Extended covariance identities and inequalities

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Abstract

We state an abstract version of covariance identities and inequalities for normal martingales, which uses any gradient operator that satisfies a Clark formula. This extends and makes more precise some results of C. Houdré and V. Pérez-Abreu in *Ann. Probab.* 23, 1995, with simplified proofs.

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1 Introduction

Covariance identities for functionals on the Wiener and Poisson spaces have been obtained in Houdré and Pérez-Abreu (1995). In this note we provide a simplified proof of these identities in the general context of normal martingales, and we make precise some statements in Houdré and Pérez-Abreu (1995) on covariance identities with the gradient of Carlen and Pardoux (1990) on Poisson space. The identity proved in Houdré and Pérez-Abreu (1995) is

$$\begin{aligned}\mathrm{Cov}(F,G) &= \sum_{k=1}^{k=n}\frac{(-1)^{k+1}}{k!}E\left[\int_{\mathbb{R}_+^k}(D_{t_1}\cdots D_{t_k}F)(D_{t_1}\cdots D_{t_k}G)dt_1\cdots dt_k\right] \quad (1.1)\\ &\quad +\frac{(-1)^n}{n!}E\left[\int_{\mathbb{R}_+^n}\int_{\max(t_1,\ldots,t_n)}E\left[D_{t_1}\cdots D_{t_n}D_sF\mid\mathcal{F}_s\right]\right.\\ &\quad \left.\times\, E\left[D_{t_1}\cdots D_{t_n}D_sG\mid\mathcal{F}_s\right]ds t_1\cdots dt_n\right],\end{aligned}$$

where D is the operator defined by lowering the degree of multiple stochastic integrals with respect to Brownian motion, or with respect to the standard Poisson process, and $F, G \in \bigcap_{k=1}^{k=n+1} I\!D_{k,2}$, where $I\!D_{k,2}$ is the L^2 domain of D^k, defined by completion with respect to the norm

$$\|F\|^2_{I\!D(\Delta_n)} = E[F^2] + E\left[\int_{\mathbb{R}^n_+} (D_{t_n} \cdots D_{t_1} F)^2 dt_1 \cdots dt_n\right].$$

Since $D_{t_k} \cdots D_{t_1} F$ is symmetric in $(t_1, \ldots, t_k)$, Relation (1.1) can be rewritten as

$$\begin{aligned}\mathrm{Cov}(F, G) &= \sum_{k=1}^{k=n} (-1)^{k+1} E\left[\int_{\Delta_k} (D_{t_k} \cdots D_{t_1} F)(D_{t_k} \cdots D_{t_1} G) dt_1 \cdots dt_k\right] \qquad (1.2)\\ &+(-1)^n E\left[\int_{\Delta_{n+1}} E\left[D_{t_{n+1}} \cdots D_{t_1} F \mid \mathcal{F}_{t_{n+1}}\right] E\left[D_{t_{n+1}} \cdots D_{t_1} G \mid \mathcal{F}_{t_{n+1}}\right] dt_1 \cdots dt_{n+1}\right],\end{aligned}$$

with $\Delta_n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n_+ \ : \ 0 \le t_1 < \cdots < t_n\}$, $n \ge 1$. The proof of Houdré and Pérez-Abreu (1995) relies in particular on the commutation relation

$$D_t \delta(u) = u_t + \delta(D_t u) \qquad (1.3)$$

between D and its adjoint δ. It is claimed in Houdré and Pérez-Abreu (1995), that (1.1) also holds on Poisson space if one takes for D the gradient operator of Carlen and Pardoux (1990) defined as

$$\hat{D}_t f(T_1, \ldots, T_n) = -\sum_{k=1}^{k=n} \partial_k f(T_1, \ldots, T_n)(1_{[0,T_k]}(t) - T_k), \qquad (1.4)$$

where $(T_k)_{k\ge1}$ denote the jump times of a standard Poisson process. However this is incorrect and the proof given there is only formal since

a) the operator $\hat{D}$ can not be iterated due to the non-differentiability in T_k of the indicator function $1_{[0,T_k]}(t)$, hence (1.1) can not hold for $\hat{D}$,

b) the commutation relation (1.3) is not satisfied by $\hat{D}$ (see Relation (4.4) below),

c) the adjoint of $\hat{D}$ coincides with the compensated Poisson integral only on processes of zero integral on $[0, 1]$ (see Th. 3.1 in Carlen and Pardoux (1990)), and as a consequence the Clark formula does not hold for $\hat{D}$.

We show that (1.3) is in fact not needed in order to prove the covariance identity (1.1), in particular such an identity still holds for a modification $\tilde{D}$ of $\hat{D}$ (see (4.1 below) which does not satisfy (1.3). For this we develop an abstract derivation of the covariance identity (1.2) with a shortened proof (see Th. 1 below) requiring essentially the Clark formula as assumption on the operator D.

We proceed as follows: first in Sect. 2 we provide a simplified proof of covariance identities that relies on minimal hypothesis, in particular it does not require (1.3) and it applies to any normal martingale with the chaos representation property. Apart from the the Wiener and Poisson processes, examples of such martingales are given by the family of Azéma martingales. Second, in Sect. 3 we show that this extended covariance identity also applies to a modification $\tilde{D}$ of the gradient of Carlen and Pardoux (1990), which is by nature completely different from D.

2 Covariance identities and inequalities

Let $(M_t)_{t\in\mathbb{R}_+}$ be a martingale on $(\Omega, \mathcal{F}, P)$ such that

(i) $d\langle M_t, M_t\rangle = dt$, i.e. $(M_t)_{t\in\mathbb{R}_+}$ is a *normal* martingale.

Let $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ denote the filtration generated by $(M_t)_{t\in\mathbb{R}_+}$. Let $\mathcal{S}$ be a dense subspace of $L^2(\Omega, \mathcal{F}, dP)$, and let

$$\mathcal{U} = \left\{ \sum_{i=1}^{i=n} 1_{[a_i,b_i]} F_i \ : \ F_i \in \mathcal{S},\ 0 \le a_i < b_i,\ i = 1, \ldots, n,\ n \ge 1 \right\}.$$

Let $D : L^2(\Omega, dP) \to L^2(\Omega \times \mathbb{R}_+, dP \times dt)$ and $\delta : L^2(\Omega \times \mathbb{R}_+, dP \times dt) \to L^2(\Omega, dP)$ be linear operators defined respectively on $\mathcal{S}$ and $\mathcal{U}$ with the duality relation

$$E[\langle DF, u\rangle_{L^2(\mathbb{R}_+)}] = E[F\delta(u)], \quad F \in \mathcal{S},\ u \in \mathcal{U}, \tag{2.1}$$

which implies the closability of D and δ. Let $\mathbb{D}([a,\infty[)$, $a > 0$, denote the completion of $\mathcal{S}$ under the norm

$$\|F\|^2_{\mathbb{D}([a,\infty[)} = E[F^2] + \int_a^\infty (D_t F)^2 dt,$$

i.e. $(D_t F)_{t\in[a,\infty[}$ is defined in $L^2(\Omega \times [a,\infty[)$ for $F \in \mathbb{D}([a,\infty[)$. We assume that

(ii) D satisfies a Clark representation formula, i.e. every $F \in \mathbb{D}([0,\infty[)$ has a representation

$$F = E[F] + \int_0^\infty E[D_t F \mid \mathcal{F}_t] dM_t. \qquad (2.2)$$

We further assume that

(iii) For all $F \in \mathcal{S}$ and a.e. $(t_1, \ldots, t_n) \in \Delta_n$, $D_{t_n} \cdots D_{t_1} F \in \mathbb{D}([t_n, \infty[)$.

Let $\mathbb{D}(\Delta_n)$ be the completion of $\mathcal{S}$ under the norm

$$\|F\|^2_{\mathbb{D}(\Delta_n)} = E[F^2] + E\left[\int_{\Delta_n} (D_{t_n} \cdots D_{t_1} F)^2 dt_1 \cdots dt_n\right].$$

Examples of operators satisfying (ii) and (iii) will be given in Sect. 3 by annihilation operators on multiple stochastic integrals with respect to normal martingales, and also in Sect. 4 by the operator $\tilde{D}$ which is of a completely different nature. Condition (ii) implies

(iv) δ coincides with the Itô integral with respect to $(M_t)_{t\in\mathbb{R}_+}$ on the adapted processes in $\mathcal{U}$, since

$$\begin{aligned} E[\langle DF, u\rangle_{L^2(\mathbb{R}_+)}] &= E\left[\int_0^\infty E[D_t F \mid \mathcal{F}_t] u(t) dt\right] \\ &= E\left[\int_0^\infty E[D_t F \mid \mathcal{F}_t] dM_t \int_0^\infty u(t) dM_t\right] \\ &= E\left[(F - E[F]) \int_0^\infty u(t) dM_t\right] = E\left[F \int_0^\infty u(t) dM_t\right] = E[F\delta(u)]. \end{aligned}$$

And (iv) reciprocally implies (ii) if $\{\delta(u) \ : \ u \in \mathcal{U}, \ u \text{ adapted}\}$ is dense in $L^2(\Omega, \mathcal{F}, dP)$:

$$\begin{aligned} E[(F - E[F])\delta(u)] &= E[F\delta(u)] = E[\langle DF, u\rangle_{L^2(\mathbb{R}_+)}] \\ &= E\left[\int_0^\infty E[D_t F \mid \mathcal{F}_t] dM_t \int_0^\infty u(t) dM_t\right] \\ &= E\left[\int_0^\infty E[D_t F \mid \mathcal{F}_t] dM_t \delta(u)\right]. \end{aligned}$$

A representation result for $F \in \mathbb{D}([a,\infty[)$ can be stated as a consequence of the Clark formula:

Lemma 1 *Let $a \in \mathbb{R}_+$ and $F \in I\!D([a,\infty[)$. We have*

$$F = E[F \mid \mathcal{F}_a] + \int_a^\infty E[D_t F \mid \mathcal{F}_t] dM_t. \tag{2.3}$$

Proof. Relation (2.3) holds if $F \in \mathcal{S}$, because

$$F - \int_a^\infty E[D_t F \mid \mathcal{F}_t] dM_t = E[F] + \int_0^a E[D_t F \mid \mathcal{F}_t] dM_t$$

is $\mathcal{F}_a$-measurable, and

$$\int_a^\infty E[D_t F \mid \mathcal{F}_t] dM_t$$

is orthogonal to $L^2(\Omega, \mathcal{F}_a, dP)$ in $L^2(\Omega, \mathcal{F}, dP)$. A density argument concludes the proof. □

The following Lemma is an immediate consequence of (2.3) with the fact that $(M_t)_{t\in\mathbb{R}_+}$ is a normal martingale.

Lemma 2 *We have for $F \in I\!D([a,\infty[)$:*

$$E[(E[F \mid \mathcal{F}_a])^2] = E[F^2] - E\left[\int_a^\infty (E[D_b F \mid \mathcal{F}_b])^2 db\right].$$

Next we prove the extension of the covariance identity of Houdré and Pérez-Abreu (1995), with a shortened proof.

Theorem 1 *Let $n \in \mathbb{N}$ and $F, G \in \bigcap_{k=1}^{k=n+1} I\!D(\Delta_k)$. We have*

$$\begin{aligned} \mathrm{Cov}(F,G) &= \sum_{k=1}^{k=n} (-1)^{k+1} E\left[\int_{\Delta_k} (D_{t_k}\cdots D_{t_1}F)(D_{t_k}\cdots D_{t_1}G) dt_1\cdots dt_k\right] \qquad (2.4) \\ &+(-1)^n E\left[\int_{\Delta_{n+1}} E\left[D_{t_{n+1}}\cdots D_{t_1}F \mid \mathcal{F}_{t_{n+1}}\right] E\left[D_{t_{n+1}}\cdots D_{t_1}G \mid \mathcal{F}_{t_{n+1}}\right] dt_1\cdots dt_{n+1}\right]. \end{aligned}$$

Proof. By bilinearity we may take $F = G$. For $n = 0$, (2.4) is a consequence of the Clark formula. Let $n \geq 1$. Applying Lemma 2 to $D_{t_n}\cdots D_{t_1}F$ with $a = t_n$ and $db = dt_{n+1}$, and integrating on $(t_1,\cdots,t_n) \in \Delta_n$, we obtain

$$\begin{aligned} E\left[\int_{\Delta_n} (E[D_{t_n}\cdots D_{t_1}F \mid \mathcal{F}_{t_n}])^2 dt_1\cdots dt_n\right] &= E\left[\int_{\Delta_n} (D_{t_n}\cdots D_{t_1}F)^2 dt_1\cdots dt_n\right] \\ &\quad - E\left[\int_{\Delta_{n+1}} \left(E\left[D_{t_{n+1}}\cdots D_{t_1}F \mid \mathcal{F}_{t_{n+1}}\right]\right)^2 dt_1\cdots dt_{n+1}\right], \end{aligned}$$

which concludes the proof by induction. □

We note the inclusion $I\!D_{k,2} \subset I\!D(\Delta_k)$, $k \geq 1$, which shows that even on Wiener space, Th. 1 extends (1.1). This fact will also allow us to apply Th. 1 to the gradient of Carlen and Pardoux (1990), see Sect. 3. The variance inequality

$$\sum_{k=1}^{k=2n} (-1)^{k+1} \|D^k F\|^2_{L^2(\Delta_k)} \leq \mathrm{Var}(F) \leq \sum_{k=1}^{k=2n-1} (-1)^{k+1} \|D^k F\|^2_{L^2(\Delta_k)},$$

$F \in \bigcap_{k=1}^{k=2n} I\!D(\Delta_k)$, is a consequence of Th. 1, and extends (2.15) in Houdré and Pérez-Abreu (1995).

3 Normal martingales

In this section we show that the hypothesis (ii) and (iii) are satisfied by the gradient operator D defined by lowering the degree of multiple stochastic integrals with respect a family of martingales that includes Brownian motion and the compensated Poisson process as particular cases. Let $(M_t)_{t\in\mathbb{R}_+}$ be a normal martingale, i.e. $(M_t)_{t\in\mathbb{R}_+}$ satisfies (i). The multiple stochastic integral $I_n(f_n)$ of the symmetric square-integrable function $f_n \in L^2(\mathbb{R}_+)^{\circ n}$ is defined as

$$I_n(f_n) = n! \int_{\Delta_n} f_n(t_1, \ldots, t_n) dM_{t_1} \cdots dM_{t_n}, \quad n \geq 1,$$

with

$$E[I_n(f_n) I_m(g_m)] = n! 1_{\{n=m\}} \langle f_n, g_m \rangle_{L^2(\mathbb{R}_+)^{\circ n}}, \tag{3.1}$$

and

$$E[I_n(f_n) \mid \mathcal{F}_t] = I_n(f_n 1_{[0,t]}^{\circ n}), \quad f_n \in L^2(\mathbb{R}_+)^{\circ n},\ n \geq 1,\ t \in \mathbb{R}_+.$$

Note that since $(M_t)_{t\in\mathbb{R}_+}$ is normal it is equivalent to define $I_n(f_n)$ in $L^2(\Omega, \mathcal{F}, dP)$ by integration on Δ_n or on $\{(t_1, \ldots, t_n) \ : \ 0 \leq t_1 \leq \cdots \leq t_n\}$. We assume that $(M_t)_{t\in\mathbb{R}_+}$ has the chaos representation property, i.e. every $F \in L^2(\Omega, \mathcal{F}, dP)$ has a decomposition $F = \sum_{n=0}^{\infty} I_n(f_n)$. Let

$$\mathcal{S} = \left\{ \sum_{k=0}^{k=n} I_k(f_k) \ : \ f_i \in \mathcal{C}_c(\mathbb{R}_+^k) \cap L^2(\mathbb{R}_+)^{\circ k},\ i = 0, \ldots, n\ n \geq 0 \right\}.$$

Let $D : L^2(\Omega, dP) \longrightarrow L^2(\Omega \times \mathbb{R}_+, dP \times dt)$ be defined on $\mathcal{S}$ as

$$D_t I_n(f_n) = n I_{n-1}(f_n(*, t)), \quad t \in \mathbb{R}_+,$$

and let δ be defined on $\mathcal{U}$ by

$$\delta(I_n(f_n \otimes 1_{[a,b]})) = I_{n+1}(f_n \circ 1_{[a,b]}), \quad 0 \leq a < b,$$

where $f_n \circ 1_{[a,b]}$ denotes the symmetrization of $f_n \otimes 1_{[a,b]}$. It is well-known that from (3.1) the duality relation (2.1) is satisfied by D and δ. The Clark formula (ii) is a consequence of the chaos representation property:

$$\begin{aligned} F &= E[F] + \sum_{n=1}^{\infty} n! \int_{\Delta_n} f_n(t_1, \ldots, t_n) dM_{t_1} \cdots dM_{t_n} \\ &= E[F] + \sum_{n=1}^{\infty} n \int_0^{\infty} I_{n-1}(f_n(*, t_n) 1_{\{* \leq t_n\}}) dM_{t_n} = E[F] + \int_0^{\infty} E[D_t F \mid \mathcal{F}_t] dM_t. \end{aligned}$$

Corollary 1 *Let $F, G \in \bigcap_{k=1}^{k=2n} \mathbb{D}(\Delta_k)$ have chaos expansions $F = \sum_{n=0}^{\infty} I_n(f_n)$ and $G = \sum_{n=0}^{\infty} I_n(g_n)$ then (2.4) can be written as*

$$\begin{aligned} \mathrm{Cov}(F, G) &= \sum_{k=1}^{k=n} (-1)^{k+1} \sum_{l=k}^{\infty} \frac{(l-k)!}{k!} \langle f_l, g_l \rangle_{L^2(\mathbb{R}_+^l)} \\ &\quad + (-1)^n \sum_{l=n+1}^{\infty} \frac{(l-n-1)!}{(n+1)! l} \langle f_l, g_l \rangle_{L^2(\mathbb{R}_+^l)}. \end{aligned}$$

Proof. We have

$$E[(D_{t_k} \cdots D_{t_1} F)(D_{t_k} \cdots D_{t_1} G)] = \sum_{l=k}^{\infty} (l-k)! \langle f_l(t_1, \ldots, t_k, *), g_l(t_1, \ldots, t_k, *) \rangle_{L^2(\mathbb{R}_+^{l-k})},$$

and

$$\begin{aligned} & E\left[E\left[D_{t_{n+1}} \cdots D_{t_1} F \mid \mathcal{F}_{t_{n+1}} \right] E\left[D_{t_{n+1}} \cdots D_{t_1} G \mid \mathcal{F}_{t_{n+1}} \right] \right] \\ &\quad = \sum_{l=n+1}^{\infty} (l-n-1)! \langle f_l(t_1, \ldots, t_{n+1}, *), g_l(t_1, \ldots, t_{n+1}, *) \rangle_{L^2([0, t_{n+1}]^{l-n-1})}. \end{aligned}$$

□

This corollary has a particular interpretation when $(M_t)_{t \in \mathbb{R}_+}$ is Brownian motion, resp. the Poisson process, and multiple stochastic integrals are written as generalized Hermite, resp. Charlier, polynomials.

We close this section with other examples of normal martingales with the chaos representation property. Let $(\phi_t)_{t\in\mathbb{R}_+}$ be a real-valued deterministic function with $i_t = 1_{\{\phi_t=0\}}$ and $\lambda_t = (1-i_t)1/\phi_t^2$, $t \in \mathbb{R}_+$. Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion, and let $(N_t)_{t\in\mathbb{R}_+}$ be a standard Poisson process independent of $(B_t)_{t\in\mathbb{R}_+}$, with intensity $\nu_t = \int_0^t \lambda_s ds$, $t \in \mathbb{R}_+$. If $(M_t)_{t\in\mathbb{R}_+}$ is written as

$$dM_t = i_t dB_t + \phi_t(dN_t - \lambda_t dt), \quad t \in \mathbb{R}_+, \quad M_0 = 0, \tag{3.2}$$

then it satisfies the hypothesis of this section. In this case $i_t D_t$ is still a derivation operator, we have the product rule

$$D_t(FG) = F D_t G + G D_t F + \phi_t D_t F D_t G, \quad t \in \mathbb{R}_+, \tag{3.3}$$

cf. Prop. 1.3 of Privault (1999a), and D_t can be written as

$$D_t = \frac{j_t}{\phi_t}\Delta_t^\phi + i_t D_t,$$

where Δ_t^ϕ is the finite difference operator defined on random functionals by addition at time t of a jump of height ϕ_t to $(M_t)_{t\in\mathbb{R}_+}$ This provides an extension of Cor. 4.3 in Houdré and Pérez-Abreu (1995). Other examples of normal martingales with the chaos representation property are given by the family of Azéma martingales.

4 Covariance identities for the local gradient on Poisson space

In this section we consider a modification of the gradient introduced by Carlen and Pardoux (1990) (see also Elliott and Tsoi (1993)), and show that it satisfies the hypothesis *(ii)* and *(iii)*. Let $(T_k)_{k\geq 1}$ denote the jump times of the standard Poisson process $(N(t))_{t\in\mathbb{R}_+}$, with $T_0 = 0$. Let $\mathcal{S}$ denote the set of smooth random functionals F of the form

$$F = f(T_1, \ldots, T_n), \quad f \in \mathcal{C}_c^1(\mathbb{R}_+^n), \quad n \geq 1.$$

Let $\tilde{D}$ denote the closable gradient defined as

$$\tilde{D}_t F = -\sum_{k=1}^{k=n} 1_{[0,T_k]}(t)\partial_k f(T_1, \ldots, T_n), \quad t \in \mathbb{R}_+, \quad F \in \mathcal{S}. \tag{4.1}$$

Remarks:

a) (4.1) is the definition of Bouleau and Hirsch (1991), p. 236, and Privault (1994). It differs from the definition (1.4) of Carlen and Pardoux (1990) used in Houdré and Pérez-Abreu (1995).

b) The adjoint $\tilde{\delta}$ of $\tilde{D}$, defined by the duality

$$E[F\tilde{\delta}(u)] = E[\langle \tilde{D}F, u\rangle_{L^2(\mathbb{R}_+)}], \quad F \in \mathrm{Dom}(\tilde{D}),\ u \in \mathrm{Dom}(\tilde{\delta}),$$

also coincides with the stochastic integral with respect to the compensated Poisson process $(\tilde{N}(t))_{t\in\mathbb{R}_+}$ on the square-integrable adapted processes, (see Prop. 9 of Privault (1994)) whereas for the adjoint of $\hat{D}$ defined in (1.4) this property holds only on square-integrable adapted processes $(u_t)_{t\in[0,1]}$ with P-a.s. zero mean on $[0,1]$, see Th. 3.1 in Carlen and Pardoux (1990).

c) As a consequence of (b), the Clark formula is not satisfied by the gradient (1.4).

The Clark formula (ii) is satisfied by $\tilde{D}$, see Prop. 5.3.3 in Bouleau and Hirsch (1991) and Th. 1 in Privault (1994). Note that $\tilde{D}$ can not be iterated, hence the covariance identity can not hold for $\tilde{D}$ as stated in (1.1). However we have for all $\mathcal{F}_a$-measurable $F \in \mathcal{S}$:

$$\tilde{D}_t F = 0, \qquad t \geq a,$$

and since $\tilde{D}$ has the derivation property this shows that $FG \in I\!D([a,\infty[)$ for all $F \in L^2(\Omega, \mathcal{F}_a, dP)$, $G \in \mathcal{S}$, with

$$\tilde{D}_t(FG) = F\tilde{D}_t G, \qquad t \geq a. \tag{4.2}$$

Although $\tilde{D}_{t_n} \cdots \tilde{D}_{t_1} F$ is in general not defined in $L^2(\Omega, dP)$ even for $F \in \mathcal{S}$, we have $1_{[0,T_k]}(t_n) \in L^2(\Omega, \mathcal{F}_{t_n}, dP)$ for all $t_n \in \mathbb{R}_+$ and $k \geq 1$. Hence Relation (4.2) shows by induction that if $(t_1, \ldots, t_n) \in \Delta_n$ then $\tilde{D}_{t_n} \cdots \tilde{D}_{t_1} F \in I\!D([t_n, \infty[)$, with

$$\tilde{D}_{t_n} \cdots \tilde{D}_{t_1} F = (-1)^n \sum_{1 \leq k_1, \ldots, k_n \leq m} 1_{[0,T_{k_1}]}(t_1) \cdots 1_{[0,T_{k_n}]}(t_n) \partial_{k_1} \cdots \partial_{k_n} f(T_1, \ldots, T_m),$$

$F = f(T_1, \ldots, T_m)$, hence $\mathcal{S} \subset I\!D(\Delta_n)$, and hypothesis ($iii$) is satisfied. The following is then an application of Th. 1.

Corollary 2 *Let $n \in \mathbb{N}$ and $F, G \in \bigcap_{k=1}^{k=n+1} I\!D(\Delta_k)$. We have*

$$\mathrm{Cov}(F,G) = \sum_{k=1}^{k=n}(-1)^{k+1}E\left[\int_{\Delta_k}(\tilde{D}_{t_k}\cdots\tilde{D}_{t_1}F)(\tilde{D}_{t_k}\cdots\tilde{D}_{t_1}G)dt_1\cdots dt_k\right] \quad (4.3)$$
$$+(-1)^n E\left[\int_{\Delta_{n+1}} E\left[\tilde{D}_{t_{n+1}}\cdots\tilde{D}_{t_1}F \mid \mathcal{F}_{t_{n+1}}\right] E\left[\tilde{D}_{t_{n+1}}\cdots\tilde{D}_{t_1}G \mid \mathcal{F}_{t_{n+1}}\right] dt_1\cdots dt_{n+1}\right].$$

As a corollary we obtain a covariance identity for functionals of the jump times $(T_k)_{k\geq 1}$ of the standard Poisson process which are gamma distributed random variables. This corrects (4.20) in Houdré and Pérez-Abreu (1995).

Corollary 3 *Let $f, g \in \mathcal{C}_c^2(\mathbb{R}_+^2)$. We have*

$$\mathrm{Cov}(f(T_1,\ldots,T_m), g(T_1,\ldots,T_m)) = \sum_{i,j=1}^{m} E[T_i\wedge T_j \partial_i f(T_1,\ldots,T_m)\partial_j g(T_1,\ldots,T_m)]$$
$$-\sum_{i,j,k,l=1}^{m} E\left[\int_0^{T_i\wedge T_k} T_j\wedge T_l\wedge t E\left[\partial_i\partial_j f(T_1,\ldots,T_m)\mid\mathcal{F}_t\right] E\left[\partial_k\partial_l g(T_1,\ldots,T_m)\mid\mathcal{F}_t\right] dt\right].$$

Proof. We have

$$\int_0^\infty \tilde{D}_t f(T_1,\ldots,T_m)\tilde{D}_t g(T_1,\ldots,T_m)dt = \sum_{i,j=1}^{k=m} T_i\wedge T_j\partial_i f(T_1,\ldots,T_m)\partial_j g(T_1,\ldots,T_m),$$

and

$$\begin{aligned}
&\int_0^\infty\int_0^t E\left[\tilde{D}_t\tilde{D}_s f(T_1,\ldots,T_m)\mid\mathcal{F}_t\right] E\left[\tilde{D}_t\tilde{D}_s f(T_1,\ldots,T_m)\mid\mathcal{F}_t\right] dsdt\\
&= \sum_{i,j,k,l=1}^{m}\int_0^\infty\int_0^t E\left[1_{[0,T_i]}(t)1_{[0,T_j]}(s)\partial_i\partial_j f(T_1,\ldots,T_m)\mid\mathcal{F}_t\right]\\
&\quad\times E\left[1_{[0,T_k]}(t)1_{[0,T_l]}(s)\partial_k\partial_l g(T_1,\ldots,T_m)\mid\mathcal{F}_t\right] dsdt\\
&= \sum_{i,j,k,l=1}^{m}\int_0^\infty\int_0^t 1_{[0,T_i]}(t)1_{[0,T_j]}(s)1_{[0,T_k]}(t)1_{[0,T_l]}(s)E\left[\partial_i\partial_j f(T_1,\ldots,T_m)\mid\mathcal{F}_t\right]\\
&\quad\times E\left[\partial_k\partial_l g(T_1,\ldots,T_m)\mid\mathcal{F}_t\right] dsdt\\
&= \sum_{i,j,k,l=1}^{m}\int_0^\infty T_j\wedge T_l\wedge t 1_{[0,T_i\wedge T_k]}(t)E\left[\partial_i\partial_j f(T_1,\ldots,T_m)\mid\mathcal{F}_t\right]\\
&\quad\times E\left[\partial_k\partial_l g(T_1,\ldots,T_m)\mid\mathcal{F}_t\right] dt.
\end{aligned}$$

□

In particular we have

$$\mathrm{Cov}(T_m, f(T_1,\dots,T_m)) = \sum_{i=1}^{i=m} E[T_i \partial_i f(T_1,\dots,T_m)],$$

which compares to (4.21) in Houdré and Pérez-Abreu (1995), and

$$\mathrm{Cov}(T_m, f(T_m)) = E[T_m f'(T_m)].$$

This last identity is easily checked by integration by parts on $\mathbb{R}_+$. Let $p_k(s) = 1_{\{k\geq 0\}} e^{-s} s^k/k!$, $s \geq 0$, denote the density function of T_{k+1} on $\mathbb{R}_+$, $k \geq 0$, with $\partial_s p_k(s) = p_{k-1}(s) - p_k(s)$, $s \in \mathbb{R}_+$, $k \geq 1$.

$$\begin{aligned}
\mathrm{Cov}(T_m, f(T_m)) &= E[T_m f(T_m)] - mE[f(T_m)] \\
&= \int_0^\infty x f(x) p_{m-1}(x)dx - m\int_0^\infty f(x)p_{m-1}(x)dx \\
&= m\int_0^\infty f(x)(p_m(x) - p_{m-1}(x))dx = -m\int_0^\infty f(x)\partial_x p_m(x)dx \\
&= m\int_0^\infty f'(x)p_m(x)dx = \int_0^\infty x f'(x) p_{m-1}(x)dx = E[T_m f'(T_m)].
\end{aligned}$$

In the general setting of normal martingales with the chaos representation property, the function f_n in the development of $F = \sum_{n=0}^\infty I_n(f_n)$ is given by

$$f_n(t_1,\dots,t_n) = \frac{1}{n!} E[D_{t_n}\cdots D_{t_1} F], \quad a.e.\ t_1,\dots,t_n,$$

and this is true for the compensated Poisson process in particular. However on Poisson space, $\tilde{D}_{t_n}\cdots\tilde{D}_{t_1}$, $(t_1,\dots,t_n) \in \Delta_n$, can not be used as $D_{t_n}\cdots D_{t_1}$ to give the chaos decomposition of a random variable, for example we have

$$\tilde{D}_{t_n}\cdots\tilde{D}_{t_1} f(T_k) = (-1)^n 1_{[0,T_k]}(t_n) f^{(n)}(T_k), \quad (t_1,\dots,t_n) \in \Delta_n,$$

and

$$\begin{aligned}
E[\tilde{D}_{t_n}\cdots\tilde{D}_{t_1} f(T_k)] &= (-1)^n E[1_{[0,T_k]}(t_n) f^{(n)}(T_k)] \\
&= (-1)^n \int_{t_n}^\infty f^{(n)}(t) p_{k-1}(t)dt, \quad (t_1,\dots,t_n) \in \Delta_n,
\end{aligned}$$

which differs from

$$E[D_{t_n}\cdots D_{t_1} f(T_k)] = -\int_{t_n}^\infty f'(t) P_k^{(n)}(t)dt,$$

computed in Th. 1 of Privault (1999b), where $P_k(t) = \int_0^t p_{k-1}(s)ds$ is the distribution function of T_k. In particular, for $n \geq 2$ and $(t_1, \ldots, t_n) \in \Delta_n$,

$$E[\tilde{D}_{t_n} \cdots \tilde{D}_{t_1} F \mid \mathcal{F}_{t_n}] \neq E[D_{t_n} \cdots D_{t_1} F \mid \mathcal{F}_{t_n}],$$

although we have for $n = 1$:

$$E[\tilde{D}_t F \mid \mathcal{F}_t] = E[D_t F \mid \mathcal{F}_t], \quad t \in \mathbb{R}_+, \quad F \in \mathcal{S},$$

cf. Prop. 20 of Privault (1994), hence (2.4) is truly different from (4.3), except if $n = 0$: in this case both identities coincide on $\mathrm{Dom}(D) \cap \mathrm{Dom}(\tilde{D})$.
Finally we note that the correct form of the commutation relation between $\tilde{D}$ and $\tilde{\delta}$ is

$$\tilde{D}_t \tilde{\delta}(u) = -\tilde{\delta}(1_{[t,\infty[}(\cdot)u'(\cdot)) + u(t), \quad u \in \mathcal{C}_c^\infty(\mathbb{R}_+),$$

which can be proved as follows:

$$\begin{aligned} \tilde{D}_t \tilde{\delta}(u) &= \tilde{D}_t \int_0^\infty u(s)(dN_s - ds) = \sum_{k=1}^\infty \tilde{D}_t u(T_k) \\ &= -\sum_{k=1}^\infty 1_{[0,T_k]}(t) u'(T_k) = -\int_0^\infty 1_{[0,s]}(t) u'(s) dN_s \\ &= -\int_0^\infty 1_{[0,s]}(t) u'(s)(dN_s - ds) - \int_0^\infty 1_{[0,s]}(t) u'(s) ds \\ &= -\tilde{\delta}(1_{[t,\infty[}(\cdot)u'(\cdot)) + u(t). \end{aligned} \tag{4.4}$$

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