Cumulant operators and moments of the Itô and Skorohod integrals

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Abstract - We propose a formula for the computation of the moments of all orders of Itô and Skorohod stochastic integrals with respect to Brownian motion, based on cumulant operators defined by the Malliavin calculus. Some characterizations of Gaussian distributions for stochastic integrals are recovered as a consequence.

Opérateurs cumulants et moments des intégrales d'Itô et de Skorohod

Résumé - Nous proposons une formule de calcul des moments d'intégrales d'Itô et de Skorohod par rapport au mouvement Brownien à l'aide d'opérateurs cumulants définis par le calcul de Malliavin. On retrouve ainsi certaines caractérisations de la loi gaussienne pour les intégrales stochastiques.

Key words: Moments; cumulants; Itô integral; Skorohod integral, Malliavin calculus.
Mathematics Subject Classification: 60H05, 60H07.

1 Introduction

The moments of a random variable X are linked to its cumulants $(\kappa_n^X)_{n\geq 1}$ by the combinatorial identity

$$E[X^n] = \sum_{a=1}^{n} \sum_{B_1,\ldots,B_a} \kappa_{|B_1|}^X \cdots \kappa_{|B_a|}^X, \tag{1.1}$$

where the sum runs over the partitions $B_1, \ldots, B_a$ of $\{1, \ldots, n\}$ with cardinal $|B_i|$ by the Faà di Bruno formula, cf. [5], [6] and references therein for background on combinatorial probability. When X is centered Gaussian, e.g. X is the Wiener integral of a deterministic function with respect to a standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, we have $\kappa_n^X = 0$, $n \neq 2$, and (1.1) reads as Wick's theorem for the computation of Gaussian moments of X counting the pair partitions of $\{1, \ldots, n\}$, cf. [1].

When $X = \int_0^\infty u_t dB_t$ is the (centered) stochastic integral of a square-integrable adapted process $(u_t)_{t\in\mathbb{R}_+}$ the second moment of X is given by the Itô isometry, and higher order moments can be evaluated by decomposing the power $\left(\int_0^\infty u_t dB_t\right)^n$ into a sum of multiple integrals with vanishing expectation plus a remainder term. In this Note we derive a moment formula for X by computing the expectation of this remainder term using cumulant operators defined through the duality relation between the gradient D and divergence δ of the Malliavin calculus. Being based on the Skorohod extension of the adapted Itô integral, our results also include the case where the process u is anticipating with respect

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to the Brownian filtration.

A different approach to cumulants using the Malliavin calculus has been developed in [3], based on the inverse L^{-1} of the Ornstein-Uhlenbeck operator $L = \delta D$ on the Wiener space. The present representation is different and complementary as it is specially adapted to the stochastic integral $\delta(u)$ and it does not involve L^{-1} as in [3].

This Note is a special case on the Wiener space of a more general construction presented in [10], that includes the Lie-Wiener path space and the Poisson space.

2 Cumulant operators

We work on the Wiener space $(\Omega, \mathcal{F}, \mu)$ of a d-dimensional Brownian motion, on which is defined the Skorohod stochastic integral operator (or divergence) δ which coincides with the stochastic integral with respect to $(B_t)_{t\in\mathbb{R}_+}$ on the square-integrable adapted processes with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by $(B_t)_{t\in\mathbb{R}_+}$. The operator δ admits an adjoint gradient operator D that satisfies the duality relation

$$E[F\delta(v)] = E[\langle DF, v\rangle_H], \quad F \in \mathrm{Dom}(D), \quad v \in \mathrm{Dom}(\delta), \tag{2.1}$$

where $H = L^2(\mathbb{R}_+; \mathbb{R}^d)$. We let $\mathbb{D}_{p,k}$, resp. $\mathbb{D}_{p,k}(H)$, $p, k \geq 1$, denote the standard Sobolev spaces of real-valued, resp. H-valued, functionals on the Wiener space, cf. [4] for a definition. The composition $(Du)^l$ and the adjoint D^* are defined in the sense of matrix powers on with continuous indices, cf. e.g. § 7 of [10] for details.

Definition 1. *Given $k \geq 1$ and $u \in \mathbb{D}_{k,2}(H)$, the cumulant operator $\Gamma_k^u : \mathbb{D}_{2,1} \longrightarrow L^2(\Omega)$ is defined by $\Gamma_1^u \mathbf{1} = 0$ and*

$$\Gamma_k^u \mathbf{1} = \langle (Du)^{k-2}u, u\rangle_H + \langle D^*u, D((Du)^{k-2}u)\rangle_{H\otimes H}, \qquad k \geq 2, \tag{2.2}$$

and is extended to all $F \in \mathbb{D}_{2,1}$ by the definition

$$\Gamma_k^u F := F\Gamma_k^u \mathbf{1} + \langle (Du)^{k-1}u, DF\rangle_H, \qquad k \geq 1. \tag{2.3}$$

By (2.2) we have $\Gamma_2^u \mathbf{1} = \langle u, u\rangle_H + \langle D^*u, Du\rangle_{H\otimes H}$, which from (3.2) below yields the Skorohod isometry

$$E[\delta(u)^2] = E[\Gamma_2^u \mathbf{1}] = E[\langle u, u\rangle_H] + E[\langle D^*u, Du\rangle_{H\otimes H}].$$

Proposition 1. *Letting $u \in \mathbb{D}_{2,2}(H)$, for all $k \geq 3$ we have*

$$\Gamma_k^u \mathbf{1} = \frac{1}{2}\langle (Du)^{k-3}u, D\langle u, u\rangle_H\rangle_H + \mathrm{trace}\,(Du)^k + \sum_{i=2}^{k-1} \frac{1}{i}\langle (Du)^{k-1-i}u, D\mathrm{trace}\,(Du)^i\rangle_H. \tag{2.4}$$

Proof. Letting $k \geq 3$ and $u \in \mathbb{D}_{k,k}(H)$, we apply the relation

$$\langle (Du)^k v, u\rangle_H = \frac{1}{2}\langle (Du)^{k-1}v, D\langle u,u\rangle_H\rangle_H, \qquad v \in H, \tag{2.5}$$

cf. e.g. (2.3) in [7], to the first term in the right hand side of (2.2). Next, by the proof of Lemma 3.1 in [7] we have

$$\langle D^*u, D((Du)^k v)\rangle_{H\otimes H} = \mathrm{trace}\,((Du)^{k+1}Dv) + \sum_{i=2}^{k+1}\frac{1}{i}\langle (Du)^{k+1-i}v, D\mathrm{trace}\,(Du)^i\rangle_H, \tag{2.6}$$

$u \in \mathbb{D}_{2,2}(H)$, $v \in \mathbb{D}_{2,1}(H)$, $k \in \mathbb{N}$. $\square$

By (2.6) we have

$$\langle D^*u, D((Du)^{k-2}u)\rangle_{H\otimes H} = 0, \qquad k \geq 2, \tag{2.7}$$

under the quasi-nilpotence condition

$$\mathrm{trace}\,(Du)^n = 0, \qquad n \geq 2, \tag{2.8}$$

which is satisfied in particular when the process u is adapted with respect to the Brownian filtration, $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Indeed, in this case for almost all $t_1,\dots,t_n \in \mathbb{R}_+$ there exists $i \in \{1,\dots,n\}$ such that $t_i > t_{i+1 \bmod n}$, which gives $D_{t_i}u_{t_{i+1 \mathrm{mod}\ n}} = 0$ by Corollary 1.2.1 of [4] since $u_{t_{i+1\mathrm{mod}\ n}}$ is $\mathcal{F}_{t_{i+1\mathrm{mod}\ n}}$-measurable. In this case we find

$$\Gamma_k^u \mathbf{1} = \mathbf{1}_{\{k=2\}}\int_0^\infty |u_t|^2 dt + \mathbf{1}_{\{k\geq 3\}}\frac{1}{2}\left\langle (Du)^{k-3}u, D\int_0^\infty |u_t|^2 dt\right\rangle, \qquad k \geq 1. \tag{2.9}$$

3 Moment identities

Our covariance-moment relation (3.2) is established in the next Proposition, and can be seen as a non-linear (polynomial) extension of the integration by parts formula (2.1) between D and δ, where $\Gamma_1^h F = \langle h, DF\rangle_H$, $F \in \mathbb{D}_{2,1}$, $h \in H$. By inversion of the classical cumulant formula (1.1), cf. [2] Theorem 1, the cumulant κ_n^X can be computed from the moments μ_n^X of X.

Theorem 1. *Let $F \in \mathbb{D}_{2,1}$ and $u \in \mathbb{D}_{2,1}(H)$, $n \geq 1$, and assume that*

$$\Gamma_{l_1}^u \cdots \Gamma_{l_k}^u F \in \mathbb{D}_{2,1}, \tag{3.1}$$

for all $l_1 + \cdots + l_k \leq n$, $k = 1,\dots,n$. Then we have

$$E[F\delta(u)^n] = n!\sum_{a=1}^n \lambda^a \sum_{\substack{l_1+\cdots+l_a=n \\ l_1\geq 1,\dots,l_a\geq 1}} \frac{E\left[\Gamma_{l_1}^u\cdots\Gamma_{l_a}^u F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_{a+1})}. \tag{3.2}$$

Proof. Our proof will use an induction argument based on the identity

$$E[F\delta(u)^n] = \sum_{l=0}^{n-1} \frac{(n-1)!}{l!} E\left[\delta(u)^l \Gamma^u_{n-l} F\right], \tag{3.3}$$

that follows from (2.3) above and Lemma 2.2 of [9], or Theorem 2.1 of [7] in case $F = 1$, and can be seen as a stochastic version of the Thiele [11] recursion formula between moments and cumulants of random variables, cf. e.g. § 1.3.2 of [6]. For $n = 1$, (3.1) is the duality relation (2.1). Next, assuming that (3.2) holds up to the rank $n \geq 1$, we have, at the order $n+1$,

$$\begin{aligned}
E[F\delta(u)^{n+1}] &= \lambda \sum_{k=1}^{n+1} \frac{n!}{(n+1-k)!} E\left[\delta(u)^{n+1-k}\Gamma^u_k F\right] \\
&= \lambda n! \sum_{k=1}^{n+1} \sum_{a=1}^{n+1-k} \lambda^a \sum_{\substack{l_1+\cdots+l_a=n+1-k \\ l_1\geq 1,\ldots,l_a\geq 1}} \frac{E\left[\Gamma^u_{l_1}\cdots\Gamma^u_{l_a}\Gamma^u_k F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)} \\
&= n! \sum_{a=1}^{n+1} \lambda^{a+1} \sum_{\substack{l_1+\cdots+l_a=n+1-l_{a+1} \\ l_1\geq 1,\ldots,l_a\geq 1}} \sum_{l_{a+1}=1}^{n+1} \frac{E\left[\Gamma^u_{l_1}\cdots\Gamma^u_{l_{a+1}} F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)} \\
&= n! \sum_{a=1}^{n+1} \lambda^{a+1} \sum_{\substack{l_1+\cdots+l_{a+1}=n+1 \\ l_1\geq 1,\ldots,l_{a+1}\geq 1}} \frac{E\left[\Gamma^u_{l_1}\cdots\Gamma^u_{l_{a+1}} F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)} \\
&= (n+1)! \sum_{a=1}^{n+1} \lambda^a \sum_{\substack{l_1+\cdots+l_a=n+1 \\ l_1\geq 1,\ldots,l_a\geq 1}} \frac{E\left[\Gamma^u_{l_1}\cdots\Gamma^u_{l_a} F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_{a+1})}.
\end{aligned}$$

□

When $h \in H$ is deterministic, Definition 1 reads

$$\Gamma^h_k F = \mathbf{1}_{\{k=2\}} F \langle h, h\rangle_H + \mathbf{1}_{\{k=1\}} \langle h, DF\rangle_H, \qquad k \geq 1,$$

and (3.1) becomes

$$\begin{aligned}
E[F\delta(h)^n] &= \sum_{a=1}^{n} \lambda^a \sum_{\substack{l_1+\cdots+l_a=n \\ 1\leq l_1\leq 2,\ldots,1\leq l_a\leq 2}} \mathcal{N}_{\mathfrak{L}_a} (l_a-1)!\cdots(l_1-1)! E\left[\Gamma^h_{l_1}\cdots\Gamma^h_{l_a} F\right] \\
&= \sum_{k=0}^{n} \binom{n}{k} E[(\langle h, DF\rangle_H)^k] E\left[\left(\int_0^\infty h(t)dB_t\right)^{n-k}\right].
\end{aligned}$$

If, in addition to (2.7), $\int_0^\infty |u_t|^2 dt$ is deterministic we find

$$\Gamma^u_{l_k}\cdots\Gamma^u_{l_1}\mathbf{1} = \mathbf{1}_{\{l_1=2\}}\cdots\mathbf{1}_{\{l_k=2\}} \left(\int_0^\infty |u_t|^2 dt\right)^k, \qquad l_1,\ldots,l_k \geq 1,$$

and $\delta(u)$ has cumulants

$$\Gamma_l^u \mathbf{1} = \mathbf{1}_{\{l=2\}} \langle u, u \rangle_H, \qquad l \geq 1, \tag{3.4}$$

i.e. $\delta(u)$ becomes a centered Gaussian random variable with variance $\langle u, u \rangle_H$. This applies in particular when $u = Rh$ is given from a random adapted isometry $R : L^p(\mathbb{R}_+) \longrightarrow L^p(\mathbb{R}_+)$, $p \geq 1$, cf. [12] Theorem 2.1-b), in which case the Skorohod integral $\delta(Rh)$ on the Wiener space has a Gaussian law when $h \in H$ and R is a random isometry of H with quasi-nilpotent gradient DRh.

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