Stochastic dynamics of determinantal processes by integration by parts

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Abstract

We derive an integration by parts formula for functionals of determinantal processes on compact sets, completing the arguments of [5]. This is used to show the existence of a configuration-valued diffusion process which is non-colliding and admits the distribution of the determinantal process as reversible law. In particular, this approach allows us to build a concrete example of the associated diffusion process, providing an illustration of the results of [5] and [32].

Keywords: Dirichlet forms; diffusion processes; integration by parts; Malliavin calculus; determinantal processes.

Mathematics Subject Classification: 60G55; 60J60; 60G60; 60H07; 60K35.

1 Introduction

Determinantal processes are point processes that exhibit repulsion, and were introduced to represent the configuration of fermions, cf. [18, 23, 27]. They are known to be connected with the zeros of analytic functions (cf. [15] and references therein) as well as with the theory of random matrices (cf. [3]). To the best of our knowledge, the first use of determinantal processes as models in applications trace back to [4]. More recently, in [9, 19, 28, 31],

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different authors have used determinantal processes to model phenomena arising in telecommunication networks.

The Markov process associated to the Ornstein-Uhlenbeck operator on the Poisson space has been constructed in [26]. In [1], using the Dirichlet forms theory (see [10, 16]), the diffusion whose symmetrizing measure is the law of a Poisson process over a Riemannian manifold is constructed. This result has been extended to Gibbs processes on $\mathbb{R}^d$ in [2]. Using such methods, the Dirichlet form and diffusion process associated to determinantal processes has been constructed in [32].

In this paper, by completing the arguments of [5] we prove an integration by parts formula for functionals of determinantal processes on compact subsets of $\mathbb{R}^d$, and we recover the closability of the associated Dirichlet form. This provides a novel proof of the existence of interacting diffusion processes properly associated to determinantal processes. In addition, our approach based on integration by parts exhibits the generator of the diffusion process, and allows in turn to provide an explicit example of a diffusion process satisfying our hypotheses.

As a preliminary step, we derive an integration by parts formula for functionals of a determinantal process on a compact set $D \subset \mathbb{R}^d$, by completing the result established in [5]. In comparison with [5], the integration by parts formula on compact sets is extended to closed gradient and divergence operators by the use of a different set of test functionals, cf. (3.1) and Theorem 3.8. Our construction of the diffusion processes follows the lines of [1], and it differs from the one of [11] which is based on sample-path identities. Our gradient and divergence operators also differ from those of [21], which also deals with compact subsets of $\mathbb{R}^d$. Nevertheless the integration by parts formula of Theorem 3.8 can also be applied to density estimation and sensitivity analysis for functionals of determinantal processes along the same lines.

In Theorem 4.1 we construct the Dirichlet form corresponding to a determinantal process on a compact set $D \subset \mathbb{R}^d$. In Theorem 5.1 we show the existence of the diffusion properly associated to a determinantal process on a compact set $D \subset \mathbb{R}^d$. Note that, as in the other constructions (cf. [1, 2, 32]), the associated diffusion process admits the distribution of the determinantal process as a reversible law. We prove the non-collision property of the diffusion in Theorem 5.3. Finally, in Section 6, we provide an example of a determinantal process satisfying our integration by parts formula, and for which the aforementioned properly associated diffusion process exists.

Some definitions related to point processes theory and more in particular to determinantal processes are recalled in Section 2 based on [6, 7, 15, 20, 30]. Some notions from the Dirichlet forms theory are given in Sections 4 and 5 based on [10, 16]. We also refer the reader to [8, 24] for the required background on functional analysis.

2 Preliminaries

Locally finite point processes

Let S be a Polish space, and denote by $\mathcal{B}(S)$ the associated Borel σ-algebra. For any subset $B \subset S$, let $\sharp B$ denote the cardinality of B, setting $\sharp B = \infty$ if B is not finite. We denote by N_{lf} the set of locally finite point configurations on S:

$$\mathrm{N}_{lf} := \{B \subset S \,:\, \sharp(B \cap D) < \infty, \text{ for any compact } D \subset S\}.$$

We identify locally finite configurations with $\mathbb{N}$-valued simple Radon measures, equip N_{lf} with the vague topology (see Appendix 2 in [6]), and we denote the corresponding Borel σ-algebra by $\mathcal{N}_{lf}$. We recall that a non-negative simple Radon measure is a Radon measure which is less than or equal to 1 on singletons. We define similarly N_f the set of finite point configurations on S:

$$\mathrm{N}_f := \{B \subset S \,:\, \sharp B < \infty\},$$

which is naturally equipped with the trace σ-algebra $\mathcal{N}_f = \mathcal{N}_{lf}|_{\mathrm{N}_f}$. For any measurable set $B \subset S$, let N_f^B be the space of finite configurations on B, and $\mathcal{N}_f^B$ the associated trace-σ-algebra.

By a locally finite and simple point process $\mathbf{X}$ on S we mean a measurable mapping defined on some probability space $(\Omega, \mathcal{F}, P)$ taking values on $(\mathrm{N}_{lf}, \mathcal{N}_{lf})$. We denote by $\mathbf{X}(B)$ the number of points of $\mathbf{X}$ in a measurable set $B \subset S$, i.e. $\mathbf{X}(B) := \sharp(\mathbf{X} \cap B)$, and by

$$\mathbf{X}^B = \mathbf{X} \cap B = \{X_1, \ldots, X_{\mathbf{X}(B)}\}$$

the restriction to B of the point process $\mathbf{X} \equiv \{X_n\}_{1\le n\le \mathbf{X}(S)}$. In the following, we shall denote by $\mathbb{P}$ the law of $\mathbf{X}$ and by $\mathbb{P}_B$ the law of $\mathbf{X}^B$.

The correlation functions of $\mathbf{X}$, with respect to (*w.r.t.*) a given Radon measure ν on $(S, \mathcal{B}(S))$, are (if they exist) symmetric measurable functions $\rho_n : S^n \longrightarrow \mathbb{R}_+$ such that

$$\mathbb{E}\left[\prod_{i=1}^{n} \mathbf{X}(B_i)\right] = \int_{B_1\times\ldots\times B_n} \rho_n(x_1,\ldots,x_n)\,\nu(\mathrm{d}x_1)\cdots\nu(\mathrm{d}x_n),$$

for any family of mutually disjoint bounded subsets $B_1,\ldots,B_n$ of S, $n \ge 1$. We require in addition that $\rho_n(x_1,\ldots,x_n) = 0$ whenever $x_i = x_j$ for some $1 \le i \ne j \le n$. When ρ_1 exists, the measure $\rho_1(x)\,\nu(\mathrm{d}x)$ is known as the intensity measure of $\mathbf{X}$.

As in [13], we define for any Radon measure ν on $(S, \mathcal{B}(S))$ the ν-sample measure L^ν on $(\mathrm{N}_f, \mathcal{N}_f)$ by

$$\int_{\mathrm{N}_f} f(\alpha)\,L^\nu(\mathrm{d}\alpha) := \sum_{n\ge 0}\frac{1}{n!}\int_{S^n} f(\{x_1,\ldots,x_n\})\,\nu(\mathrm{d}x_1)\cdots\nu(\mathrm{d}x_n), \tag{2.1}$$

for any measurable $f : \mathrm{N}_f \to \mathbb{R}_+$. For any compact subset $D \subset S$, the Janossy densities of $\mathbf{X}$ *w.r.t.* ν are (if they exist) measurable symmetric functions $j_D^n : D^n \to \mathbb{R}$ satisfying, for all measurable $f : \mathrm{N}_f^D \to \mathbb{R}_+$,

$$\mathbb{E}\left[f(\mathbf{X}^D)\right] = \sum_{n\ge 0}\frac{1}{n!}\int_{D^n} f(\{x_1,\ldots,x_n\})\,j_D^n\,(x_1,\ldots,x_n)\;\nu(\mathrm{d}x_1)\cdots\nu(\mathrm{d}x_n), \tag{2.2}$$

i.e. defining $j_D(\mathbf{x}) := j_D^{\mathbf{x}(D)}(x_1,\ldots,x_{\mathbf{x}(D)})$ for $\mathbf{x} = \{x_1,\ldots,x_{\mathbf{x}(D)}\} \in \mathrm{N}_f^D$, j_D is the density of $\mathbb{P}_D$ with respect to L_D^ν (the restriction to $\mathcal{N}_f^D$ of L^ν), when $\mathbb{P}_D \ll L_D^\nu$.

Kernels and integral operators

Let ν be a Radon measure on $(S, \mathcal{B}(S))$. For any compact set $D \subset S$, we denote by $\mathrm{L}^2(D,\nu)$ the Hilbert space of complex-valued square integrable functions *w.r.t.* the restriction of the Radon measure ν to D, equipped with the inner product

$$\langle f, g\rangle_{\mathrm{L}^2(D,\nu)} := \int_D f(x)\overline{g(x)}\,\nu(\mathrm{d}x), \quad f, g \in \mathrm{L}^2(D,\nu)$$

where $\overline{z}$ denotes the complex conjugate of a complex $z \in \mathbb{C}$. By definition, an integral operator $\mathcal{T} : \mathrm{L}^2(S,\nu) \to \mathrm{L}^2(S,\nu)$ with kernel $T : S^2 \to \mathbb{C}$ is a bounded operator defined by

$$\mathcal{T}f(x) := \int_S T(x,y)f(y)\,\nu(\mathrm{d}y), \quad \text{for } \nu\text{-almost all } x \in S.$$

Letting $\mathcal{P}_D$ denote the projection operator from $\mathrm{L}^2(S,\nu)$ onto $\mathrm{L}^2(D,\nu)$, we set $\mathcal{T}_D = \mathcal{P}_D\mathcal{T}\mathcal{P}_D$ and note that its kernel is $T_D(x,y) := \mathbf{1}_D(x)T(x,y)\mathbf{1}_D(y)$, for ν-almost all $x,y \in S$. It can be shown that $\mathcal{T}_D$ is a compact operator. The operator $\mathcal{T}$ is said to be Hermitian or self-adjoint if

$$T(x,y) = \overline{T(y,x)}, \quad \text{for } \nu^{\otimes 2}\text{-almost all } (x,y) \in S^2. \tag{2.3}$$

Equivalently, this means that the integral operator $\mathcal{T}_D$ is self-adjoint for any compact set $D \subset S$. If $\mathcal{T}_D$ is self-adjoint, by the spectral theorem we have that $\mathrm{L}^2(D,\nu)$ has an orthonormal basis $(\varphi_j^D)_{j\geq 1}$ of eigenfunctions of $\mathcal{T}_D$. The corresponding eigenvalues $(\lambda_j^D)_{j\geq 1}$ have finite multiplicity (except possibly the zero eigenvalue) and the only possible accumulation point of the eigenvalues is the eigenvalue zero. Then, the kernel T_D of $\mathcal{T}_D$ can be written as

$$T_D(x,y) = \sum_{j\geq 1} \lambda_j^D \varphi_j^D(x)\overline{\varphi_j^D(y)}, \tag{2.4}$$

for ν-almost all $x,y \in D$. We say that an operator $\mathcal{T}$ is positive (respectively non-negative) if its spectrum is included in $(0,+\infty)$ (respectively $[0,+\infty)$). For two operators $\mathcal{T}$ and $\mathcal{U}$, we will say that $\mathcal{T} > \mathcal{U}$ (respectively $\mathcal{T} \geq \mathcal{U}$) in the operator ordering if $\mathcal{T} - \mathcal{U}$ is a positive operator (respectively a non-negative operator).

We say that a self-adjoint integral operator $\mathcal{T}_D$, with kernel T_D as in (2.4), is of trace class if

$$\sum_{j\geq 1} |\lambda_j^D| < \infty,$$

and we define the trace of the operator $\mathcal{T}_D$ as

$$\mathrm{Tr}\,(\mathcal{T}_D) := \sum_{j\geq 1} \lambda_j^D.$$

If $\mathcal{T}_D$ is of trace class for every compact subset $D \subset S$, then we say that $\mathcal{T}$ is locally of trace class. It is easily seen that $\mathcal{T}^n$ is locally of trace class, for all $n \geq 2$, if $\mathcal{T}$ is locally of trace class. Finally, we define the Fredholm determinant of $\mathbf{Id} + \mathcal{T}_D$, when $\|\mathcal{T}_D\|_{op} < 1$, as

$$\mathrm{Det}(\mathbf{Id} + \mathcal{T}_D) := \exp\left(\sum_{n\geq 1} \frac{(-1)^{n-1}}{n}\mathrm{Tr}(\mathcal{T}_D^n)\right). \tag{2.5}$$

Here, $\mathbf{Id}$ denotes the identity operator on $\mathrm{L}^2(S,\nu)$ and $\|\cdot\|_{op}$ denotes the operator norm.

Determinantal processes on S

Let μ be a Radon measure on $(S, \mathcal{B}(S))$. A locally finite and simple point process $\mathbf{X}$ on S is said to be a determinantal process with kernel K and reference measure μ if its correlation functions *w.r.t.* μ exist and satisfy

$$\rho_n(x_1, \dots, x_n) = \det(K(x_i, x_j))_{1 \le i,j \le n},$$

for any $n \ge 1$ and μ-a.e. $x_1, \dots, x_n \in S$, where $K(\cdot, \cdot)$ is a measurable function. Throughout this paper we shall work under the following hypothesis.

(H1): *The integral operator $\mathcal{K}$ on $\mathrm{L}^2(S, \mu)$ with kernel K is locally of trace class, self-adjoint, and* $0 \le \mathcal{K} < \mathbf{Id}$.

The existence and uniqueness in law of a determinantal process with kernel K is guaranteed under **(H1)** by the results in [18, 23, 25]. See also Lemma 4.2.6 and Theorem 4.5.5 in [15].

We define the global interaction operator $\mathcal{J} := (\mathbf{Id} - \mathcal{K})^{-1}\mathcal{K}$. As proved in [13], we also define the local interaction operator $\mathcal{J}[D]$ as

$$\mathcal{J}[D] := (\mathbf{Id} - \mathcal{K}_D)^{-1}\mathcal{K}_D,$$

and we emphasize that $\mathcal{J}[D]$ is not the projection of $\mathcal{J}$ onto $\mathrm{L}^2(D, \mu)$. We refer the reader to [13] for the following properties of $\mathcal{J}[D]$. First, $\mathcal{J}[D]$ is a self-adjoint integral operator and, letting $J[D]$ denote its kernel, as a consequence of (2.4) we have

$$J[D](x, y) = \sum_{j \ge 1} \frac{\lambda_j^D}{1 - \lambda_j^D} \varphi_j^D(x) \overline{\varphi_j^D(y)}, \tag{2.6}$$

for μ-almost all $x, y \in D$. Second, $\mathcal{J}[D]$ is a trace class operator. Third, denoting by $\det J[D](\{x_1, \dots, x_n\}) := \det \left(J[D](x_i, x_j)\right)_{1 \le i,j \le n}$, the function

$$(x_1, \dots, x_n) \mapsto \det J[D](\{x_1, \dots, x_n\})$$

is $\mu^{\otimes n}$-a.e. non-negative and symmetric in $x_1, \dots, x_n$, and we set $\det J[D](\{x_1, \dots, x_n\}) := \det J[D](x_1, \dots, x_n)$. The local interaction operator is related to the Janossy densities of a determinantal process by the following proposition.

Proposition 2.1 ([23]) *Under* **(H1)**, *for any compact $D \subset S$, the Janossy densities $j_D^n(\mathbf{x})$ of $\mathbf{X}$ are given by*

$$j_D^n(x_1, \ldots, x_n) = \mathrm{Det}(\mathbf{Id} - \mathcal{K}_D) \det J[D](x_1, \ldots, x_n), \qquad x_1, \ldots, x_n \in D, \quad n \geq 1. \quad (2.7)$$

Moreover, $P(\mathbf{X}^D = \emptyset)$ is given by $j_D^0(\emptyset) = \mathrm{Det}(\mathbf{Id} - \mathcal{K}_D)$.

3 Differential calculus and integration by parts

In this section we derive an integration by parts formula for functionals of a determinantal point process, and we extend it by closability. Hereafter we assume that S is a domain of $\mathbb{R}^d$ equipped with the Euclidean distance, μ is a Radon measure on $(S, \mathcal{B}(S))$ and $D \subset S$ is a fixed compact set. We denote by $\| \cdot \|$ the Euclidean norm on $\mathbb{R}^d$, by $x \cdot y$ the usual inner product of $x, y \in \mathbb{R}^d$, and by $x^{(i)}$ the i-th component of $x \in \mathbb{R}^d$, $i = 1, \ldots, d$.

Differential calculus

We denote by $\mathcal{C}^\infty(D, \mathbb{R}^d)$ the set of all $\mathcal{C}^\infty$-vector fields $v : D \longrightarrow \mathbb{R}^d$ and by $\mathcal{C}^\infty(S^k)$ the set of all $\mathcal{C}^\infty$-functions on S^k.

Definition 1 *A function $F : \mathrm{N}_f^D \to \mathbb{R}$ is said to be in $\mathcal{S}_D$ if*

$$F(\mathbf{x}) = f_0 \mathbf{1}_{\{\mathbf{x}(D)=0\}} + \sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{x}(D)=k\}} f_k(x_1, \ldots, x_k), \qquad \textit{for } \mathbf{x} = \{x_1, \ldots, x_{\mathbf{x}(D)}\} \in \mathrm{N}_f^D, \quad (3.1)$$

where $f_0 \in \mathbb{R}$ is a constant, $n \geq 1$ is an integer and, for any $k = 1, \ldots, n$, $f_k \in \mathcal{C}^\infty(D^k)$ is a symmetric function.

The set of test functions $\mathcal{S}_D$ is dense in $\mathrm{L}^2_D := \mathrm{L}^2(\mathrm{N}_f^D, \mathbb{P}_D)$ (indeed, it contains the space $\tilde{\mathcal{S}}_D$ defined in Definition 2 below which is dense in L^2_D, see e.g. [16] p.54).

The gradient of $F \in \mathcal{S}_D$ as in (3.1) is defined by

$$\nabla_x^{\mathrm{N}_{lf}} F(\mathbf{x}) := \sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{x}(D)=k\}} \sum_{i=1}^{k} \mathbf{1}_{\{x_i\}}(x) \nabla_{x_i} f_k(x_1, \ldots, x_k), \qquad x \in D, \ \mathbf{x} \in \mathrm{N}_f^D, \quad (3.2)$$

where ∇_y denotes the usual gradient on $\mathbb{R}^d$ with respect to y. For $v \in \mathcal{C}^\infty(D, \mathbb{R}^d)$, we also let

$$\nabla_v^{\mathrm{N}_{lf}} F(\mathbf{x}) := \sum_{k=1}^{\mathbf{x}(D)} \nabla_{x_k}^{\mathrm{N}_{lf}} F(\mathbf{x}) \cdot v(x_k) = \sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{x}(D)=k\}} \sum_{i=1}^{k} \nabla_{x_i} f_k(x_1, \ldots, x_k) \cdot v(x_i), \quad (3.3)$$

where we recall that the symbol $\cdot$ denotes the inner product on $\mathbb{R}^d$.

Quasi-invariance

Next we recall some results from [5], with some complements that make the proofs more precise. Let $\mathrm{Diff}_0(D)$ be the set of all diffeomorphisms from D into itself. For $\phi \in \mathrm{Diff}_0(D)$ and a Radon measure ν on D, ν_ϕ denotes the image measure of ν by ϕ. For such a ϕ, we define the map

$$\begin{aligned} \mathcal{I}_\phi \,:\, \mathrm{L}^2(D,\nu_\phi) &\longrightarrow \mathrm{L}^2(D,\nu), \\ f &\longmapsto f \circ \phi, \end{aligned}$$

whose inverse is given by $\mathcal{I}_\phi^{-1} = \mathcal{I}_{\phi^{-1}}$. Note that $\mathcal{I}_\phi$ is an isometry. Given an operator $\mathcal{T}$ on $\mathrm{L}^2(D,\nu)$, we define the operator on $\mathrm{L}^2(D,\nu_\phi)$

$$\mathcal{T}^\phi = \mathcal{I}_\phi^{-1}\,\mathcal{T}\,\mathcal{I}_\phi.$$

Lastly, for any $\mathbf{x} = \{x_n\}_{1\le n\le \mathbf{x}(S)} \in \mathrm{N}_{lf}$, we denote by Φ the map:

$$\begin{aligned} \Phi \,:\, \mathrm{N}_{lf} &\longrightarrow \mathrm{N}_{lf}, \\ \{x_n\}_{1\le n\le \mathbf{x}(S)} &\longmapsto \{\phi(x_n)\}_{1\le n\le \mathbf{x}(S)}. \end{aligned}$$

The following lemma is proved in [5] and [29].

Lemma 3.1 *Assume* **(H1)**, *and take* $\phi \in \mathrm{Diff}_0(D)$. *The following properties hold.*

a) $\mathcal{K}_D^\phi$ *and* $\mathcal{J}[D]^\phi$ *are integral operators on* $\mathrm{L}^2(D,\mu_\phi)$ *with kernels given respectively by* $K_D^\phi(x,y) = K\left(\phi^{-1}(x),\phi^{-1}(y)\right)$ *and* $J[D]^\phi(x,y) = J[D]\left(\phi^{-1}(x),\phi^{-1}(y)\right)$.

b) $\mathcal{K}_D^\phi$ *is of trace class and* $\mathrm{Tr}(\mathcal{K}_D^\phi) = \mathrm{Tr}(\mathcal{K}_D)$.

c) $\mathrm{Det}(\mathbf{Id} - \mathcal{K}_D^\phi) = \mathrm{Det}(\mathbf{Id} - \mathcal{K}_D)$. *This translates into the fact that* $P(\mathbf{X}(D) = 0) = P(\Phi(\mathbf{X})(D) = 0)$ *which is expected since* ϕ *is a diffeomorphism.*

d) $\mathcal{J}[D]^\phi = \mathcal{J}^\phi[D] := (\mathbf{Id} - \mathcal{K}_D^\phi)^{-1}\mathcal{K}_D^\phi$ *is the local interaction operator associated with* $\mathcal{K}^\phi$.

The following mapping theorem holds, see Theorem 7 in [5].

Lemma 3.2 *Assume* **(H1)**, *and let* $\phi \in \mathrm{Diff}_0(D)$. *Then,* $\Phi(\mathbf{X}^D)$ *is a determinantal process with integral operator* $\mathcal{K}_D^\phi$ *and reference measure* μ_ϕ.

To prove the quasi-invariance of the determinantal measure restricted to a compact set $D \subset S$ with respect to the group of diffeomorphisms on D, we state one last result.

Lemma 3.3 *Under* **(H1)**, *we have* $\det J[D](\mathbf{x}) > 0$, *for* $\mathbb{P}_D$*-a.e.* $\mathbf{x} \in \mathrm{N}_f^D$. *However, it does not in general hold that* $\det J[D](\mathbf{x}) > 0$, *for* L_D^μ*-a.e.* $\mathbf{x} \in \mathrm{N}_f^D$.

Proof. Recall that under **(H1)**, we have $\mathbb{P}_D \ll L_D^\mu$ and

$$j_D(\mathbf{x}) = \frac{\mathrm{d}\mathbb{P}_D}{\mathrm{d}L_D^\mu}(\mathbf{x}) = \mathrm{Det}(\mathbf{Id} - \mathcal{K}_D) \det J[D](\mathbf{x}),$$

for $\mathbf{x} \in \mathrm{N}_f^D$. Since j_D is a density, we clearly have $j_D(\mathbf{x}) > 0$, for $\mathbb{P}_D$-a.e. $\mathbf{x} \in \mathrm{N}_f^D$. Hence, since $\|\mathcal{K}_D\| < 1$, we have $\det J[D](\mathbf{x}) > 0$, for $\mathbb{P}_D$-a.e. $\mathbf{x} \in \mathrm{N}_f^D$. As to the concluding part of the lemma, we notice that in general, one does not have that $\mathbb{P}_D$ is equivalent to L_D^μ. Indeed, consider for example the case where the rank of $\mathcal{K}_\mathcal{D}$ is less than or equal to $N \geq 1$. Then, $j_D^{N+1}(x_1, \ldots, x_{N+1}) = 0$, for $\mu^{\otimes(N+1)}$-a.e. $(x_1, \ldots, x_{N+1}) \in D^{N+1}$ (since $\mathbf{X}^D$ has less than $N+1$ points almost surely, see [25] for details). It suffices to define the set

$$A := \{B \subset D \ : \ \sharp B = N+1\},$$

which verifies $\mathbb{P}(A) = 0$ but $L_D^\mu(A) = \frac{1}{(N+1)!}\mu(D)$. □

Remark 3.4 *If we assume that, for any* $n \geq 1$, *the function*

$$(x_1, \ldots, x_n) \longmapsto \det J[D](x_1, \ldots, x_n)$$

is strictly positive $\mu^{\otimes n}$*-a.e. on* D^n, *then we have that* $\mathbb{P}_D$ *and* L_D^μ *are equivalent, and it follows that* $\det J[D](\mathbf{x}) > 0$, *for* L_D^μ*-a.a.* $\mathbf{x} \in \mathrm{N}_f^D$.

The next Proposition 3.5 is similar to its analog in [5], however the proof given there implicitly uses the fact that $\det J[D](\mathbf{x}) > 0$, for L_D^μ-a.a. $\mathbf{x} \in \mathrm{N}_f^D$, which has been shown to be false in general. In order to prove Proposition 3.5, we assume the following technical condition.

(**H2**) : *The Radon measure* μ *is absolutely continuous* w.r.t. *the Lebesgue measure* ℓ *on* D, *with Radon-Nikodym derivative* $\rho = \frac{\mathrm{d}\mu}{\mathrm{d}\ell}$ *which is strictly positive and continuously differentiable on* D.

Under (**H2**), for any $\phi \in \mathrm{Diff}_0(D)$, μ_ϕ is absolutely continuous with respect to μ with density given by

$$p_\phi^\mu(x) = \frac{\mathrm{d}\mu_\phi(x)}{\mathrm{d}\mu(x)} = \frac{\rho(\phi^{-1}(x))}{\rho(x)} \mathrm{Jac}(\phi^{-1})(x), \tag{3.4}$$

where $\mathrm{Jac}(\phi^{-1})(x)$ is the Jacobian of ϕ^{-1} at a point $x \in S$. We draw attention to the fact that it is indeed $\mathrm{Jac}(\phi^{-1})(x)$ that appears in (3.4), which differs from equation (2.11) of [1]. We are now in a position to state and prove the main result of this section.

Proposition 3.5 *Assume* **(H1)**, **(H2)**, *and let* $\phi \in \mathrm{Diff}_0(D)$, *for any measurable non-negative* f *on* D *we have*

$$\mathbb{E}\left[\exp\left(-\sum_{k=1}^{\mathbf{X}(D)} f\circ\phi(X_k)\right)\right] = \mathbb{E}\left[\exp\left(-\sum_{k=1}^{\mathbf{X}(D)} f(X_k) + \sum_{k=1}^{\mathbf{X}(D)} \ln(p_\phi^\mu(X_k))\right)\frac{\det J^\phi[D](\mathbf{X}^D)}{\det J[D](\mathbf{X}^D)}\right].$$

Proof. For any measurable non-negative f on D, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\sum_{k=1}^{\mathbf{X}(D)} f(X_k) + \sum_{k=1}^{\mathbf{X}(D)} \ln(p_\phi^\mu(X_k))\right)\frac{\det J^\phi[D](\mathbf{X}^D)}{\det J[D](\mathbf{X}^D)}\right]\\
&= \sum_{n\geq 0}\frac{1}{n!}\int_{D^n} \mathrm{e}^{-\sum_{k=1}^n f(x_k)}\prod_{k=1}^n p_\phi^\mu(x_k)\\
&\qquad\qquad \times \frac{\det J^\phi[D](x_1,\dots,x_n)}{\det J[D](x_1,\dots,x_n)}\, j_D\,(x_1,\dots,x_n)\,\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_n)\\
&= \sum_{n\geq 0}\frac{1}{n!}\int_{D^n} \mathrm{e}^{-\sum_{k=1}^n f(x_k)}\prod_{k=1}^n p_\phi^\mu(x_k)\det J^\phi[D](x_1,\dots,x_n)\,\mathrm{Det}(\mathbf{Id}-\mathcal{K}_D)\,\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_n)\\
&= \sum_{n\geq 0}\frac{1}{n!}\int_{D^n} \mathrm{e}^{-\sum_{k=1}^n f(x_k)}\,\mathrm{Det}(\mathbf{Id}-\mathcal{K}_D^\phi)\det J^\phi[D](x_1,\dots,x_n)\,\mu_\phi(\mathrm{d}x_1)\cdots\mu_\phi(\mathrm{d}x_n),
\end{aligned}$$

where we have used (2.7) and (3.4). Then, we conclude by Lemma 3.2. Indeed, $\mathrm{Det}(\mathbf{Id}-\mathcal{K}_D^\phi)\det J^\phi[D](x_1,\dots,x_n)$ is the Janossy density of $\Phi(\mathbf{X}^D)$ with respect to μ_ϕ (see Lemma 3.1, *e*)). □

Integration by parts and closability

We close this section with the statement and proof of the integration by parts formula for determinantal processes which is based on the closed gradient and divergence operators, cf. Theorem 3.8 below. The integration by parts formula is proved on the set of test functionals $\mathcal{S}_D$ introduced in Definition 1, extending and making more precise the argument and proof of Theorem 10 page 289 of [5].

$(\mathbf{H3})$: *The function* $(x_1,\dots,x_n) \longmapsto \det J[D](x_1,\dots,x_n)$ *is continuously differentiable on* D^n.

Assuming that $(\mathbf{H1})$ and $(\mathbf{H3})$ hold, the potential energy is the function $U[D] : \mathrm{N}_f^D \longrightarrow \mathbb{R}$ defined by

$$U[D](\mathbf{x}) := -\log \det J[D](\mathbf{x}).$$

We insist that since $\det J[D](\mathbf{x}) > 0$ for $\mathbb{P}_D$-a.e. $\mathbf{x} \in \mathrm{N}_f^D$, U is well defined for $\mathbb{P}_D$-a.e. $\mathbf{x} \in \mathrm{N}_f^D$. For any $v \in \mathcal{C}^\infty(D,\mathbb{R}^d)$ and $\mathbf{x} = \{x_1,\dots,x_{\mathbf{x}(D)}\} \in \mathrm{N}_f^D$, we set

$$\begin{aligned} \nabla_v^{\mathrm{N}_{lf}} U[D](\mathbf{x}) &:= -\sum_{k=1}^\infty \mathbf{1}_{\{\mathbf{x}(D)=k\}} \sum_{i=1}^k \frac{\nabla_{x_i} \det J[D](x_1,\dots,x_k)}{\det J[D](x_1,\dots,x_k)} \cdot v(x_i) \\ &= \sum_{k=1}^\infty \mathbf{1}_{\{\mathbf{x}(D)=k\}} \sum_{i=1}^k U_{i,k}(x_1,\dots,x_k) \cdot v(x_i). \end{aligned} \tag{3.5}$$

Under Conditions $(\mathbf{H1})$ and $(\mathbf{H2})$ we define the vector field

$$\beta^\mu(x) := \frac{\nabla \rho(x)}{\rho(x)},$$

as well as

$$B_v^\mu(\mathbf{x}) := \sum_{k=1}^{\mathbf{x}(D)} (-\beta^\mu(x_k) \cdot v(x_k) + \operatorname{div} v(x_k)),$$

where div denotes the adjoint of the gradient ∇ on D.

Lemma 3.6 *Assume* $(\mathbf{H1})$, $(\mathbf{H2})$ *and* $(\mathbf{H3})$. *Then, for any* $F, G \in \mathcal{S}_D$ *and* $v \in \mathcal{C}^\infty(D,\mathbb{R}^d)$, *we have*

$$\mathbb{E}[G(\mathbf{X}^D)\nabla_v^{\mathrm{N}_{lf}} F(\mathbf{X}^D)] = \mathbb{E}[F(\mathbf{X}^D)\nabla_v^{\mathrm{N}_{lf}*} G(\mathbf{X}^D)], \tag{3.6}$$

where

$$\nabla_v^{\mathrm{N}_{lf}*} G(\mathbf{x}) := -\nabla_v^{\mathrm{N}_{lf}} G(\mathbf{x}) + G(\mathbf{x}) \left(-B_v^\mu(\mathbf{x}) + \nabla_v^{\mathrm{N}_{lf}} U[D](\mathbf{x}) \right), \qquad \mathbf{x} \in \mathrm{N}_f^D.$$

Proof. For $v \in \mathcal{C}^\infty(D,\mathbb{R}^d)$, consider the flow $\phi_t^v : D \longrightarrow D$, $t \in \mathbb{R}$, where for a fixed $x \in D$, the curve $t \mapsto \phi_t^v(x)$ is defined as the solution to the Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^v(x) = v(\phi_t^v(x)), \quad \phi_0^v(x) = x.$$

We define the mapping $\Phi_t^v : \mathrm{N}_f^D \longrightarrow \mathrm{N}_f^D$ by $\Phi_t^v(\mathbf{x}) := \{\phi_t^v(x) : \ x \in \mathbf{x}\}$. Following [1], for a function $R : \mathrm{N}_f^D \to \mathbb{R}$, we define the gradient $\nabla_v^{\mathrm{N}_{lf}} R(\mathbf{x})$ as the directional derivative along v i.e. for $\mathbf{x} \in \mathrm{N}_f^D$,

$$\nabla_v^{\mathrm{N}_{lf}} R(\mathbf{x}) := \frac{\mathrm{d}}{\mathrm{d}t} R(\Phi_t^v(\mathbf{x}))\Big|_{t=0},$$

provided that the derivative exists. It is easy to check that formulas (3.3) and (3.5) are consistent with this definition. Note that by (3.4), the image measure $\mu_{\phi_t^v}$ is absolutely continuous with respect to μ on D, with Radon-Nikodym derivative

$$\frac{\rho(\phi_{-t}^v(x))}{\rho(x)} \operatorname{Jac}\left(\phi_{-t}^v\right)(x).$$

Note also that

$$\operatorname{Jac}\left(\phi_t^v\right)(x) = \exp\left(-\int_0^t \operatorname{div} v(\phi_z^v(x))\,\mathrm{d}z\right), \tag{3.7}$$

and therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho(\phi_{-t}^v(x))}{\rho(x)} \operatorname{Jac}\left(\phi_{-t}^v\right)(x)\right) = -\exp\left(-\int_0^t \operatorname{div} v(\phi_z^v(x))\,\mathrm{d}z\right)\Big[\frac{\nabla\rho(\phi_{-t}^v(x))}{\rho(x)} \cdot v(\phi_{-t}^v(x))\\ + \frac{\rho(\phi_{-t}^v(x))}{\rho(x)} \operatorname{div} v(\phi_t^v(x))\Big].\end{aligned} \tag{3.8}$$

Using Proposition 3.5, for any $t \in \mathbb{R}$ and $F, G \in \mathcal{S}_D$, we have

$$\begin{aligned}\mathbb{E}[F(\Phi_t^v(\mathbf{X}^D))G(\mathbf{X}^D)] = \mathbb{E}\Bigg[F(\mathbf{X}^D)G(\Phi_{-t}^v(\mathbf{X}^D)) \\ \times \left(\prod_{k=1}^{\mathbf{X}(D)} \frac{\rho(\phi_{-t}^v(X_k))}{\rho(X_k)} \operatorname{Jac}\left(\phi_{-t}^v\right)(X_k)\right) \frac{\det J[D](\phi_{-t}^v(X_1), \ldots, \phi_{-t}^v(X_{\mathbf{X}(D)}))}{\det J[D](X_1, \ldots, X_{\mathbf{X}(D)})}\Bigg].\end{aligned} \tag{3.9}$$

We now differentiate this relation with respect to t, and exchange $\mathrm{d}/\mathrm{d}t$ with $\mathbb{E}$. This exchange will be justified later on after (3.14). Writing, for ease of notation, $\operatorname{Jac}^{\phi_t^v} := \operatorname{Jac}(\phi_t^v)$, we have

$$\mathbb{E}\left[G(\mathbf{X}^D)\frac{\mathrm{d}}{\mathrm{d}t}F(\Phi_t^v(\mathbf{X}^D))\right] \tag{3.10}$$

$$\begin{aligned}= \mathbb{E}\Bigg[\left(\frac{\mathrm{d}}{\mathrm{d}t}G(\Phi_{-t}^v(\mathbf{X}^D))\right) F(\mathbf{X}^D)\left(\prod_{k=1}^{\mathbf{X}(D)} \frac{\rho(\phi_{-t}^v(X_k))}{\rho(X_k)} \operatorname{Jac}^{\phi_{-t}^v}(X_k)\right) \\ \times \frac{\det J[D](\phi_{-t}^v(X_1), \ldots, \phi_{-t}^v(X_{\mathbf{X}(D)}))}{\det J[D](X_1, \ldots, X_{\mathbf{X}(D)})}\Bigg]\end{aligned} \tag{3.11}$$

$$+\mathbb{E}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\prod_{k=1}^{\mathbf{X}(D)}\frac{\rho(\phi^v_{-t}(X_k))}{\rho(X_k)}\,\mathrm{Jac}^{\phi^v_{-t}}(X_k)\right)F(\mathbf{X}^D)G(\Phi^v_{-t}(\mathbf{X}^D))\right.$$
$$\left.\times\frac{\det J[D](\phi^v_{-t}(X_1),\dots,\phi^v_{-t}(X_{\mathbf{X}(D)}))}{\det J[D](X_1,\dots,X_{\mathbf{X}(D)})}\right] \tag{3.12}$$
$$+\mathbb{E}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\det J[D](\phi^v_{-t}(X_1),\dots,\phi^v_{-t}(X_{\mathbf{X}(D)}))}{\det J[D](X_1,\dots,X_{\mathbf{X}(D)})}\right)F(\mathbf{X}^D)G(\Phi^v_{-t}(\mathbf{X}^D))\right.$$
$$\left.\times\prod_{k=1}^{\mathbf{X}(D)}\frac{\rho(\phi^v_{-t}(X_k))}{\rho(X_k)}\,\mathrm{Jac}^{\phi^v_t}(X_k)\right]. \tag{3.13}$$

The claimed integration by parts formula follows by evaluating the above relation at $t=0$. In particular, we use (3.8) to evaluate (3.12), and we use the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\det J[D](\phi^v_{-t}(X_1),\dots,\phi^v_{-t}(X_{\mathbf{X}(D)}))}{\det J[D](X_1,\dots,X_{\mathbf{X}(D)})}$$
$$=-\sum_{i=1}^{\mathbf{X}(D)}\frac{\nabla_{X_i}\det J[D](\phi^v_{-t}(X_1),\dots,\phi^v_{-t}(X_{\mathbf{X}(D)}))}{\det J[D](X_1,\dots,X_{\mathbf{X}(D)})}\cdot v(\phi^v_{-t}(X_i)) \tag{3.14}$$

to evaluate (3.13). Using the definition of $\mathcal{S}_D$, one checks that for $\mathbb{P}_D$-a.e. $\mathbf{x}\in\mathrm{N}^D_f$,

$$t\mapsto G(\mathbf{x})\frac{\mathrm{d}}{\mathrm{d}t}F(\Phi^v_t(\mathbf{x})),$$

is uniformly bounded by a positive constant in an neighborhood of zero. By the assumptions (**H2**) and (**H3**) and the form (3.1) of the functionals in $\mathcal{S}_D$, one may easily check that (3.11), (3.12) and (3.13) can be uniformly bounded in an neighborhood of zero by $\mathbb{P}_D$-integrable functions. This justifies the exchange of derivative and expectation in (3.9). We check this fact only for (3.13). Take

$$F(\mathbf{x})=f_0\mathbf{1}_{\{\mathbf{x}(D)=0\}}+\sum_{k=1}^{n}\mathbf{1}_{\{\mathbf{x}(D)=k\}}f_k(x_1,\dots,x_k),\qquad \mathbf{x}=\{x_1,\dots,x_{\mathbf{x}(D)}\}\in\mathrm{N}^D_f,$$

of the form (3.1). By (3.14) we easily see that, up to a positive constant, the modulus of the r.v.

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\det J[D](\phi^v_{-t}(X_1),\dots,\phi^v_{-t}(X_{\mathbf{X}(D)}))}{\det J[D](X_1,\dots,X_{\mathbf{X}(D)})}\right)F(\mathbf{X}^D)G(\Phi^v_{-t}(\mathbf{X}^D))\prod_{k=1}^{\mathbf{X}(D)}\frac{\rho(\phi^v_{-t}(X_k))}{\rho(X_k)}\,\mathrm{Jac}^{\phi^v_{-t}}(X_k)$$

is bounded above by

$$\sum_{k=1}^{n}\mathbf{1}_{\{\mathbf{X}(D)=k\}}\left(\prod_{i=1}^{k}\frac{\rho(\phi_{-t}^{v}(X_i))}{\rho(X_i)}\,\mathrm{Jac}^{\phi_{-t}^{v}}(X_i)\right)$$
$$\times\sum_{i=1}^{k}\left|\frac{\nabla_{X_i}\det J[D](\phi_{-t}^{v}(X_1),\ldots,\phi_{-t}^{v}(X_k))}{\det J[D](X_1,\ldots,X_k)}\cdot v(\phi_{-t}^{v}(X_i))\right|. \quad (3.15)$$

By assumptions (**H2**), (**H3**) and equation (3.7), it follows that, up to a positive constant, (3.15) is bounded above by

$$\sum_{k=1}^{n}\mathbf{1}_{\{\mathbf{X}(D)=k\}}\frac{\rho(X_1)^{-1}\cdots\rho(X_k)^{-1}}{\det J[D](X_1,\ldots,X_k)},$$

for any t in a neighborhood of zero. To conclude the proof, we only need to check that the mean of this r.v. is finite. We have by definition of the Janossy densities, and since $\det J[D](\mathbf{x})>0$, for $\mathbb{P}_D$-a.e. $\mathbf{x}\in N_f^D$:

$$\mathbb{E}\left[\mathbf{1}_{\{\mathbf{X}(D)=k\}}\frac{\rho(X_1)^{-1}\cdots\rho(X_k)^{-1}}{\det J[D](X_1,\ldots,X_k)}\right]=\frac{1}{k!}\int_{D^k}\frac{j_D^k(x_1,\ldots,x_k)}{\det J[D](x_1,\ldots,x_k)}\mathbf{1}_{\{j_D^k(x_1,\ldots,x_k)>0\}}\,\mathrm{d}x_1\cdots\mathrm{d}x_k$$
$$=\frac{\mathrm{Det}(\mathbf{Id}-\mathcal{K}_D)}{k!}\,\ell(D^k)<\infty,$$

where ℓ denotes the Lebesgue measure. □

Remark 3.7 *We remark that there is a sign change in (3.6), as compared to the results of [5], which is justified by the corrected formula for (3.4). This corrected version is also more in line with the corresponding integration by parts on the Poisson space given in [1].*

Next, we extend the integration by parts formula by closability to a larger class of functions. For $v\in\mathcal{C}^\infty(D,\mathbb{R}^d)$, we consider the closability of the linear operators $\nabla_v^{\mathrm{N}_{lf}}:\mathcal{S}_D\longrightarrow\mathrm{L}_D^2$ and $\nabla_v^{\mathrm{N}_{lf}*}:\mathcal{S}_D\longrightarrow\mathrm{L}_D^2$. In the following, we denote by $\overline{A}$ the minimal closed extension of a closable linear operator A, and by $\mathrm{Dom}(\overline{A})$ the domain of $\overline{A}$.

In Theorem 3.8 below we assume, in addition to (**H1**), (**H2**) and (**H3**), the following condition.

(**H4**): *For any $n\geq 1$, $1\leq i,j\leq n$, and $1\leq h,k\leq d$, we have*

$$\int_{D^n}\left|\frac{\partial_{x_i^{(h)}}\det J[D](x_1,\ldots,x_n)\partial_{x_j^{(k)}}\det J[D](x_1,\ldots,x_n)}{\det J[D](x_1,\ldots,x_n)}\right|$$
$$\mathbf{1}_{\{\det J[D](x_1,\ldots,x_n)>0\}}\,\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_n)<\infty.$$

Theorem 3.8 *Assume* (**H1**) – (**H4**).

(*i*) *For any* $v \in \mathcal{C}^\infty(D, \mathbb{R}^d)$*, the linear operators* $\nabla_v^{\mathrm{N}_{lf}}$ *and* $\nabla_v^{\mathrm{N}_{lf}*}$ *are well-defined, i.e.*

$$\nabla_v^{\mathrm{N}_{lf}}(\mathcal{S}_D) \subset \mathrm{L}^2_D \qquad \text{and} \qquad \nabla_v^{\mathrm{N}_{lf}*}(\mathcal{S}_D) \subset \mathrm{L}^2_D,$$

and closable.

(*ii*) *For any* $v \in \mathcal{C}^\infty(D, \mathbb{R}^d)$*, we have*

$$\mathbb{E}\left[G(\mathbf{X}^D)\overline{\nabla_v^{\mathrm{N}_{lf}}}F(\mathbf{X}^D)\right] = \mathbb{E}\left[F(\mathbf{X}^D)\overline{\nabla_v^{\mathrm{N}_{lf}*}}G(\mathbf{X}^D)\right]$$

for all $F \in \mathrm{Dom}\left(\overline{\nabla_v^{\mathrm{N}_{lf}}}\right)$, $G \in \mathrm{Dom}\left(\overline{\nabla_v^{\mathrm{N}_{lf}*}}\right)$.

Note that under (**H1**), (**H2**) and (**H3**), condition **(H4)** is satisfied if, for any $n \geq 1$, the function

$$(x_1, \ldots, x_n) \longmapsto \det J[D](x_1, \ldots, x_n),$$

is strictly positive on the compact D^n.

Proof of Theorem 3.8. (*i*) Let $v \in \mathcal{C}_c^\infty(D, \mathbb{R}^d)$ and $F \in \mathcal{S}_D$. For ease of notation, throughout this proof we write ∇_v in place of $\nabla_v^{\mathrm{N}_{lf}}$ and ∇_v^* in place of $\nabla_v^{\mathrm{N}_{lf}*}$. We clearly have

$$|\nabla_v F(\mathbf{x})| \leq C$$

for some constant $C > 0$, $\mathbb{P}_D$-a.e., and therefore $\nabla_v(\mathcal{S}_D) \subset \mathrm{L}^2_D$. The claim $\nabla_v^*(\mathcal{S}_D) \subset \mathrm{L}^2_D$ follows if we check that $\|G(\mathbf{x})\nabla_v U[D](\mathbf{x})\|_{\mathrm{L}^2_D} < \infty$ and $\|G(\mathbf{x})B_v^\mu(\mathbf{x})\|_{\mathrm{L}^2_D} < \infty$ for any $G \in \mathcal{S}_D$. The latter relation easily follows noticing that

$$|G(\mathbf{x})B_v^\mu(\mathbf{x})| \leq C$$

for some constant $C > 0$, $\mathbb{P}_D$-a.e.. Taking

$$G(\mathbf{x}) = g_0 \mathbf{1}_{\{\mathbf{x}(D)=0\}} + \sum_{k=1}^m \mathbf{1}_{\{\mathbf{x}(D)=k\}} g_k(x_1, \ldots, x_k)$$

of the form (3.1), by (3.5) we have

$$G(\mathbf{x})\nabla_v U[D](\mathbf{x}) \;=\; -\sum_{k=1}^m \mathbf{1}_{\{\mathbf{x}(D)=k\}} g_k(x_1, \ldots, x_k) \sum_{i=1}^k \frac{\nabla_{x_i}\det J[D](x_1, \ldots, x_k)}{\det J[D](x_1, \ldots, x_k)} \cdot v(x_i),$$

and for some positive constant $C > 0$,

$$
\begin{aligned}
&\|G\nabla_v U[D]\|^2_{\mathrm{L}^2_D} \\
&= \sum_{k=1}^m \frac{1}{k!}\int_{D^k} g_k^2(x_1,\ldots,x_k)\mathbf{1}_{\{\det J[D](x_1,\ldots,x_k)>0\}} \\
&\quad \left(\sum_{i=1}^k \frac{\nabla_{x_i}\det J[D](x_1,\ldots,x_k)}{\det J[D](x_1,\ldots,x_k)}\cdot v(x_i)\right)^2 j_D^k(x_1,\ldots,x_k)\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_k) \\
&= \mathrm{Det}(\mathbf{Id}-\mathcal{K}_D)\sum_{k=1}^m \frac{1}{k!}\int_{D^k}\frac{g_k^2(x_1,\ldots,x_k)}{\det J[D](x_1,\ldots,x_k)} \\
&\quad \mathbf{1}_{\{\det J[D](x_1,\ldots,x_k)>0\}}\left(\sum_{i=1}^k \nabla_{x_i}\det J[D](x_1,\ldots,x_k)\cdot v(x_i)\right)^2 \mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_k) \\
&\le C\,\mathrm{Det}(\mathbf{Id}-\mathcal{K}_D)\sum_{k=1}^m \frac{1}{k!}\sum_{1\le i,j\le k}\int_{D^k}\mathbf{1}_{\{\det J[D](x_1,\ldots,x_k)>0\}} \\
&\quad \frac{\nabla_{x_i}\det J[D](x_1,\ldots,x_k)\cdot v(x_i)\,\nabla_{x_j}\det J[D](x_1,\ldots,x_k)\cdot v(x_j)}{\det J[D](x_1,\ldots,x_k)}\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_k) \\
&< \infty,
\end{aligned}
$$

where the latter integral is finite by **(H4)**.

To conclude, we only need to show that ∇_v is closable (the closability of ∇_v^* can be proved similarly). Let $(F_n)_{n\ge1}$ be a sequence in $\mathcal{S}_D$ converging to 0 in L^2_D and such that $\nabla_v F_n$ converges to V in L^2_D as n goes to infinity. We need to show that $V = 0$ $\mathbb{P}_D$-a.e.. We have

$$
\begin{aligned}
\left|\langle G, V\rangle_{\mathrm{L}^2_D}\right| &= \lim_{n\to\infty}|\mathbb{E}[G(\mathbf{X}^D)\nabla_v F_n(\mathbf{X}^D)]| = \lim_{n\to\infty}|\mathbb{E}[F_n(\mathbf{X}^D)\nabla_v^* G(\mathbf{X}^D]| \\
&\le \|\nabla_v^* G\|_{\mathrm{L}^2_D}\lim_{n\to\infty}\|F_n\|_{\mathrm{L}^2_D} = 0, \qquad G\in\mathcal{S}_D.
\end{aligned} \tag{3.16}
$$

Here, the second inequality in (3.16) follows by the integration by parts formula (3.6). The conclusion follows noticing that $\langle G, V\rangle_{\mathrm{L}^2_D} = 0$ for all $G\in\mathcal{S}_D$ implies $V = 0$ $\mathbb{P}_D$-a.e. due to the density of $\mathcal{S}_D$ in L^2_D.

(ii) By *(i)* both operators ∇_v and ∇_v^* are closable. Take $F\in\mathrm{Dom}(\overline{\nabla}_v)$, $G\in\mathrm{Dom}(\overline{\nabla_v^*})$ and let $(F_n)_{n\ge1}$, $(G_n)_{n\ge1}$ be sequences in $\mathcal{S}_D$ such that F_n converges to F, G_n converges to G, $\nabla_v F_n$ converges to $\overline{\nabla_v}F$ and $\nabla_v^* G_n$ converges to $\overline{\nabla_v^*}G$ in L^2_D as n goes to infinity. By Lemma 3.6 the integration by parts formula applies to r.v.'s in $\mathcal{S}_D$, therefore we have $\mathbb{E}[G_n(\mathbf{X}^D)\nabla_v F_n(\mathbf{X}^D)] = \mathbb{E}[F_n(\mathbf{X}^D)\nabla_v^* G_n(\mathbf{X}^D)]$ for all $n\ge1$. The claim follows if we prove

that

$$\lim_{n\to\infty} \mathbb{E}[G_n(\mathbf{X}^D)\nabla_v F_n(\mathbf{X}^D)] = \mathbb{E}[G(\mathbf{X}^D)\overline{\nabla_v} F(\mathbf{X}^D)]$$

and

$$\lim_{n\to\infty} \mathbb{E}[F_n(\mathbf{X}^D)\nabla_v^* G_n(\mathbf{X}^D)] = \mathbb{E}[F(\mathbf{X}^D)\overline{\nabla_v^*} G(\mathbf{X}^D)].$$

We only show the first limit above; the second limit being proved similarly. We have

$$\begin{aligned}
&|\mathbb{E}[G_n(\mathbf{X}^D)\nabla_v F_n(\mathbf{X}^D)] - \mathbb{E}[G(\mathbf{X}^D)\overline{\nabla_v} F(\mathbf{X}^D)]| \\
&= |\mathbb{E}[G_n(\mathbf{X}^D)\nabla_v F_n(\mathbf{X}^D)] - \mathbb{E}[G_n(\mathbf{X}^D)\overline{\nabla_v} F(\mathbf{X}^D)] + \mathbb{E}[G_n(\mathbf{X}^D)\overline{\nabla_v} F(\mathbf{X}^D)] \\
&\qquad - \mathbb{E}[G(\mathbf{X}^D)\overline{\nabla_v} F(\mathbf{X}^D)]| \\
&\le |\mathbb{E}[G_n(\mathbf{X}^D)(\nabla_v F_n(\mathbf{X}^D) - \overline{\nabla_v} F(\mathbf{X}^D))]| + |\mathbb{E}[(G_n(\mathbf{X}^D) - G(\mathbf{X}^D))\overline{\nabla_v} F(\mathbf{X}^D)]| \\
&\le \|G_n\|_{\mathrm{L}^2_D} \|\nabla_v F_n - \overline{\nabla_v} F\|_{\mathrm{L}^2_D} + \|G_n - G\|_{\mathrm{L}^2_D} \|\overline{\nabla_v} F\|_{\mathrm{L}^2_D},
\end{aligned}$$

which tends to 0 as n goes to infnity. □

4 Dirichlet forms corresponding to determinantal processes on $D \subset S$

In this section we construct the Dirichlet form associated to a determinantal process, cf. Theorem 4.1 below. We start by recalling some definitions related to bilinear forms (see [16] for details). Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \times \mathrm{Dom}(\mathcal{A}) \longrightarrow \mathbb{R}$ a bilinear form defined on a dense subspace $\mathrm{Dom}(\mathcal{A})$ of H, the domain of $\mathcal{A}$. The form $\mathcal{A}$ is said to be symmetric if $\mathcal{A}(F, G) = \mathcal{A}(G, F)$, for any $F, G \in \mathrm{Dom}(\mathcal{A})$, and non-negative definite if $\mathcal{A}(F, F) \ge 0$, for any $F \in \mathrm{Dom}(\mathcal{A})$. Let $\mathcal{A}$ be symmetric and non-negative definite, $\mathcal{A}$ is said to be closed if $\mathrm{Dom}(\mathcal{A})$ equipped with the norm

$$\|F\|_{\mathcal{A}} := \sqrt{\mathcal{A}(F, F) + \langle F, F \rangle}, \quad F \in \mathrm{Dom}(\mathcal{A}),$$

is a Hilbert space. A symmetric and non-negative definite bilinear form $\mathcal{A}$ is said to be closable if, for any sequence $(F_n)_{n\ge1} \subset \mathrm{Dom}(\mathcal{A})$ such that F_n goes to 0 in H and $(F_n)_{n\ge1}$ is Cauchy *w.r.t.* $\|\cdot\|_{\mathcal{A}}$ it holds that $\mathcal{A}(F_n, F_n)$ converges to 0 in $\mathbb{R}$ as n goes to infinity. Let $\mathcal{A}$ be closable and denote by $\mathrm{Dom}(\overline{\mathcal{A}})$ the completion of $\mathrm{Dom}(\mathcal{A})$ *w.r.t.* the norm $\|\cdot\|_{\mathcal{A}}$. It turns out that $\mathcal{A}$ is uniquely extended to $\mathrm{Dom}(\overline{\mathcal{A}})$ by the closed, symmetric and non-negative definite bilinear form

$$\overline{\mathcal{A}}(F, G) = \lim_{n\to\infty} \mathcal{A}(F_n, G_n), \quad (F, G) \in \mathrm{Dom}(\overline{\mathcal{A}}) \times \mathrm{Dom}(\overline{\mathcal{A}}),$$

where $\{(F_n, G_n)\}_{n\geq 1}$ is any sequence in $\mathrm{Dom}(\mathcal{A}) \times \mathrm{Dom}(\mathcal{A})$ such that (F_n, G_n) converges to $(F, G) \in \mathrm{Dom}(\overline{\mathcal{A}}) \times \mathrm{Dom}(\overline{\mathcal{A}})$ *w.r.t.* the norm $\|\cdot\|_{\overline{\mathcal{A}}} + \|\cdot\|_{\overline{\mathcal{A}}}$. Suppose $H = \mathrm{L}^2(B, \beta)$ where $(B, \mathcal{F}, \beta)$ is a σ-finite measure space. A symmetric, non-negative definite and closed bilinear form $\mathcal{A}$ is said to be a Dirichlet form if

$$\mathcal{A}(\min\{F^+, 1\}, \min\{F^+, 1\}) \leq \mathcal{A}(F, F), \qquad F \in \mathrm{Dom}(\mathcal{A}),$$

where F^+ denotes the positive part of F. Suppose that B is a Hausdorff topological space, $\mathcal{F} = \mathcal{B}(B)$ the corresponding Borel σ-algebra, and let $\mathcal{A}$ be a Dirichlet form. Throughout this paper, we shall use the following notions. The generator $\mathcal{H}^{\mathrm{gen}}$ of the Dirichlet form is the unique symmetric non-negative definite operator on H such that

$$\mathcal{A}(F, G) = \langle (-\mathcal{H}^{\mathrm{gen}} F), G \rangle,$$

for $F, G \in \mathrm{Dom}(-\mathcal{H}^{\mathrm{gen}}) \subset \mathrm{Dom}(\mathcal{A})$. The symmetric semi-group of $\mathcal{A}$ is the linear operator on H defined by $T_t F := \exp(-t\mathcal{H}^{\mathrm{gen}})F$, $t > 0$. An $\mathcal{A}$-nest is an increasing sequence $(C_n)_{n\geq 1}$ of closed subsets of B such that

$$\bigcup_{n\geq 1} \{F \in \mathrm{Dom}(\mathcal{A}) \colon \ F = 0 \ \beta\text{-a.e. on } B \setminus C_n\}$$

is dense in $\mathrm{Dom}(\mathcal{A})$ *w.r.t.* the norm $\|\cdot\|_{\mathcal{A}}$. Throughout this paper, we say that a subset $B' \subset B$ is $\mathcal{A}$-exceptional if there exists an $\mathcal{A}$-nest $(C_n)_{n\geq 1}$ with $B' \subset B \setminus \bigcup_{n\geq 1} C_n$. We say that a property holds $\mathcal{A}$-almost everywhere ($\mathcal{A}$-a.e.) if it holds up to an $\mathcal{A}$-exceptional set. We say that a function $f : B \to \mathbb{R}$ is $\mathcal{A}$-almost continuous ($\mathcal{A}$-a.c.) if there exists an $\mathcal{A}$-nest $(C_n)_{n\geq 1}$ such that the restriction $f_{|C_n}$ of f to C_n is continuous for each $n \geq 1$. We say that a Dirichlet form $\mathcal{A}$ is quasi-regular if:
(i) There exists an $\mathcal{A}$-nest $(C_n)_{n\geq 1}$ consisting of compact sets.
(ii) There exists a $\|\cdot\|_{\mathcal{A}}$-dense subset of $\mathrm{Dom}(\mathcal{A})$ whose elements have $\mathcal{A}$-a.c. β-versions.
(iii) There exist $F_k \in \mathrm{Dom}(\mathcal{A})$, $k \geq 1$, having $\mathcal{A}$-a.c. β-versions $\tilde{F}_k$, $k \geq 1$, such that $(\tilde{F}_k)_{k\geq 1}$ is a separating set for $B \setminus N$ (i.e. for any $x, y \in B \setminus N$, $x \neq y$, there exists $k^* \geq 1$ such that $\tilde{F}_{k^*}(x) \neq \tilde{F}_{k^*}(y)$), where N is a subset of B which is $\mathcal{A}$-exceptional.

After these general considerations, we move to the situation at hand. Assume (**H1**), (**H2**) and (**H3**). In particular, we recall that D is a fixed compact set of S which is in turn included in $\mathbb{R}^d$. We denote by $\ddot{\mathrm{N}}_f^D$ the set of $\mathbb{N}$-valued Radon measures on D. We equip $\ddot{\mathrm{N}}_f^D$ with the vague topology and the corresponding Borel σ-algebra $\mathcal{B}(\ddot{\mathrm{N}}_f^D)$. Note that N_f^D is contained in $\ddot{\mathrm{N}}_f^D$. In the following, we consider the subspace $\tilde{\mathcal{S}}_D$ of $\mathcal{S}_D$ made of cylindrical functions.

Definition 2 *A function* $F : \mathrm{N}_f^D \to \mathbb{R}$ *is said to be in* $\tilde{\mathcal{S}}_D$ *if it is of the form*

$$F(\mathbf{x}) = f\left(\sum_{k=1}^{\mathbf{x}(D)} \varphi_1(x_k), \dots, \sum_{k=1}^{\mathbf{x}(D)} \varphi_m(x_k)\right) \mathbf{1}_{\{\mathbf{x}(D)\leq n\}}, \qquad \mathbf{x} = \{x_1, \dots, x_{\mathbf{x}(D)}\} \in \mathrm{N}_f^D,$$

for some integers $m, n \geq 1$, $\varphi_1, \dots, \varphi_m \in \mathcal{C}^\infty(D)$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^m)$.

Note that, as already mentioned, $\tilde{\mathcal{S}}_D$ is dense in L_D^2. We consider the bilinear map $\mathcal{E}_D$ defined on $\tilde{\mathcal{S}}_D \times \tilde{\mathcal{S}}_D$ by

$$\mathcal{E}_D(F, G) := \mathbb{E}\left[\sum_{i=1}^{\mathbf{X}(D)} \nabla_{X_i}^{\mathrm{N}_{lf}} F(\mathbf{X}^D) \cdot \nabla_{X_i}^{\mathrm{N}_{lf}} G(\mathbf{X}^D)\right].$$

For $F \in \mathcal{S}_D$ of the form (3.1), i.e.

$$F(\mathbf{x}) = f_0 \mathbf{1}_{\{\mathbf{x}(D)=0\}} + \sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{x}(D)=k\}} f_k(x_1, \dots, x_k), \qquad \mathbf{x} = \{x_1, \dots, x_{\mathbf{x}(D)}\} \in \mathrm{N}_f^D$$

we also define the Laplace operator $\mathcal{H}_D$ by

$$\mathcal{H}_D F(\mathbf{x}) = \sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{x}(D)=k\}}$$

$$\sum_{i=1}^{k} \left(-\beta^\mu(x_i) \cdot \nabla_{x_i} f_k(x_1, \dots, x_k) - \Delta_{x_i} f_k(x_1, \dots, x_k) + U_{i,k}(x_1, \dots, x_k) \cdot \nabla_{x_i} f_k(x_1, \dots, x_k)\right),$$

where $\Delta = -\mathrm{div}\nabla$ denotes the Laplacian. The next theorem provides the Dirichlet form associated to a determinantal process.

Theorem 4.1 *Under conditions* (**H1**) – (**H4**) *we have:*
(i) The Laplace operator $\mathcal{H}_D : \tilde{\mathcal{S}}_D \longrightarrow \mathrm{L}_D^2$ *is linear, symmetric, non-negative definite and well-defined, i.e.* $\mathcal{H}_D(\tilde{\mathcal{S}}_D) \subset \mathrm{L}_D^2$. *In particular the operator square root* $\mathcal{H}_D^{1/2}$ *of* $\mathcal{H}_D$ *exists.*
(ii) The bilinear form $\mathcal{E}_D : \tilde{\mathcal{S}}_D \times \tilde{\mathcal{S}}_D \longrightarrow \mathbb{R}$ *is symmetric, non-negative definite and well-defined, i.e.* $\mathcal{E}_D(\tilde{\mathcal{S}}_D \times \tilde{\mathcal{S}}_D) \subset \mathbb{R}$.
(iii) $\mathcal{H}_D^{1/2}$ *and* $\mathcal{E}_D$ *are closable and the following relation holds:*

$$\overline{\mathcal{E}_D}(F, G) = \mathbb{E}\left[\overline{\mathcal{H}_D^{1/2}} F(\mathbf{X}^D) \overline{\mathcal{H}_D^{1/2}} G(\mathbf{X}^D)\right], \qquad \forall\ F, G \in \mathrm{Dom}(\overline{\mathcal{H}_D^{1/2}}). \tag{4.1}$$

(iv) The bilinear form $(\overline{\mathcal{E}_D}, \mathrm{Dom}(\overline{\mathcal{H}_D^{1/2}}))$ *is a Dirichlet form.*

The proof of the theorem is based on the following lemma.

Lemma 4.2 *Under conditions* (**H1**) – (**H4**), *for any* $F, G \in \tilde{\mathcal{S}}_D$, *we have*

$$\mathbb{E}\left[\sum_{i=1}^{\mathbf{X}(D)} \nabla_{X_i}^{\mathrm{N}_{lf}} F(\mathbf{X}^D) \cdot \nabla_{X_i}^{\mathrm{N}_{lf}} G(\mathbf{X}^D)\right] = \mathbb{E}[G(\mathbf{X}^D)\mathcal{H}_D F(\mathbf{X}^D)] \tag{4.2}$$

$$= \mathbb{E}[\mathcal{H}_D^{1/2} F(\mathbf{X}^D)\mathcal{H}_D^{1/2} G(\mathbf{X}^D)]. \tag{4.3}$$

Proof of Theorem 4.1.
(i) By Relation (4.2) in Lemma 4.2 we easily deduce that, for any $F, G \in \tilde{\mathcal{S}}_D$ we have

$$\mathbb{E}[G(\mathbf{X}^D)\mathcal{H}_D F(\mathbf{X}^D)] = \mathbb{E}[F(\mathbf{X}^D)\mathcal{H}_D G(\mathbf{X}^D)] \quad \text{and} \quad \mathbb{E}[F(\mathbf{X}^D)\mathcal{H}_D F(\mathbf{X}^D)] \geq 0.$$

Therefore, $\mathcal{H}_D$ is symmetric and non-negative definite. It remains to check that, under the foregoing assumptions, $\mathcal{H}_D$ is well-defined. Let $F \in \tilde{\mathcal{S}}_D$ be of the form

$$\begin{aligned} F(\mathbf{x}) &= \sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{x}(D)=k\}} f\left(\sum_{i=1}^{k} \varphi_1(x_i), \dots, \sum_{i=1}^{k} \varphi_m(x_i)\right) \\ &= \sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{x}(D)=k\}} f_k(x_1, \dots, x_k) \end{aligned}$$

for some integers $m, n \geq 1$, $\varphi_1, \dots, \varphi_m \in \mathcal{C}^\infty(D)$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^m)$. For the well-definiteness of $\mathcal{H}_D$ we only need to check that, for $\mathbf{x} = \{x_1, \dots, x_{\mathbf{x}(D)}\} \in \mathrm{N}_f^D$,

$$F_1(\mathbf{x}) := \mathbf{1}_{\{\mathbf{x}(D)=k\}} \left(\beta^\mu(x_i)\right)^{(j)}, \text{ and } F_2(\mathbf{x}) := \mathbf{1}_{\{\mathbf{x}(D)=k\}} \left(U_{i,k}(x_1, \dots, x_k)\right)^{(j)},$$

are in L_D^2 for all $k \in \{1, \dots, n\}$, $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, d\}$. One may easily check that $F_1 \in \mathrm{L}_D^2$ due to (**H2**) and $F_2 \in \mathrm{L}_D^2$ due to (**H4**).

(ii) The symmetry and non-negative definiteness of $\mathcal{E}_D$ follow from Lemma 4.2. It remains to check that, under the foregoing assumptions, $\mathcal{E}_D$ is well-defined. By Step (i), for any $F \in \tilde{\mathcal{S}}_D$, we have $\mathcal{H}_D F \in \mathrm{L}_D^2$. We conclude the proof by noting that, by Lemma 4.2, for any $F, G \in \tilde{\mathcal{S}}_D$ and some positive constant $c > 0$, we have

$$|\mathcal{E}_D(F, G)| = |\mathbb{E}[G(\mathbf{X}^D)\mathcal{H}_D F(\mathbf{X}^D)]| \leq c\|\mathcal{H}_D F(\mathbf{X}^D)\|_{\mathrm{L}_D^2} < \infty.$$

(iii) We first show that $\mathcal{E}_D$ is closable. By Lemma 3.4 page 29 in [16], we have to check that if $(F_n)_{n\geq 1} \subset \tilde{\mathcal{S}}_D$ is such that F_n converges to 0 in L_D^2, then $\mathcal{E}_D(G, F_n)$ converges to 0, for any $G \in \tilde{\mathcal{S}}_D$. This easily follows by Lemma 4.2, the Cauchy-Schwarz inequality and the fact that $\mathcal{H}_D G$ is square integrable (see Step (i)). The closability of $\mathcal{H}_D^{1/2}$ follows by the closability of $\mathcal{E}_D$, relation (4.3) in Lemma 4.2 and Remark 3.2 (i) page 29 in [16]. Finally, we prove relation (4.1). Take $F, G \in \mathrm{Dom}(\overline{\mathcal{H}_D^{1/2}})$ and let $(F_n)_{n\geq 1}$, $(G_n)_{n\geq 1}$ be sequences in $\tilde{\mathcal{S}}_D$ such that F_n converges to F, G_n converges to G, $\mathcal{H}_D^{1/2}F_n$ converges to $\overline{\mathcal{H}_D^{1/2}}\,F$, and $\mathcal{H}_D^{1/2}G_n$ converges to $\overline{\mathcal{H}_D^{1/2}}\,G$ in L_D^2 as n goes to infinity. By Lemma 4.2 we have

$$\mathcal{E}_D(F_n, G_n) = \mathbb{E}[\mathcal{H}_D^{1/2}F_n(\mathbf{X}^D)\mathcal{H}_D^{1/2}G_n(\mathbf{X}^D)], \quad \text{for all } n \geq 1.$$

The claim follows if we prove that

$$\lim_{n\to\infty} \mathbb{E}[\mathcal{H}_D^{1/2}F_n(\mathbf{X}^D)\mathcal{H}_D^{1/2}G_n(\mathbf{X}^D)] = \mathbb{E}[\overline{\mathcal{H}_D^{1/2}}\,F(\mathbf{X}^D)\overline{\mathcal{H}_D^{1/2}}\,G(\mathbf{X}^D)].$$

Indeed we have:

$$\begin{aligned}
&|\mathbb{E}[\mathcal{H}_D^{1/2}G_n(\mathbf{X}^D)\mathcal{H}_D^{1/2}F_n(\mathbf{X}^D)] - \mathbb{E}[\overline{\mathcal{H}_D^{1/2}}G(\mathbf{X}^D)\,\overline{\mathcal{H}_D^{1/2}}\,F(\mathbf{X}^D)]| \\
&= |\mathbb{E}[\mathcal{H}_D^{1/2}G_n(\mathbf{X}^D)\mathcal{H}_D^{1/2}F_n(\mathbf{X}^D)] - \mathbb{E}[\mathcal{H}_D^{1/2}G_n(\mathbf{X}^D)\,\overline{\mathcal{H}_D^{1/2}}\,F(\mathbf{X}^D)] \\
&\qquad\qquad + \mathbb{E}[\mathcal{H}_D^{1/2}G_n(\mathbf{X}^D)\,\overline{\mathcal{H}_D^{1/2}}\,F(\mathbf{X}^D)] - \mathbb{E}[\overline{\mathcal{H}_D^{1/2}}G(\mathbf{X}^D)\,\overline{\mathcal{H}_D^{1/2}}\,F(\mathbf{X}^D)]| \\
&\leq |\mathbb{E}[\mathcal{H}_D^{1/2}G_n(\mathbf{X}^D)(\mathcal{H}_D^{1/2}F_n(\mathbf{X}^D) - \overline{\mathcal{H}_D^{1/2}}\,F(\mathbf{X}^D))]| \\
&\qquad\qquad + |\mathbb{E}[(\mathcal{H}_D^{1/2}G_n(\mathbf{X}^D) - \overline{\mathcal{H}_D^{1/2}}G(\mathbf{X}^D))\overline{\mathcal{H}_D^{1/2}}\,F(\mathbf{X}^D)]| \\
&\leq \|\mathcal{H}_D^{1/2}G_n\|_{\mathrm{L}_D^2}\|\mathcal{H}_D^{1/2}F_n - \overline{\mathcal{H}_D^{1/2}}\,F\|_{\mathrm{L}_D^2} \\
&\qquad\qquad + \|\mathcal{H}_D^{1/2}G_n - \overline{\mathcal{H}_D^{1/2}}G\|_{\mathrm{L}_D^2}\|\overline{\mathcal{H}_D^{1/2}}\,F\|_{\mathrm{L}_D^2} \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

(iv) The bilinear form $(\overline{\mathcal{E}_D}, \mathrm{Dom}(\overline{\mathcal{H}_D^{1/2}}))$ defined by (4.1) is clearly symmetric, non-negative definite, and closed. We conclude the proof by applying Proposition 4.10 page 35 in [16]. First, note that $\tilde{\mathcal{S}}_D$ is dense in $\mathrm{Dom}(\overline{\mathcal{H}_D^{1/2}})$ ($w.r.t.$ the norm $\|\cdot\|_{\overline{\mathcal{E}}_D}$). By Exercise 2.7 page 47 in [16], for any $\varepsilon > 0$ there exists an infinitely differentiable function $\varphi_\varepsilon : \mathbb{R} \longrightarrow [-\varepsilon, 1+\varepsilon]$ (which shall not be confused with the functions $\varphi_1, \ldots, \varphi_m$ involved in Definition 2) such that $\varphi_\varepsilon(t) = t$ for any $t \in [0,1]$, $0 \leq \varphi_\varepsilon(t) - \varphi_\varepsilon(s) \leq t - s$ for all $t, s \in \mathbb{R}$, $t \geq s$, $\varphi_\varepsilon(t) = 1+\varepsilon$ for $t \in [1+2\varepsilon, \infty)$ and $\varphi_\varepsilon(t) = -\varepsilon$ for $t \in (-\infty, -2\varepsilon]$. Note that $|\varphi_\varepsilon'(t)|^2 \leq 1$ for any $\varepsilon > 0$, $t \in \mathbb{R}$ and φ_ε is in $\mathcal{C}_b^\infty$, for any $\varepsilon > 0$. Consider the function

$$F(\mathbf{x}) = f\left(\sum_{k=1}^{\mathbf{x}(D)} \varphi_1(x_k), \ldots, \sum_{k=1}^{\mathbf{x}(D)} \varphi_m(x_k)\right)\mathbf{1}_{\{\mathbf{x}(D)\leq n\}}, \qquad \mathbf{x} = \{x_1, \ldots, x_{\mathbf{x}(D)}\} \in \mathrm{N}_f^D,$$

for some integers $m, n \geq 1$, $\varphi_1, \ldots, \varphi_m \in \mathcal{C}^\infty(D)$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^m)$. Note that $\varphi_\varepsilon \circ F \in \tilde{\mathcal{S}}_D$. Indeed we have

$$\varphi_\varepsilon \circ F(\mathbf{x}) = \varphi_\varepsilon\left(f\left(\sum_{k=1}^{\mathbf{x}(D)} \varphi_1(x_k), \ldots, \sum_{k=1}^{\mathbf{x}(D)} \varphi_m(x_k)\right)\mathbf{1}_{\{\mathbf{x}(D)\leq n\}}\right)$$

$$= \varphi_\varepsilon \left(f \left(\sum_{k=1}^{\mathbf{x}(D)} \varphi_1(x_k), \dots, \sum_{k=1}^{\mathbf{x}(D)} \varphi_m(x_k) \right) \right) \mathbf{1}_{\{\mathbf{x}(D) \le n\}},$$

because $\varphi_\varepsilon(0) = 0$. Next, we have

$$\begin{aligned}
\overline{\mathcal{E}_D}(\varphi_\varepsilon \circ F, \varphi_\varepsilon \circ F) &= \mathbb{E}\left[\sum_{i=1}^{\mathbf{X}(D)} \nabla_{X_i}^{\mathrm{N}_{lf}} \varphi_\varepsilon \circ F(\mathbf{X}^D) \cdot \nabla_{X_i}^{\mathrm{N}_{lf}} \varphi_\varepsilon \circ F(\mathbf{X}^D) \right] \\
&= \mathbb{E}\left[\sum_{i=1}^{\mathbf{X}(D)} \|\nabla_{X_i}^{\mathrm{N}_{lf}} \varphi_\varepsilon \circ F(\mathbf{X}^D)\|^2 \right] \\
&= \mathbb{E}\left[\sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{i=1}^{k} \|\nabla_{X_i}^{\mathrm{N}_{lf}} \varphi_\varepsilon \circ F(\mathbf{X}^D)\|^2 \right] \\
&= \mathbb{E}\Bigg[\sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{i=1}^{k} \Bigg\| \sum_{j=1}^{m} \varphi_\varepsilon' \circ f \left(\sum_{l=1}^{k} \varphi_1(X_l), \dots, \sum_{l=1}^{k} \varphi_m(X_l) \right) \\
&\qquad \times \partial_j f \left(\sum_{l=1}^{k} \varphi_1(X_l), \dots, \sum_{l=1}^{k} \varphi_m(X_l) \right) \nabla \varphi_j(X_i) \Bigg\|^2 \Bigg] \\
&\le \mathbb{E}\left[\sum_{k=1}^{n} \mathbf{1}_{\{\mathbf{X}(D)=k\}} \sum_{i=1}^{k} \left\| \sum_{j=1}^{m} \partial_j f \left(\sum_{l=1}^{k} \varphi_1(X_l), \dots, \sum_{l=1}^{k} \varphi_m(X_l) \right) \nabla \varphi_j(X_i) \right\|^2 \right] \qquad (4.4) \\
&= \overline{\mathcal{E}_D}(F, F),
\end{aligned}$$

where in (4.4) we used the fact that $|\varphi_\varepsilon'(t)|^2 \le 1$, $t \in \mathbb{R}$. By this inequality we have, for any $F \in \tilde{\mathcal{S}}_D$,

$$\liminf_{\varepsilon \to 0} \overline{\mathcal{E}_D}(F \pm \varphi_\varepsilon \circ F, F \mp \varphi_\varepsilon \circ F) \ge 0$$

and the proof is completed (since, as required by Proposition 4.10 page 35 in [16], we checked condition (4.6) page 34 in [16]). Indeed, for any $\varepsilon > 0$, by the above inequality and Lemma 4.2, we have

$$\begin{aligned}
&\overline{\mathcal{E}_D}(F + \varphi_\varepsilon \circ F, F - \varphi_\varepsilon \circ F) \\
&= \overline{\mathcal{E}_D}(F - \varphi_\varepsilon \circ F, F + \varphi_\varepsilon \circ F) \\
&= \mathbb{E}[(F(\mathbf{X}^D) - \varphi_\varepsilon \circ F(\mathbf{X}^D))\mathcal{H}_D(F(\mathbf{X}^D) + \varphi_\varepsilon \circ F(\mathbf{X}^D))] \\
&= \mathbb{E}[F(\mathbf{X}^D)\mathcal{H}_D F(\mathbf{X}^D) + F(\mathbf{X}^D)\mathcal{H}_D \varphi_\varepsilon \circ F(\mathbf{X}^D) \\
&\qquad - \varphi_\varepsilon \circ F(\mathbf{X}^D)\mathcal{H}_D F(\mathbf{X}^D) - \varphi_\varepsilon \circ F(\mathbf{X}^D)\mathcal{H}_D \varphi_\varepsilon \circ F(\mathbf{X}^D)] \\
&\ge \mathbb{E}[F(\mathbf{X}^D)\mathcal{H}_D \varphi_\varepsilon \circ F(\mathbf{X}^D) - \varphi_\varepsilon \circ F(\mathbf{X}^D)\mathcal{H}_D F(\mathbf{X}^D)] \\
&= 0.
\end{aligned}$$

$\square$

Proof of Lemma 4.2. Let $F, G \in \tilde{\mathcal{S}}_D$ be, respectively, defined for $\mathbf{x} = \{x_1, \dots, x_{\mathbf{x}(D)}\} \in \mathrm{N}_f^D$ as

$$F(\mathbf{x}) = f\left(\sum_{k=1}^{\mathbf{x}(D)} \varphi_1(x_k), \dots, \sum_{k=1}^{\mathbf{x}(D)} \varphi_{m_1}(x_k)\right) \mathbf{1}_{\{\mathbf{x}(D) \leq n_1\}},$$

$$G(\mathbf{x}) = g\left(\sum_{k=1}^{\mathbf{x}(D)} \gamma_1(x_k), \dots, \sum_{k=1}^{\mathbf{x}(D)} \gamma_{m_2}(x_k)\right) \mathbf{1}_{\{\mathbf{x}(D) \leq n_2\}},$$

for some integers $m_1, m_2, n_1, n_2 \geq 1$, $\varphi_1, \dots, \varphi_{m_1}, \gamma_1, \dots, \gamma_{m_2} \in \mathcal{C}^\infty(D)$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^{m_1})$, $g \in \mathcal{C}_b^\infty(\mathbb{R}^{m_2})$. Define

$$F_i(\mathbf{x}) = \partial_i f\left(\sum_{k=1}^{\mathbf{x}(D)} \varphi_1(x_k), \dots, \sum_{k=1}^{\mathbf{x}(D)} \varphi_{m_1}(x_k)\right) \mathbf{1}_{\{\mathbf{x}(D) \leq n_1\}},$$

and

$$v_i(x) = \nabla \varphi_i(x), \quad x \in D.$$

By direct computation we find

$$\begin{aligned}
\mathcal{H}_D F(\mathbf{x}) = & -\mathbf{1}_{\{\mathbf{x}(D) \leq n_1\}} \sum_{i=1}^{m_1} \sum_{k=1}^{\mathbf{x}(D)} \beta^\mu(x_k) \cdot v_i(x_k) \partial_i f\left(\sum_{k=1}^{\mathbf{x}(D)} \varphi_1(x_k), \dots, \sum_{k=1}^{\mathbf{x}(D)} \varphi_{m_1}(x_k)\right) \\
& -\mathbf{1}_{\{\mathbf{x}(D) \leq n_1\}} \sum_{i,j=1}^{m_1} \sum_{k=1}^{\mathbf{x}(D)} v_i(x_k) \sum_{l=1}^{\mathbf{x}(D)} v_j(x_l) \partial_i \partial_j f\left(\sum_{k=1}^{\mathbf{x}(D)} \varphi_1(x_k), \dots, \sum_{k=1}^{\mathbf{x}(D)} \varphi_{m_1}(x_k)\right) \\
& +\mathbf{1}_{\{\mathbf{x}(D) \leq n_1\}} \sum_{i=1}^{m_1} \sum_{k=1}^{\mathbf{x}(D)} \operatorname{div} v_i(x_k) \partial_i f\left(\sum_{k=1}^{\mathbf{x}(D)} \varphi_1(x_k), \dots, \sum_{k=1}^{\mathbf{x}(D)} \varphi_{m_1}(x_k)\right) \\
& +\mathbf{1}_{\{\mathbf{x}(D) \leq n_1\}} \sum_{i=1}^{m_1} \sum_{k=1}^{\mathbf{x}(D)} U_{k,\mathbf{x}(D)}(x_1, \dots, x_{\mathbf{x}(D)}) \cdot v_i(x_k) \partial_i f\left(\sum_{k=1}^{\mathbf{x}(D)} \varphi_1(x_k), \dots, \sum_{k=1}^{\mathbf{x}(D)} \varphi_{m_1}(x_k)\right) \\
= & -\sum_{i=1}^{m_1} F_i(\mathbf{x}) \sum_{k=1}^{\mathbf{x}(D)} \beta^\mu(x_k) \cdot v_i(x_k) \\
& -\mathbf{1}_{\{\mathbf{x}(D) \leq n_1\}} \sum_{i,j=1}^{m_1} \sum_{k=1}^{\mathbf{x}(D)} v_i(x_k) \sum_{l=1}^{\mathbf{x}(D)} v_j(x_l) \partial_i \partial_j f\left(\sum_{k=1}^{\mathbf{x}(D)} \varphi_1(x_k), \dots, \sum_{k=1}^{\mathbf{x}(D)} \varphi_{m_1}(x_k)\right) \\
& +\sum_{i=1}^{m_1} F_i(\mathbf{x}) \sum_{k=1}^{\mathbf{x}(D)} \operatorname{div} v_i(x_k)
\end{aligned}$$

$$+\sum_{i=1}^{m_1} F_i(\mathbf{x})\nabla_{v_i}^{\mathrm{N}_{lf}} U[D](\mathbf{x}),$$

which yields

$$\begin{aligned}\mathcal{H}_D F(\mathbf{x}) &= \sum_{i=1}^{m_1} \Big(-\nabla_{v_i}^{\mathrm{N}_{lf}} F_i(\mathbf{x}) + (B_{v_i}^{\mu}(\mathbf{x}) + \nabla_{v_i}^{\mathrm{N}_{lf}} U[D](\mathbf{x}))F_i(\mathbf{x})\Big) \\ &= \sum_{i=1}^{m_1} \nabla_{v_i}^{\mathrm{N}_{lf}*} F_i(\mathbf{x}).\end{aligned}$$

So, by Lemma 3.6 and since $\tilde{\mathcal{S}}_D \subset \mathcal{S}_D$, using obvious notation we have

$$\begin{aligned}\mathbb{E}[G(\mathbf{X}^D)\mathcal{H}_D F(\mathbf{X}^D)] &= \sum_{i=1}^{m_1} \mathbb{E}\left[G(\mathbf{X}^D)\nabla_{v_i}^{\mathrm{N}_{lf}*} F_i(\mathbf{X}^D)\right] = \sum_{i=1}^{m_1} \mathbb{E}\left[F_i(\mathbf{X}^D)\nabla_{v_i}^{\mathrm{N}_{lf}} G(\mathbf{X}^D)\right] \\ &= \sum_{i=1}^{m_1} \mathbb{E}\left[F_i(\mathbf{X}^D) \sum_{l=1}^{\mathbf{X}(D)} \sum_{j=1}^{m_2} \partial_j g\left(\sum_{m=1}^{\mathbf{X}(D)} \gamma_1(X_m), \ldots, \sum_{m=1}^{\mathbf{X}(D)} \gamma_{m_2}(X_m)\right) \nabla\gamma_j(X_l)\cdot\nabla\varphi_i(X_l)\right] \\ &= \mathbb{E}\left[\sum_{l=1}^{\mathbf{X}(D)} \sum_{i=1}^{m_1} F_i(\mathbf{X}^D)\nabla\varphi_i(X_l)\cdot\sum_{j=1}^{m_2} G_j(\mathbf{X}^D)\nabla\gamma_j(X_l)\right] \\ &= \mathbb{E}\left[\sum_{i=1}^{\mathbf{X}(D)} \nabla_{X_i}^{\mathrm{N}_{lf}} F(\mathbf{X}^D)\cdot\nabla_{X_i}^{\mathrm{N}_{lf}} G(\mathbf{X}^D)\right].\end{aligned}$$

Finally, since $\mathcal{H}_D$ is symmetric and non-negative definite the square root operator $\mathcal{H}_D^{1/2}$ is well-defined. Relation (4.3) follows by the properties of $\mathcal{H}_D^{1/2}$.

□

Let $\mathcal{F}C_b^\infty(D)$ denote the set of functionals $F : \mathrm{N}_f^D \to \mathbb{R}$ of the form

$$F(\mathbf{x}) = f\left(\sum_{k=1}^{\mathbf{x}(D)} \varphi_1(x_k), \ldots, \sum_{k=1}^{\mathbf{x}(D)} \varphi_m(x_k)\right), \qquad \mathbf{x} = \{x_1, \ldots, x_{\mathbf{x}(D)}\} \in \mathrm{N}_f^D,$$

for some integer $m \geq 1$, $\varphi_1, \ldots, \varphi_m \in \mathcal{C}^\infty(D)$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^m)$. For $F \in \mathcal{F}C_b^\infty(D)$, we naturally define

$$\nabla_x^{\mathrm{N}_{lf}} F(\mathbf{x}) := \sum_{l=1}^{m} \partial_l f\left(\sum_{k=1}^{\mathbf{x}(D)} \varphi_1(x_k), \ldots, \sum_{k=1}^{\mathbf{x}(D)} \varphi_m(x_k)\right) \nabla\varphi_l(x), \qquad \mathbf{x} = \{x_1, \ldots, x_{\mathbf{x}(D)}\} \in \mathrm{N}_f^D.$$

We conclude this section with the following proposition.

Proposition 4.3 *Under* (**H1**) − (**H4**), *we have that* $\mathcal{F}C_b^\infty(D) \subset \mathrm{Dom}(\overline{\mathcal{H}_D^{1/2}})$ *and*

$$\overline{\mathcal{E}}_D(F,G) = \mathbb{E}\left[\sum_{i=1}^{\mathbf{X}(D)} \nabla_{X_i}^{\mathrm{N}_{lf}} F(\mathbf{X}^D) \cdot \nabla_{X_i}^{\mathrm{N}_{lf}} G(\mathbf{X}^D)\right], \qquad F,G \in \mathcal{F}C_b^\infty(D). \tag{4.5}$$

Proof. For $F \in \mathcal{F}C_b^\infty(D)$, we clearly have $F^n(\mathbf{x}) := F(\mathbf{x})\mathbf{1}_{\{|\mathbf{x}|(D)\le n\}} \in \tilde{\mathcal{S}}_D$, for any positive integer n. By a straightforward computation, $F^m \to F$ in L_D^2 as n goes to infinity. By the standard construction of the smallest closed extension of $\mathcal{E}_D$ (see e.g. [10]), to get that $F \in \mathrm{Dom}(\overline{\mathcal{E}}_D)$ and that $\overline{\mathcal{E}}_D(F,F) = \lim_{n\to\infty} \mathcal{E}_D(F^n,F^n)$, it suffices to prove that $\mathcal{E}_D(F^n - F^m, F^n - F^m)$ tends to zero as m,n go to infinity. This easily follows by the dominated convergence theorem. Indeed for some positive $C > 0$ and $m < n$,

$$\begin{aligned}\mathcal{E}_D(F^n - F^m, F^n - F^m) &= \mathbb{E}\left[\sum_{i=1}^{\mathbf{X}(D)} \|\nabla_{X_i}^{\mathrm{N}_{lf}} F(\mathbf{X}^D)\|^2 \mathbf{1}_{\{m<\mathbf{X}(D)\le n\}}\right] \\ &\le C\,\mathbb{E}\left[\mathbf{X}(D)\mathbf{1}_{\{m<\mathbf{X}(D)\le n\}}\right] \xrightarrow[m,n\to\infty]{} 0.\end{aligned}$$

Now, noticing that by the monotone convergence theorem,

$$\mathcal{E}_D(F^n,F^n) = \mathbb{E}\left[\sum_{i=1}^{\mathbf{X}(D)} \|\nabla_{X_i}^{\mathrm{N}_{lf}} F(\mathbf{X}^D)\|^2 \mathbf{1}_{\{\mathbf{X}(D)\le n\}}\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[\sum_{i=1}^{\mathbf{X}(D)} \|\nabla_{X_i}^{\mathrm{N}_{lf}} F(\mathbf{X}^D)\|^2\right],$$

we have (4.5) with $G = F$ and we conclude by polarization. □

5 Diffusions associated to determinantal processes on $D \subset S$

We start recalling some notions, see Chapters IV and V in [16]. Given π in the set $\mathcal{P}(\ddot{\mathrm{N}}_f^D)$ of the probability measures on $(\ddot{\mathrm{N}}_f^D, \mathcal{B}(\ddot{\mathrm{N}}_f^D))$, we call a π-stochastic process with state space $\ddot{\mathrm{N}}_f^D$ the collection

$$\mathbf{M}_{D,\pi} = (\mathbf{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, (\mathbf{M}_t)_{t\ge 0}, (\mathbb{P}_\mathbf{x})_{\mathbf{x}\in\ddot{\mathrm{N}}_f^D}, \mathbb{P}_\pi)$$

where $\mathcal{F} := \bigvee_{t\ge 0} \mathcal{F}_t$ is a σ-algebra on the set $\mathbf{\Omega}$, $(\mathcal{F}_t)_{t\ge 0}$ is the $\mathbb{P}_\pi$-completed filtration generated by the process $\mathbf{M}_t : \mathbf{\Omega} \longrightarrow \ddot{\mathrm{N}}_f^D$, $\mathbb{P}_\mathbf{x}$ is a probability measure on $(\mathbf{\Omega}, \mathcal{F})$ for all $\mathbf{x} \in \ddot{\mathrm{N}}_f^D$, and $\mathbb{P}_\pi$ is the probability measure on $(\mathbf{\Omega}, \mathcal{F})$ defined by

$$\mathbb{P}_\pi(A) := \int_{\ddot{\mathrm{N}}_f^D} \mathbb{P}_\mathbf{x}(A)\,\pi(\mathrm{d}\mathbf{x}), \quad A \in \mathcal{F}.$$

A collection $(\mathbf{M}_{D,\pi},(\theta_t)_{t\geq 0})$ is called a π-time homogeneous Markov process with state space $\ddot{\mathrm{N}}_f^D$ if $\theta_t:\mathbf{\Omega}\longrightarrow\mathbf{\Omega}$ is a shift operator, i.e. $\mathbf{M}_s\circ\theta_t=\mathbf{M}_{s+t}$, $s,t\geq 0$; the map $\mathbf{x}\mapsto\mathbb{P}_{\mathbf{x}}(A)$ is measurable for all $A\in\mathcal{F}$; the time homogeneous Markov property

$$\mathbb{P}_{\mathbf{x}}(\mathbf{M}_t\in A\,|\,\mathcal{F}_s)=\mathbb{P}_{\mathbf{M}_s}(\mathbf{M}_{t-s}\in A),\qquad \mathbb{P}_{\mathbf{x}}-a.s.,\quad A\in\mathcal{F},\quad 0\leq s\leq t,\quad \mathbf{x}\in\ddot{\mathrm{N}}_f^D$$

holds. A π-time homogeneous Markov process $(\mathbf{M}_{D,\pi},(\theta_t)_{t\geq 0})$ with state space $\ddot{\mathrm{N}}_f^D$ is said to be π-tight on $\ddot{\mathrm{N}}_f^D$ if $(\mathbf{M}_t)_{t\geq 0}$ is right-continuous with left limits $\mathbb{P}_\pi$-almost surely; $\mathbb{P}_{\mathbf{x}}(\mathbf{M}_0=\mathbf{x})=1$, $\mathbf{x}\in\ddot{\mathrm{N}}_f^D$; the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right continuous; the following strong Markov property holds:

$$\mathbb{P}_{\pi'}(\mathbf{M}_{t+\tau}\in A\,|\,\mathcal{F}_\tau)=\mathbb{P}_{\mathbf{M}_\tau}(\mathbf{M}_t\in A)$$

$\mathbb{P}_{\pi'}$-almost surely for all $\mathcal{F}_t$-stopping time τ, $\pi'\in\mathcal{P}(\ddot{\mathrm{N}}_f^D)$, $A\in\mathcal{F}$ and $t\geq 0$, cf. Theorem IV.1.15 in [16]. In addition, a π-tight process on $\ddot{\mathrm{N}}_f^D$ is said to be a π-special standard process on $\ddot{\mathrm{N}}_f^D$ if for any $\pi'\in\mathcal{P}(\ddot{\mathrm{N}}_f^D)$ which is equivalent to π and all $\mathcal{F}_t$-stopping times τ, $(\tau_n)_{n\geq 1}$ such that $\tau_n\uparrow\tau$ then $\mathbf{M}_{\tau_n}$ converges to $\mathbf{M}_\tau$, $\mathbb{P}_{\pi'}$-almost surely.

In the following theorem, $\mathbb{E}_{\mathbf{x}}$ denotes the expectation under $\mathbb{P}_{\mathbf{x}}$, $\mathbf{x}\in\ddot{\mathrm{N}}_f^D$.

Theorem 5.1 *Assume* $(\mathbf{H1})-(\mathbf{H4})$*. Then there exists a* $\mathbb{P}_D$*-tight special standard process* $(\mathbf{M}_{D,\mathbb{P}_D},(\theta_t)_{t\geq 0})$ *on* $\ddot{\mathrm{N}}_f^D$ *such that:*

1. $\mathbf{M}_{D,\mathbb{P}_D}$ *is a diffusion, in the sense that:*

$$\mathbb{P}_{\mathbf{x}}(\{\omega\in\mathbf{\Omega}:\ t\mapsto\mathbf{M}_t(\omega)\text{ is continuous on }[0,+\infty)\})=1,\qquad \overline{\mathcal{E}}_D\text{-a.e. }\mathbf{x}\in\ddot{\mathrm{N}}_f^D;\tag{5.1}$$

2. *the transition semigroup of* $\mathbf{M}_{D,\mathbb{P}_D}$ *is given by*

$$p_tF(\xi):=\mathbb{E}_{\mathbf{x}}[F(\mathbf{M}_t)],\quad \mathbf{x}\in\ddot{\mathrm{N}}_f^D,\quad F:\ddot{\mathrm{N}}_f^D\to\mathbb{R}\quad\text{square integrable,}$$

   and it is properly associated with the Dirichlet form $(\overline{\mathcal{E}}_D,\mathrm{Dom}(\overline{\mathcal{H}_D^{1/2}}))$ *in the sense that* p_tF *is an* $\overline{\mathcal{E}}_D$*-a.c.,* $\mathbb{P}_D$*-version of* $\exp(-t\mathcal{H}_D^{gen})F$*, for all square integrable* $F:\ddot{\mathrm{N}}_f^D\longrightarrow\mathbb{R}$ *and* $t>0$ *(where* $\mathcal{H}_D^{gen}$ *is the generator of* $\overline{\mathcal{E}}_D$*);*

3. $\mathbf{M}_{D,\mathbb{P}_D}$ *is unique up to* $\mathbb{P}_D$*-equivalence (we refer the reader to Definition 6.3 page 140 in [16] for the meaning of this notion);*

4. $\mathbf{M}_{D,\mathbb{P}_D}$ *is* $\mathbb{P}_D$*-symmetric, i.e.*

$$\mathbb{E}\left[G(\mathbf{X}^D)\,p_t F(\mathbf{X}^D)\right] = \mathbb{E}\left[F(\mathbf{X}^D)\,p_t G(\mathbf{X}^D)\right],$$

for $F, G \in \mathrm{L}^2_D$*;*

5. $\mathbf{M}_{D,\mathbb{P}_D}$ *has* $\mathbb{P}_D$ *as invariant measure.*

Proof. We apply Theorem 4.13 of [17, p. 308]. Using the notation of [17], set

$$\mathrm{S}(f,g)(x) := \nabla f(x) \cdot \nabla g(x), \qquad x \in \mathbb{R}^d,\ f, g \in \mathcal{C}^\infty(D),$$

where ∇ is the usual gradient on $\mathbb{R}^d$, and

$$\mathrm{S}^\Gamma(F,G)(\mathbf{x}) := \sum_{i=1}^{\mathbf{x}(D)} \nabla_{x_i}^{\mathrm{N}_{lf}} F(\mathbf{x}) \cdot \nabla_{x_i}^{\mathrm{N}_{lf}} G(\mathbf{x}), \qquad \mathbf{x} = \{x_1, \ldots, x_{\mathbf{x}(D)}\} \in \mathrm{N}_f^D,\ F, G \in \tilde{\mathcal{S}}_D.$$

Then, it is readily seen that $(\mathrm{S}, \mathcal{C}^\infty(D))$ satisfies conditions $(\mathcal{D}.1)$, (S.1), (S.2) and (S.3) of [17], and $(\mathrm{S}^\Gamma, \tilde{\mathcal{S}}_D)$ satisfies condition $(\mathrm{S}^\Gamma.\mu)$ of [17, p. 282]. Furthermore, $\mathbb{P}_D$ satisfies condition $(\mu.1)$ of [17, p. 282] and condition (Q) of [17] holds since $\mathbb{R}^d$ is complete (see Example 4.5.1 in [17]). The assumptions of Theorem 4.13 are therefore verified and the proof is completed. □

Non-collision property of the associated diffusions

In the following, we will show the non-collision property of the diffusion constructed in the previous theorem which, roughly speaking, means that the diffusion takes values on N_f^D.

We start by recalling the following lemma, which is borrowed from [22].

Lemma 5.2 *Assume* **(H1)** – **(H4)** *and let* $(\mathbf{M}_{D,\mathbb{P}_D}, (\theta_t)_{t\geq 0})$ *be the diffusion given by Theorem 5.1. Let* $u_n \in \mathrm{Dom}(\overline{\mathcal{E}}_D)$*,* $n \geq 1$*, be such that:* $u_n : \ddot{\mathrm{N}}_f^D \to \mathbb{R}$ *is continuous,* $u_n \to u$ *point-wise in* $\ddot{\mathrm{N}}_f^D$*,*

$$\sup_{n\geq 1} \overline{\mathcal{E}}_D(u_n, u_n) < \infty. \tag{5.2}$$

Then u *is* $\overline{\mathcal{E}}_D$*-a.c. and, in particular,*

$$\mathbb{P}_{\mathbf{x}}(\{\omega \in \mathbf{\Omega} :\ t \mapsto u(\mathbf{M}_t)(\omega) \text{ is continuous on } [0, +\infty)\}) = 1, \qquad \overline{\mathcal{E}}_D\text{-a.e. } \mathbf{x} \in \mathrm{N}_f^D.$$

The next theorem provides the non-collision property.

Theorem 5.3 *Assume $d \geq 2$, and* $(\mathbf{H1}) - (\mathbf{H4})$*. Then*

$$\mathbb{P}_{\mathbf{x}}(\{\omega \in \mathbf{\Omega}: \ \mathbf{M}_t(\omega) \in \mathrm{N}_f^D \quad \forall\, 0 \leq t < \infty\}) = 1, \qquad \overline{\mathcal{E}}_D\text{-}a.e.\ \mathbf{x} \in \mathrm{N}_f^D.$$

Proof. Since the proof is similar to the proof of Proposition 1 in [22] we skip some details. For every positive integer a, define $u := \mathbf{1}_N$, where

$$N := \{\mathbf{x} \in \ddot{\mathrm{N}}_f^D \ : \ \sup_{x \in [-a,a]^d} \mathbf{x}(\{x\}) \geq 2\}.$$

The claim follows if we prove that u is $\overline{\mathcal{E}}_D$-a.c.. For this we are going to apply Lemma 5.2. Define

$$u_n(\mathbf{x}) = \Psi\left(\sup_{i \in A_n} \sum_{x \in \mathbf{x}} \phi_i(x)\right), \qquad n \geq 1,$$

where $\Psi \in \mathcal{C}_b^\infty(\mathbb{R})$ and $\phi_i \in \mathcal{C}^\infty(D)$ are chosen as in the proof of Proposition 1 in [22], and $A_n := \mathbb{Z}^d \cap [-na, na]^d$. Note that $u_n \in \mathrm{Dom}(\overline{\mathcal{H}_D^{1/2}})$ by Proposition 4.3. Furthermore, $u_n : \ddot{\mathrm{N}}_f^D \to \mathbb{R}$ is continuous and $u_n \to u$ point-wise by the proof of Proposition 1 in [22]. It remains to check (5.2). For $i = (i^{(1)}, \ldots, i^{(d)}) \in \mathbb{Z}^d$ and $n \geq 1$, we denote by $I_i^{(n)}$ the function defined by

$$I_i^{(n)}(x) := \prod_{k=1}^d \mathbf{1}_{[-1/2, 3/2]}(nx^{(k)} - i^{(k)}), \qquad x = (x^{(1)}, \ldots, x^{(d)}) \in D.$$

As proved in Proposition 1 in [22], the following upper bound holds:

$$\overline{\mathcal{E}}_D(u_n, u_n) \leq Cn^2 \sum_{i \in A_n} \mathbb{E}\left[\mathbf{1}_{\left\{\sum_{j=1}^{\mathbf{X}(D)} I_i^{(n)}(X_j) \geq 2\right\}} \sum_{j=1}^{\mathbf{X}(D)} I_i^{(n)}(X_j)\right], \tag{5.3}$$

where $C > 0$ is a positive constant.

Now, we upper-bound the r.h.s. of (5.3) by stochastic domination. In [13], it is proved that

$$c[D](x, \mathbf{x}) \leq J[D](x, x), \quad x \in D, \mathbf{x} \in \mathrm{N}_f^D,$$

where we denote by $c[D](x, \mathbf{x})$ the Papangelou conditional intensity of $\mathbf{X}^D$, see [6]. Note that a Poisson process $\mathbf{Y}^D$ of mean measure $J[D](x,x)\mu(\mathrm{d}x)$ has Papangelou conditional intensity $J[D](x,x)$. Therefore, by Theorem 1.1 in [12], we have that $\mathbf{X}^D$ is stochastically dominated by $\mathbf{Y}^D$, in the sense that

$$\mathbb{E}[f(\mathbf{X}^D)] \leq \mathbb{E}[f(\mathbf{Y}^D)] \tag{5.4}$$

for any integrable $f : \mathrm{N}_{lf} \to \mathbb{R}$ such that $f(\mathbf{x}) \leq f(\mathbf{y})$ whenever $\mathbf{x} \subset \mathbf{y} \in \mathrm{N}_{lf}$. Since for any $i \in A_n$, the mapping $\mathbf{x} \mapsto \mathbf{1}_{\left\{\sum_{j=1}^{\mathbf{x}(D)} I_i^{(n)}(x_j) \geq 2\right\}} \sum_{j=1}^{\mathbf{x}(D)} I_i^{(n)}(x_j)$ is increasing, we then have

$$\overline{\mathcal{E}}_D(u_n, u_n) \leq C n^2 \sum_{i \in A_n} \mathbb{E}\left[\mathbf{1}_{\left\{\sum_{j=1}^{\mathbf{Y}(D)} I_j^{(n)}(Y_j) \geq 2\right\}} \sum_{j=1}^{\mathbf{Y}(D)} I_i^{(n)}(Y_j)\right]. \tag{5.5}$$

By the properties of the Poisson process, the right hand side of (5.5) is equal to

$$C\, n^2 \sum_{i \in A_n} \left(1 - \mathrm{e}^{-\int_D I_i^{(n)}(x) J[D](x,x)\, \mu(\mathrm{d}x)}\right) \int_D I_i^{(n)}(x) J[D](x,x)\, \mu(\mathrm{d}x),$$

which is bounded above by

$$C\, n^2 \sum_{i \in A_n} \left(\int_D I_i^{(n)}(x) J[D](x,x) \rho(x)\, \mathrm{d}x\right)^2.$$

By using the Cauchy-Schwarz inequality, this term is further bounded by

$$C\, n^2 \sum_{i \in A_n} \left(\int_D I_i^{(n)}(x)\, \mathrm{d}x\right) \left(\int_D I_i^{(n)}(x) J[D](x,x)^2 \rho(x)^2\, \mathrm{d}x\right). \tag{5.6}$$

We have on the one hand that

$$\int_{\mathbb{R}^d} I_i^{(n)}(x)\, \mathrm{d}x = (2/n)^d$$

and on the other hand for some constant $C' > 0$,

$$\int_D I_i^{(n)}(x) J[D](x,x)^2 \rho(x)^2\, \mathrm{d}x \leq C' n^{-d},$$

since J and ρ are bounded on D. Moreover, $\sharp(A_n) \leq (2an)^d$. Consequently, the quantity (5.6) is in turn bounded by

$$C''\, n^{2-d}, \qquad \text{for some constant } C'' > 0.$$

The claim follows by the assumption $d \geq 2$. □

6 An illustrating example

Let $S := B(0,1)$ and $D := B(0,R) \subset \mathbb{R}^2$ be the closed ball centered at the origin with radius $R \in (0,1)$ and $(\varphi_k^{(R)})_{1 \leq k \leq 3}$, the orthonormal subset of $\mathrm{L}^2(B(0,R), \ell)$ defined by

$$\varphi_k^{(R)}(x) := \frac{1}{R}\sqrt{\frac{k+1}{\pi}} \left(\frac{x^{(1)}}{R} + i\frac{x^{(2)}}{R}\right)^k, \quad x = (x^{(1)}, x^{(2)}) \in B(0,R),\ k = 1,2,3,$$

where ℓ is the Lebesgue measure and $i := \sqrt{-1}$ denotes the complex unit. In this example, we consider the truncated Bergman kernel with at most 3 points (see e.g. [15]) restricted to $D = B(0, R)$:

$$K_D(x, y) := \sum_{k=1}^{3} R^{2(k+1)} \varphi_k^{(R)}(x) \overline{\varphi_k^{(R)}(y)}, \qquad x, y \in D,$$

and denote by $\mathcal{K}_D$ the associated integral operator, which is easily seen to be Hermitian and trace class with non-zero eigenvalues $\kappa_k := R^{2(k+1)}$, $k = 1, 2, 3$. As a consequence, the spectrum of $\mathcal{K}_D$ is contained in $[0, 1)$ and the triplet $(\mathcal{K}_D, K_D, \ell)$ satisfies (**H1**). In addition, (**H2**) is trivially satisfied since the reference measure is the Lebesgue measure. The Janossy densities of $\mathbf{X}^D$ defined in (2.2) are given by

$$j_D^n(x_1, \dots, x_n) = \mathrm{Det}(\mathbf{Id} - \mathcal{K}_D)\, \det J[D](x_1, \dots, x_n), \qquad n = 1, 2, 3, \quad (x_1, \dots, x_n) \in D^n,$$

where the kernel $J[D]$ is given by

$$J[D](x, y) := \sum_{h=1}^{3} \frac{R^{2(h+1)}}{1 - R^{2(h+1)}} \varphi_h^{(R)}(x) \overline{\varphi_h^{(R)}(y)},$$

cf. (2.6). Since $\mathbf{X}^D$ has at most 3 points, see e.g. [25], we have $j_D^n = 0$, for $n \geq 4$. To prove condition (**H3**) it suffices to remark that the function

$$(x_1, \dots, x_n) \to \det(J[D](x_p, x_q))_{1 \leq p,q \leq n}$$

is continuously differentiable on D^n, for $n = 1, 2, 3$. To show that **(H4)** is verified, we first consider the case $n = 3$. Note that

$$(J[D](x_p, x_q))_{1 \leq p,q \leq 3} = A(x_1, x_2, x_3) A(x_1, x_2, x_3)^*,$$

where the matrix $A(x_1, x_2, x_3) := (A_{ph})_{1 \leq p,h \leq 3}$ is given by

$$A_{ph} := \frac{R^{h+1}}{\sqrt{1 - R^{2(h+1)}}} \varphi_h^{(R)}(x_p)$$

and $A(x_1, x_2, x_3)^*$ denotes the transpose conjugate of $A(x_1, x_2, x_3)$. Hence,

$$\det J[D](x_1, x_2, x_3) = |\det A(x_1, x_2, x_3)|^2,$$

and since the previous determinant can be rewritten involving a Vandermonde determinant, we have

$$\det A(x_1, x_2, x_3) = \prod_{p=1}^{3} \sqrt{\frac{1+p}{\pi(1 - R^{2(p+1)})}} \left(\prod_{p=1}^{3} (x_p^{(1)} + i x_p^{(2)}) \right) \prod_{1 \leq p < q \leq 3} ((x_p^{(1)} - x_q^{(1)}) + i(x_p^{(2)} - x_q^{(2)})).$$

Note that **(H4)** with $n = 3$ is exactly

$$\int_{D^3} \left| \frac{\partial_{x_i^{(h)}} |\det A(x_1, x_2, x_3)|^2 \partial_{x_j^{(k)}} |\det A(x_1, x_2, x_3)|^2}{|\det A(x_1, x_2, x_3)|^2} \right| \mathbf{1}_{\{|\det A(x_1,x_2,x_3)|>0\}} \, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 < \infty,$$

for all $1 \le i, j \le 3$ and $1 \le h, k \le 2$, and so it suffices to check

$$\int_{B(0,R)^3} \left| \frac{\partial_{x_1^{(1)}} |\det A(x_1, x_2, x_3)|^2}{|\det A(x_1, x_2, x_3)|^2} \right| \mathbf{1}_{\{|\det A(x_1,x_2,x_3)|>0\}} \, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 < \infty.$$

This latter integral reduces to

$$\int_{B(0,R)^3} \left| \frac{2x_1^{(1)}}{(x_1^{(1)})^2 + (x_1^{(2)})^2} + 2 \sum_{j=2}^{3} \frac{x_1^{(1)} - x_j^{(1)}}{(x_1^{(1)} - x_j^{(1)})^2 + (x_1^{(2)} - x_j^{(2)})^2} \right| \mathrm{d}x_1 \mathrm{d}x_3 \mathrm{d}x_3,$$

which is indeed finite. Consequently, we proved that **(H4)** is verified for $n \ge 3$ (indeed it is trivially satisfied for $n > 3$). Now, consider $n = 1, 2$. We have again

$$J[D](x_1, \dots, x_n) = A(x_1, \dots, x_n) A(x_1, \dots, x_n)^*,$$

where this time, $A(x_1, \dots, x_n)$ is a rectangular $n \times 3$ matrix given by $A(x_1, \dots, x_n) := (A_{ph})_{1 \le p \le n, 1 \le h \le 3}$ is given by

$$A_{ph} := \frac{R^{h+1}}{\sqrt{1 - R^{2(h+1)}}} \varphi_h^{(R)}(x_p).$$

Recall the Cauchy-Binet formula:

$$\det J[D](x_1, \dots, x_n) = \sum_{1 \le i_1 < i_2 < \dots < i_n \le 3} |\det A^{i_1, \dots, i_n}(x_1, \dots, x_n)|^2, \tag{6.1}$$

where

$$A_{ph}^{i_1, \dots, i_n} := \frac{R^{i_h+1}}{\sqrt{1 - R^{2(i_h+1)}}} \varphi_{i_h}^{(R)}(x_p), \qquad 1 \le p, h \le n,$$

defines a square matrix. We now consider fixed $1 \le i_1 < i_2 < \dots < i_n \le 3$ and evaluate $|\det A^{i_1, \dots, i_n}(x_1, \dots, x_n)|^2$. We note that

$$|\det A^{i_1, \dots, i_n}(x_1, \dots, x_n)|^2 = \prod_{p=1}^{n} \frac{1 + i_p}{\pi(1 - R^{2(i_p+1)})} \left| V_{i_1, \dots, i_n}(x_1, \dots, x_n) \right|^2, \tag{6.2}$$

where

$$V_{i_1, \dots, i_n}(x_1, \dots, x_n) := \det \left(\left(x_h^{i_p} \right)_{1 \le p, h \le n} \right)$$

is known in the literature as the generalized Vandermonde determinant. By definition, the Schur polynomial s_λ is the ratio between the generalized Vandermonde determinant and the classical Vandermonde determinant. More precisely,

$$\det\left(\left(x_h^{i_p}\right)_{1\le p,h\le n}\right) = \det\left(\left(x_h^{p-1}\right)_{1\le p,h\le n}\right) s_{\lambda(i_1,\ldots,i_n)}(x_1,\ldots,x_n), \tag{6.3}$$

where $\lambda(i_1,\ldots,i_n) := (i_n - n + 1, \ldots, i_2 - 1, i_1)$. Combining (6.1), (6.2) and (6.3), we have

$$\det J[D](x_1,\ldots,x_n) = \left|\prod_{1\le p<q\le n}((x_p^{(1)} - x_q^{(1)}) + i(x_p^{(2)} - x_q^{(2)}))\right|^2$$
$$\sum_{1\le i_1<i_2<\cdots<i_n\le 3}\left(\prod_{p=1}^{n}\frac{1+i_p}{\pi(1-R^{2(i_p+1)})}\right)|s_{\lambda(i_1,\ldots,i_n)}(x_1,\ldots,x_n)|^2.$$

For $n=1$, one has $s_{\lambda(1)}(x) = x$, $s_{\lambda(2)}(x) = x^2$ and $s_{\lambda(3)}(x) = x^3$, see e.g. [14], and therefore

$$\det J[D](x) = \frac{2}{\pi(1-R^4)}|x|^2 + \frac{3}{\pi(1-R^6)}|x|^4 + \frac{4}{\pi(1-R^8)}|x|^6.$$

Thus,

$$\begin{aligned}\partial_{x^{(p)}}\ln\left(\det J[D](x)\right) = {} & \frac{2}{\pi(1-R^4)}\frac{2x^{(p)}}{\frac{2}{\pi(1-R^4)}|x|^2 + \frac{3}{\pi(1-R^6)}|x|^4 + \frac{4}{\pi(1-R^8)}|x|^6} \\ & + \frac{3}{\pi(1-R^6)}\frac{4x^{(p)}|x|^2}{\frac{2}{\pi(1-R^4)}|x|^2 + \frac{3}{\pi(1-R^6)}|x|^4 + \frac{4}{\pi(1-R^8)}|x|^6} \\ & + \frac{4}{\pi(1-R^8)}\frac{6x^{(p)}|x|^4}{\frac{2}{\pi(1-R^4)}|x|^2 + \frac{3}{\pi(1-R^6)}|x|^4 + \frac{4}{\pi(1-R^8)}|x|^6},\end{aligned}$$

and therefore

$$|\partial_{x^{(p)}}\ln\left(\det J[D](x)\right)| \le \frac{2\left|x^{(p)}\right|}{|x|^2} + \frac{4\left|x^{(p)}\right|}{|x|^2} + \frac{6\left|x^{(p)}\right|}{|x|^2},$$

which is integrable on D. For $n=2$, one has $s_{\lambda(1,2)}(x,y) = 1$, $s_{\lambda(1,3)}(x,y) = x+y$ and $s_{\lambda(2,3)}(x,y) = xy$, see e.g. [14], and therefore

$$\begin{aligned}&\det J[D](x,y) = |x-y|^2 \\ &\times\left(\frac{2}{\pi(1-R^4)}\frac{3}{\pi(1-R^6)} + \frac{2}{\pi(1-R^4)}\frac{4}{\pi(1-R^8)}|x+y|^2 + \frac{3}{\pi(1-R^6)}\frac{4}{\pi(1-R^8)}|xy|^2\right).\end{aligned}$$

Note that the differential of the logarithm of $|x-y|^2$ gives rise to a locally integrable term. So it remains to check that the differential of the logarithm of the second term, hereafter denoted by $\Psi_D(x,y)$, is integrable on D^2. By symmetry of the Schur polynomials, it suffices to check that the derivative of $\Psi_D(x,y)$ with respect to $x^{(p)}$ is integrable on D^2. We have

$$|\partial_{x^{(p)}}\Psi_D(x,y)| \le \frac{2\left|x^{(p)}+y^{(p)}\right|}{|x+y|^2} + \frac{2\left|x^{(p)}\right||y|^2}{|xy|^2} = \frac{2\left|x^{(p)}+y^{(p)}\right|}{|x+y|^2} + \frac{2x^{(p)}}{|x|^2},$$

and the claim follows by noticing that the r.h.s. is integrable on D^2.

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