

# Sensitivity analysis and density estimation for finite-time ruin probabilities

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## Abstract

The goal of this paper is to obtain probabilistic representation formulas that are suitable for the numerical computation of the (possibly non-continuous) density functions of infima of reserve processes commonly used in insurance. In particular we show, using Monte Carlo simulations, that these representation formulas perform better than standard finite difference methods. Our approach differs from Malliavin probabilistic representation formulas which generally require more smoothness on random variables and entail the continuity of their density functions.

*Key words:* Ruin probability, Malliavin calculus, integration by parts, insurance mathematics.

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## 1 Introduction

In ruin theory, computational methods for finite-time ruin probabilities have received considerable attention in the last decade. The reader is referred to the books by Gerber [7], Grandell [8], Panjer and Willmot [13], Asmussen [1],

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and Kaas et al. [10] for general results on ruin-related issues.

Consider the classical compound Poisson risk model, in which the surplus process  $(R_x(t))_{t \geq 0}$  is defined as

$$R_x(t) = x + f(t) - S(t), \quad t \geq 0, \quad (1.1)$$

where  $x \geq 0$  is the amount of initial reserves and  $f(t)$  is the premium income received between time 0 and time  $t > 0$ . Here, the aggregate claim amount up to time  $t$  is described by the compound Poisson process

$$S(t) = \sum_{k=1}^{N(t)} W_k,$$

where the claim amounts  $W_k$ ,  $k \geq 1$ , are non-negative independent, identically distributed random variables, with  $S(t) = 0$  if  $N(t) = 0$ . The number of claims  $N(t)$  until  $t \geq 0$  is modeled by a homogeneous Poisson process  $(N(t))_{t \geq 0}$  with intensity  $\lambda > 0$ . We do not make any assumption on the claim amount distribution, which are nevertheless assumed to be independent of the arrival times.

Given  $T$  some fixed time horizon, the ruin probability

$$\psi(x, T) = \mathbb{P} \left( \inf_{0 \leq t \leq T} R_x(t) < 0 \right)$$

in the classical Crámer-Lundberg risk model has been analyzed by many authors, in particular by way of the Picard and Lefèvre formula [14], discussed by De Vylder [5] and Ignatov et al. [9], and compared to a Prabhu or Seal-type formula by Rullière and Loisel [18]. Further analysis and extensions have been proposed more recently by Lefèvre and Loisel [11].

Another important practical problem is to obtain numerical values for the sensitivity

$$\frac{\partial \psi}{\partial x}(x, T)$$

of the finite-time ruin probability  $\psi(x, t)$  with respect to the initial reserve  $x$ , in particular due to new solvency regulations in Europe. This problem is closely related to that of density estimation since  $-\frac{\partial \psi}{\partial x}(x, T)$  is also the probability density at  $-x < 0$  of the infimum

$$\mathcal{M}_{[0, T]} = \inf \{ f(t) - S(t) : t \in [0, T] \}.$$

In [17], Privault and Wei used the Malliavin calculus to compute the sensitivity

of the probability

$$\mathbb{P}(R_x(T) < 0)$$

that the terminal surplus is negative with respect to parameters such as the initial reserve  $x$  or the interest rate of the model.

However the problem of computing the corresponding sensitivity for the finite-time ruin probability  $\psi(x, T)$  has not been covered in [17] because  $\inf_{0 \leq t \leq T} R_x(t)$  does not satisfy the smoothness conditions imposed, see Remark 5.2 therein.

We proceed in two steps. First, in Sections 2 and 3 we review the main features of the Malliavin calculus applied to density estimation, in relation to the discontinuity of probability densities. In particular, in Section 3 we use the Malliavin calculus on the Poisson space to show in Proposition 4 that the infimum

$$\inf_{0 \leq t \leq T} R_x(t)$$

admits a probability density under certain conditions. We also note that the probability density of  $\inf_{0 \leq t \leq T} R_x(t)$  is not continuous, and that this infimum actually fails to satisfy the second order Malliavin differentiability conditions that would ensure the continuity of its density.

Second, in Section 4 we develop an alternative approach to the problem of existence and smoothness of the density of  $\inf_{0 \leq t \leq T} R_x(t)$ , based on a direct integration by parts. In particular this technique yields, in Proposition 5 below, an explicit probabilistic representation formulas suitable for the computation of the sensitivity

$$\frac{\partial \psi}{\partial x}(x, T)$$

by numerical simulation. We also treat the case of jump-diffusion processes (with an independent Brownian component that models investment of the surplus into a risky asset), using the density of the Brownian bridge.

Finally in Section 5 we present several simulation examples (for unit valued, exponential, and Pareto distributed claim amounts) that demonstrate the stability of our method compared to classical finite difference schemes. Our results are general and operational for light- or heavy-tailed, discrete or continuous claim amount distributions.

## 2 Malliavin calculus for density estimation

This section is a preparation for the next one where we apply the Malliavin calculus on Poisson space to show that although the random variable

$$\inf_{0 \leq t \leq T} R_x(t),$$

has an absolutely continuous law with respect to the Lebesgue measure, it does not satisfy the stronger differentiability conditions that would lead to the continuity of this density.

Our goal in particular is to determine more precisely the range of application of these techniques to the suprema of compensated jump processes. Here we work in an abstract setting before turning to the Poisson space in Section 3.

### Existence of densities

Here we state conditional versions of classical results on the existence of probability densities, see e.g. § 3.1 of Nualart [12] or Corollary 5.2.3 of Bouleau and Hirsch [3]. We work on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

**Proposition 1** *Let  $A \in \mathcal{F}$  such that  $\mathbb{P}(A) > 0$  and let  $F, G$  be two random variables satisfying the relation*

$$\mathbb{E}[Gf'(F)|A] = \mathbb{E}[\Lambda_{F,G}f(F)|A], \quad f \in \mathcal{C}_b^1(\mathbb{R}), \quad (2.1)$$

where  $\Lambda_{F,G}$  is an integrable random variable depending on  $F$  and  $G$ , and independent of  $f \in \mathcal{C}_b^1(\mathbb{R})$ .

Then:

*i) If  $G$  is (strictly) positive a.s. on  $A$  then the law of  $F$  has a conditional density  $\varphi_{F|A}$  given  $A$  with respect to the Lebesgue measure.*

*ii) If in addition  $G = \mathbf{1}$  a.s. on  $A$  then this density is given by*

$$\varphi_{F|A}(y) = \mathbb{E}[\Lambda_{F,\mathbf{1}}\mathbf{1}_{\{y \leq F\}}|A], \quad y \in \mathbb{R}. \quad (2.2)$$

*Proof.*

*i) The bound*

$$\mathbb{E}[Gf'(F)|A] = \mathbb{E}[f(F)\Lambda_{F,G}|A] \leq \|f\|_\infty \mathbb{E}[|\Lambda_{F,G}| | A], \quad f \in \mathcal{C}_b^1(\mathbb{R}),$$

extends to  $f' = \mathbf{1}_B$  for any bounded Borel subset  $B$  of  $\mathbb{R}$ , to yield

$$\mathbb{E}[G\mathbf{1}_B(F)|A] \leq m(B)\mathbb{E}[|\Lambda_{F,G}| | A],$$

where  $m(B)$  denotes the Lebesgue measure of  $B$ , hence the law of  $F$  is absolutely continuous with respect to the Lebesgue measure since  $G > 0$  a.s. on  $A$ .

*ii) In the case  $G = \mathbf{1}$  a.s. on  $A$  we get*

$$\begin{aligned}
\mathbb{E}[f(F)|A] &= \mathbb{E} \left[ \int_{-\infty}^F f'(y) dy \middle| A \right] \\
&= \int_{-\infty}^0 \mathbb{E} \left[ f'(y + F) \middle| A \right] dy \\
&= \int_{-\infty}^0 \mathbb{E} \left[ \Lambda_{F,1} f(y + F) \middle| A \right] dy \\
&= \int_{-\infty}^{\infty} f(y) \mathbb{E} \left[ \mathbf{1}_{\{F \geq y\}} \Lambda_{F,1} \middle| A \right] dy.
\end{aligned}$$

□

In what follows, any relation of the form (2.2) will be termed an integration by parts formula, and the random variable  $\Lambda_{F,G}$  will be called a *weight*.

### Continuity of densities

Proposition 1 ensures the existence of the density  $\varphi_{F|A}$  but not its smoothness. The next proposition provides a more precise statement.

**Proposition 2** *Assume that the hypotheses of Proposition 1 hold with  $G = \mathbf{1}$  a.s. on  $A$ , and suppose in addition that  $\Lambda_{F,1} \in L^p(A)$  for some  $p > 1$ .*

*Then the conditional probability density  $\varphi_{F|A}$  is continuous on  $\mathbb{R}$ .*

*Proof.* Use the bound

$$|\varphi_{F|A}(y) - \varphi_{F|A}(z)| \leq \frac{1}{\mathbb{P}(A)} \|\Lambda_{F,1}\|_{L^p(A)} (\mathbb{E}[\mathbf{1}_{[z,y]}(F)])^{1/q}, \quad y, z \in \mathbb{R}, \quad (2.3)$$

that follows from (2.2), with  $1/p + 1/q = 1$ . □

The integrability of  $\Lambda_{F,1}$  in  $L^p(A)$  for  $p > 1$  can be obtained under strong (second order) differentiability conditions in the Malliavin sense as a consequence of Corollary 2 and Proposition 3 below.

### Non-continuous densities

In Section 4 we will replace (2.1) by an expression of the form

$$\mathbb{E}[f'(F)|A] = \mathbb{E} \left[ \sum_{j=1}^Z \Lambda_j f(F_j) \middle| A \right]$$

where  $Z, F_j, \Lambda_j, j \geq 1$ , are random variables, which also implies the existence of a conditional density of  $F$  given  $A$  as

$$\varphi_{F|A}(y) = \mathbb{E} \left[ \sum_{j=1}^Z \Lambda_j \mathbf{1}_{\{y \leq F_j\}} \middle| A \right]. \quad (2.4)$$

However, Relation (2.4) no longer ensures the continuity of  $\varphi_{F|A}$  as the bound (2.3) is no longer valid. Such expressions will be obtained in Section 4, Proposition 5, for the infimum

$$\mathcal{M}_{[0,T]} = \inf_{t \in [0,T]} R_x(t).$$

### 3 Malliavin calculus on the Poisson space

In this section we consider the application of the Malliavin calculus on the Poisson space to the infimum

$$\inf_{0 \leq t \leq T} R_x(t).$$

In Corollaries 1 and 2 below we implement the results of Section 2. For this we will use an unbounded linear derivation operator

$$D : L^2(\Omega) \rightarrow L^2(\Omega \times \mathbb{R}_+)$$

admitting an adjoint

$$\delta : L^2(\Omega \times \mathbb{R}_+) \rightarrow L^2(\Omega),$$

with respective domains  $\text{Dom}(D|A) \subset L^2(\Omega)$  and  $\text{Dom}(\delta|A) \subset L^2(\Omega \times \mathbb{R}_+)$ , such that

$$\mathbb{E}[\langle DF, u \rangle | A] = \mathbb{E}[F \delta(u) | A], \quad F \in \text{Dom}(D|A), \quad u \in \text{Dom}(\delta|A), \quad (3.1)$$

where  $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2([0,T])}$  denotes the scalar product in  $L^2([0, T])$ . A concrete example of operator  $D$  will be given in Definition 2 below.

First, we treat the existence of densities in the next corollary of Proposition 1, using the duality (3.1) between  $D$  and  $\delta$ .

**Corollary 1** *Let  $F \in \text{Dom}(D|A)$  and  $w \in \text{Dom}(\delta|A)$  such that*

$$\langle DF, w \rangle > 0, \quad \text{a.s. on } A. \quad (3.2)$$

*Then the law of  $F$  has a conditional density  $\varphi_{F|A}$  given  $A$  with respect to the Lebesgue measure.*

*Proof.* Letting  $G = \langle DF, w \rangle$  we get

$$\mathbb{E}[\langle DF, w \rangle f'(F) | A] = \mathbb{E} \left[ \langle Df(F), w \rangle \Big| A \right] = \mathbb{E} \left[ f(F) \delta(w) \Big| A \right],$$

hence it suffices to apply Proposition 1 with  $\Lambda_{F,G} = \delta(w)$ .  $\square$

As a consequence, the existence of density of the random variable  $F$  can be obtained under first order Malliavin  $D$ -differentiability conditions, see below for an implementation in the setting of jump processes.

Next we recall how the operators  $D$  and  $\delta$  can be applied to the representation and continuity of densities.

**Corollary 2** *Let  $F \in \text{Dom}(D|A)$  and  $w \in L^2(\Omega \times \mathbb{R}_+)$  such that*

$$\langle DF, w \rangle > 0, \quad \text{a.s. on } A, \quad \text{and} \quad \frac{wG}{\langle DF, w \rangle} \in \text{Dom}(\delta|A).$$

*Then:*

*i) if  $G$  is (strictly) positive a.s. on  $A$  then the law of  $F$  has a conditional density  $\varphi_{F|A}$  given  $A$  with respect to the Lebesgue measure.*

*ii) if in addition  $G = \mathbf{1}$  a.s. on  $A$  then this density is continuous and given by (2.2), with the weight*

$$\Lambda_{F,G} = \delta \left( G \frac{w}{\langle DF, w \rangle} \right). \quad (3.3)$$

*Proof.* Using the relation

$$f'(F) = \frac{\langle Df(F), w \rangle}{\langle DF, w \rangle}, \quad f \in \mathcal{C}_b^1(\mathbb{R}),$$

we get

$$\mathbb{E}[Gf'(F)|A] = \mathbb{E} \left[ G \frac{\langle Df(F), w \rangle}{\langle DF, w \rangle} \Big| A \right] = \mathbb{E} \left[ f(F) \delta \left( \frac{wG}{\langle DF, w \rangle} \right) \Big| A \right],$$

hence the existence of a conditional density follows from Proposition 1. The continuity of  $\varphi_{F|A}$  in the case  $G = \mathbf{1}$  a.s. on  $A$  follows from Proposition 2 and the fact that  $\delta$  is  $L^2(\Omega)$ -valued on  $\text{Dom}(\delta|A)$ .  $\square$

In order to apply the above results to functionals of jump processes, we now turn to a specific implementation of the Malliavin calculus on Poisson space, cf. Carlen and Pardoux [4], Privault [15]. Here,  $(\Omega, \mathcal{F}, \mathbb{P})$  denotes the canonical

probability space of the Poisson process  $(N(t))_{t \in \mathbb{R}_+}$  with intensity  $\lambda > 0$  whose jumps are denoted by  $(T_k)_{k \geq 1}$ , with  $T_0 = 0$ .

**Definition 1** Given  $m \in \mathbb{N}$  we denote by  $\mathcal{S}_m$  the space of Poisson functionals of the form

$$F = h(T_1 \wedge T, \dots, T_n \wedge T) \quad (3.4)$$

for some  $h \in \mathcal{C}^1([0, T]^n)$  and  $n \geq m$ , with the boundary condition  $F = 0$  on  $\{N(T) < m\}$ , i.e.

$$h(t_1, \dots, t_{m-1}, T, \dots, T) = 0, \quad t_1, \dots, t_{m-1} \in [0, T]. \quad (3.5)$$

Every  $F \in \mathcal{S}_m$  can be written as

$$F = \sum_{k=m}^{\infty} \mathbf{1}_{\{N(T)=k\}} f_k(T_1, \dots, T_k), \quad (3.6)$$

where  $f_0 \in \mathbb{R}$  and  $f_k \in \mathcal{C}^1([0, T]^k)$  satisfies

$$f_k(T_1, \dots, T_k) = h(T_1, \dots, T_{n \wedge k}, T, \dots, T), \quad k \geq m, \text{ on } \{N(T) = k\}. \quad (3.7)$$

Note that Condition (3.5) is void when  $m = 0$ .

**Definition 2** Let  $D_t F$ ,  $t \in \mathbb{R}_+$ , denote the gradient of  $F \in \mathcal{S}_m$ , defined as

$$D_t F = - \sum_{k=1}^n \mathbf{1}_{[0, T_k]}(t) \partial_k h(T_1 \wedge T, \dots, T_n \wedge T),$$

for  $F \in \mathcal{S}_m$  of the form (3.4), where  $\partial_k h$  denotes the partial derivative of  $h$  with respect to its  $k$ -th variable.

For  $F$  of the form (3.6) we have:

$$D_t F = - \sum_{n=m \vee 1}^{\infty} \mathbf{1}_{\{N(T)=n\}} \sum_{k=1}^n \mathbf{1}_{[0, T_k]}(t) \partial_k f_n(T_1, \dots, T_n).$$

From now on we consider  $A$  of the form  $A = \{N(T) \geq m\}$  for some  $m \in \mathbb{N}$ , and let  $\text{Dom}_m(D)$ ,  $\text{Dom}_m(\delta)$  respectively denote  $\text{Dom}(D|_{N(T) \geq m})$  and  $\text{Dom}(\delta|_{N(T) \geq m})$ . Similarly we will denote  $\mathbb{E}[F|N(T) \geq m]$  by  $\mathbb{E}_m[F]$  for simplicity of notation.

**Lemma 1** The operator  $D$  can be extended to its closed domain  $\text{Dom}_m(D)$  and admits an adjoint  $\delta$  with domain  $\text{Dom}_m(\delta)$  such that

$$\mathbb{E}_m[\langle DF, u \rangle] = \mathbb{E}_m[F \delta(u)], \quad F \in \text{Dom}_m(D), \quad u \in \text{Dom}_m(\delta). \quad (3.8)$$



Moreover for all  $u \in L^2([0, T])$  and  $F \in \text{Dom}_m(D)$  we have

$$\delta(Fu) = F \int_0^T u(t) d(N(t) - \lambda t) - \int_0^\infty u(t) D_t F dt. \quad (3.9)$$

*Proof.* This proposition is a conditional version of the classical integration by parts formula on the Poisson space. For completeness its proof is given in the Appendix Section 6.  $\square$

In order to check that  $\Lambda_{F,G}$  defined in (3.3) belongs to  $L^p$  as required in Corollary 2, we can proceed as follows.

Let  $\mathcal{U}$  denote the space of processes of the form

$$u = \sum_{k=1}^n F_k h_k, \quad h_1, \dots, h_n \in \mathcal{C}_c^1((0, T)), \quad F_1, \dots, F_n \in \mathcal{S}_m, \quad n \geq 1, \quad (3.10)$$

and let the operator  $\nabla$  be defined as

$$\nabla_s u(t) = D_s u(t) - \mathbf{1}_{[0,t]}(s) \dot{u}(t), \quad s, t \in \mathbb{R}_+, \quad u \in \mathcal{U}.$$

We remark that the operator  $\nabla$  plays the role of a covariant derivative in the framework of the Malliavin calculus, cf. [16].

**Proposition 3** *For all  $u \in \mathcal{U}$  we have the identity*

$$\mathbb{E}_m[|\delta(u)|^2] = \mathbb{E}_m[\|u\|_{L^2([0,T])}^2] + \mathbb{E}_m \left[ \int_0^T \int_0^T \nabla_s u(t) \nabla_t u(s) ds dt \right]. \quad (3.11)$$

*Proof.* cf. the Appendix Section 6.  $\square$

The identity (3.11) is called the Skorohod isometry and implies the bound

$$\mathbb{E}_m[|\delta(u)|^2] \leq \mathbb{E}_m[\|u\|_{L^2([0,T])}^2] + \mathbb{E}_m[\|\nabla u\|_{L^2([0,T]^2)}^2], \quad (3.12)$$

which provides sufficient conditions for a process  $u \in \mathcal{U}$  to belong to  $\text{Dom}(\delta)$ .

As an example of application of Propositions 1 and 2 (resp. Corollaries 1 and 2) in this context, consider a constant premium income rate  $f(t) = \alpha$ ,  $t \in [0, T]$ , with deterministic claim amounts equal to 1, and consider the infimum

$$\mathcal{M}_{[0,T]} = \inf_{0 \leq t \leq T} (\alpha t - N(t)) = \inf_{T_k \leq T, k \geq 0} (\alpha T_k - k) = \mathbf{1}_{\{N(T) \geq 1\}} \inf_{T_k \leq T, k \geq 1} (\alpha T_k - k). \quad (3.13)$$

**Proposition 4** *Assume that  $0 < \alpha T \leq 1$ . Then the probability law of  $\mathcal{M}_{[0,T]}$  admits a density conditionally to  $\{\mathcal{M}_{[0,T]} < 0\}$  with respect to the Lebesgue*

measure.

*Proof.* First, note that  $\mathcal{M}_{[0,T]}$  has the form (3.6) with  $f_0 = 0$  and

$$f_n(t_1, \dots, t_n) = \inf_{1 \leq k \leq n} (\alpha t_k - k), \quad n \geq 1,$$

and we have  $\{\mathcal{M}_{[0,T]} < 0\} = \{N(T) \geq 1\}$  since  $\alpha T \leq 1$ . Hence taking  $m = 1$ ,

$$\mathcal{M}_{[0,T]} = \sum_{k=1}^{N(T)} (\alpha T_k - k) \mathbf{1}_{\{\mathcal{M}_{[0,T]} = \alpha T_k - k\}}$$

belongs to  $\text{Dom}_1(D)$  with

$$D_t \mathcal{M}_{[0,T]} = -\alpha \sum_{k=1}^{N(T)} \mathbf{1}_{[0, T_k]}(t) \mathbf{1}_{\{\mathcal{M}_{[0,T]} = \alpha T_k - k\}},$$

and the gradient norm

$$\langle D \mathcal{M}_{[0,T]}, D \mathcal{M}_{[0,T]} \rangle = \alpha \sum_{k=1}^{N(T)} T_k \mathbf{1}_{\{\mathcal{M}_{[0,T]} = \alpha T_k - k\}}$$

is a.e. positive on  $A = \{N(T) \geq 1\}$ , thus ensuring the existence of the density of  $\mathcal{M}_{[0,T]}$  conditionally to  $\{\mathcal{M}_{[0,T]} < 0\}$  from Corollary 1.  $\square$

The application of Corollary 2 to obtain the continuity of the density of  $F = \mathcal{M}_{[0,T]} < 0$  and its representation formula (2.2) with the weight (3.3) would require  $\Lambda_{F,1} \in L^p$  for some  $p > 1$ . In order to check this condition one can apply the divergence formula (3.9) to  $G = 1/\langle DF, w \rangle$ , however from (3.12) this would require a second order  $D$ -differentiability as a function of the Poisson process jump times, a property not satisfied by  $F = \mathcal{M}_{[0,T]}$ .

It is actually natural that such differentiability conditions do not hold here since they would ensure the continuity of the probability density of  $\mathcal{M}_{[0,T]}$ , a property which is not satisfied, cf. Relation (5.1) and Figure 1 below.

#### 4 Calculation of densities by integration by parts

In this section we develop a direct integration by parts method as a way around the difficulties noted in Section 3 with the application of the Malliavin calculus to  $\mathcal{M}_{[0,T]}$ . In particular, in Proposition 5 we obtain a probabilistic representation formula for non-continuous densities that replaces (2.2). We consider both deterministic and random drifts.

## Monotone deterministic drift

Assume that  $(S(t))_{t \in \mathbb{R}_+}$  has the form

$$S(t) = Y_{N(t)}, \quad t \in \mathbb{R}_+,$$

where  $Y_0 = 0$  and  $(Y_k)_{k \geq 1}$  is a sequence of random variables, independent of  $(N(t))_{t \in \mathbb{R}_+}$ , i.e. in the compound Poisson risk model,  $S(t)$  represents the aggregate claim amount and

$$Y_k = \sum_{j=1}^k W_j, \quad k \in \mathbb{N}.$$

Let  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be an increasing function mapping  $t \geq 0$  to the premium income  $f(t)$  received between time 0 and time  $t$ , such that  $f(0) = 0$ , and consider the infimum

$$\mathcal{M}_{[0,T]} = \inf_{0 \leq t \leq T} (f(t) - S(t)).$$

Clearly we have  $\mathcal{M}_{[0,T]} \leq f(0) - S(0) = 0$  hence the law of  $\mathcal{M}_{[0,T]}$  is carried by  $(-\infty, 0]$ . On the other hand, we have  $\mathcal{M}_{[0,T]} = 0$  if and only if  $N(T) = 0$  or  $f(T_k) - Y_k > 0$  for all  $k = 1, \dots, N(T)$ . Hence the law of  $\mathcal{M}_{[0,T]}$  has a Dirac mass  $\mathbb{P}(\mathcal{M}_{[0,T]} = 0)$  at 0, equal to

$$\begin{aligned} \mathbb{P}(\mathcal{M}_{[0,T]} = 0) &= \mathbb{P}(N(T) = 0) + \mathbb{P}(\{\mathcal{M}_{[0,T]} \geq 0\} \cap \{N(T) \geq 1\}) \\ &= e^{-\lambda T} + e^{-\lambda T} \mathbb{E} \left[ \sum_{k=1}^{\infty} \lambda^k \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \mathbf{1}_{\{f(t_1) > Y_1\}} \cdots \mathbf{1}_{\{f(t_k) > Y_k\}} dt_1 \cdots dt_k \right]. \end{aligned}$$

In the next proposition we compute the density of  $\mathcal{M}_{[0,T]}$ , and provide a probabilistic representation which is suitable for simulation purposes.

**Proposition 5** *Assume that  $f$  is  $\mathcal{C}^1$  on  $\mathbb{R}_+$  with  $f'(t) > c > 0$  for all  $t \in \mathbb{R}_+$ . Then the probability density at  $y < 0$  of  $\mathcal{M}_{[0,T]}$  given that  $\{\mathcal{M}_{[0,T]} < 0\}$  is equal to*

$$\begin{aligned} \varphi_{\mathcal{M}_{[0,T]} | \mathcal{M}_{[0,T]} < 0}(y) &= \\ &= \frac{\lambda (f^{-1})'(y)}{P(\mathcal{M}_{[0,T]} < 0)} \mathbb{E} \left[ \sum_{j=1}^{N(T)} \mathbf{1}_{\{y \leq \inf_{1 \leq l \leq j} (f(T_l) - Y_l)\}} \mathbf{1}_{\{f(T_{j-1}) - Y_j < y\}} \mathbf{1}_{\{y \leq \inf_{j \leq l \leq N(T)} (f(T_l) - Y_{l+1})\}} \right] \\ &+ \frac{\lambda (f^{-1})'(y)}{P(\mathcal{M}_{[0,T]} < 0)} \mathbb{E} \left[ \mathbf{1}_{\{0 < Y_{N(T)+1} + y < f(T)\}} \mathbf{1}_{\{f(T_{N(T)}) < Y_{N(T)+1} + y\}} \mathbf{1}_{\{\inf_{1 \leq l \leq N(T)} (f(T_l) - Y_l) > y\}} \right], \end{aligned}$$

$y < 0$ , where we use the convention  $\inf \emptyset = +\infty$ .

*Proof.* Since  $f$  is increasing we have, on  $\{\mathcal{M}_{[0,T]} < 0\}$ ,

$$\mathcal{M}_{[0,T]} = \inf_{T_k \leq T, k \geq 0} (f(T_k) - Y_k) = \mathbf{1}_{\{N(T) \geq 1\}} \inf_{T_k \leq T, k \geq 1} (f(T_k) - Y_k).$$

Hence for  $y < 0$ ,

$$\begin{aligned} & \mathbb{P}(\{\mathcal{M}_{[0,T]} > y\}) \tag{4.1} \\ &= \mathbb{P}(\{N(T) = 0\}) + e^{-\lambda T} \mathbb{E} \left[ \sum_{k=1}^{\infty} \lambda^k \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \mathbf{1}_{\{y < \inf_{1 \leq l \leq k} (f(t_l) - Y_l)\}} dt_1 \cdots dt_k \right] \\ &= e^{-\lambda T} + \lambda e^{-\lambda T} \mathbb{E} \left[ \sum_{k=0}^{\infty} \lambda^k \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} \mathbf{1}_{\{f(t_1) > Y_1 + y\}} \cdots \mathbf{1}_{\{f(t_{k+1}) > Y_{k+1} + y\}} dt_1 \cdots dt_{k+1} \right]. \end{aligned}$$

Now, using the relation

$$d \left( \mathbf{1}_{\{f(t_1) > Y_1 + y\}} \cdots \mathbf{1}_{\{f(t_{k+1}) > Y_{k+1} + y\}} \right) = - \sum_{j=1}^{k+1} \prod_{l=1, l \neq j}^{k+1} \mathbf{1}_{\{f(t_l) > Y_l + y\}} \delta_{(f(t_j) - Y_j)}(dy)$$

we have, for any  $g \in \mathcal{C}_c((-\infty, 0))$ :

$$\begin{aligned} E[g(\mathcal{M}_{[0,T]})] &= - \int_{-\infty}^0 g(y) dP(\mathcal{M}_{[0,T]} > y) \\ &= \lambda e^{-\lambda T} \mathbb{E} \left[ \sum_{k=0}^{\infty} \lambda^k \sum_{j=1}^{k+1} \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} \int_{-\infty}^0 \prod_{l=1, l \neq j}^{k+1} \mathbf{1}_{\{f(t_l) > Y_l + y\}} g(y) \delta_{(f(t_j) - Y_j)}(dy) dt_1 \cdots dt_{k+1} \right] \\ &= \lambda e^{-\lambda T} \mathbb{E} \left[ \sum_{k=0}^{\infty} \lambda^k \sum_{j=1}^{k+1} \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} \prod_{l=1, l \neq j}^{k+1} \mathbf{1}_{\{f(t_l) > Y_l + f(t_j) - Y_j\}} g(f(t_j) - Y_j) dt_1 \cdots dt_{k+1} \right] \\ &= \lambda e^{-\lambda T} \mathbb{E} \left[ \sum_{k=0}^{\infty} \lambda^k \sum_{j=1}^k \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} g(f(t_j) - Y_j) \mathbf{1}_{\{f(t_j) - Y_j < \inf_{1 \leq l \neq j \leq k+1} (f(t_l) - Y_l)\}} dt_1 \cdots dt_{k+1} \right] \\ &+ \lambda e^{-\lambda T} \mathbb{E} \left[ \sum_{k=0}^{\infty} \lambda^k \int_0^T g(f(t_{k+1}) - Y_{k+1}) \int_0^{t_k} \cdots \int_0^{t_2} \mathbf{1}_{\{f(t_{k+1}) - Y_{k+1} < \inf_{1 \leq l \leq k} (f(t_l) - Y_l)\}} dt_1 \cdots dt_{k+1} \right] \\ &= \lambda e^{-\lambda T} \mathbb{E} \left[ \sum_{k=0}^{\infty} \lambda^k \int_{-\infty}^0 \sum_{j=1}^k \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_{j+1}} \int_0^{t_{j-1}} \cdots \int_0^{t_2} \frac{g(y)}{f'(f^{-1}(y))} \right. \\ &\quad \left. \mathbf{1}_{\{y < \inf_{1 \leq l < j} (f(t_l) - Y_l)\}} \mathbf{1}_{\{f(t_{j-1}) < y + Y_j < f(t_{j+1})\}} \mathbf{1}_{\{y < \inf_{j < l \leq k+1} (f(t_l) - Y_l)\}} dt_1 \cdots dt_{j-1} dt_{j+1} \cdots dt_{k+1} dy \right] \\ &+ \lambda e^{-\lambda T} \mathbb{E} \left[ \sum_{k=0}^{\infty} \lambda^k \int_{-\infty}^0 \mathbf{1}_{\{0 < Y_{k+1} + y < f(T)\}} \frac{g(y)}{f'(f^{-1}(y))} \right. \\ &\quad \left. \int_0^T \mathbf{1}_{\{f(t_k) < f(t_{k+1})\}} \int_0^{t_k} \cdots \int_0^{t_2} \mathbf{1}_{\{y < \inf_{1 \leq l \leq k} (f(t_l) - Y_l)\}} dt_1 \cdots dt_{k+1} dy \right] \\ &= \lambda e^{-\lambda T} \int_{-\infty}^0 \frac{g(y)}{f'(f^{-1}(y))} \mathbb{E} \left[ \sum_{k=0}^{\infty} \lambda^k \sum_{j=1}^k \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_{j+1}} \int_0^{t_{j-1}} \cdots \int_0^{t_2} \right. \end{aligned}$$

$$\begin{aligned}
& \mathbf{1}_{\{y < \inf_{1 \leq l < j} (f(t_l) - Y_l)\}} \mathbf{1}_{\{f(t_{j-1}) < Y_j + y < f(t_{j+1})\}} \mathbf{1}_{\{y < \inf_{j < l \leq k+1} (f(t_l) - Y_l)\}} dt_1 \cdot dt_{j-1} dt_{j+1} \cdot dt_{k+1} \Big] dy \\
& + \lambda e^{-\lambda T} \int_{-\infty}^0 \frac{g(y)}{f'(f^{-1}(y))} \mathbb{E} \left[ \sum_{k=0}^{\infty} \lambda^k \mathbf{1}_{\{0 < Y_{k+1} + y < f(T)\}} \right. \\
& \left. \int_0^T \mathbf{1}_{\{f(t_k) < Y_{k+1} + y\}} \int_0^{t_k} \cdots \int_0^{t_2} \mathbf{1}_{\{y < \inf_{1 \leq l \leq k} (f(t_l) - Y_l)\}} dt_1 \cdots dt_k \right] dy \\
& = \lambda e^{-\lambda T} \int_{-\infty}^0 \frac{g(y)}{f'(f^{-1}(y))} \mathbb{E} \left[ \sum_{k=0}^{\infty} \lambda^k \sum_{j=1}^k \int_0^T \int_0^{t_{k-1}} \cdots \int_0^{t_2} \right. \\
& \mathbf{1}_{\{y \leq \inf_{1 \leq l < j} (f(t_l) - Y_l)\}} \mathbf{1}_{\{f(t_{j-1}) < y + Y_j < f(t_j)\}} \mathbf{1}_{\{y \leq \inf_{j \leq l \leq k} (f(t_l) - Y_{l+1})\}} dt_1 \cdots dt_k \\
& \left. + \sum_{k=0}^{\infty} \lambda^k \mathbf{1}_{\{0 < Y_{k+1} + y < f(T)\}} \int_0^T \mathbf{1}_{\{f(t_k) < Y_{k+1} + y\}} \int_0^{t_k} \cdots \int_0^{t_2} \mathbf{1}_{\{y < \inf_{1 \leq l \leq k} (f(t_l) - Y_l)\}} dt_1 \cdots dt_k \right] dy \\
& = \lambda e^{-\lambda T} \int_{-\infty}^0 \frac{g(y)}{f'(f^{-1}(y))} \mathbb{E} \left[ \sum_{k=0}^{\infty} \lambda^k \sum_{j=1}^k \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \right. \\
& \mathbf{1}_{\{y \leq \inf_{1 \leq l \leq j} (f(t_l) - Y_l)\}} \mathbf{1}_{\{f(t_{j-1}) - Y_j < y\}} \mathbf{1}_{\{y \leq \inf_{j \leq l \leq k} (f(t_l) - Y_{l+1})\}} dt_1 \cdots dt_k \\
& \left. + \sum_{k=0}^{\infty} \lambda^k \mathbf{1}_{\{Y_{k+1} + y < f(T)\}} \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \mathbf{1}_{\{f(t_k) < Y_{k+1} + y\}} \mathbf{1}_{\{y < \inf_{1 \leq l \leq k} (f(t_l) - Y_l)\}} dt_1 \cdots dt_k \right] dy \\
& = \int_{-\infty}^0 \frac{\lambda g(y)}{f'(f^{-1}(y))} \mathbb{E} \left[ \sum_{j=1}^{N(T)} \mathbf{1}_{\{y \leq \inf_{1 \leq l \leq j} (f(T_l) - Y_l)\}} \mathbf{1}_{\{f(T_{j-1}) - Y_j < y\}} \mathbf{1}_{\{y \leq \inf_{j \leq l \leq N(T)} (f(T_l) - Y_{l+1})\}} \right] dy \\
& + \int_{-\infty}^0 \frac{\lambda g(y)}{f'(f^{-1}(y))} \mathbb{E} \left[ \mathbf{1}_{\{0 < Y_{N(T)+1} + y < f(T)\}} \mathbf{1}_{\{f(T_{N(T)}) < Y_{N(T)+1} + y\}} \mathbf{1}_{\{\inf_{1 \leq l \leq N(T)} (f(T_l) - Y_l) > y\}} \right] dy.
\end{aligned}$$

□

Note that other analytic expressions for the density of  $\mathcal{M}_{[0,T]}$  can be obtained in some cases. For example, when  $(Y_k)_{k \geq 1}$  are independent, exponentially distributed random variables with parameter  $\mu > 0$  and  $f(t) = \alpha t$  is linear,  $\alpha \geq 0$ , Theorem 4.1 and Relation (4.6) of Dozzi and Vallois [6] show that

$$\begin{aligned}
& \mathbb{P}(\mathcal{M}_{[0,T]} < x) \\
& = \lambda \int_0^T \left( x \sum_{n=0}^{\infty} \frac{(\lambda \mu t (x + \alpha t))^n}{(n!)^2} + \alpha t \sum_{n=0}^{\infty} \frac{(\lambda \mu t (x + \alpha t))^n}{n!(n+1)!} \right) \frac{e^{-\mu(x+\alpha t) - \lambda t}}{x + \alpha t} dt,
\end{aligned}$$

which provides another expression for the density of  $\mathcal{M}_{[0,T]}$  by differentiation with respect to  $x$ .

Note that other series expansions for  $\sup_{0 \leq t < 1} X(t)$  have been recently obtained by Bernyk, Dalang and Peskir [2] when  $X(t)$  is a stable Lévy process with no negative jump.

We can use Proposition 5 to derive an expression for the sensitivity of the expectation

$$\mathbb{E}[h(R_x(T)) | \mathcal{M}_{[0,T]} < 0]$$

with respect to the initial reserve  $x$ .

**Corollary 3** *Assume that  $f(t) = \alpha t$ ,  $t \in \mathbb{R}_+$ , for some  $\alpha > 0$ . We have for all  $h \in \mathcal{C}_b^1(\mathbb{R})$ :*

$$\begin{aligned} & \frac{\partial}{\partial x} \mathbb{E}[h(R_x(T)) | \mathcal{M}_{[0,T]} < 0] \\ &= \frac{\lambda}{\alpha} \mathbb{E} \left[ \sum_{j=1}^{N(T)} h \left( x + \min \left( \inf_{1 \leq l \leq j} (\alpha T_l - Y_l), \inf_{j \leq l \leq N(T)} (\alpha T_l - Y_{l+1}) \right) \right) \right. \\ & \quad \left. - \sum_{j=1}^{N(T)} h(x + \alpha T_{j-1} - Y_j) | \mathcal{M}_{[0,T]} < 0 \right] \\ &+ \frac{\lambda}{\alpha} \mathbb{E} \left[ h \left( x + \min \left( \inf_{1 \leq l \leq N(T)} (\alpha T_l - Y_l), \alpha T - Y_{N(T)+1} \right) \right) | \mathcal{M}_{[0,T]} < 0 \right] \\ & \quad - \frac{\lambda}{\alpha} \mathbb{E} \left[ h(x + \alpha T_{N(T)} - Y_{N(T)+1}) | \mathcal{M}_{[0,T]} < 0 \right]. \end{aligned}$$

*Proof.* We apply Proposition 5 and the relation

$$\begin{aligned} \frac{\partial}{\partial x} \mathbb{E}[h(R_x(T)) | \mathcal{M}_{[0,T]} < 0] &= \mathbb{E}[h'(R_x(T)) | \mathcal{M}_{[0,T]} < 0] \\ &= \int_{-\infty}^{\infty} h'(x+z) \varphi_{\mathcal{M}_{[0,T]} | \mathcal{M}_{[0,T]} < 0}(z) dz, \end{aligned}$$

□

Note that the above formula has the form (2.4) (with constant weights  $\Lambda_j$ ) and, as noted in Section 2, it does not ensure the continuity of the probability density of  $\mathcal{M}_{[0,T]}$ .

### Random drift

In this section we study the effect of replacing the drift  $f(t)$  by a random process  $Z(t)$ . Now consider the infimum

$$\mathcal{M}_{[0,T]} = \inf_{0 \leq t \leq T} (Z(t) - S(t))$$

where  $(Z(t))_{t \in \mathbb{R}_+}$  is a stochastic process with independent increments and  $Z(0) = 0$ , independent of  $(S(t))_{t \in \mathbb{R}_+}$ , and such that

$$\inf_{t \in [a, b]} Z(t), \quad 0 \leq a < b,$$

has a density denoted by  $\varphi_{a,b}(x)$ . For example, if  $(Z(t))_{t \in \mathbb{R}_+}$  is a standard Brownian motion then  $\varphi_{a,b}(x)$  is given by

$$\begin{aligned} \int_x^\infty \varphi_{a,b}(z) dz &= \mathbb{P} \left( \inf_{t \in [a, b]} Z(t) \geq x \right) \\ &= \mathbb{E} \left[ \mathbf{1}_{\{Z(a) < x\}} \mathbb{P} \left( \inf_{t \in [a, b]} Z(t) \geq x \mid Z(a) \right) \right] + \mathbb{E} \left[ \mathbf{1}_{\{Z(a) \geq x\}} \mathbb{P} \left( \inf_{t \in [a, b]} Z(t) \geq x \mid Z(a) \right) \right] \\ &= \mathbb{E} \left[ \mathbf{1}_{\{Z(a) < x\}} \mathbb{P} \left( \inf_{t \in [0, b-a]} B(t) \geq x - Z(a) \mid Z(a) \right) \right] + \mathbb{P}(Z(a) \geq x) \\ &= 2\mathbb{E} \left[ \mathbf{1}_{\{Z(a) < x\}} \mathbb{P}(B(b-a) \geq x - Z(a) \mid Z(a)) \right] + \mathbb{P}(Z(a) \geq x) \\ &= \frac{1}{\pi \sqrt{a(b-a)}} \int_0^\infty e^{-(x-y)^2/(2a)} \int_y^\infty e^{-z^2/(2(b-a))} dz dy + \frac{1}{\sqrt{2\pi a}} \int_x^\infty e^{-z^2/(2a)} dz. \end{aligned}$$

We have  $\mathcal{M}_{[0, T]} \leq Z(0) = 0$  a.s., hence the law of  $\mathcal{M}_{[0, T]}$  is carried by  $(-\infty, 0]$ .

**Proposition 6** *The probability density of  $\mathcal{M}_{[0, T]}$  at  $y < 0$  is equal to*

$$\begin{aligned} \varphi_{\mathcal{M}_{[0, T]}}(y) &= -\lambda e^{-\lambda T} \mathbb{E} \left[ \sum_{k=0}^\infty \lambda^k \sum_{j=1}^{k+1} \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} \varphi_{t_{j-1}, t_j}(y + S(t_{j-1})) \right. \\ &\quad \left. \mathbb{P} \left( y + S(t_{k+1}) < \inf_{t \in [t_{k+1}, T]} Z(t) \mid S \right) \prod_{\substack{l=1 \\ l \neq j}}^{k+1} \mathbb{P} \left( y + S(t_{l-1}) < \inf_{t \in [t_{l-1}, t_l]} Z(t) \mid S \right) dt_1 \cdots dt_{k+1} \right] \\ &= -\lambda e^{-\lambda T} \mathbb{E} \left[ \sum_{k=0}^\infty \lambda^k \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} \right. \\ &\quad \left. \varphi_{t_{k+1}, T}(y + S(t_{k+1})) \prod_{l=1}^{k+1} \mathbb{P} \left( y + S(t_{l-1}) < \inf_{t \in [t_{l-1}, t_l]} Z(t) \mid S \right) dt_1 \cdots dt_{k+1} \right]. \end{aligned}$$

*Proof.* We have

$$\mathcal{M}_{[0, T]} = \min \left( \min_{T_k \leq T, k \geq 1} \inf_{t \in [T_{k-1}, T_k]} (Z(t) - S(T_{k-1})), \inf_{t \in [T_{N(T)}, T]} (Z(t) - S(T_{N(T)})) \right).$$

Hence

$$\begin{aligned}
\mathbb{P}(\mathcal{M}_{[0,T]} \geq y) &= e^{-\lambda T} \mathbb{E} \left[ \sum_{k=1}^{\infty} \lambda^k \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \right. \\
&\quad \left. \mathbf{1}_{\{y+S(t_k) < \inf_{t \in [t_k, T]} Z(t)\}} \prod_{l=1}^k \mathbf{1}_{\{y+S(t_{l-1}) < \inf_{t \in [t_{l-1}, t_l]} Z(t)\}} dt_1 \cdots dt_k \right] \\
&= \lambda e^{-\lambda T} \mathbb{E} \left[ \sum_{k=0}^{\infty} \lambda^k \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} \right. \\
&\quad \left. \mathbf{1}_{\{y+S(t_{k+1}) < \inf_{t \in [t_{k+1}, T]} Z(t)\}} \prod_{l=1}^{k+1} \mathbf{1}_{\{y+S(t_{l-1}) < \inf_{t \in [t_{l-1}, t_l]} Z(t)\}} dt_1 \cdots dt_{k+1} \right] \\
&= \lambda e^{-\lambda T} \mathbb{E} \left[ \sum_{k=0}^{\infty} \lambda^k \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} \right. \\
&\quad \left. \mathbb{P} \left( y + S(t_{k+1}) < \inf_{t \in [t_{k+1}, T]} Z(t) \middle| S \right) \prod_{l=1}^{k+1} \mathbb{P} \left( y + S(t_{l-1}) < \inf_{t \in [t_{l-1}, t_l]} Z(t) \middle| S \right) dt_1 \cdots dt_{k+1} \right],
\end{aligned} \tag{4.2}$$

and in order to determine the density  $\varphi_{\mathcal{M}_{[0,T]}}$  of  $\mathcal{M}_{[0,T]}$  it remains to compute the derivative  $-\frac{\partial}{\partial y} \mathbb{P}(\mathcal{M}_{[0,T]} \geq y)$ .  $\square$

By a simple change of variable this also allows one to treat the infima of exponential jump-diffusion processes, such as

$$\inf_{0 \leq t \leq T} e^{Z(t) - S(t)}.$$

## 5 Numerical simulations

We present an example of simulation when  $f(t) = t$  and  $W_k = 1$ ,  $k \in \mathbb{N}$ , i.e. for the infimum

$$\mathcal{M}_{[0,T]} = \inf_{0 \leq t \leq T} (t - N(t)) = \inf_{T_k \leq T, k \geq 0} (T_k - k) = \mathbf{1}_{\{N(T) \geq 1\}} \inf_{T_k \leq T, k \geq 1} (T_k - k).$$

In this case the (unconditional) density function found in Proposition 5 rewrites as

$$\begin{aligned}
&-\frac{\partial}{\partial y} \mathbb{P}(\{\mathcal{M}_{[0,T]} \geq y\}) \\
&= \lambda \mathbb{E} \left[ \sum_{j=1}^{N(T)} \mathbf{1}_{\{y \leq \inf_{1 \leq l \leq j-1} (T_l - l)\}} \mathbf{1}_{\{T_{j-1} - j < y\}} \mathbf{1}_{\{y \leq \inf_{j \leq l \leq N(T)} (T_l - l - 1)\}} \right] \\
&\quad + \lambda \mathbb{E} \left[ \mathbf{1}_{\{T_{N(T)} < N(T) + 1 + y < T\}} \mathbf{1}_{\{y < \inf_{1 \leq l \leq N(T)} (T_l - l)\}} \right] \\
&= \lambda \mathbb{E} \left[ \sum_{j=1}^{N(T)} \mathbf{1}_{\{y \leq \inf_{1 \leq l \leq j-1} (T_l - l)\}} \mathbf{1}_{\{T_{j-1} - j < y\}} \mathbf{1}_{\{y \leq \inf_{j \leq l \leq N(T)} (T_l - l - 1)\}} \right]
\end{aligned}$$



$$\begin{aligned}
& +\lambda \sum_{k=0}^{\infty} \mathbb{E} \left[ \mathbf{1}_{\{N(T)=k\}} \mathbf{1}_{\{T_k < k+1+y < T\}} \mathbf{1}_{\{y < \inf_{1 \leq l \leq k} (T_l - l)\}} \right] \\
& = \lambda \mathbb{E} \left[ \sum_{j=1}^{N(T)} \mathbf{1}_{\{y \leq \inf_{1 \leq l \leq j-1} (T_l - l)\}} \mathbf{1}_{\{T_{j-1} - j < y\}} \mathbf{1}_{\{y \leq \inf_{j \leq l \leq N(T)} (T_l - l - 1)\}} \right] \\
& +\lambda \sum_{i=0}^{\infty} \mathbf{1}_{[T-i-2, T-i-1]}(y) \sum_{k=0}^i \mathbb{E} \left[ \mathbf{1}_{\{N(T)=k\}} \mathbf{1}_{\{T_k < k+1+y\}} \mathbf{1}_{\{y < \inf_{1 \leq l \leq k} (T_l - l)\}} \right] \quad (5.1) \\
& = \lambda \mathbb{E} \left[ \sum_{j=1}^{N(T)+1} \mathbf{1}_{\{T_{j-1} - j < y \leq \min(\inf_{1 \leq l \leq j} (T \wedge T_l - l), \inf_{j \leq l \leq N(T)} (T_l - l - 1))\}} \right].
\end{aligned}$$

Note that the non-continuous component of the density appears explicitly in (5.1) of the above expression. For the purpose of sensitivity analysis, the result of Corollary 3 becomes:

$$\begin{aligned}
& \mathbb{E}[g'(y + \mathcal{M}_{[0, T]})] \\
& = \lambda \mathbb{E} \left[ \sum_{j=1}^{N(T)} g \left( \min \left( \inf_{1 \leq l \leq j-1} (T_l - l), \inf_{j \leq l \leq N(T)} (T_l - l - 1) \right) \right) - g(T_{j-1} - j) \right] \\
& + \lambda \mathbb{E} \left[ g \left( \min \left( \inf_{1 \leq l \leq N(T)} (T_l - l), T - N(T) - 1 \right) \right) - g(T_{N(T)} - N(T) - 1) \right] \\
& = \lambda \mathbb{E} \left[ \sum_{j=1}^{N(T)+1} g \left( \min \left( \inf_{1 \leq l \leq j} (T \wedge T_l - l), \inf_{j \leq l \leq N(T)} (T_l - l - 1) \right) \right) - g(T_{j-1} - j) \right].
\end{aligned}$$

For a same number of iterations, the integration by parts algorithm is not significantly slower than the finite differences method, because it only involves the computation of two infima instead of one. However it yields a much greater level of precision: one can check in Figure 1 that our results are much less noisy than the ones of the finite difference method. Besides, the density at each point is obtained independently from other points, which is not the case with finite difference or kernel estimation methods. This is especially important for non-continuous densities, for which kernel estimators will introduce some form of unwanted smoothing.

In Figure 2 we illustrate the fact that our method requires much fewer trials to accurately estimate the target value.

After this simple example, we also illustrate the case of exponentially and Pareto distributed claim amounts in Figures 3, 4 and 5 below, to show that our method is operational for typical light- and heavy-tailed insurance models. The respective computation times to obtain the graph of Figure 3 above are 2m35s for the finite difference method and 4m5s for the integration by parts method.

In Figure 4 we present a density estimate obtained via the integration by

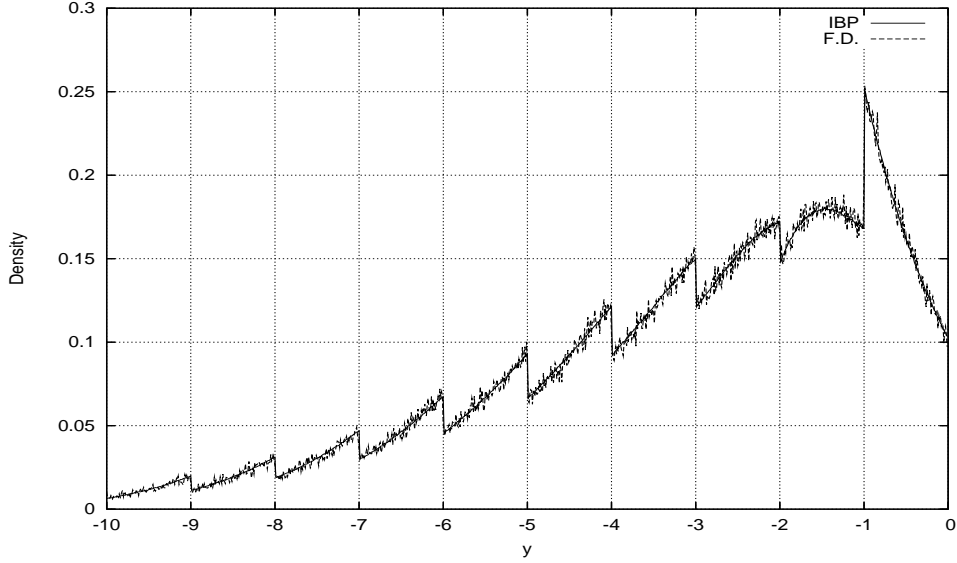


Figure 1. Estimation of the probability density of  $\mathcal{M}_{[0,T]}$  by our method (IBP) and by finite differences (FD) with  $N = 100000$  trials.

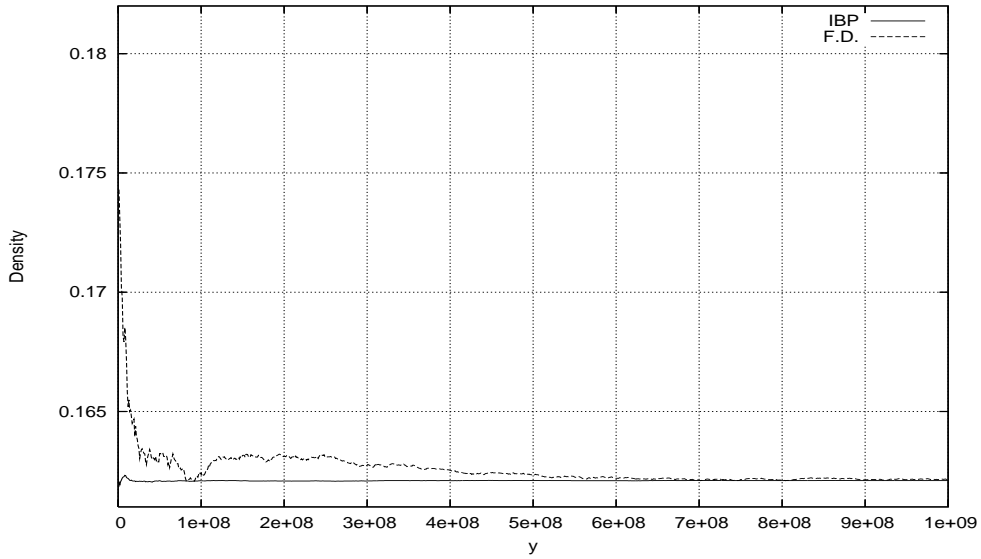


Figure 2. Estimation of the probability density of  $\mathcal{M}_{[0,T]}$  at  $y = -0.5$  vs number of trials by our method (IBP) and by finite differences (FD).

parts method with  $N = 1000$  samples and a computation time of 2.6s, to be compared with the similar level of precision reached in Figure 3 by a finite difference method with  $N = 100000$  samples and a computation time of 4m5s. Finally, in Figure 5 below we consider the case of Pareto distributed claims.

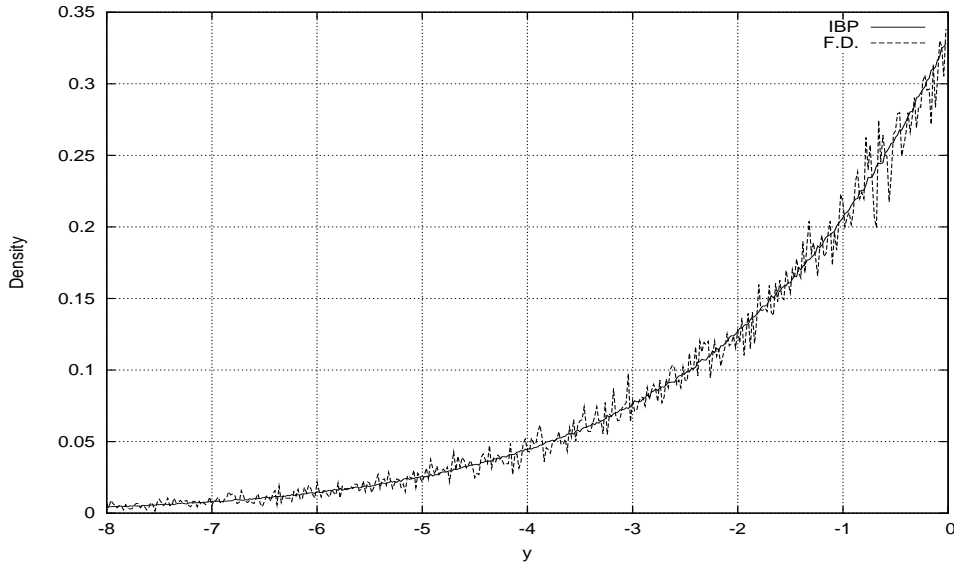


Figure 3. Probability density of  $\mathcal{M}_{[0,T]}$  by finite differences and integration by parts for exponentially distributed claim amounts with  $N = 100000$  trials.

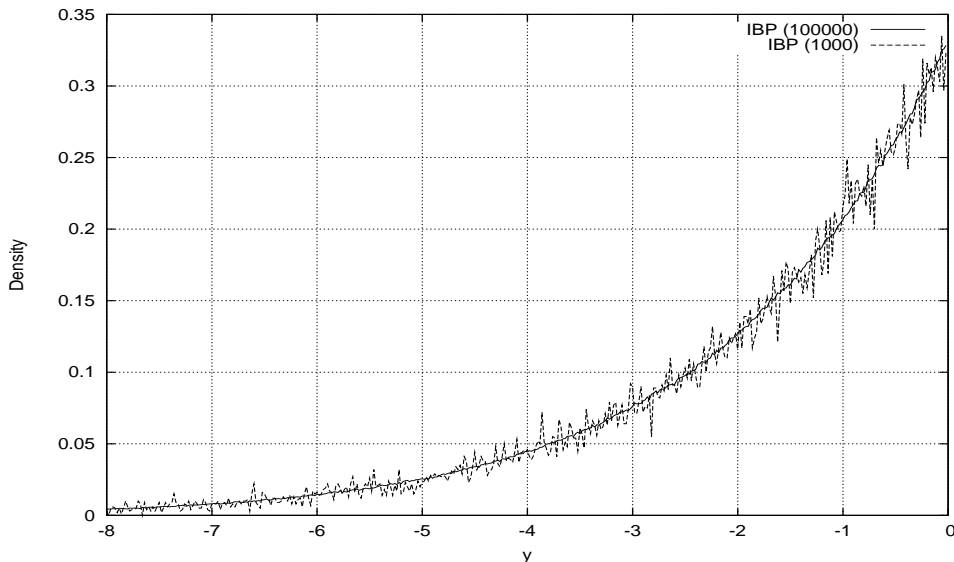


Figure 4. Comparison of density estimates of  $\mathcal{M}_{[0,T]}$  for exponentially distributed claim amounts by integration by parts with  $N = 1000$  and  $N = 100000$  trials.

## 6 Appendix

For completeness, in this appendix we provide the proofs of Lemma 1 and Proposition 3, which are conditional versions of existing results, see e.g. [16], [17], and the references therein.

*Proof of Lemma 1.* Recall that for all  $F \in \mathcal{S}_m$  of the form (3.6) we have:

$$\mathbb{E}_m[F] = e^{-\lambda T} \sum_{n=m}^{\infty} \lambda^n \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \dots, t_n) dt_1 \cdots dt_n.$$

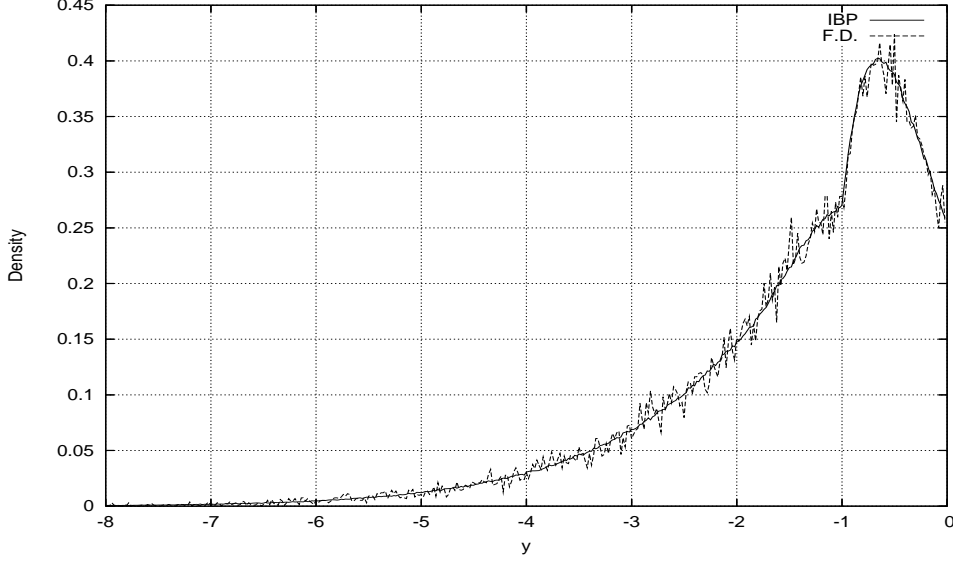


Figure 5. Probability density of  $\mathcal{M}_{[0,T]}$  by finite differences and integration by parts for Pareto distributed claim amounts with  $N = 100000$  trials.

By standard integration by parts we first prove (3.8) when  $u \in L^2([0, T])$  is deterministic:

$$\begin{aligned}
& \mathbb{E}_m[\langle DF, u \rangle] \\
&= -e^{-\lambda T} \sum_{n=m \vee 1}^{\infty} \frac{\lambda^n}{n!} \sum_{k=1}^n \int_0^T \cdots \int_0^T \int_0^{t_k} u(s) ds \partial_k f_n(t_1, \dots, t_n) dt_1 \cdots dt_n \\
&= e^{-\lambda T} \sum_{n=m \vee 1}^{\infty} \frac{\lambda^n}{n!} \sum_{k=1}^n \int_0^T \cdots \int_0^T f_n(t_1, \dots, t_n) u(t_k) ds dt_1 \cdots dt_n \\
&- e^{-\lambda T} \sum_{n=m \vee 1}^{\infty} \frac{\lambda^n}{(n-1)!} \int_0^T u(s) ds \int_0^T \cdots \int_0^T f_n(t_1, \dots, t_{n-1}, T) dt_1 \cdots dt_{n-1}.
\end{aligned}$$

From (3.7) we have the continuity condition

$$f_{n-1}(t_1, \dots, t_{n-1}) = f_n(t_1, \dots, t_{n-1}, T), \quad n \geq m, \quad (6.1)$$

hence

$$\begin{aligned}
\mathbb{E}_m[\langle DF, u \rangle] &= e^{-\lambda T} \sum_{n=m \vee 1}^{\infty} \frac{\lambda^n}{n!} \int_0^T \cdots \int_0^T f_n(t_1, \dots, t_n) \sum_{k=1}^n u(t_k) dt_1 \cdots dt_n \\
&\quad - \lambda e^{-\lambda T} \int_0^T u(s) ds \sum_{n=m}^{\infty} \frac{\lambda^n}{n!} \int_0^T \cdots \int_0^T f_n(t_1, \dots, t_n) dt_1 \cdots dt_n \\
&= \mathbb{E}_m \left[ F \left( \sum_{k=1}^{N(T)} u(T_k) - \lambda \int_0^T u(s) ds \right) \right] \\
&= \mathbb{E}_m \left[ F \int_0^T u(t) d(N(t) - \lambda t) \right].
\end{aligned}$$

Next we define  $\delta(uG)$ ,  $G \in \mathcal{S}_m$ , by (3.9), i.e.

$$\delta(uG) := G \int_0^T u(t) d(N(t) - \lambda dt) - \langle u, DG \rangle,$$

with for all  $F \in \mathcal{S}_m$ :

$$\begin{aligned} \mathbb{E}_m[G \langle DF, u \rangle] &= \mathbb{E}_m[\langle D(FG), u \rangle - F \langle DG, u \rangle] \\ &= \mathbb{E}_m \left[ F \left( G \int_0^T u(t) dN(t) - \langle DG, u \rangle \right) \right] \\ &= \mathbb{E}_m[F \delta(uG)], \end{aligned}$$

which proves (3.8). The closability of  $D$  then follows from the integration by parts formula (3.8): if  $(F_n)_{n \in \mathbb{N}} \subset \mathcal{S}_m$  is such that  $F_n \rightarrow 0$  in  $L^2(\Omega)$  and  $DF_n \rightarrow U$  in  $L^2(\Omega)$ , then (3.8) implies

$$\begin{aligned} |\mathbb{E}_m[\langle U, Gu \rangle_{L^2([0,T])}]| &\leq |\mathbb{E}_m[F_n \delta(uG)] - \mathbb{E}_m[UG]| + |\mathbb{E}_m[F_n \delta(uG)]| \\ &= |\mathbb{E}_m[\langle DF_n, u \rangle - U]G| + |\mathbb{E}_m[F_n \delta(uG)]| \\ &\leq \|\langle DF_n, u \rangle - U\|_{L^2(\{N(T) \geq m\})} \|G\|_{L^2(\{N(T) \geq m\})} \\ &\quad + \|F_n\|_{L^2(\{N(T) \geq m\})} \|\delta(uG)\|_{L^2(\{N(T) \geq m\})}, \quad n \in \mathbb{N}, \end{aligned}$$

hence  $\mathbb{E}_m[UG] = 0$ ,  $G \in \mathcal{S}_m$ , i.e.  $U = 0$  in  $L^2(\{N(T) \geq m\})$ , which implies  $U = 0$  in  $L^2(\Omega)$  by construction of  $\mathcal{S}_m$ .

As a consequence of (3.8) the operator  $D$  can be extended to its closed domain  $\text{Dom}_m(D)$  of functionals  $F \in L^2(\{N(T) \geq m\})$  for which there exists a sequence  $(F_n)_{n \in \mathbb{N}} \subset \mathcal{S}_m$  converging to  $F$  such that  $(DF_n)_{n \in \mathbb{N}}$  converges in  $L^2(\Omega \times \mathbb{R}_+)$ , by letting

$$DF = \lim_{n \rightarrow \infty} DF_n,$$

for all such  $F \in \text{Dom}_m(D)$ , and  $DF$  is well-defined due to the closability of  $D$ . The argument is similar for  $\delta$ .  $\square$

*Proof of Proposition 3.* For simplicity of notation, let

$$D_u F = \langle DF, u \rangle, \quad F \in \text{Dom}_m(D), \quad u \in L^2(\Omega \times [0, T]).$$

and

$$\nabla_u v(t) = \int_0^T \nabla_s v(t) ds, \quad u \in \mathcal{C}_c^1((0, T)).$$

For all  $u, v \in \mathcal{C}_c^2((0, T))$  we have

$$(D_u D_v - D_v D_u) T_n = -D_u \int_0^{T_n} v(s) ds + D_v \int_0^{T_n} u(s) ds$$

$$\begin{aligned}
&= v(T_n) \int_0^{T_n} u(s) ds - u(T_n) \int_0^{T_n} v(s) ds \\
&= \int_0^{T_n} \left( \dot{v}(t) \int_0^t u(s) ds - \dot{u}(t) \int_0^t v(s) ds \right) dt \\
&= D_{\nabla_u v - \nabla_v u} T_n,
\end{aligned}$$

hence

$$(D_u D_v - D_v D_u)F = D_{\nabla_u v - \nabla_v u} F, \quad F \in \mathcal{S}_m. \quad (6.2)$$

On the other hand we have

$$\begin{aligned}
D_u \delta(v) &= - \sum_{k=1}^{\infty} \dot{v}(T_k) \int_0^{T_k} u(s) ds \\
&= -\delta \left( v(\cdot) \int_0^\cdot u(s) ds \right) - \int_0^\infty \dot{v}(t) \int_0^t u(s) ds dt \\
&= \delta(\nabla_u v) + \langle u, v \rangle_{L^2(\mathbb{R}_+)},
\end{aligned}$$

hence the commutation relation

$$D_u \delta(v) = \delta(\nabla_u v) + \langle u, v \rangle_{L^2(\mathbb{R}_+)}, \quad u, v \in \mathcal{C}_c^2((0, T)), \quad (6.3)$$

between  $D$  and  $\delta$ .

Next, note that for  $u = \sum_{i=1}^n h_i F_i \in \mathcal{U}$  of the form (3.10) we have  $\delta(u) \in \text{Dom}_m(D)$  and

$$\begin{aligned}
&\mathbb{E}_m [\delta(h_i F_i) \delta(h_j F_j)] = \mathbb{E}_m [F_i D_{h_i} \delta(h_j F_j)] \\
&= \mathbb{E}_m [F_i D_{h_i} (F_j \delta(h_j) - D_{h_j} F_j)] \\
&= \mathbb{E}_m [(F_i F_j D_{h_i} \delta(h_j) + F_i \delta(h_j) D_{h_i} F_j - F_i D_{h_i} D_{h_j} F_j)] \\
&= \mathbb{E}_m [(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_i F_j \delta(\nabla_{h_i} h_j) + F_i \delta(h_j) D_{h_i} F_j - F_i D_{h_i} D_{h_j} F_j)] \\
&= \mathbb{E}_m [(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + D_{\nabla_{h_i} h_j} (F_i F_j) + D_{h_j} (F_i D_{h_i} F_j) - F_i D_{h_i} D_{h_j} F_j)] \\
&= \mathbb{E}_m [(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + D_{\nabla_{h_i} h_j} (F_i F_j) + D_{h_j} F_i D_{h_i} F_j + F_i (D_{h_j} D_{h_i} F_j - D_{h_i} D_{h_j} F_j)] \\
&= \mathbb{E}_m [(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + D_{\nabla_{h_i} h_j} (F_i F_j) + D_{h_j} F_i D_{h_i} F_j + F_i D_{\nabla_{h_j} h_i - \nabla_{h_i} h_j} F_j)] \\
&= \mathbb{E}_m [(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_j D_{\nabla_{h_i} h_j} F_i + F_i D_{\nabla_{h_j} h_i} F_j + D_{h_j} F_i D_{h_i} F_j)] \\
&= \mathbb{E}_m \left[ F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_j \int_0^T D_s F_i \int_0^T \nabla_t h_j(s) h_i(t) dt ds \right. \\
&\quad \left. + F_i \int_0^T D_t F_j \int_0^T \nabla_s h_i(t) h_j(s) ds dt + \int_0^T h_i(t) D_t F_j \int_0^T h_j(s) D_s F_i ds dt \right],
\end{aligned}$$

where we used the commutation relations (6.2) and (6.3).  $\square$

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