin{aligned}\mathrm{ad}u &= \sum_{k=1}^{m} \langle u, e_k\rangle_{\mathcal{G}} \mathrm{ad}e_k \\ &= \sum_{i,j,k=1}^{m} \langle u, e_k\rangle_{\mathcal{G}} (e_i \otimes e_j) \langle e_i \otimes e_j, \mathrm{ad}e_k\rangle_{\mathcal{G}\otimes\mathcal{G}}\end{aligned}$$

$$= \sum_{i,j,k=1}^{m} \langle u, e_k \rangle_{\mathcal{G}} (e_i \otimes e_j) A_{i,j,k},$$

where the matrix $A = (A_{i,j,k})_{1 \le i,j,k \le m}$ is the 3-tensor given by

$$A_{i,j,k} = \langle e_i \otimes e_j, \mathrm{ad} e_k \rangle_{\mathcal{G} \otimes \mathcal{G}} = \langle e_j, \mathrm{ad}(e_i) e_k \rangle_{\mathcal{G}} = \langle e_j, [e_i, e_k] \rangle_{\mathcal{G}},$$

$1 \le i, j, k \le m$.

Note that Assumption (B1) differs from the vanishing of curvature in e.g. Theorem 2.3-(ii) of [9] in the path group case.

6 Clark-Ocone representation formula

In this section we take $d = 1$ and $X = \mathbb{R}_+$, and we consider a normal martingale $(M_t)_{t \in \mathbb{R}_+}$ generating a filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ on the probability space $(\Omega, \mathcal{F}, P)$. We assume that D satisfies the following Assumptions (C1) and (C2), in addition to (A1)-(A5) .

(C1) The operator D satisfies the Clark-Ocone formula

$$F = E[F \mid \mathcal{F}_t] + \int_t^\infty E[D_r F \mid \mathcal{F}_r] dM_r, \qquad t \in \mathbb{R}_+, \tag{C1}$$

for $F \in \mathrm{Dom}\,(D)$.

(C2) The operator D satisfies the commutation relation

$$D_s E[F \mid \mathcal{F}_t] = \mathbf{1}_{[0,t]}(s) E[D_s F \mid \mathcal{F}_t], \qquad s, t \in \mathbb{R}_+, \tag{C2}$$

for $F \in \mathrm{Dom}\,(D)$,

(A4') The operator ∇ satisfies the condition

$$\nabla_s f(t) = 0, \qquad 0 \le t < s, \qquad f \in H, \tag{A4'}$$

We note that (A4') is stronger than (A4), and that (C2) implies

$$\nabla_s E[u(t) \mid \mathcal{F}_t] = \mathbf{1}_{[0,t]}(s) E[\nabla_s u(t) \mid \mathcal{F}_t], \qquad s, t \in \mathbb{R}_+,$$

for $u \in \mathrm{Dom}\,(\nabla)$. In addition, under the duality assumption (A2), Assumption (C1) is equivalent to stating that $(M_t)_{t\in\mathbb{R}_+}$ has the predictable representation property and δ coincides with the stochastic integral with respect to $(M_t)_{t\in\mathbb{R}_+}$ on the square-integrable predictable processes, cf. Corollary 3.2.8 and Propositions 3.3.1 and 3.3.2 of [18]. Also it is sufficient to assume that (C1) holds for $t = 0$, cf. Proposition 3.2.3 of [18].

Clark-Ocone formula for n-forms

In this section extend the Clark-Ocone formula for differential forms of [22] to the general framework of this paper.

Theorem 6.1. *Under the Assumptions (A1)-(A5) and (C1)-(C2), for $u_n \in \mathrm{Dom}\,(\mathrm{d}^n)$, we have, for a.e. $t_1,\dots,t_n \in \mathbb{R}_+$,*

$$\begin{aligned} u_n(t_1,\dots,t_n) &= \mathrm{d}^{n-1}_{t_n} \int_{t_1\vee\cdots\vee t_{n-1}}^{\infty} E[u_n(r,t_1,\dots,t_{n-1}) \mid \mathcal{F}_r] dM_r \\ &\quad + \int_{t_1\vee\cdots\vee t_n}^{\infty} E[\mathrm{d}^n_{t_n} u_n(r,t_1,\dots,t_{n-1}) \mid \mathcal{F}_r] dM_r. \end{aligned} \tag{6.1}$$

In particular, Theorem 6.1 shows that any closed form $u_n \in \mathrm{Dom}\,(\mathrm{d}^n)$ can be written as

$$u_n(t_1,\dots,t_n) = \mathrm{d}^{n-1}_{t_n} \int_{t_1\vee\cdots\vee t_{n-1}}^{\infty} E[u_n(r,t_1,\dots,t_{n-1}) \mid \mathcal{F}_r] dM_r,$$

$t_1,\dots,t_n \in \mathbb{R}_+$. As a consequence of Theorem 6.1 the range of the exterior derivative d^n is closed, and similarly for its adjoint d^{n*}, for all $n \geq 1$. In this way we recover the fact that the Hodge Laplacian Δ_n has a closed range as well, so it has a spectral gap, cf. Theorem 6.6 and Corollary 6.7 of [7]. However this does not yield an explicit Poincaré inequality and lower bound for the spectral gap, unlike for the classical Clark-Ocone formula cf. e.g. Proposition 3.2.7 of [18]. Note that the Weitzeböck formula (5.7) also shows that all eigenvalues λ_n of the Bochner Laplacian $L = -\nabla^*\nabla$ on n-forms satisfy $\lambda_n \geq n$, $n \geq 1$.

A quick proof of the identity (6.1) for one-forms is instructive while we delay the proof for general n-forms to the appendix. When $n = 1$, for $u \in \mathcal{S} \otimes H$, we have, by (A4') and the Clark-Ocone formula (C1) for $t \in \mathbb{R}_+$,

$$
\begin{aligned}
u(t) &= E[u(t) \mid \mathcal{F}_t] + \int_t^\infty E[D_r u(t) \mid \mathcal{F}_r] dM_r \\
&= E[u(t) \mid \mathcal{F}_t] + \int_t^\infty E[\nabla_r u(t) \mid \mathcal{F}_r] dM_r,
\end{aligned}
$$

and by (A5) and (C2) we find

$$
\begin{aligned}
D_t \int_0^\infty E[u(r) \mid \mathcal{F}_r] dM_r &= E[u(t) \mid \mathcal{F}_t] + \int_0^\infty \nabla_t E[u(r) \mid \mathcal{F}_r] dM_r \qquad (6.2) \\
&= E[u(t) \mid \mathcal{F}_t] + \int_t^\infty E[\nabla_t u(r) \mid \mathcal{F}_r] dM_r,
\end{aligned}
$$

hence

$$
\begin{aligned}
u(t) &= D_t \int_0^\infty E[u(r) \mid \mathcal{F}_r] dM_r + \int_t^\infty E[\nabla_r u(t) - \nabla_t u(r) \mid \mathcal{F}_r] dM_r \\
&= D_t \int_0^\infty E[u(r) \mid \mathcal{F}_r] dM_r + \int_t^\infty E[\mathrm{d}_t^1 u(r) \mid \mathcal{F}_r] dM_r.
\end{aligned}
$$

Note that in (6.2) above we applied Assumption (A5) to an adapted process v, in which case the condition simply reads

$$
D_t \delta(v) = v(t) + \delta(\nabla_t v), \qquad t \in \mathbb{R}_+, \qquad \textbf{(A5")}
$$

since when $v(\cdot) = \mathbf{1}_{[t,\infty)}(\cdot) F \otimes a$ is a simple adapted process, where $a \in \mathbb{R}^d$ and F is $\mathcal{F}_t$-measurable, we have $\mathbf{1}_{[t,\infty)}(r) D_r F = 0$, $r \in \mathbb{R}_+$, as follows from (C2), i.e.

$$
D_r F = D_r E[F \mid \mathcal{F}_t] = \mathbf{1}_{[0,t]}(r) E[D_r F \mid \mathcal{F}_t], = 0, \qquad r \geq t.
$$

The Clark-Ocone formula Theorem 6.1 allows us in particular to recover the de Rham-Hodge-Kodaira decomposition (5.8).

Corollary 6.2. *We have* $\operatorname{Im} \mathrm{d}^n = \operatorname{Ker} \mathrm{d}^{n+1}$, $n \in \mathbb{N}$, *and the de Rham-Hodge-Kodaira decomposition* (3.7) *reads*

$$
L^2(\Omega; H^{\wedge n}) = \operatorname{Im} \mathrm{d}^{n-1} \oplus \operatorname{Im} \mathrm{d}^{n*}, \qquad n \geq 1.
$$

Proof. By Theorem 6.1 we have $\operatorname{Im} \mathrm{d}^n \supset \operatorname{Ker} \mathrm{d}^{n+1}$, which shows by (2.13) that $\operatorname{Im} \mathrm{d}^n = \operatorname{Ker} \mathrm{d}^{n+1}$, $n \in \mathbb{N}$. □

As a consequence of Corollary 6.2, we also get the exactness of the sequence

$$\mathrm{Dom}\,(\mathrm{d}^n) \xrightarrow{\mathrm{d}^n} \mathrm{Im}\,(\mathrm{d}^n) = \mathrm{Ker}\,(\mathrm{d}^{n+1}) \xrightarrow{\mathrm{d}^{n+1}} \mathrm{Im}\,(\mathrm{d}^{n+1}), \qquad n \in \mathbb{N}, \tag{6.3}$$

as in Theorem 3.2 of [20]. By duality of (6.3) we also find by Corollary 6.2 that

$$\mathrm{Im}\,\mathrm{d}^{(n+1)*} = \mathrm{Ker}\,\mathrm{d}^{n*}, \qquad n \in \mathbb{N},$$

and the following sequence is also exact:

$$\mathrm{Im}\,(\mathrm{d}^{n*}) \xleftarrow{\mathrm{d}^{n*}} \mathrm{Ker}\,(\mathrm{d}^{n*}) = \mathrm{Im}\,(\mathrm{d}^{(n+1)*}) \xleftarrow{\mathrm{d}^{(n+1)*}} \mathrm{Dom}\,(\mathrm{d}^{(n+1)*}), \qquad n \in \mathbb{N}.$$

Next, we consider some examples to which the above framework applies.

Commutative examples - chaos expansions

As written at the beginning of this section, we take $X = \mathbb{R}_+$ in all cases due to the need of a time scale in order to state the Clark-Ocone formula.

Example 3.1-a) - Normal martingales

As in Section 5 we have $X = \mathbb{R}_+$ and $\nabla = 0$ on H, i.e. $\nabla = D$ and Assumption (A4') is automatically satisfied. Let us check that Assumptions (C1) and (C2) are satisfied in the framework of normal martingales that have the chaos representation property (CRP).

(C1) Since the normal martingale $(M_t)_{t\in\mathbb{R}_+}$ has the chaos representation property, the Clark-Ocone formula holds for any $F \in \mathrm{Dom}(D) \subset L^2(\Omega, \mathcal{F}, P)$ as

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F \mid \mathcal{F}_t] dM_t, \tag{6.4}$$

cf. Proposition 4.2.3 of [18] for a proof via the chaos expansion of F.

(C2) This condition is satisfied from the definition (4.1) of D and e.g. Lemma 2.7.2 page 88 of [18] or Proposition 1.2.8 page 34 of [14] in the Wiener case.

Example 3.1-b) - Discrete-time chaos expansions

The Clark-Ocone formula (C1) holds in the discrete-time case as

$$F = \mathbb{E}[F] + \sum_{k=0}^{\infty} \mathbb{E}[D_k F \mid \mathcal{F}_{k-1}] Z_k, \tag{6.5}$$

cf. Proposition 1.7.1 of [18] and references therein, hence Assumption (C2) is also satisfied here, as it is satisfied for normal martingales. However, Theorem 6.1 does not hold here because (A5) is not satisfied.

Noncommutative example

Example 3.2) - Lie-Wiener path space (Example 1.3 continued)

(C1) On the classical Wiener space, when $(u(t))_{t\in\mathbb{R}_+}$ is square-integrable and adapted to the Brownian filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, $\delta(u)$ coincides with the Itô integral of $u \in L^2(\Omega; H)$ with respect to the underlying Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, i.e.

$$\delta(u) = \int_0^\infty u(t) dB_t, \tag{6.6}$$

and this shows that Assumption (C1) is satisfied, cf. e.g. Proposition 3.3.2 of [18].

(C2) Assumption (C2) is satisfied here as in the case of normal martingales as in e.g. Lemma 2.7.2 page 88 of [18], or for Brownian motion as in Proposition 1.2.8 page 34 of [14].

On the Lie-Wiener path space we note that we have the relation

$$\langle DF, h\rangle_H = \langle \hat{D}F, h\rangle_H + \hat{\delta}\left(\int_0^{\cdot} \mathrm{ad}(h(s)) ds \hat{D}.F\right), \qquad F \in \mathcal{S}, \tag{6.7}$$

where $\hat{D}$ and $\hat{\delta}$ denote here the gradient and divergence appearing in (4.4) on the underlying standard Wiener space with Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ in (4.8), cf. e.g. Lemma 4.1 of [17] and references therein, or Corollary 5.2.1 of [16] for the more general setting of Riemannian manifolds. Relation (6.7) shows that

$$E[D_s F \mid \mathcal{F}_r] = E[\hat{D}_s F \mid \mathcal{F}_r], \qquad 0 \le r \le s, \tag{6.8}$$

cf. also Relation (5.7.5) page 191 of [18] on Riemannian manifolds, hence (C1) is satisfied for D because it holds for $\hat{D}$ as noted above.

Consequently, (C2) holds on the Lie-Wiener path space since we have

$$\begin{aligned}\langle DE[F \mid \mathcal{F}_t], h\rangle_H &= \langle \hat{D}E[F \mid \mathcal{F}_t], h\rangle_H + \hat{\delta}\left(\int_0^{\cdot} \mathrm{ad}(h(s))ds\hat{D}.E[F \mid \mathcal{F}_t]\right) \\ &= \langle \mathbf{1}_{[0,t]}(\cdot)E[\hat{D}.F \mid \mathcal{F}_t], h\rangle_H + \hat{\delta}\left(\int_0^{\cdot} \mathrm{ad}(h(s))ds\mathbf{1}_{[0,t]}(\cdot)E[\hat{D}.F \mid \mathcal{F}_t]\right) \\ &= \langle \mathbf{1}_{[0,t]}(\cdot)E[\hat{D}.F \mid \mathcal{F}_t], h\rangle_H + E\left[\hat{\delta}\left(\int_0^{\cdot} \mathbf{1}_{[0,t]}(s)\mathrm{ad}(h(s))ds\hat{D}.F\right)\Big|\mathcal{F}_t\right] \\ &= \langle \mathbf{1}_{[0,t]}(\cdot)E[D.F \mid \mathcal{F}_t], h\rangle_H, \qquad t \in \mathbb{R}_+.\end{aligned}$$

Assumption (A4') is also clearly satisfied by the definition (4.11) of ∇.

Hence Theorems 6.1 covers Theorems 3.1 of [22] on the Wiener space as well as its extension to the path space using the diffeomorphism approach of [11].

Appendix

In this section we state the proofs of Proposition 3.1, Theorems 5.2 and 6.1, by extension of the original arguments of [20], [10], [11], and [22] to our framework.

Proof of Proposition 3.1 (Duality relation). Assuming that $u_n \in \mathcal{S} \otimes H^{\wedge n}$ and $v_{n+1} \in \mathcal{S} \otimes H^{\wedge(n+1)}$ have the form (2.6) and using the definition (2.4) of d^n and the duality assumption (A2) we have, using the antisymmetry of g_{n+1},

$$\begin{aligned}&\langle \mathrm{d}^n_{t_{n+1}}(Ff_n(t_1,\dots,t_n)), Gg_{n+1}(t_1,\dots,t_{n+1})\rangle_{L^2(\Omega,H^{\wedge(n+1)})} \\ &= \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{(n+1)!}\langle \nabla_{t_j}(Ff_n(t_1,\dots,t_{j-1},t_{j+1},\dots,t_{n+1})), Gg_{n+1}(t_1,\dots,t_{n+1})\rangle_{L^2(\Omega,H^{\otimes(n+1)})} \\ &= \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{(n+1)!}\langle D_{t_j}Ff_n(t_1,\dots,t_{j-1},t_{j+1},\dots,t_{n+1}), Gg_{n+1}(t_1,\dots,t_{n+1})\rangle_{L^2(\Omega,H^{\otimes(n+1)})}\end{aligned}$$

$$+\sum_{j=1}^{n+1}\frac{(-1)^{j-1}}{(n+1)!}\langle F\nabla_{t_j}f_n(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1}),Gg_{n+1}(t_1,\ldots,t_{n+1})\rangle_{L^2(\Omega,H^{\otimes(n+1)})}$$

$$=\frac{1}{(n+1)!}\sum_{j=1}^{n+1}(-1)^{j-1}\langle Ff_n(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1}),$$
$$\delta(Gg_{n+1}(t_1,\ldots,t_{j-1},\cdot,t_{j+1},\ldots,t_{n+1}))\rangle_{L^2(\Omega,H^{\otimes n})}$$
$$-\frac{1}{(n+1)!}\sum_{j=1}^{n+1}(-1)^{j-1}\langle Ff_n(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1}),$$
$$G\int_0^\infty \operatorname{Tr}\nabla_t g_{n+1}(t_1,\ldots,t_{j-1},t,t_{j+1},\ldots,t_{n+1})dt\rangle_{L^2(\Omega,H^{\otimes n})}$$
$$=\frac{1}{(n+1)!}\sum_{j=1}^{n+1}\langle Ff_n(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1}),$$
$$\delta(Gg_{n+1}(\cdot,t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1}))\rangle_{L^2(\Omega,H^{\otimes n})}$$
$$-\frac{1}{(n+1)!}\sum_{j=1}^{n+1}\langle Ff_n(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1}),$$
$$G\int_0^\infty \operatorname{Tr}\nabla_t g_{n+1}(t,t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1})dt\rangle_{L^2(\Omega,H^{\otimes n})}$$
$$=\langle Ff_n(t_1,\ldots,t_n),\mathrm{d}^{n*}(Gg_{n+1})(t_1,\ldots,t_n)\rangle_{L^2(\Omega,H^{\wedge n})},$$

where we applied the antisymmetry condition (A3) and the definition (3.2) of d^{n*}.

□

Proof of Lemma 5.1 (Intertwinning relation). When $n=1$, by (A5) or (3.8) we have

$$\begin{aligned}\mathrm{d}^0_{x_1}\mathrm{d}^{0*}u_1 &= D_{x_1}\delta(u_1)\\ &= u_1(x_1)+\delta(\nabla_{x_1}u_1)+\langle DD_{x_1}F,f_1\rangle_H-\langle D_{x_1}DF,f_1\rangle_H+\langle D.F,\nabla_{x_1}f_1(\cdot)\rangle_H,\end{aligned}$$

for $u_1=F\otimes f_1\in\mathcal{S}\otimes H$. Next, by the definition (2.4) of d^n and (A5) or (3.8) we have

$$\begin{aligned}&\mathrm{d}^{n-1}_{t_n}\delta(F\otimes f_n(\cdot,x_1,\ldots,x_{n-1}))\\ &\quad=\sum_{j=1}^{n}(-1)^{j-1}\nabla_{x_j}\delta(F\otimes f_n(\cdot,x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n))\\ &\quad=\sum_{j=1}^{n}(-1)^{j-1}F\otimes f_n(x_j,x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n)\end{aligned}$$

$$
\begin{aligned}
& + \sum_{j=1}^{n} (-1)^{j-1} \delta(\nabla_{x_j}(F \otimes f_n(\cdot, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n))) \\
& + \sum_{j=1}^{n} (-1)^{j-1} \langle D.D_{x_j} F - D_{x_j} D.F, f_n(\cdot, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \rangle_H \\
& + \sum_{j=1}^{n} (-1)^{j-1} \langle D.F, \nabla^{(1)}_{x_j} f_n(\cdot, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \rangle_H \\
= \ & nF \otimes f_n(x_1, \ldots, x_n) \\
& + \sum_{j=1}^{n} \delta(\nabla_{x_j}(F \otimes f_n(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n))) \\
& + \sum_{j=1}^{n} \int_X \langle D_x F \otimes f_n(x_1, \ldots, x_n), \nabla^{(j)}_{x_j} g_n(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_n) \rangle_{H^{\wedge n}} \sigma(dx).
\end{aligned}
$$

where we applied (2.9). We conclude by the definition (3.2) of d^{n*} which states that

$$
\begin{aligned}
\mathrm{d}^{(n-1)*}(F \otimes f_n)(x_1, \ldots, x_{n-1}) \ = \ & \delta(F \otimes f_n(\cdot, x_1, \ldots, x_{n-1})) \\
& - F \int_X \mathrm{Tr}\, \nabla_x f_n(x, x_1, \ldots, x_{n-1}) \sigma(dx).
\end{aligned}
$$

□

Proof of Theorem 5.2 (Weitzenböck identity). We will show that

$$
\begin{aligned}
& \| \mathrm{d}^{(n-1)*} u_n \|^2_{L^2(\Omega, H^{\otimes(n-1)})} + \frac{1}{n+1} \| \mathrm{d}^n u_n \|^2_{L^2(\Omega, H^{\otimes(n+1)})} \\
& \qquad = n \| u_n \|^2_{L^2(\Omega, H^{\otimes n})} + \| \nabla u_n \|^2_{L^2(\Omega, H^{\otimes(n+1)})},
\end{aligned}
$$

for $u_n \in \mathcal{S} \otimes H^{\wedge n}$. By the intertwining relation of Lemma 5.1 combined with the use of Assumption (B1) to reach (6.9) below, we have

$$
\begin{aligned}
& \langle \mathrm{d}^{n-1} \mathrm{d}^{(n-1)*} u_n(x_1, \ldots, x_n), g_n(x_1, \ldots, x_n) \rangle_{H^{\wedge n}} = nF \langle f_n(x_1, \ldots, x_n), g_n(x_1, \ldots, x_n) \rangle_{H^{\wedge n}} \\
& + \sum_{j=1}^{n} \langle \delta((D_{x_j} F) \otimes f_n(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n)), g_n(x_1, \ldots, x_n) \rangle_{H^{\wedge n}} \\
& + \sum_{j=1}^{n} \langle \delta(F \otimes \nabla_{x_j} f_n(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n), g_n(x_1, \ldots, x_n) \rangle_{H^{\wedge n}} \\
& - \sum_{j=1}^{n} (-1)^{j-1} \qquad (6.9)
\end{aligned}
$$

$$\sum_{l=2}^{n}\Big\langle (D_{x_j}F)\otimes\int_X \operatorname{Tr}\nabla_x^{(l)} f_n(x,x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)\sigma(dx), g_n(x_1,\dots,x_n)\Big\rangle_{H^{\wedge n}}$$
$$-F\left\langle\int_X \operatorname{Tr}\nabla_x \mathrm{d}_{x_n}^{n-1} f_n(x,x_1,\dots,x_{n-1})\sigma(dx), g_n(x_1,\dots,x_n)\right\rangle_{H^{\wedge n}} \tag{6.10}$$
$$+\sum_{j=1}^{n}\int_X \langle (D_xF)\otimes f_n(x_1,\dots,x_n), \nabla_{x_j}^{(j)} g_n(x_1,\dots,x_{j-1},x,x_{j+1},\dots,x_n)\rangle_{H^{\wedge n}}\sigma(dx).$$

Hence, applying Assumption (A4) from (6.9)-(6.10) to (6.11)-(6.12), and Assumption (A4) from (6.12) to (6.13), we find

$$\langle \mathrm{d}_{x_n}^{n-1}\mathrm{d}^{(n-1)*}u_n(\cdot,x_1,\dots,x_{n-1}), v_n(x_1,\dots,x_n)\rangle_{L^2(\Omega,H^{\wedge n})}$$
$$= n\langle u_n(x_1,\dots,x_n), v_n(x_1,\dots,x_n)\rangle_{L^2(\Omega,H^{\wedge n})}$$
$$+\sum_{j=1}^{n}\langle \delta((D_{x_j}F)\otimes f_n(x_1,\dots,x_{j-1},\cdot,x_{j+1},\dots,x_n)), G\otimes g_n(x_1,\dots,x_n)\rangle_{L^2(\Omega,H^{\wedge n})}$$
$$+\sum_{j=1}^{n}\langle \delta(F\otimes\nabla_{x_j} f_n(x_1,\dots,x_{j-1},\cdot,x_{j+1},\dots,x_n)), G\otimes g_n(x_1,\dots,x_n)\rangle_{L^2(\Omega,H^{\wedge n})}$$
$$+\frac{1}{n}\sum_{j=1}^{n}(-1)^{n-j}\sum_{l=1}^{n-1}\int_X\int_X \langle (D_yF)\otimes f_n(x_1,\dots,x_{j-1},x,x_{j+1},\dots,x_n), \tag{6.11}$$
$$G\otimes\nabla_x^{(l)} g_n(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n,y)\rangle_{L^2(\Omega,H^{\wedge(n-1)})}\sigma(dx)\sigma(dy)$$
$$+\frac{1}{n}\sum_{l=1}^{n}\sum_{\substack{j=1\\ j\neq l}}^{n}\int_X\int_X \langle F\otimes\nabla_y^{(j)} f_n(x_1,\dots,x_{l-1},x,x_{l+1},\dots,x_n), \tag{6.12}$$
$$G\otimes\nabla_x^{(l)} g_n(x_1,\dots,x_{j-1},y,x_{j+1},\dots,x_n)\rangle_{L^2(\Omega,H^{\wedge(n-1)})}\sigma(dx)\sigma(dy)$$
$$+\sum_{l=1}^{n}(-1)^{n-j}\int_X \big\langle (D_yF)\otimes f_n(x_1,\dots,x_n), G\otimes\nabla_{x_l}^{(n)} g_n(x_1,\dots,x_{l-1},x_{l+1},\dots,x_n,y)\big\rangle_{L^2(\Omega,H^{\wedge n})}\sigma(dy)$$
$$= n\langle u_n(x_1,\dots,x_n), v_n(x_1,\dots,x_n)\rangle_{L^2(\Omega,H^{\wedge n})}$$
$$+\frac{1}{n}\sum_{j=1}^{n}\int_X\int_X \langle (D_xF)\otimes f_n(x1,\dots,x_{j-1},y,x_{j+1},\dots,x_n),$$
$$(D_yG)\otimes g_n(x_1,\dots,x_{j-1},x,x_{j+1},\dots,x_n)\rangle_{L^2(\Omega,H^{\wedge(n-1)})}\sigma(dx)\sigma(dy)$$
$$+\frac{1}{n}\sum_{j=1}^{n}\int_X\int_X \langle F\otimes\nabla_x f_n(x_1,\dots,x_{j-1},y,x_{j+1},\dots,x_n),$$
$$(D_yG)\otimes g_n(x_1,\dots,x_{j-1},x,x_{j+1},\dots,x_n)\rangle_{L^2(\Omega,H^{\wedge(n-1)})}\sigma(dx)\sigma(dy)$$

$$+\frac{1}{n}\sum_{j=1}^{n}(-1)^{n-j}\sum_{l=1}^{n}\int_X\int_X\langle(D_yF)\otimes f_n(x_1,\ldots,x_{j-1},x,x_{j+1},\ldots,x_n),$$
$$G\otimes\nabla_x^{(l)}g_n(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n,y)\rangle_{L^2(\Omega,H^{\wedge(n-1)})}\sigma(dx)\sigma(dy)$$
$$+\frac{1}{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\int_X\int_X\langle F\otimes\nabla_y^{(j)}f_n(x_1,\ldots,x_{l-1},x,x_{l+1},\ldots,x_n),\tag{6.13}$$
$$G\otimes\nabla_x^{(l)}g_n(x_1,\ldots,x_{j-1},y,x_{j+1},\ldots,x_n)\rangle_{L^2(\Omega,H^{\wedge(n-1)})}\sigma(dx)\sigma(dy)$$
$$=n\langle u_n(x_1,\ldots,x_n),v_n(x_1,\ldots,x_n)\rangle_{L^2(\Omega,H^{\wedge n})}$$
$$+\frac{1}{n}\sum_{l=1}^{n}\int_X\int_X\langle(D_xF)\otimes f_n(x_1,\ldots,x_{l-1},y,x_{l+1},\ldots,x_n),$$
$$(D_yG)\otimes g_n(x_1,\ldots,x_{j-1},x,x_{j+1},\ldots,x_n)\rangle_{L^2(\Omega,H^{\wedge(n-1)})}\sigma(dx)\sigma(dy)$$
$$+\frac{1}{n}\sum_{j=1}^{n}\int_X\int_X\langle F\otimes\nabla_xf_n(x_1,\ldots,x_{j-1},y,x_{j+1},\ldots,x_n),$$
$$(D_yG)\otimes g_n(x_1,\ldots,x_{j-1},x,x_{j+1},\ldots,x_n)\rangle_{L^2(\Omega,H^{\wedge(n-1)})}\sigma(dx)\sigma(dy)$$
$$+\frac{1}{n}\sum_{j=1}^{n}(-1)^{n-j}\sum_{l=1}^{n}\int_X\int_X\langle(D_yF)\otimes f_n(x_1,\ldots,x_{j-1},x,x_{j+1},\ldots,x_n),$$
$$G\otimes\nabla_x^{(l)}g_n(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n,y)\rangle_{L^2(\Omega,H^{\wedge(n-1)})}\sigma(dx)\sigma(dy)$$
$$+\frac{1}{n}\sum_{j=1}^{n}(-1)^{n-j}\sum_{l=1}^{n}\int_X\int_X\langle F\otimes\nabla_y^{(j)}f_n(x_1,\ldots,x_{l-1},x,x_{l+1},\ldots,x_n),$$
$$G\otimes\nabla_x^{(l)}g_n(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n,y)\rangle_{L^2(\Omega,H^{\wedge(n-1)})}\sigma(dx)\sigma(dy)$$
$$=n\langle u_n(x_1,\ldots,x_n),v_n(x_1,\ldots,x_n)\rangle_{L^2(\Omega,H^{\wedge n})}$$
$$+\frac{1}{n!}\Big\langle(D_{x_{n+1}}F)\otimes f_n(x_1,\ldots,x_n)+F\otimes\nabla_{x_{n+1}}f_n(x_1,\ldots,x_n),$$
$$\sum_{j=1}^{n}(-1)^{n-j}(D_{x_j}G)\otimes g_n(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n,x_{n+1})$$
$$+\sum_{j=1}^{n}(-1)^{n-j}\sum_{l=1}^{n}G\otimes\nabla_{x_j}^{(l)}g_n(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n,x_{n+1})\Big\rangle_{L^2(\Omega,H^{\otimes(n+1)})}$$
$$=n\langle u_n(x_1,\ldots,x_n),v_n(x_1,\ldots,x_n)\rangle_{L^2(\Omega,H^{\wedge n})}$$
$$-\frac{(-1)^n}{n!}\left\langle(D_{x_{n+1}}F)\otimes f_n(x_1,\ldots,x_n)+F\otimes\nabla_{x_{n+1}}f_n(x_1,\ldots,x_n),\right.$$
$$\sum_{j=1}^{n+1}(-1)^{j-1}(D_{x_j}G)\otimes g_n(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{n+1})$$

$$+\sum_{j=1}^{n+1}(-1)^{j-1}\sum_{l=1}^{n} G\otimes\nabla^{(l)}_{x_j} g_n(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{n+1})\Bigg\rangle_{L^2(\Omega,H^{\otimes(n+1)})}$$

$$+\frac{1}{n!}\Big\langle (D_{x_{n+1}}F)\otimes f_n(x_1,\ldots,x_n)+F\otimes\nabla_{x_{n+1}} f_n(x_1,\ldots,x_n),$$

$$(D_{x_{n+1}}G)\otimes g_n(x_1,\ldots,x_n)+G\otimes\sum_{l=1}^{n}\nabla^{(l)}_{x_{n+1}} g_n(x_1,\ldots,x_n)\Bigg\rangle_{L^2(\Omega,H^{\otimes(n+1)})}$$

$$= n\langle u_n(x_1,\ldots,x_n), v_n(x_1,\ldots,x_n)\rangle_{L^2(\Omega,H^{\wedge n})}$$

$$-\frac{(-1)^n}{n!}\Big\langle (D_{x_{n+1}}F)\otimes f_n(x_1,\ldots,x_n)+F\otimes\nabla_{x_{n+1}} f_n(x_1,\ldots,x_n),$$

$$\sum_{j=1}^{n+1}(-1)^{j-1}(D_{x_j}G)\otimes g_n(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{n+1})$$

$$+\sum_{j=1}^{n+1}(-1)^{j-1} G\otimes\nabla_{x_j} g_n(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{n+1})\Bigg\rangle_{L^2(\Omega,H^{\otimes(n+1)})}$$

$$+\frac{1}{n!}\Big\langle (D_{x_{n+1}}F)\otimes f_n(x_1,\ldots,x_n)+F\otimes\nabla_{x_{n+1}} f_n(x_1,\ldots,x_n),$$

$$(D_{x_{n+1}}G)\otimes g_n(x_1,\ldots,x_n)+G\otimes\nabla_{x_{n+1}} g_n(x_1,\ldots,x_n)\Big\rangle_{L^2(\Omega,H^{\otimes(n+1)})}$$

$$= n\langle u_n(x_1,\ldots,x_n), v_n(x_1,\ldots,x_n)\rangle_{L^2(\Omega,H^{\wedge n})}$$

$$-\frac{1}{(n+1)!}\Bigg\langle\sum_{j=1}^{n+1}(-1)^{j-1}(D_{x_j}F)\otimes f_n(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{n+1})$$

$$+\sum_{j=1}^{n+1}(-1)^{j-1} F\otimes\nabla_{x_j} f_n(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{n+1}),$$

$$\sum_{j=1}^{n+1}(-1)^{j-1}(D_{x_j}G)\otimes g_n(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{n+1})$$

$$+\sum_{j=1}^{n+1}(-1)^{j-1} G\otimes\nabla_{x_j} g_n(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{n+1})\Bigg\rangle_{L^2(\Omega,H^{\otimes(n+1)})}$$

$$+\frac{1}{n!}\Big\langle (D_{x_{n+1}}F)\otimes f_n(x_1,\ldots,x_n)+F\otimes\nabla_{x_{n+1}} f_n(x_1,\ldots,x_n),$$

$$(D_{x_{n+1}}G)\otimes g_n(x_1,\ldots,x_n)+G\otimes\nabla_{x_{n+1}} g_n(x_1,\ldots,x_n)\Big\rangle_{L^2(\Omega,H^{\otimes(n+1)})}$$

$$= n\langle u_n(x_1,\ldots,x_n), v_n(x_1,\ldots,x_n)\rangle_{L^2(\Omega,H^{\wedge n})}-\langle \mathrm{d}^n u_n, \mathrm{d}^n v_n\rangle_{L^2(\Omega,H^{\wedge(n+1)})}$$

$$+\frac{1}{n!}\langle\nabla_{x_{n+1}} u_n(x_1,\ldots,x_n),\nabla_{x_{n+1}} v_n(x_1,\ldots,x_n)\rangle_{L^2(\Omega,H^{\wedge(n+1)})},$$

where we used (A4). Hence we have

$$\langle \mathrm{d}^{(n-1)*}u_n, \mathrm{d}^{(n-1)*}v_n\rangle_{L^2(\Omega,H^{\wedge n})} + \langle \mathrm{d}^n u_n, \mathrm{d}^n v_n\rangle_{L^2(\Omega,H^{\wedge(n+1)})}$$
$$= n\langle u_n, v_n\rangle_{L^2(\Omega,H^{\wedge n})} + \frac{1}{n!}\langle \nabla u_n, \nabla v_n\rangle_{L^2(\Omega,H^{\otimes(n+1)})},$$

i.e.

$$\langle \mathrm{d}^{(n-1)*}u_n, \mathrm{d}^{(n-1)*}v_n\rangle_{L^2(\Omega,H^{\otimes n})} + \frac{1}{n+1}\langle \mathrm{d}^n u_n, \mathrm{d}^n v_n\rangle_{L^2(\Omega,H^{\otimes(n+1)})}$$
$$= n\langle u_n, v_n\rangle_{L^2(\Omega,H^{\otimes n})} + \langle \nabla u_n, \nabla v_n\rangle_{L^2(\Omega,H^{\otimes(n+1)})},$$

and applying the duality

$$\langle \nabla u_n, \nabla v_n\rangle_{L^2(\Omega,H^{\otimes(n+1)})} = \langle \nabla^*\nabla u_n, v_n\rangle_{L^2(\Omega,H^{\wedge n})}, \qquad u_n, v_n \in \mathcal{S}\otimes H^{\wedge n},$$

we get

$$\mathrm{d}^{n-1}\mathrm{d}^{(n-1)*} + \mathrm{d}^{n*}\mathrm{d}^n = n\mathrm{I}_{H^{\wedge n}} + \nabla^*\nabla.$$

□

Proof of Theorem 6.1 (Clark-Ocone formula).
By the Clark-Ocone formula (C1) and Assumption (A4') we have

$$\begin{aligned} u_n(t_1,\ldots,t_n) &= E[u_n(t_1,\ldots,t_n) \mid \mathcal{F}_{t_1\vee\cdots\vee t_n}] + \int_{t_1\vee\cdots\vee t_n}^\infty E[D_r u_n(t_1,\ldots,t_n) \mid \mathcal{F}_r]dM_r \\ &= E[u_n(t_1,\ldots,t_n) \mid \mathcal{F}_{t_1\vee\cdots\vee t_n}] + \int_{t_1\vee\cdots\vee t_n}^\infty E[\nabla_r u_n(t_1,\ldots,t_n) \mid \mathcal{F}_r]dM_r, \quad (6.14)\end{aligned}$$

$t_1,\ldots,t_n \in \mathbb{R}_+$. Next, by the definition (2.4) of d^n and (3.8) applied to adapted processes we have

$$\begin{aligned} &\mathrm{d}^{n-1}_{t_n}\int_{t_1\vee\cdots\vee t_{n-1}}^\infty E[u_n(r,t_1,\ldots,t_{n-1}) \mid \mathcal{F}_r]dM_r \\ &= \sum_{j=1}^n (-1)^{j-1}\nabla_{t_j}\int_{t_1\vee\cdots\vee t_{j-1}\vee t_{j+1}\vee\cdots\vee t_n}^\infty E[u_n(r,t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n) \mid \mathcal{F}_r]dM_r \\ &= \sum_{j=1}^n (-1)^{j-1}\mathbf{1}_{[t_1\vee\cdots\vee t_n,\infty)}(t_j)E[u_n(t_j,t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n) \mid \mathcal{F}_{t_j}] \\ &+ \sum_{j=1}^n (-1)^{j-1}\int_{t_1\vee\cdots\vee t_{j-1}\vee t_{j+1}\cdots\vee t_n}^\infty \nabla_{t_j}E[u_n(t_{n+1},t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n) \mid \mathcal{F}_{t_{n+1}}]dM_{t_{n+1}}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \mathbf{1}_{[t_1\vee\cdots\vee t_n,\infty)}(t_j) E[u_n(t_1,\ldots,t_n) \mid \mathcal{F}_{t_j}] \\
&+ \sum_{j=1}^{n} (-1)^{j-1} \int_{t_1\vee\cdots\vee t_{j-1}\vee t_{j+1}\cdots\vee t_n}^{\infty} \nabla_{t_j} E[u_n(t_{n+1},t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n) \mid \mathcal{F}_{t_{n+1}}] dM_{t_{n+1}} \\
&= E[u_n(t_1,\ldots,t_n) \mid \mathcal{F}_{t_1\vee\cdots\vee t_n}] \\
&+ \sum_{j=1}^{n} (-1)^{j-1} \int_{t_1\vee\cdots\vee t_n}^{\infty} E[\nabla_{t_j} u_n(t_{n+1},t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n) \mid \mathcal{F}_{t_{n+1}}] dM_{t_{n+1}}, \quad (6.15)
\end{aligned}
$$

where on the last line we used the fact that by (A4), $\nabla_{t_j} u_n(t_{n+1},t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n)$ vanishes when $t_1\vee\cdots\vee t_{j-1}\vee t_{j+1}\cdots\vee t_n < t_{n+1} < t_j$, hence by taking the difference of (6.14) and (6.15) we find

$$
\begin{aligned}
u_n(t_1,\ldots,t_n) &= \mathrm{d}_{t_n}^{n-1} \int_{t_1\vee\cdots\vee t_{n-1}}^{\infty} E[u_n(r,t_1,\ldots,t_{n-1}) \mid \mathcal{F}_r] dM_r \\
&\quad - \sum_{j=1}^{n} (-1)^{j-1} \int_{t_1\vee\cdots\vee t_n}^{\infty} E[\nabla_{t_j} u_n(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1}) \mid \mathcal{F}_{t_{n+1}}] dM_{t_{n+1}} \\
&\quad + \int_{t_1\vee\cdots\vee t_n}^{\infty} E[\nabla_r u_n(t_1,\ldots,t_n) \mid \mathcal{F}_r] dM_r \\
&= \mathrm{d}_{t_n}^{n-1} \int_{t_1\vee\cdots\vee t_{n-1}}^{\infty} E[u_n(r,t_1,\ldots,t_{n-1}) \mid \mathcal{F}_r] dM_r \\
&\quad + \int_{t_1\vee\cdots\vee t_n}^{\infty} E[\mathrm{d}_{t_{n+1}}^{n} u_n(t_1,\ldots,t_n) \mid \mathcal{F}_{t_{n+1}}] dM_{t_{n+1}},
\end{aligned}
$$

$t_1,\ldots,t_n \in \mathbb{R}_+$. □

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