Convex ordering for random vectors using predictable representation *

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Abstract

We prove convex ordering results for random vectors admitting a predictable representation in terms of a Brownian motion and a non-necessarily independent jump component. Our method uses forward-backward stochastic calculus and extends the results proved in [4] in the one-dimensional case. We also study a geometric interpretation of convex ordering for discrete measures in connection with the conditions set on the jump heights and intensities of the considered processes.

Keywords: Convex ordering, forward-backward stochastic calculus, deviation inequalities, Brownian motion, jump processes.

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1 Introduction

Given two finite measures μ and ν on $\mathbb{R}^d$ we say that μ is convex dominated by ν, and we write $\mu \preceq_{\rm cx} \nu$, if

$$\int_{\mathbb{R}^d} \phi(x)\mu(dx) \leq \int_{\mathbb{R}^d} \phi(x)\nu(dx) \tag{1.1}$$

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for all sufficiently integrable convex functions $\phi : \mathbb{R}^d \to \mathbb{R}$. In case μ and ν are the respective probability distributions of two random variables F and G, Relation (1.1) is interpreted as the convex concentration inequality

$$\mathbb{E}[\phi(F)] \leq \mathbb{E}[\phi(G)]. \tag{1.2}$$

Such concentration inequalities have applications in mathematical finance where they can be interpreted in terms of bounds on option prices on multidimensional underlying assets, see e.g. [1] and references therein.

If F is more convex concentrated than G and G is integrable, then $\mathbb{E}[F] = \mathbb{E}[G]$ as follows from taking successively $\phi(x) = x_i$ and $\phi(x) = -x_i$, $i = 1, \ldots, d$, and this amounts to saying that the distributions of F and G have same barycenter. On the other hand, applying (1.1) to the convex function $y \mapsto \phi_x(y) = |\langle y - \mathbb{E}[F], x\rangle|^2$ shows that the matrix $\operatorname{Var} G - \operatorname{Var} F$ is positive semidefinite, where

$$\operatorname{Var} F = (\operatorname{Cov}(F_i, F_j))_{1 \leq i,j \leq d} \quad \text{and} \quad \operatorname{Var} G = (\operatorname{Cov}(G_i, G_j))_{1 \leq i,j \leq d}$$

denote the covariance matrices of F and of G.

In case F and G are Gaussian random vectors with covariance matrices Σ and $\tilde{\Sigma}$, these conditions become necessary and sufficient. More precisely, if $\mathbb{E}[F] = \mathbb{E}[G]$ and $\tilde{\Sigma} - \Sigma$ is positive semidefinite then there exists a centered Gaussian random variable Z with covariance $\tilde{\Sigma} - \Sigma$, independent of F and such that $G = F + Z$ in distribution, hence F is more convex concentrated than G from Jensen's inequality:

$$\mathbb{E}[\phi(F)] = \mathbb{E}[\phi(\mathbb{E}[F + Z \mid F])] \leq \mathbb{E}[\mathbb{E}[\phi(F + Z) \mid Z]] = \mathbb{E}[\phi(F + Z)] = \mathbb{E}[\phi(G)]. \tag{1.3}$$

In this paper we aim at obtaining sufficient conditions for the convex ordering of vector-valued random variables, based on their predictable representation as the sum of a diffusion and a jump part. Our main tool of proof consists in the inequality

$$\mathbb{E}[\phi(M(t) + M^*(t))] \leq \mathbb{E}[\phi(M(s) + M^*(s))], \quad 0 \leq s \leq t,$$

for all convex functions $\phi : \mathbb{R}^d \to \mathbb{R}$, where $(M(t))_{t\in\mathbb{R}_+}$ and $(M^*(t))_{t\in\mathbb{R}_+}$ are respectively a forward and a backward d-dimensional martingale with jumps and continuous parts whose local characteristics satisfy the comparison inequalities assumed in Theorem 3.5 below. Such an inequality has been proved in [4] for real-valued random variables. We stress however that the arguments of [4] are particular to the one-dimensional case and in general they can not be applied to the vector valued setting considered in this paper, for which specific methods have to be developed.

Note also that by a classical argument, the application of (1.1) to $\phi(x) = \exp(\lambda\|x\|)$, $\lambda > 0$, entails the deviation bound

$$P(\|F\| \geq x) \leq \inf_{\lambda>0} \mathbb{E}[e^{\lambda(\|F\|-x)}\mathbf{1}_{\{\|F\|\geq x\}}] \leq \inf_{\lambda>0} \mathbb{E}[e^{\lambda(\|F\|-x)}] \leq \inf_{\lambda>0} \mathbb{E}[e^{\lambda(\|G\|-x)}],$$

$x > 0$, hence the deviation probabilities for F can be estimated via the Laplace transform of $\|G\|$.

We will prove the following type of result. Let $(W(t))_{t\in\mathbb{R}_+}$ and $Z(t) = (Z_1(t), \dots, Z_n(t))$ be respectively a standard n-dimensional Brownian motion and a vector of independent real point processes with compensator $(\lambda_1(t), \dots, \lambda_n(t))_{t\in\mathbb{R}_+}$ generating a filtration $\mathcal{F}^M$. Let $\mathcal{M}^{d\times n}$ denote the set of $d\times n$ real matrices, with $\mathcal{M}^d = \mathcal{M}^{d\times d}$. Consider F and G two random variables with the predictable representations

$$F = \int_0^\infty A(t)dW(t) + \int_0^\infty J(t)(dZ(t) - \lambda(t)dt)$$

where $(A(t))_{t\in\mathbb{R}_+}$, $(J(t))_{t\in\mathbb{R}_+}$ are square-integrable $\mathcal{M}^{d\times n}$-valued $\mathcal{F}_t^M$-predictable processes, and

$$G = \int_0^\infty \hat{A}(t)d\tilde{W}(t) + \int_0^\infty \hat{J}(t)(d\tilde{Z}(t) - \tilde{\lambda}(t)dt)$$

where $(\hat{A}(t))_{t\in\mathbb{R}_+}$, $(\hat{J}(t))_{t\in\mathbb{R}_+}$ are $\mathcal{M}^{d\times n}$-valued square-integrable $\mathcal{F}_t^M$-predictable processes and $\tilde{W}(t)$ and $\tilde{Z}(t) = (\tilde{Z}_1(t), \dots, \tilde{Z}_n(t))$, $t \in \mathbb{R}_+$, are a n-dimensional Brownian motion and a vector of real point processes with respective intensities $(\tilde{\lambda}_i(t))_{t\in\mathbb{R}_+}$, $i = 1, \dots, n$, independent of $\mathcal{F}^M$. In terms of the convex orders cxp, cxpi and psd introduced in Definitions 3.3 and 3.2 below, we have for example the following corollary of Theorem 4.1. In the sequel, the † symbol stands for matrix transposition.

Corollary 1.1. *The convex concentration inequality* (1.2) *holds provided*

$$A^\dagger(t)A(t) \le_{\mathrm{psd}} \hat{A}^\dagger(t)\hat{A}(t), \qquad dPdt - a.e.,$$

and for almost all $t \in \mathbb{R}_+$, *we have either:*

i) $\displaystyle\sum_{j=1}^n \lambda_j(t)\delta_{(J_{1,j}(t),\ldots,J_{d,j}(t))} \preceq_{\mathrm{cxp}} \sum_{j=1}^n \tilde{\lambda}_j(t)\delta_{(\hat{J}_{1,j}(t),\ldots,\hat{J}_{d,j}(t))}$,

or:

ii) $J_{i,j}(t) \ge 0$, $\hat{J}_{i,j}(t) \ge 0$, $i = 1,\ldots,d$, $j = 1,\ldots,n$, *and*

$$\sum_{j=1}^n \lambda_j(t)\delta_{(J_{1,j}(t),\ldots,J_{d,j}(t))} \preceq_{\mathrm{cxpi}} \sum_{j=1}^n \tilde{\lambda}_j(t)\delta_{(\hat{J}_{1,j}(t),\ldots,\hat{J}_{d,j}(t))}.$$

Condition (ii) above will hold in particular if

$$\lambda_i(t) \le \tilde{\lambda}_i(t), \qquad \text{and} \qquad 0 \le J_{i,j}(t) \le \hat{J}_{i,j}(t),$$

$1 \le i \le n$, $1 \le j \le d$, for dt-almost all $t \in \mathbb{R}_+$. In Theorem 5.2, we provide a geometric interpretation of the convex ordering Condition (i) for finitely supported measures.

In case F and G are Gaussian random vectors with covariance matrices Σ and $\tilde{\Sigma}$, we recover (1.3) from Corollary 1.1 by taking $\lambda(t) = \tilde{\lambda}(t) = 0$, $A(t) = \mathbf{1}_{[0,T]}(t)\sqrt{\Sigma/T}$ and $\hat{A}(t) = \mathbf{1}_{[0,T]}(t)\sqrt{\tilde{\Sigma}/T}$, $t \in \mathbb{R}_+$.

Note that related convex comparison results have also been obtained in [1], [2] for diffusions with jumps, under different hypotheses. Namely, it is assumed therein that G is given by the value at time T of a diffusion with jumps. Convex ordering then holds under similar assumptions on the process characteristics, provided the generator of this diffusion satisfies the propagation of convexity property.

This paper is organized as follows. In Section 2 we introduce the notation of multi-dimensional forward-backward stochastic calculus with jumps, which will be used in the next sections. In Section 3 we prove some convex ordering results for the sums of forward and backward martingales, and in Section 4 we apply those results to random variables given by their predictable representation in terms of a diffusion and a point process. Section 5 is devoted to a geometric interpretation of convex ordering for discrete measures on $\mathbb{R}^d$, which gives a better understanding of the conditions set of the jump heights and intensities of the considered point processes.

2 Notation

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with an increasing filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and a decreasing filtration $(\mathcal{F}_t^*)_{t\in\mathbb{R}_+}$. Consider $(M(t))_{t\in\mathbb{R}_+} = (M_1(t), \dots, M_d(t))_{t\in\mathbb{R}_+}$ a d-dimensional $\mathcal{F}_t$-forward martingale and $(M^*(t))_{t\in\mathbb{R}_+} = (M_1^*(t), \dots, M_d^*(t))_{t\in\mathbb{R}_+}$ a d-dimensional $\mathcal{F}_t^*$-backward martingale, such that $(M(t))_{t\in\mathbb{R}_+}$ has right-continuous paths with left limits, and $(M^*(t))_{t\in\mathbb{R}_+}$ has left-continuous paths with right limits. Denote respectively by $(M^c(t))_{t\in\mathbb{R}_+}$ and $(M^{*c}(t))_{t\in\mathbb{R}_+}$ the continuous parts of $(M(t))_{t\in\mathbb{R}_+}$ and of $(M^*(t))_{t\in\mathbb{R}_+}$, and by

$$\Delta M(t) = M(t) - M(t^-), \qquad \Delta^* M^*(t) = M^*(t) - M^*(t^+),$$

their forward and backward jumps. The processes $(M(t))_{t\in\mathbb{R}_+}$ and $(M^*(t))_{t\in\mathbb{R}_+}$ have jump measures

$$\mu(dt, dx) = \sum_{s>0} \mathbf{1}_{\{\Delta M(s)\neq 0\}} \delta_{(s,\Delta M(s))}(dt, dx),$$

and

$$\mu^*(dt, dx) = \sum_{s>0} \mathbf{1}_{\{\Delta^* M^*(s)\neq 0\}} \delta_{(s,\Delta^* M^*(s))}(dt, dx),$$

where $\delta_{(s,x)}$ denotes the Dirac measure at $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Denote by $\nu(dt, dx)$ and $\nu^*(dt, dx)$ the $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and $(\mathcal{F}_t^*)_{t\in\mathbb{R}_+}$-dual predictable projections of $\mu(dt, dx)$ and of $\mu^*(dt, dx)$. The quadratic variations $([M, M]_t)_{t\in\mathbb{R}_+}$, $([M^*, M^*]_t)_{t\in\mathbb{R}_+}$ are the $\mathcal{M}^d$-valued processes defined as the limits in uniform convergence in probability

$$[M, M]_t = \lim_{n\to\infty} \sum_{k=1}^{n} (M(t_k^n) - M(t_{k-1}^n))(M(t_k^n) - M(t_{k-1}^n))^\dagger,$$

and

$$[M^*, M^*]_t = \lim_{n\to\infty} \sum_{k=0}^{n-1} (M^*(t_k^n) - M^*(t_{k+1}^n))(M^*(t_k^n) - M^*(t_{k+1}^n))^\dagger,$$

for all refining sequences $\{0 = t_0^n \le t_1^n \le \cdots \le t_n^n = t\}$, $n \ge 1$, of partitions of $[0,t]$ tending to the identity. We let $M^J(t) = M(t) - M^c(t)$, $M^{*J}(t) = M^*(t) - M^{*c}(t)$,

$$[M^J, M^J]_t = \sum_{0<s\le t} \Delta M(s)\Delta M(s)^\dagger, \qquad [M^{*J}, M^{*J}]_t = \sum_{0\le s<t} \Delta^* M^*(s)(\Delta^* M^*(s))^\dagger,$$

with

$$\langle M^c, M^c\rangle_t = [M,M]_t - [M^J, M^J]_t,$$

and

$$\langle M^{*c}, M^{*c}\rangle_t = [M^*, M^*]_t - [M^{*J}, M^{*J}]_t,$$

$t \in \mathbb{R}_+$. Denote by $(\langle M^J, M^J\rangle_t)_{t\in\mathbb{R}_+}$, $(\langle M^{*J}, M^{*J}\rangle_t)_{t\in\mathbb{R}_+}$ the conditional quadratic variations of $(M^J(t))_{t\in\mathbb{R}_+}$ and of $(M^{*J}(t))_{t\in\mathbb{R}_+}$, with

$$d\langle M^J, M^J\rangle_t = \int_{\mathbb{R}^d} xx^\dagger \nu(dt,dx) \quad \text{and} \quad d\langle M^{*J}, M^{*J}\rangle_t = \int_{\mathbb{R}^d} xx^\dagger \nu^*(dt,dx).$$

The conditional quadratic variations $(\langle M, M\rangle_t)_{t\in\mathbb{R}_+}$, $(\langle M^*, M^*\rangle_t)_{t\in\mathbb{R}_+}$ of $(M(t))_{t\in\mathbb{R}_+}$ and of $(M^*(t))_{t\in\mathbb{R}_+}$ satisfy

$$\langle M, M\rangle_t = \langle M^c, M^c\rangle_t + \langle M^J, M^J\rangle_t, \quad \text{and} \quad \langle M^*, M^*\rangle_t = \langle M^{*c}, M^{*c}\rangle_t + \langle M^{*J}, M^{*J}\rangle_t,$$

$t \in \mathbb{R}_+$. Theorem 3.5 below and its corollaries are based on the following forward-backward Itô type change of variable formula, for $(M(t), M^*(t))_{t\in\mathbb{R}_+}$, in which Conditions (3.1) and (3.2) below are assumed in order to make sense of the integrals with respect to $dM(t)$ and $d^*M^*(t)$. This formula is proved in dimension one in Theorem 8.1 of [4], and its extension to dimension $d \ge 2$ can be proved similarly. For all $f \in \mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}^d)$ of the form $f(x,y) = f(x_1,\ldots,x_d,y_1,\ldots,y_d)$ we have

$$\begin{aligned}
&f(M(t), M^*(t)) = f(M(s), M^*(s)) \qquad (2.1)\\
&+\sum_{i=1}^d \int_{s^+}^t \frac{\partial f}{\partial x_i}(M(u^-), M^*(u))dM_i(u) + \frac{1}{2}\sum_{i,j=1}^d \int_s^t \frac{\partial^2 f}{\partial x_i \partial x_j}(M(u), M^*(u))d\langle M_i^c, M_j^c\rangle_u\\
&+\sum_{s<u\le t}\left(f(M(u), M^*(u)) - f(M(u^-), M^*(u)) - \sum_{i=1}^d \Delta M_i(u)\frac{\partial f}{\partial x_i}(M(u^-), M^*(u))\right)\\
&-\sum_{i=1}^d \int_s^{t^-} \frac{\partial f}{\partial y_i}(M(u), M^*(u^+))d^*M_i^*(u) - \frac{1}{2}\sum_{i,j=1}^d \int_s^t \frac{\partial^2 f}{\partial y_i \partial y_j}(M(u), M^*(u))d\langle M_i^{*c}, M_j^{*c}\rangle_u
\end{aligned}$$

$$-\sum_{s\le u<t}\left(f(M(u),M^*(u))-f(M(u),M^*(u^+))-\sum_{i=1}^{d}\Delta^*M_i^*(u)\frac{\partial f}{\partial y_i}(M(u),M^*(u^+))\right),$$

$0 \le s \le t$, where d and d^* denote the forward and backward Itô differential, respectively defined as the limits of the Riemann sums

$$\sum_{k=1}^{n}(M_i(t_k^n)-M_i(t_{k-1}^n))\frac{\partial f}{\partial x_i}(M(t_{k-1}^n),M^*(t_{k-1}^n))$$

and

$$\sum_{k=0}^{n-1}(M_i^*(t_k^n)-M_i^*(t_{k+1}^n))\frac{\partial f}{\partial y_i}(M(t_{k+1}^n),M^*(t_{k+1}^n))$$

for all refining sequences $\{s=t_0^n\le t_1^n\le\cdots\le t_n^n=t\}$, $n\ge 1$, of partitions of $[s,t]$ tending to the identity.

Here, $\int_0^t \eta(u)dM_i(u)$, resp. $\int_t^\infty \eta^*(u)d^*M_i^*(u)$, refer to the right, resp. left, continuous version of the indefinite stochastic integrals of the forward, resp. backward, adapted and sufficiently integrable processes $(\eta(u))_{u\in\mathbb{R}_+}$, resp. $(\eta^*(u))_{u\in\mathbb{R}_+}$.

3 Convex ordering for martingales

We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the usual Euclidean scalar product and norm on $\mathbb{R}^d$. Let $\mathcal{M}^d$ be the space of real matrices with the scalar product

$$\langle A,B\rangle := \mathrm{Tr}\,(AB^\dagger)=\sum_{i,j=1}^{d}A_{i,j}B_{i,j},\qquad A,B\in\mathcal{M}^d,$$

where we recall that $A^\dagger$ stands for the transpose $(A_{j,i})_{1\le i,j\le d}$ of $A=(A_{i,j})_{1\le i,j\le d}$, and let $\mathcal{M}_+^d$ be the subset of $\mathcal{M}^d$ made of positive semidefinite matrices.

Lemma 3.1. *Let A be a symmetric $d\times d$ matrix. Then the following statements are equivalent:*

i) A is positive semidefinite,

ii) for all positive semidefinite matrices B we have $\langle A,B\rangle\ge 0$.

Proof. Since A is symmetric, if it is positive semidefinite then its spectral decomposition is given as

$$A = \sum_{k=1}^{d} \lambda_k e_k e_k^\dagger,$$

where the eigenvalues $(\lambda_k)_{k=1,\ldots,d}$ of A are non-negative and $(e_1, \ldots, e_d)$ denote the eigenvectors of A. Hence we have

$$\mathrm{Tr}\,(AB^\dagger) = \sum_{k=1}^{d} \lambda_k \langle e_k, Be_k \rangle \geq 0$$

if B is positive semidefinite. The converse follows by choosing $B = x^\dagger x$, $x \in \mathbb{R}^d$, and noting that

$$\langle x, Ax \rangle = \mathrm{Tr}\,(AB^\dagger) \geq 0.$$

□

Definition 3.2. *Given $A, B \in \mathcal{M}^d$, we will write $A \leq_{\mathrm{psd}} B$ if $B - A$ is positive semidefinite, i.e.*

$$\langle x, Ax \rangle \leq \langle x, Bx \rangle, \qquad x \in \mathbb{R}^d.$$

In the sequel, a function $f : \mathbb{R}^d \to \mathbb{R}$ will be said to be *non-decreasing* if

$$f(x_1, \ldots, x_d) \leq f(y_1, \ldots, y_d)$$

for all $x_1, \ldots, x_d \in \mathbb{R}$ and $y_1, \ldots, y_d \in \mathbb{R}$ such that $x_i \leq y_i$, $i = 1, \ldots, d$. We will need the following orders between positive measures μ, ν on $\mathbb{R}^d$.

Definition 3.3.

i) We say that $\mu \preceq_{\mathrm{cxp}} \nu$ if

$$\int_{\mathbb{R}^d} \phi(x)\mu(dx) \leq \int_{\mathbb{R}^d} \phi(x)\nu(dx)$$

for all non-negative *convex functions $\phi : \mathbb{R}^d \to \mathbb{R}^+$.*

ii) We say that $\mu \preceq_{\mathrm{cxpi}} \nu$ if

$$\int_{\mathbb{R}^d} \phi(x)\mu(dx) \leq \int_{\mathbb{R}^d} \phi(x)\nu(dx)$$

for all non-negative *and* non-decreasing *convex functions $\phi : \mathbb{R}^d \to \mathbb{R}^+$.*

If μ and ν are finite measures on $\mathbb{R}^d$, then both $\mu \preceq_{\rm cxp} \nu$ and $\mu \preceq_{\rm cxpi} \nu$ imply $\mu(\mathbb{R}^d) \leq \nu(\mathbb{R}^d)$. More precisely we have the following result.

Proposition 3.4. *Assume that μ and ν are finite measures on $\mathbb{R}^d$. Then $\mu \preceq_{\rm cx} \nu$ is equivalent to $\mu \preceq_{\rm cxp} \nu$ and $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$.*

Proof. Assume that $\mu \preceq_{\rm cxp} \nu$ and $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$, and let $\phi \in L^1(\mu) \bigcap L^1(\nu)$. For all $a \in \mathbb{R}$ we have

$$\begin{aligned} &\int_{\mathbb{R}^d} \phi(x)\mu(dx) - \int_{\{\phi<a\}} \phi(x)\mu(dx) + a\mu(\{\phi < a\}) \\ &= \int_{\{\phi\geq a\}} (\phi(x)-a)^+ \mu(dx) + a\mu(\mathbb{R}^d) \\ &\leq \int_{\{\phi\geq a\}} (\phi(x)-a)^+ \nu(dx) + a\nu(\mathbb{R}^d) \\ &= \int_{\mathbb{R}^d} \phi(x)\nu(dx) - \int_{\{\phi<a\}} \phi(x)\nu(dx) + a\nu(\{\phi < a\}), \end{aligned}$$

and for $a \leq 0$,

$$\int_{\{\phi<a\}} \phi(x)\mu(dx) \leq a\mu(\{\phi < a\}) \leq 0, \qquad \int_{\{\phi<a\}} \phi(x)\nu(dx) \leq a\nu(\{\phi < a\}) \leq 0,$$

hence letting a tend to $-\infty$ yields

$$\int_{\mathbb{R}^d} \phi(x)\mu(dx) \leq \int_{\mathbb{R}^d} \phi(x)\nu(dx).$$

Conversely we note that $\mu \preceq_{\rm cxp} \nu$ clearly implies $\mu \preceq_{\rm cx} \nu$, and we recover the identity $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$ by applying the property $\mu \preceq_{\rm cx} \nu$ successively with $\phi = 1$ and $\phi = -1$. $\square$

Consequently, $\mu \preceq_{\rm cxp} \nu$ implies

$$\int_{\mathbb{R}^d} x_i \mu(dx) \leq \int_{\mathbb{R}^d} x_i \nu(dx), \quad \text{and} \quad -\int_{\mathbb{R}^d} x_i \mu(dx) \leq -\int_{\mathbb{R}^d} x_i \nu(dx), \quad i = 1, \ldots, d,$$

i.e. μ and ν have same barycenter, provided $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$ and μ, ν are integrable. This also holds when $\mu \preceq_{\rm cxpi} \nu$, $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$ and μ, ν are supported by $\mathbb{R}^d_+$.

Let now

$$(M(t))_{t\in\mathbb{R}_+} \text{ be an } \mathcal{F}^*_t\text{-adapted, } \mathcal{F}_t\text{-forward martingale,} \tag{3.1}$$

and

$$(M^*(t))_{t\in\mathbb{R}_+} \text{ be an } \mathcal{F}_t\text{-adapted, } \mathcal{F}_t^*\text{-backward martingale,} \tag{3.2}$$

with characteristics of the form

$$\nu(dt,dx) = \nu_t(dx)dt \qquad \text{and} \qquad \nu^*(dt,dx) = \nu_t^*(dx)dt,$$

and

$$d\langle M^c, M^c\rangle_t = H(t)dt, \quad \text{and} \quad d\langle M^{*c}, M^{*c}\rangle_t = H^*(t)dt,$$

where $H(t) = (H_{i,j}(t))_{1\leq i,j\leq d}$ and $H^*(t) = (H^*_{i,j}(t))_{1\leq i,j\leq d}$ are $\mathcal{M}^d$-valued, $t \in \mathbb{R}_+$, and predictable respectively with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and to $(\mathcal{F}_t^*)_{t\in\mathbb{R}_+}$. In the sequel, we will also assume that $(H(t))_{t\in\mathbb{R}_+}$, $(H^*(t))_{t\in\mathbb{R}_+} \in L^2(\Omega\times\mathbb{R}_+)$, and that

$$\mathbb{E}\left[\int_{\mathbb{R}^d\times\mathbb{R}_+} \|x\|\nu_t(dx)dt\right] < \infty, \qquad \mathbb{E}\left[\int_{\mathbb{R}^d\times\mathbb{R}_+} \|x\|\nu_t^*(dx)dt\right] < \infty. \tag{3.3}$$

The hypotheses on $(H(t))_{t\in\mathbb{R}_+}$ and $(H^*(t))_{t\in\mathbb{R}_+}$ imply that M_t^c and M_t^{*c} are in $L^2(\Omega)$, $t \in \mathbb{R}_+$, and Condition 3.3 is a technical integrability assumption.

Theorem 3.5. *Assume that*

$$H(t) \leq_{\text{psd}} H^*(t), \quad dPdt - a.e.$$

and that for almost all $t \in \mathbb{R}_+$ we have either:

i) $\nu_t \preceq_{\text{cxp}} \nu_t^*$,

or:

ii) $\nu_t \preceq_{\text{cxpi}} \nu_t^*$ *and* ν_t, ν_t^* *are supported by* $(\mathbb{R}_+)^d$.

Then we have

$$\mathbb{E}[\phi(M(s) + M^*(s))] \geq \mathbb{E}[\phi(M(t) + M^*(t))], \qquad 0 \leq s \leq t, \tag{3.4}$$

for all convex functions $\phi : \mathbb{R}^d \to \mathbb{R}$.

Proof. We start by assuming that ϕ is a $\mathcal{C}^2$, convex Lipschitz function and we apply Itô's formula (2.1) for forward-backward martingales to $f(x,y) = \phi(x+y)$. Taking expectations on both sides of Itô's formula we get

$$
\begin{aligned}
&\mathbb{E}[\phi(M(t) + M^*(t))] \\
&= \mathbb{E}[\phi(M(s) + M^*(s))] + \frac{1}{2}\sum_{i,j=1}^{d} \mathbb{E}\left[\int_s^t \phi''_{i,j}(M(u) + M^*(u)) d(\langle M_i^c, M_j^c\rangle_u - \langle M_i^{*c}, M_j^{*c}\rangle_u)\right] \\
&+ \mathbb{E}\left[\int_s^t \int_{\mathbb{R}^d} (\phi(M(u) + M^*(u) + x) - \phi(M(u) + M^*(u)) - \langle x, \nabla\phi(M(u) + M^*(u))\rangle)\nu_u(dx)du\right] \\
&- \mathbb{E}\left[\int_s^t \int_{\mathbb{R}^d} (\phi(M(u) + M^*(u) + x) - \phi(M(u) + M^*(u)) - \langle x, \nabla\phi(M(u) + M^*(u))\rangle)\nu_u^*(dx)du\right] \\
&= \mathbb{E}[\phi(M(s) + M^*(s))] + \frac{1}{2}\mathbb{E}\left[\int_s^t \langle \nabla^2\phi(M(u) + M^*(u)), H(u) - H^*(u)\rangle du\right] \qquad (3.5) \\
&\quad + \mathbb{E}\left[\int_s^t \int_{\mathbb{R}^d} \Psi(x, M(u) + M^*(u))(\nu_u(dx) - \nu_u^*(dx))du\right],
\end{aligned}
$$

where

$$
\Psi(x,y) \quad = \quad \phi(x+y) - \phi(y) - \sum_{i=1}^{d} x_i \frac{\partial\phi}{\partial y_i}(y), \qquad x, y \in \mathbb{R}^d.
$$

Due to the convexity of ϕ, the Hessian $\nabla^2\phi$ is positive semidefinite hence Lemma 3.1 yields

$$
\begin{aligned}
\mathbb{E}[\phi(M(t) + M^*(t))] &\leq \mathbb{E}[\phi(M(s) + M^*(s))] \\
&+ \mathbb{E}\left[\int_s^t \int_{\mathbb{R}^d} \Psi(x, M(u) + M^*(u))(\nu_u(dx) - \nu_u^*(dx))du\right], \qquad (3.6)
\end{aligned}
$$

since $H^*(u) - H(u)$ is positive semidefinite for fixed $(\omega, u) \in \Omega \times \mathbb{R}_+$.

Finally we examine the consequences of hypotheses (i) and (ii) on (3.6).

i) By convexity of ϕ, $x \mapsto \Psi(x,y)$ is non-negative and convex on $\mathbb{R}^d$ for all fixed $y \in \mathbb{R}^d$, hence the second term in (3.6) is non-positive.

ii) When ν_u and ν_u^* are supported by $\mathbb{R}_+^d$, (3.6) is also non-positive since for all y, $x \mapsto \Psi(x,y)$ is non-decreasing in $x \in \mathbb{R}_+^d$.

The extension to convex non $\mathcal{C}^2$ functions ϕ follows by approximation of ϕ by an increasing sequence of $\mathcal{C}^2$ convex Lipschitz functions, and by application of the monotone convergence theorem. $\square$

Remark 3.6. *When $\phi \in C^2$, the hypothesis on the diffusion part and on the jump part can be mixed together. Indeed, in order for the conclusion of Theorem 3.5 to hold it suffices that*

$$\begin{aligned}\mathrm{Tr}\,(\nabla^2\phi(y)H_t) &+ \int_{\mathbb{R}^d} \mathrm{Tr}\,(\nabla^2\phi(y+\tau x)xx^\dagger)\nu_t(dx) \\ &\leq \mathrm{Tr}\,(\nabla^2\phi(y)H_t^*) + \int_{\mathbb{R}^d} \mathrm{Tr}\,(\nabla^2\phi(y+\tau x)xx^\dagger)\nu_t^*(dx), \qquad (3.7)\end{aligned}$$

$y \in \mathbb{R}^d$, $\mathbf{1}_{[0,1]}(\tau)d\tau dt$-a.e.

Proof. Using the following version of Taylor's formula

$$\phi(y+x) = \phi(y) + \sum_{i=1}^d x_i\phi_i'(y) + \int_0^1 (1-\tau)\sum_{i,j=1}^d x_ix_j\phi_{i,j}''(y+\tau x)d\tau, \qquad x,y\in\mathbb{R}^d,$$

we have

$$\Psi(x,y) = \int_0^1 (1-\tau)\langle\nabla^2\phi(y+\tau x)x,x\rangle d\tau = \int_0^1 (1-\tau)\mathrm{Tr}\,(\nabla^2\phi(y+\tau x)xx^\dagger)d\tau$$

and (3.5) rewrites as

$$\begin{aligned}&\mathbb{E}[\phi(M_t+M_t^*)] - \mathbb{E}[\phi(M_s+M_s^*)] \\ &= \frac{1}{2}\mathbb{E}\left[\int_s^t (\mathrm{Tr}\,(\nabla^2\phi(M_u+M_u^*)H_u) - \mathrm{Tr}\,(\nabla^2\phi(M_u+M_u^*)H_u^*))du\right] \\ &\quad +\mathbb{E}\left[\int_0^1\int_s^t\int_{\mathbb{R}^d}(1-\tau)\mathrm{Tr}\,(\nabla^2\phi(M_u+M_u^*+\tau x)xx^\dagger)(\nu_u(dx)-\nu_u^*(dx))dud\tau\right]\end{aligned}$$

which is non-positive from (3.7). $\square$

Let now $(\mathcal{F}_t^M)_{t\in\mathbb{R}_+}$ and $(\mathcal{F}_t^{M^*})_{t\in\mathbb{R}_+}$, denote the forward and backward filtrations generated by $(M(t))_{t\in\mathbb{R}_+}$ and by $(M^*(t))_{t\in\mathbb{R}_+}$. The proof of the following corollary of Theorem 3.5 is identical to that of Corollary 3.7 in [4].

Corollary 3.7. *If* (3.4) *holds and if in addition $\mathbb{E}[M^*(t) \mid \mathcal{F}_t^M] = 0$, $t\in\mathbb{R}_+$, then*

$$\mathbb{E}[\phi(M(s)+M^*(s))] \geq \mathbb{E}[\phi(M(t))], \qquad 0\leq s\leq t.$$

In particular, if $M_0 = \mathbb{E}[M(t)]$ is deterministic (or if $\mathcal{F}_0^M$ is the trivial σ-field), Corollary 3.7 shows that $M(t) - \mathbb{E}[M(t)]$ is more convex concentrated than M_0^*, i.e.:

$$\mathbb{E}[\phi(M(t) - \mathbb{E}[M(t)])] \leq \mathbb{E}[\phi(M_0^*)], \qquad t \geq 0,$$

for all sufficiently integrable convex functions ϕ on $\mathbb{R}^d$. In applications to convex concentration inequalities the independence of $(M(t))_{t\in\mathbb{R}_+}$ with $(M^*(t))_{t\in\mathbb{R}_+}$ will not be required, see Section 4.

Note that in case $\nu^*(dt, dx)$ has the form

$$\nu^*(dt, dx) = \lambda^*(t)\delta_k(dx)dt,$$

where $k \in \mathbb{R}^d$ and $(\lambda^*(t))_{t\in\mathbb{R}_+}$ is a positive $\mathcal{F}_t^*$-predictable process, then Condition (i) (resp. (ii)) of Theorem 3.5 is equivalent to:

$$\nu_t = \lambda(t)\delta_k \qquad \text{and} \qquad \lambda(t) \leq \lambda^*(t)$$

resp. to: $k \in (\mathbb{R}_+)^d$, $\nu_t(\mathbb{R}^d) \leq \lambda^*(t)$ and

$$\nu_t\left(\mathbb{R}^d \setminus \bigcap_{i=1}^{d}]-\infty, k_i]\right) = 0,$$

i.e. the jump $\Delta M_i(t)$ is a.s. upper bounded by k_i, $i = 1, \ldots, d$.

Theorem 3.5 applies for instance when the jump parts of $(M(t))_{t\in\mathbb{R}_+}$ and of $(M^*(t))_{t\in\mathbb{R}_+}$ are point processes. Let $(W(t))_{t\in\mathbb{R}_+}$ be a standard $\mathbb{R}^n$-valued Brownian motion and $(W^*(t))_{t\in\mathbb{R}_+}$ be a backward standard $\mathbb{R}^n$-valued Brownian motion, and let $(Z(t))_{t\in\mathbb{R}_+}$ be a point process in $\mathbb{R}^n$ given by $Z(t) = (Z_1(t), \ldots, Z_n(t))$ where $(Z_i(t))_{t\in\mathbb{R}_+}$ is a real point process with intensity $(\lambda_i(t))_{t\in\mathbb{R}_+}$, $1 \leq i \leq n$. Similarly, let $(Z^*(t))_{t\in\mathbb{R}_+}$ be a backward point process in $\mathbb{R}^n$ with intensity $\lambda^*(t) = (\lambda_1^*(t), \ldots, \lambda_n^*(t))$, $t \in \mathbb{R}_+$.

We can take

$$M(t) = M_0 + \int_0^t A(s)dW(s) + \int_0^t J(s)(dZ(s) - \lambda(s)ds), \quad t \in \mathbb{R}_+,$$

and

$$M^*(t) = \int_t^{+\infty} A^*(s) d^* W^*(s) + \int_t^{+\infty} J^*(s)(d^* Z^*(s) - \lambda^*(s) ds), \quad t \in \mathbb{R}_+,$$

where $(A(t))_{t\in\mathbb{R}_+}$, $(J(t))_{t\in\mathbb{R}_+}$, resp. $(A^*(t))_{t\in\mathbb{R}_+}$, $(J^*(t))_{t\in\mathbb{R}_+}$ are $\mathcal{M}^{d\times n}$-valued and predictable with respect to

$$\mathcal{F}_t^M := \sigma(W(s), Z(s) : s \le t), \qquad \text{resp.} \qquad \mathcal{F}_t^{M^*} := \sigma(W^*(s), Z^*(s) : s \ge t),$$

$t \in \mathbb{R}_+$, i.e.

$$M_i(t) = M_i(0) + \sum_{j=1}^n \int_0^t A_{i,j}(s) dW_j(s) + \sum_{j=1}^n \int_0^t J_{i,j}(s)(dZ_j(s) - \lambda_j(s) ds)$$

and

$$M_i^*(t) = M_i^*(0) + \sum_{j=1}^n \int_t^\infty A_{i,j}(s) dW_j^*(s) + \sum_{j=1}^n \int_t^\infty J_{i,j}(s)(d^* Z_j^*(s) - \lambda_j(s) ds),$$

$t \in \mathbb{R}_+$, $i = 1, \ldots, d$, with

$$\nu_t(dx) = \sum_{j=1}^n \lambda_j(t) \delta_{(J_{1,j}(t),\ldots,J_{d,j}(t))}(dx) \tag{3.8}$$

and

$$\nu_t^*(dx) = \sum_{j=1}^n \lambda_j^*(t) \delta_{(J_{1,j}^*(t),\ldots,J_{d,j}^*(t))}(dx). \tag{3.9}$$

As seen above, Condition (i) and (ii) of Theorem 3.5 imply

$$\sum_{j=1}^n \lambda_j(t) J_{k,j}(t) \le \sum_{j=1}^n \lambda_j^*(t) J_{k,j}^*(t), \qquad k = 1, \ldots, d,$$

and under both conditions we have

$$\sum_{j=1}^n \lambda_j(t) \le \sum_{j=1}^n \lambda_j^*(t).$$

More details will be given in Section 5 on the interpretation of Conditions (i) and (ii) of Theorem 3.5 imposed on ν_t and ν_t^* defined in (3.8) and (3.9) for the order $\preceq_{\rm cxp}$.

Conditions (3.1), (3.2) will hold in particular when

$$\mathcal{F}_t = \mathcal{F}_t^M \vee \mathcal{F}_0^{M^*} \text{ and } \mathcal{F}_t^* = \mathcal{F}_\infty^M \vee \mathcal{F}_t^{M^*}, \qquad t \in \mathbb{R}_+,$$

see Section 4.

4 Convex ordering and predictable representation

Let $(W(t))_{t\in\mathbb{R}_+}$ be a n-dimensional Brownian motions and $\mu(dx, dt)$ be a jump measure with jump characteristics of the form

$$\nu(dt, dx) = \nu_t(dx)dt, \tag{4.1}$$

generating a filtration $(\mathcal{F}_t^M)_{t\in\mathbb{R}_+}$.

Consider $(A(t))_{t\in\mathbb{R}_+}$ an $\mathcal{M}^{d\times n}$-valued, $\mathcal{F}_t^M$-predictable square-integrable process, $(t, x) \mapsto B_t(x)$ an $\mathbb{R}^d$-valued $\mathcal{F}_t^M$-predictable process in $L^1(\Omega\times\mathbb{R}^d\times\mathbb{R}_+, dP\nu_t(dx)dt)$, $(\hat{A}(t))_{t\in\mathbb{R}_+}$ a deterministic $\mathcal{M}^{d\times n}$-valued square-integrable function, and $(t, x) \mapsto \hat{B}_t(x)$ an $\mathbb{R}^d$-valued deterministic function in $L^1(\Omega \times \mathbb{R}^d \times \mathbb{R}_+, dP\nu_t(dx)dt)$.

Theorem 4.1. *Let $(\tilde{W}(t))_{t\in\mathbb{R}_+}$ be an n-dimensional Brownian motion and $\tilde{\mu}(dx, dt)$ be a jump measure with jump characteristic of the form $\tilde{\nu}(dt, dx) = \tilde{\nu}_t(dx)dt$, both independent of $\mathcal{F}^M$, and consider*

$$F = \int_0^\infty A(t)dW(t) + \int_0^\infty \int_{\mathbb{R}^n} B_t(x)(\mu(dt, dx) - \nu_t(dx)dt)$$

and

$$G = \int_0^\infty \hat{A}(t)d\tilde{W}(t) + \int_0^\infty \int_{\mathbb{R}^n} \hat{B}_t(x)(\tilde{\mu}(dt, dx) - \tilde{\nu}_t(dx)dt).$$

Assume that

$$A^\dagger(t)A(t) \leq_{\mathrm{psd}} \hat{A}^\dagger(t)\hat{A}(t), \qquad dPdt - a.e.,$$

and that for almost all $t \in \mathbb{R}_+$, we have either:

i) $\nu_t \circ B_t^{-1} \preceq_{\mathrm{cxp}} \tilde{\nu}_t \circ \hat{B}_t^{-1}$, P-a.s.,

or:

ii) B_t and $\hat{B}_t$ are componentwise non-negative and $\nu_t \circ B_t^{-1} \preceq_{\mathrm{cxpi}} \tilde{\nu}_t \circ \hat{B}_t^{-1}$, P-a.s.

Then for all convex functions $\phi : \mathbb{R}^d \to \mathbb{R}$ we have

$$\mathbb{E}[\phi(F)] \leq \mathbb{E}[\phi(G)]. \tag{4.2}$$

Proof. Again we start by assuming that ϕ is a Lipschitz convex function. Let $(M(t))_{t\in\mathbb{R}_+}$ denote the forward martingale defined as

$$M(t) = \int_0^t A(s)dW(s) + \int_0^t \int_{\mathbb{R}^n} B_s(x)(\mu(ds,dx) - \nu_s(dx)ds),$$

$t \in \mathbb{R}_+$, let $(\mathcal{F}_t^{\tilde{M}})_{t\in\mathbb{R}_+}$ denote the backward filtration generated by $\{\tilde{W}(t), \tilde{\mu}(dx,dt)\}$, and let

$$\mathcal{F}_t = \mathcal{F}_t^M \vee \mathcal{F}_\infty^{\tilde{M}} \qquad \text{and} \qquad \mathcal{F}_t^* = \mathcal{F}_\infty^M \vee \mathcal{F}_t^{\tilde{M}},$$

so that $(M(t))_{t\in\mathbb{R}_+}$ is an $\mathcal{F}_t$-forward martingale. Since $(\hat{A}(t))_{t\in\mathbb{R}_+}$, $(\hat{B}_t)_{t\in\mathbb{R}_+}$ are $\mathcal{F}_t^M$-predictable, the processes $(\hat{A}(t))_{t\in\mathbb{R}_+}$ and $(\hat{B}_t)_{t\in\mathbb{R}_+}$ are independent of $(\tilde{W}_t)_{t\in\mathbb{R}_+}$ and of $\tilde{\mu}(dt,dx)$. In this case, the forward and backward differentials coincide and the process $(M^*(t))_{t\in\mathbb{R}_+}$ defined as

$$M^*(t) = \mathbb{E}[G \mid \mathcal{F}_t^*] = \int_t^\infty \hat{A}(s)d\tilde{W}(s) + \int_t^\infty \int_{\mathbb{R}^n} \hat{B}_s(x)(\tilde{\mu}(ds,dx) - \tilde{\nu}_s(dx)ds),$$

$t \in \mathbb{R}_+$, is an $\mathcal{F}_t^*$- backward martingale with $M^*(0) = G$. Moreover the jump characteristics of $(M_t)_{t\in\mathbb{R}_+}$ and of $(M_t^*)_{t\in\mathbb{R}_+}$ are

$$\nu_M(dx) = \mathbf{1}_{\mathbb{R}^d\setminus\{0\}}(x)\nu_t \circ B_t^{-1}(dx) \qquad \text{and} \qquad \nu_{M^*}(dx) = \mathbf{1}_{\mathbb{R}^d\setminus\{0\}}(x)\tilde{\nu}_t \circ \hat{B}_t^{-1}(dx).$$

Applying Theorem 3.5 to the forward and backward martingales $(M(t))_{t\in\mathbb{R}_+}$ and $(M^*(t))_{t\in\mathbb{R}_+}$ yields

$$\mathbb{E}[\phi(M(t) + M^*(t))] \leq \mathbb{E}[\phi(M(s) + M^*(s))], \qquad 0 \leq s \leq t,$$

for all convex functions ϕ and for $0 \leq s \leq t$. Since $M(0) = 0$, $M^*(0) = G$ and $\lim_{t\to\infty} M^*(t) = 0$ in $L^2(\Omega)$, we obtain (4.2) for convex Lipschitz function ϕ by taking $s = 0$ and letting t go to infinity. Finally we extend the formula to all convex integrable functions ϕ by considering an increasing sequence of Lipschitz convex functions ϕ_n converging pointwise to ϕ. Applying the monotone convergence theorem to the non-negative sequence $\phi_n(F) - \phi_0(F)$, we have

$$\mathbb{E}[\phi(F) - \phi_0(F)] = \lim_{n\to\infty} \mathbb{E}[\phi_n(F) - \phi_0(F)],$$

which yields $\mathbb{E}[\phi(F)] = \lim_{n\to\infty} \mathbb{E}[\phi_n(F)]$ since $\phi_0(F)$ is integrable. We proceed similarly for $\phi(G)$, allowing us to extend (4.2) to the general case. □

Note that if $(\hat{A}(t))_{t\in\mathbb{R}_+}$ and $(\hat{B}_t)_{t\in\mathbb{R}_+}$ are deterministic then $(\tilde{W}(t))_{t\in\mathbb{R}_+}$ and $\tilde{\mu}(dx,dt)$ can be taken equal to $(W(t))_{t\in\mathbb{R}_+}$ and $\mu(dx,dt)$ respectively.

Example: point processes

Let $(A(t))_{t\in\mathbb{R}_+}$, $(\hat{A}(t))_{t\in\mathbb{R}_+}$, $(W(t))_{t\in\mathbb{R}_+}$ and $(\tilde{W}(t))_{t\in\mathbb{R}_+}$ be as in Theorem 4.1 above and consider

$$Z(t) = (Z_1(t),\dots,Z_n(t)) \quad \text{and} \quad \tilde{Z}(t) = (\tilde{Z}_1(t),\dots,\tilde{Z}_n(t))$$

to be two independent point processes in $\mathbb{R}^n$ with compensators

$$\sum_{i=1}^n \lambda_i(t)\delta_{e_i} \quad \text{and} \quad \sum_{i=1}^n \tilde{\lambda}_i(t)\delta_{e_i},$$

where $e_i = (0,\dots,1,\dots,0)$, $i = 1,\dots,n$, denotes the canonical basis in $\mathbb{R}^n$, and let

$$\mathcal{F}_t^M = \sigma(W(s),\ Z(s)\ :\ 0 \le s \le t), \qquad t \in \mathbb{R}_+.$$

Corollary 4.2. *Given $J(t)$ an $\mathcal{M}^{d\times n}$-valued integrable $\mathcal{F}_t^M$-predictable process and $\hat{J}(t)$ an $\mathcal{M}^{d\times n}$-valued deterministic integrable function, let*

$$F = \int_0^\infty A(t)dW(t) + \int_0^\infty J(t)(dZ(t) - \lambda(t)dt),$$

and

$$G = \int_0^\infty \hat{A}(t)d\tilde{W}(t) + \int_0^\infty \hat{J}(t)(d\tilde{Z}(t) - \tilde{\lambda}(t)dt).$$

Assume that

$$A^\dagger(t)A(t) \le_{\text{psd}} \hat{A}^\dagger(t)\hat{A}(t), \qquad dPdt - a.e.,$$

and that for almost all $t \in \mathbb{R}_+$, we have either:

i) $\sum_{j=1}^n \lambda_j(t)\delta_{(J_{1,j}(t),\dots,J_{d,j}(t))} \preceq_{\text{cxp}} \sum_{j=1}^n \tilde{\lambda}_j(t)\delta_{(\hat{J}_{1,j}(t),\dots,\hat{J}_{d,j}(t))}$, *$P$-a.s.,*

or:

ii) $J_{i,j}(t) \ge 0$, *and* $\hat{J}_{i,j}(t) \ge 0$, $i = 1,\dots,d$, $j = 1,\dots,n$, *and*

$$\sum_{j=1}^n \lambda_j(t)\delta_{(J_{1,j}(t),\dots,J_{d,j}(t))} \preceq_{\text{cxpi}} \sum_{j=1}^n \tilde{\lambda}_j(t)\delta_{(\hat{J}_{1,j}(t),\dots,\hat{J}_{d,j}(t))}, \quad P\text{-a.s.}$$

Then for all convex functions $\phi : \mathbb{R}^d \to \mathbb{R}$ we have

$$\mathbb{E}[\phi(F)] \le \mathbb{E}[\phi(G)].$$

Proof. We apply Theorem 4.1 with $B_t(x) := J(t)x$ and $\hat{B}_t(x) := \hat{J}(t)x$, $t \in \mathbb{R}_+$, and

$$\nu_t(dx) = \mathbf{1}_{\mathbb{R}^d \setminus \{0\}}(x) \sum_{j=1}^{n} \lambda_j(t) \delta_{(J_{1,j}(t),\dots,J_{d,j}(t))} \tag{4.3}$$

and

$$\tilde{\nu}_t(dx) = \mathbf{1}_{\mathbb{R}^d \setminus \{0\}}(x) \sum_{j=1}^{n} \tilde{\lambda}_j(t) \delta_{(\hat{J}_{1,j}(t),\dots,\hat{J}_{d,j}(t))}, \tag{4.4}$$

since $B_t(x)$, resp. $B_t(x)$, is componentwise non-negative $\nu_t(dx)$-a.e., resp. $\tilde{\nu}_t(dx)$-a.e. $\square$

Note that $\nu_t \preceq_{\rm cxp} \tilde{\nu}_t$ if and only if $\lambda_j(t) \le \tilde{\lambda}_j(t)$, $j = 1, \dots, n$, since $\operatorname{Supp}(\nu_t) = \operatorname{Supp}(\mu_t) = \{e_1, \dots, e_n\}$. In Section 5 we will give necessary conditions for the convex ordering $\preceq_{\rm cxp}$ to hold, with application to the conditions imposed on $\nu_t \circ B_t^{-1}$ and $\tilde{\nu}_t \circ \hat{B}_t^{-1}$ in (i) and (ii) above.

Example: Poisson random measures

Consider σ, $\tilde{\sigma}$ two atomless Radon measures on $\mathbb{R}^n$ with

$$\int_{\mathbb{R}^n} (|x|^2 \wedge 1)\sigma(dx) < \infty, \quad \text{and} \quad \int_{\mathbb{R}^n} (|x|^2 \wedge 1)\tilde{\sigma}(dx) < \infty,$$

and two mutually independent Poisson random measures

$$\omega(dt, dx) = \sum_{i \in \mathbb{N}} \delta_{(t_i, x_i)}(dt, dx) \quad \text{and} \quad \tilde{\omega}(dt, dx) = \sum_{i \in \mathbb{N}} \delta_{(\tilde{t}_i, \tilde{x}_i)}(dt, dx)$$

with respective intensities $\sigma(dx)dt$ and $\tilde{\sigma}(dx)dt$ on $\mathbb{R}^n \times \mathbb{R}_+$ under P. Let also $(W(t))_{t \in \mathbb{R}_+}$ and $(\tilde{W}(t))_{t \in \mathbb{R}_+}$ be independent n-dimensional standard Brownian motions, independent of $\tilde{\omega}(dt, dx)$ under P and let $(A(t))_{t \in \mathbb{R}_+}$, $(\hat{A}(t))_{t \in \mathbb{R}_+}$ be as in Theorem 4.1 above, with

$$\mathcal{F}_t^M = \sigma(W(s),\ \omega([0, s] \times A)\ :\ 0 \le s \le t,\ A \in \mathcal{B}_b(\mathbb{R}^n)), \qquad t \in \mathbb{R}_+,$$

where $\mathcal{B}_b(\mathbb{R}^n) = \{A \in \mathcal{B}(\mathbb{R}^n)\ :\ \sigma(A) < \infty\}$.

Corollary 4.3. *Let $(J_{t,x})_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^n}$ be an $\mathbb{R}^d$-valued $\mathcal{F}_t^M$-predictable process, integrable with respect to $dPdt\sigma(dx)$, and let $(\hat{J}_{t,x})_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^n}$ be an $\mathbb{R}^d$-valued deterministic function, integrable with respect to $dt\tilde{\sigma}(dx)$. Consider the random variables*

$$F = \int_0^\infty A(t)dW(t) + \int_0^\infty \int_{\mathbb{R}^n} J_{t^-,x}(\omega(dt,dx) - \sigma(dx)dt) \tag{4.5}$$

and

$$G = \int_0^\infty \hat{A}(t)d\tilde{W}(t) + \int_0^\infty \int_{\mathbb{R}^n} \hat{J}_{t^-,x}(\tilde{\omega}(dt,dx) - \tilde{\sigma}(dx)dt).$$

Assume that

$$A^\dagger(t)A(t) \leq_{\text{psd}} \hat{A}^\dagger(t)\hat{A}(t), \qquad dPdt\text{-a.e.},$$

and that for almost all $t \in \mathbb{R}_+$, we have either:

i) $\sigma \circ J_{t^-,\cdot}^{-1} \preceq_{\text{cxp}} \tilde{\sigma} \circ \hat{J}_{t^-,\cdot}^{-1}$, *$P$-a.s.,*

or:

ii) $J_{t^-,x} \geq 0$, *$\sigma(dx)$-a.e.,* $\hat{J}_{t^-,x} \geq 0$, *$\tilde{\sigma}(dx)$-a.e., and*

$$\sigma \circ J_{t^-,\cdot}^{-1} \preceq_{\text{cxpi}} \tilde{\sigma} \circ \hat{J}_{t^-,\cdot}^{-1}, \quad P\text{-a.s.}$$

Then for all convex functions $\phi : \mathbb{R}^d \to \mathbb{R}$ we have

$$\mathbb{E}[\phi(F)] \leq \mathbb{E}[\phi(G)].$$

Proof. We apply Theorem 4.1 with the jump characteristics

$$\nu_t(dx) = \mathbf{1}_{\mathbb{R}^d\setminus\{0\}}(x)\sigma \circ J_{t^-,\cdot}^{-1}(dx) \qquad \text{and} \qquad \tilde{\nu}_t(dx) = \mathbf{1}_{\mathbb{R}^d\setminus\{0\}}(x)\tilde{\sigma} \circ \hat{J}_{t^-,\cdot}^{-1}(dx),$$

and $B_t(x) = \hat{B}_t(x) = x$, $x \in \mathbb{R}^n$. □

Condition (i), resp. (ii) in Corollary 4.3 can be written as

$$\int_{\mathbb{R}^n} f(J_{t^-,x})\sigma(dx) \leq \int_{\mathbb{R}^n} f(\hat{J}_{t^-,x})\tilde{\sigma}(dx)$$

for all non-negative convex functions $f : \mathbb{R}^d \to \mathbb{R}$, resp. for all non-negative and non-decreasing convex functions $f : \mathbb{R}^d \to \mathbb{R}$. In particular, Corollary 4.3-ii) holds if we have $\sigma \preceq_{\text{cxpi}} \tilde{\sigma}$ and $0 \leq J_{t,x} \leq \hat{J}_{t,x}$, $dt\sigma(dx)dP$-a.e., $0 \leq \hat{J}_{t,x}$, $dt\tilde{\sigma}(dx)dP$-a.e., and if $x \mapsto J_{t,x}$, $x \mapsto \hat{J}_{t,x}$ are non-decreasing and convex on $\mathbb{R}^n$ for all $t \in \mathbb{R}_+$.

We may also apply Theorem 4.1 to F as in (4.5) with

$$G = \int_0^\infty \hat{A}(t) d\tilde{W}(t) + \int_0^\infty \hat{J}(t)(d\tilde{Z}(t) - \tilde{\lambda}(t)dt)$$

where $(\hat{A}(t))_{t\in\mathbb{R}_+}$ and $(\hat{J}(t))_{t\in\mathbb{R}_+}$ are $\mathcal{F}_t^M$-predictable $\mathcal{M}^{d\times n}$-valued processes and $(\tilde{Z}_1, \ldots, \tilde{Z}_n)$ is an $\mathbb{R}^n$-valued point process independent of $\mathcal{F}^M$, with $\nu_t = \sigma \circ J_{t^-}^{-1}$ and

$$\tilde{\nu}_t = \sum_{i=1}^n \tilde{\lambda}_i(t) \delta_{e_i} \circ \hat{J}_{t^-}^{-1} = \sum_{i=1}^n \tilde{\lambda}_i(t) \delta_{\hat{J}_{t^-} e_i}.$$

Then Condition (i), resp. (ii), of Theorem 4.1 is satisfied provided

$$\sigma \circ J_{t^-}^{-1} \preceq_{\rm cxp} \sum_{i=1}^n \tilde{\lambda}_i(t) \delta_{\hat{J}_{t^-} e_i} \qquad \text{resp.} \qquad \sigma \circ J_{t^-}^{-1} \preceq_{\rm cxpi} \sum_{i=1}^n \tilde{\lambda}_i(t) \delta_{\hat{J}_{t^-} e_i} \tag{4.6}$$

In the next section we give some criteria for the validity of comparison inequalities of this type, involving sums of Dirac measures.

5 A geometric interpretation for discrete measures

The next lemma provides a first interpretation of the order $\preceq_{\rm cxp}$.

Lemma 5.1. *If μ and ν are two measures on $\mathbb{R}^d$ with finite supports, then $\mu \preceq_{\rm cxp} \nu$ implies*

$$\mathscr{C}(\mathrm{Supp}\,(\mu)) \subset \mathscr{C}(\mathrm{Supp}\,(\nu)), \tag{5.1}$$

where $\mathscr{C}(A)$ denote the convex hull of any subset A of $\mathbb{R}^d$.

Proof. Let H be any half-space of $\mathbb{R}^d$ such that $\mathrm{Supp}\,(\nu) \subset H$. For any convex function ϕ such that $\{\phi \leq 0\} = H$ and $\phi_{|\partial H} = 0$ we have $\int_{\mathbb{R}^d} \phi^+ \nu(dx) = 0$, hence $\int_{\mathbb{R}^d} \phi^+ \mu(dx) = 0$ since $\mu \preceq_{\rm cxp} \nu$, which implies $\mathrm{Supp}\,(\mu) \subset H$. The conclusion follows from the characterization of the convex hull $\mathscr{C}(\mathrm{Supp}\,(\mu))$, resp. $\mathscr{C}(\mathrm{Supp}\,(\nu))$, as the intersections of all half-spaces containing it. $\square$

However the necessary Condition (5.1) is clearly not sufficient to ensure the convex ordering of μ and ν. Our aim in this section is to find a more precise geometric interpretation of $\mu \preceq_{\mathrm{cxp}} \nu$ in the case of finite supports, with the aim of applying this criterion to the jump measures defined in (3.8), (3.9), (4.3), (4.4) and (4.6).

For all $u \in S^{d-1}$ the unit sphere in $\mathbb{R}^d$, let $\mu_u = \mu \circ \langle u, \cdot \rangle^{-1}$ (resp. $\nu_u = \nu \circ \langle u, \cdot \rangle^{-1}$) denote the image of μ, resp. ν, on $\mathbb{R}$ by the mapping $x \mapsto \langle u, x \rangle$. We have $\mu_u \preceq_{\mathrm{cxp}} \nu_u$ and the survival function $\phi_{\mu,u}$ associated with μ_u, defined by

$$\phi_{\mu,u}(a) = \int_{\mathbb{R}} (y-a)^+ \, d\mu_u(y) = \int_{\mathbb{R}^d} (\langle y, u \rangle - a)^+ \, d\mu(y),$$

is a convex function with $\phi_{\mu,u} \le \phi_{\nu,u}$ for all $u \in S^{d-1}$. Moreover for all $a \in \mathbb{R}$ such that a is sufficiently large we have $\phi_{\mu,u}(a) = \phi_{\nu,u}(a) = 0$.

For every $x \in \mathbb{R}^d$ and $u \in S^{d-1}$ let

$$a_{x,u} = \inf \{ b \in \mathbb{R} \ : \ b \ge \langle u, x \rangle, \ \phi_{\mu,u}(b) = \phi_{\nu,u}(b) \} \tag{5.2}$$

and

$$\mathcal{D}_{x,u} = \{ y \in \mathbb{R}^d \ : \ \langle u, y \rangle \le a_{x,u} \},$$

which is a closed half-space containing x. Finally we let

$$\mathcal{C}_x := \bigcap_{u \in S^{d-1}} \mathcal{D}_{x,u}$$

which is a compact convex set containing x. On the other hand, letting

$$\tilde{\mathcal{D}}_u = \{ z \in \mathbb{R}^d \ : \ \langle u, z \rangle \le \tilde{a}_u \}, \qquad x \in \mathbb{R}^d, \quad u \in S^{d-1},$$

where

$$\tilde{a}_u = \inf \{ b \in \mathbb{R} \ : \ b \ge \langle u, y \rangle, \ \forall y \in \mathrm{Supp}\,(\nu) \},$$

we have

$$\mathscr{C}(\mathrm{Supp}\,(\nu)) = \bigcap_{u \in S^{d-1}} \tilde{\mathcal{D}}_u.$$

Note that we have $\mathcal{C}_x \subset \mathscr{C}(\mathrm{Supp}\,(\nu))$ if $\mu \preceq_{\mathrm{cxp}} \nu$, indeed we have $\mathcal{D}_{x,u} \subset \tilde{\mathcal{D}}_u$ since $a_{x,u} \leq \tilde{a}_u$, as follows from $\phi_{\mu,u}(b) = \phi_{\nu,u}(b) = 0$ for all $b \geq \tilde{a}_u$. On the other hand, if $\mu = \delta_x \preceq_{\mathrm{cxp}} \nu$ then

$$\mathcal{C}_x = \mathscr{C}(\mathrm{Supp}\,(\nu)) \tag{5.3}$$

since for all $u \in S^{d-1}$ there exists $z \in \mathrm{Supp}\,(\nu)$ such that $\langle u, z\rangle = \tilde{a}_u$ and for all $b \in (\langle u, x\rangle, \tilde{a}_u]$ we have

$$\phi_{\nu,u}(b) \geq (\tilde{a}_u - b)\nu(\{z\}) > 0$$

and $\phi_{\mu,u}(b) = 0$, implying $a_{x,u} = \tilde{a}_u$ and $\mathcal{D}_{x,u} = \tilde{\mathcal{D}}_u$. Note that in Theorem 5.2 below the existence of $x \in \mathbb{R}^d$ such that $\mu(\{x\}) > \nu(\{x\})$ is always satisfied when $\mu \preceq_{\mathrm{cx}} \nu$ and $\mu \neq \nu$.

Theorem 5.2. *Assume that μ and ν have finite supports and that $\mu \preceq_{\mathrm{cxp}} \nu$. Then for all $x \in \mathbb{R}^d$ such that $\mu(\{x\}) > \nu(\{x\})$ there exists $k \in \{2, \ldots, d+1\}$ and k elements $y_1, \ldots, y_k \in \mathrm{Supp}\,(\nu)$ distinct from x, such that*

$$\{x\} \;\subset\; \mathscr{C}(\{y_1, \ldots, y_k\}) \;\subset\; \mathcal{C}_x. \tag{5.4}$$

Proof. We only need to prove that x belongs to the convex hull of $(\mathcal{C}_x \backslash \{x\}) \cap \mathrm{Supp}\,(\nu)$. Indeed, if $x \in \mathscr{C}((\mathcal{C}_x \backslash \{x\}) \cap \mathrm{Supp}\,(\nu))$ then there exists k points $y_1, \ldots, y_k$ in this set, $k \geq 2$, such that x is the convex barycenter of $y_1, \ldots, y_k$, and the Caratheodory theorem (see e.g. [6], Theorem 17.1) shows that the conclusion holds for some $k \in \{2, \ldots, d+1\}$.

Assume now that the assertion of the theorem is true when μ and ν have disjoint supports, and let μ and ν be any measures with finite supports, such that $\mu \preceq_{\mathrm{cxp}} \nu$ and $\mu \neq \nu$. We let $\mathcal{S}^+ = \{y \in \mathbb{R}^d,\ \nu(\{y\}) \geq \mu(\{y\})\}$, $\mathcal{S}^- = \{y \in \mathbb{R}^d,\ \mu(\{y\}) > \nu(\{y\})\}$. These sets are not empty since $\mu \neq \nu$. Let μ' and ν' the measures defined by

$$\mu'(\{y\}) = \mu(\{y\}) - \nu(\{y\}), \qquad \nu'(\{y\}) = 0, \qquad y \in \mathcal{S}^-,$$

$$\mu'(\{y\}) = 0, \qquad \nu'(\{y\}) = \nu(\{y\}) - \mu(\{y\}), \qquad y \in \mathcal{S}^+.$$

If f is a function on $\mathbb{R}^d$ we have

$$\mu'(f) = \mu(f) - \sum_{y \in \mathcal{S}^-} \nu(\{y\}) f(y) - \sum_{y \in \mathcal{S}^+} \mu(\{y\}) f(y)$$

and

$$\nu'(f) = \nu(f) - \sum_{y\in\mathcal{S}^-} \nu(\{y\})f(y) - \sum_{y\in\mathcal{S}^+} \mu(\{y\})f(y)$$

which implies that $\mu \preceq_{\text{cxp}} \nu$ if and only if $\mu' \preceq_{\text{cxp}} \nu'$. It also implies that for all $u \in S^{d-1}$ and $b \in \mathbb{R}$,

$$\phi_{\mu,u}(b) = \phi_{\nu,u}(b) \iff \phi_{\mu',u}(b) = \phi_{\nu',u}(b).$$

From this together with the fact that $\phi_{\mu',u} \le \phi_{\nu',u}$, we conclude that if $\mathcal{D}'_{x,u}$ is defined as $\mathcal{D}_{x,u}$ but with (μ,ν) replaced by (μ',ν'), then $\mathcal{D}'_{x,u} = \mathcal{D}_{x,u}$. Finally remarking that the support of ν' is included in the support of ν, we proved that it is sufficient to do the proof with μ' and ν'.

So in the sequel we assume that μ and ν have disjoint supports. As a consequence of Theorem 40 in [3] applied to the cone of non-negative convex functions, there exists an admissible[1] kernel K such that $\mu K = \nu$.

Let now $x \in \mathbb{R}^d$ satisfy $\mu(\{x\}) > \nu(\{x\}) = 0$. Clearly the support of $K(x,dy)$ is included in the support of ν, and by (5.3), x is in the convex hull of the support of $K(x,dy)$. Finally we are left to prove that the support of $K(x,dy)$ is included in $\mathcal{C}_x$.

For this we let μ^x be the measure defined by

$$\mu^x(\{y\}) = \begin{cases} \mu(\{y\}) + \mu(\{x\})K(x,\{y\}), & y \neq x, \\ 0, & y = x. \end{cases}$$

Then $\mu \preceq_{\text{cxp}} \mu^x$ and $\mu^x \preceq_{\text{cxp}} \nu$, which is easily proved by the existence of admissible kernels P and P' such that $\mu P = \mu^x$ and $\mu^x P' = \nu$. More precisely they are given by

$$P(x,dy) = \sum_{z\in\text{Supp }(\nu)} K(x,\{z\})\delta_z(dy), \qquad P(z,dy) = \delta_z(dy), \quad z \neq x,$$

and

$$P'(x',dy) = \begin{cases} \displaystyle\sum_{z\in\text{Supp }(\nu)} K(x',\{z\})\delta_z(dy), & x' \in \text{Supp }(\mu)\backslash\{x\}, \\ \delta_{x'}(dy), & x' \notin \text{Supp }(\mu)\backslash\{x\}. \end{cases}$$

[1] Admissible means here that for every $x \in \mathbb{R}^d$ we have $\delta_x \preceq_{\text{cxp}} K(x,dy)$.

So for every $u \in S^{d-1}$, we have $\phi_{\mu,u} \le \phi_{\mu^x,u} \le \phi_{\nu,u}$. Let us prove that this inequality implies that any point of the support of $K(x, dy)$ belongs to $\mathcal{D}_{x,u}$. Assume that a point z of the support of $K(x, dy)$ does not. An easy calculation shows that the right derivatives $\phi'_{\mu^x,u}$ and $\phi'_{\mu,u}$ satisfy $\phi'_{\mu^x,u}(t) = -\mu_u^x(]t,\infty[)$ and $\phi'_{\mu,u}(t) = -\mu_u(]t,\infty[)$. From the definition of μ^x we see that for $t \ne \langle u, x\rangle$, $\mu_u^x(\{t\}) \ge \mu_u(\{t\})$. This implies that for $t \ge \langle u, x\rangle$,

$$-\mu_u^x(]t,\infty[) \le -\mu_u(]t,\infty[).$$

But since $\mu_u^x(\{\langle u, z\rangle\}) > \mu_u(\{\langle u, z\rangle\})$ and $\langle u, z\rangle > \langle u, x\rangle$, we have for $t \in [\langle u, x\rangle, \langle u, z\rangle[$

$$-\mu_u^x(]t,\infty[) < -\mu_u(]t,\infty[)$$

and this implies that $\phi_{\mu,u}(a_{x,u}) < \phi_{\mu^x,u}(a_{x,u})$ where $a_{x,u}$ is defined in (5.2). Since $\phi_{\mu,u} \le \phi_{\mu^x,u} \le \phi_{\nu,u}$, we obtain $\phi_{\mu,u}(a_{x,u}) < \phi_{\nu,u}(a_{x,u})$, contradicting (5.2).

We proved that for every u, any point of the support of $K(x, dy)$ belongs to $\mathcal{D}_{x,u}$. This implies that the support of $K(x, dy)$ is included in $\mathcal{D}_{x,u}$, achieving the proof.

□

Remark 5.3. *If* (5.4) *holds for all* $x \in \mathrm{Supp}\,(\mu)$ *and some* $y_1, \ldots, y_k \in \mathrm{Supp}\,(\nu)$ *then we have*

$$\mathscr{C}(\mathrm{Supp}\,(\mu)) \subset \mathscr{C}(\mathrm{Supp}\,(\nu)), \tag{5.5}$$

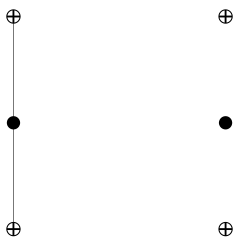
Proof. Let $x \in \mathrm{Supp}\,(\mu)$. If $\mu(x) \le \nu(x)$ then we clearly have $x \in \mathscr{C}(\mathrm{Supp}\,(\nu))$, and if $\mu(x) > \nu(x)$ then (5.4) also implies $x \in \mathscr{C}(\mathrm{Supp}\,(\nu))$. □

As a consequence of the above remark we note that the conclusion of Theorem 5.2 is stronger than (5.1), since two measures μ and ν may satisfy (5.5) without satisfying the condition $\mu \preceq_{\mathrm{cxp}} \nu$. Counterexamples are easily constructed in dimension one when μ and ν have same support.

Let us now consider some examples with $d = 2$ in the complex plane for simplicity of notation. First, let

$$\mu = \frac{1}{2}(\delta_{-1} + \delta_1) \qquad \text{and} \qquad \nu = \frac{1}{4}(\delta_{-1-i} + \delta_{-1+i} + \delta_{1-i} + \delta_{1+i}),$$

and $x = -1$. In this case we have $C_{-1} = [-1-i, -1+i]$, as illustrated in the following figure:



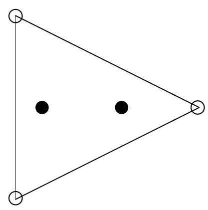
In order to see this, take $u = 1$ and then $u = -1$, and note that C_{-1} is contained in the vertical line passing through -1 and apply Theorem 5.2. Similarly we have $C_1 = [1-i, 1+i]$. Next, consider

$$\mu = \frac{1}{3}\delta_{1/2} + \frac{2}{3}\delta_{-1/4} \qquad \text{and} \qquad \nu = \frac{1}{3}(\delta_1 + \delta_j + \delta_{j^2}),$$

where $j = e^{2i\pi/3}$. Then Theorem 5.2 shows that

$$C_{1/2} = C_{-1/4} = \mathscr{C}(\mathrm{Supp}\,(\nu)),$$

since $\{y_1, y_2, y_3\}$ is necessarily equal to $\mathrm{Supp}\,(\nu)$, as illustrated below:



Finally, note that one can extend the result of Theorem 5.2 to any non empty subset E of $\mathbb{R}^d$ such that $x \in E$ implies $\mu(x) > \nu(x)$. Letting

$$a_{E,u} = \inf \left\{ b \in \left[\sup_{x \in E} \langle u, x \rangle, \infty \right[, \ \phi_{\mu,u}(b) = \phi_{\nu,u}(b) \right\}$$

for $u \in S^{d-1}$, with $a_{E,u} = \sup_{x \in E} a_{x,u}$ since E is finite,

$$\mathcal{D}_{E,u} = \{ y \in \mathbb{R}^d \ : \ \langle u, y \rangle \leq a_{E,u} \},$$

and defining

$$\mathcal{C}_E := \bigcap_{u \in S^{d-1}} \mathcal{D}_{E,u}$$

which is a compact convex set containing E, we have $\mathcal{C}_E \subset \mathrm{Supp}\,(\nu)$ (since as previously $a_{E,u} \leq \tilde{a}_u$) and:

Corollary 5.4. *Assume that μ and ν have finite supports and that $\mu \preceq_{\mathrm{cxp}} \nu$. Then for all non empty subset E of $\mathbb{R}^d$ such that $x \in E$ implies $\mu(x) > \nu(x)$, there exists $k \in \{2, \ldots, (d+1)\mathrm{card}(E)\}$ and k elements $y_1, \ldots, y_k \in \mathrm{Supp}\,(\nu)$ distinct from x, such that*

$$E \ \subset \ \mathscr{C}(\{y_1, \ldots, y_k\}) \ \subset \ \mathcal{C}_E.$$

Proof. From the definition of $\mathcal{C}_E$ it is clear that if $x \in E$ then $\mathcal{C}_x \subset \mathcal{C}_E$. Consequently applying Theorem 5.2 to every $x \in E$ gives the result. □

When μ and ν are probability measures, the existence of the admissible kernel K such that $\mu K = \nu$, used in the proof of Theorem 5.2, is also known as Strassen's theorem [7], and it is equivalent to the existence of two random variables F, G with respective laws μ and ν, and such that $F = \mathbb{E}[G|F]$. Here we used Theorem 40 of [3] which relies on the Hahn-Banach theorem. In dimension one this result has been recovered via a constructive proof in [5]. We close this paper with the following remark which concerns the $\preceq_{\mathrm{cx}}$ ordering.

Remark 5.5. *The conclusion of Theorem 5.2, associated to the condition $\mu \preceq_{\mathrm{cx}} \nu$, implies the existence of an admissible kernel K such that $\mu K = \nu$.*

Proof. We use the notation of Theorem 5.2. First we show that if $\mu(\{x\}) > \nu(\{x\})$ then there exists a kernel K_x such that $K_x(x, dy)$ is supported by $\{x, y_1, \ldots, y_k\}$ and is not equal to δ_x, $K_x(x', dy)$ is equal to $\delta_{x'}$ if $x' \neq x$, and $\mu \preceq_{\mathrm{cx}} \mu K_x \preceq_{\mathrm{cx}} \nu$. Indeed we can take

$$K_x(x, dy) = (1-\varepsilon)\delta_x(dy) + \varepsilon \sum_{i=1}^{k} a_i \delta_{y_i}(dy)$$

where the a_i's are positive, $\sum_{i=1}^k a_i = 1$, $x = \sum_{i=1}^k a_i y_i$ and $\varepsilon > 0$ is sufficiently small - the existence of ε follows from the fact that the functions $\phi_{\mu,u}$ and $\phi_{\nu,u}$ are continuous in u, together with the compactness of S^{d-1}. Now let K be a maximal[2] admissible kernel such that $\mu K \preceq_{\rm cx} \nu$ and the support of μK is included in $\mathrm{Supp}\,(\mu) \cup \mathrm{Supp}\,(\nu)$. If $\mu K \neq \nu$ then we can apply the argument above to μK, ν and x such that $\mu K(\{x\}) > \nu(\{x\})$, and find a non trivial kernel K_x such that $\mu K K_x \preceq_{\rm cx} \nu$, contradicting the maximality of K. So we conclude that $\mu K = \nu$. □

Thus an independent proof of Theorem 5.2, not relying on Theorem 40 of [3], would provide a direct construction the admissible kernel K, extending the result of [5] to higher dimensions.

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[2] Here, $K \leq K'$ means that there exists an admissible K'' such that $K \circ K'' = K'$.