Connections and curvature in the Riemannian geometry of configuration spaces

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Abstract

Torsion free connections and a notion of curvature are introduced on the infinite dimensional nonlinear configuration space Γ of a Riemannian manifold M under a Poisson measure. This allows to state identities of Weitzenböck type and energy identities for anticipating stochastic integral operators. The one-dimensional Poisson case itself gives rise to a non-trivial geometry, a de Rham-Hodge-Kodaira operator, and a notion of Ricci tensor under the Poisson measure. The methods used in this paper have been thus far applied to d-dimensional Brownian path groups, and rely on the introduction of a particular tangent bundle and associated damped gradient.

1   Introduction

The path space of Riemannian Brownian motion can be treated as an infinite-dimensional manifold via the stochastic calculus of variations. Notions of connection and curvature have been introduced in this context in [8], [9], [10], and path spaces on Lie groups have been equipped with flat connections, cf. [1], [12]. The Poisson process being another important example of stochastic process, it is natural to study Poisson spaces as examples of infinite-dimensional manifolds. The Poisson space (or configuration space) based on a Riemannian manifold is an example of infinite-dimensional nonlinear space whose geometry has been studied in [4], [25], via an integration by parts formula. In this paper we construct connections on configuration spaces using methods generally applied to Lie group valued Brownian motion. Here, the bracket of vector fields maps couples of functions on M to functions on M, and in this sense it is similar to the Poisson bracket in differential geometry. The connection constructed in this paper

has curvature if dim $M > 1$ but no torsion in general, it is not Riemannian but allows to state energy and Weitzenböck type identities. The following table gathers the basic elements of this geometry and presents an analogy between finite and infinite dimensions.

Notation	finite dimension	infinite dimension
γ	point measure on M	element of Γ
$\mathcal{C}_c^\infty(M)$	test functions on M	tangent vectors to Γ
σ	volume element of M	Riemannian metric on Γ
$\mathcal{U}_c^\infty(M)$	stochastic processes indexed by M	vector fields on Γ
$d^\Gamma u$	process indexed by $M \times M$	exterior derivative of $u \in \mathcal{U}_c^\infty(M)$
$\{\cdot,\cdot\}$	bracket on $\mathcal{U}_c^\infty(M) \times \mathcal{U}_c^\infty(M)$	bracket of vector fields on Γ
Ω^Γ	trilinear mapping on $\mathcal{U}_c^\infty(M)^3$	curvature tensor on Γ
$\tilde{D}$	damped stochastic gradient	gradient on Γ

We make use of the three basic differential structures (Fock, intrinsic and damped) on configuration spaces, and proceed as follows. In Sect. 2 we recall that the Shigekawa-Weitzenböck identity can be stated in terms Fock space and thus also applies to Poisson space. However, on Poisson space such an identity does not involve intrinsic differential geometric tools, hence the need for other constructions (the same occurs on the space of Riemannian Brownian motion compared to the flat Wiener space). In Sect. 3 we present a summary of the construction of connection and Weitzenböck type identity in the one-dimensional case ($M = \mathbb{R}_+$), which has particular properties. Some results in this section appear later as consequences of the more general framework of the following sections. Sect. 4 recalls the construction of a differential structure on configuration space according to [2], [3], [4], [5], and the proof of integration by parts formula via pointwise identities as in [20]. In Sect. 5 we state the definition of the damped gradient which will be essential here (for $M = \mathbb{R}_+$ this gradient coincides with the gradient of [6]). Functions on M are viewed as tangent vectors and a connection with vanishing torsion but non-zero curvature in general is introduced in Sect. 6. This connection is not Riemannian but it has suitable commutation properties with stochastic integrals, for this reason it will be called the Markovian connection (a Riemannian and torsion free Levi-Civita connection is also introduced). The Lie-Poisson bracket $\{\cdot,\cdot\}$ acts on functions on M and we use a notion of differential geometry in continuous indices as in [8], the indices being elements of M itself. The exterior derivative of differential one-forms (functions on M) is defined in Sect. 8. The Markovian connection is used to state energy identities and bounds for the damped anticipating integral operator $\tilde{\delta}$ in Sect. 9. The one-dimensional case is given again particular attention in Sect. 10,

where a de Rham-Hodge-Kodaira operator and a notion of Ricci curvature are defined. Sect. 11 is devoted to a linear numerical model of Poisson space in which the Ricci tensor vanishes.

2 Shigekawa identity in Fock space

Let $\Phi(L^2(M))$ denote the Fock space with inner product $\langle \cdot, \cdot \rangle_\Phi$, on a L^2 space $L^2(M, d\sigma)$. Let $D : \Phi(L^2(M)) \longrightarrow \Phi(L^2(M)) \otimes L^2(M)$ and $\delta : \Phi(L^2(M)) \otimes L^2(M) \longrightarrow \Phi(L^2(M))$ denote the unbounded gradient and Skorokhod integral operator on $\Phi(L^2(M))$, which are mutually adjoint. An energy identity for δ can be stated as

$$\|\delta(u)\|_\Phi^2 = \|u\|_{\Phi \otimes L^2(M)}^2 + \int_M \int_M \langle D_x u(y), D_y u(x) \rangle_\Phi \sigma(dx) \sigma(dy),$$

see e.g. Th. 4.1. of [18]. Its proof being dependent only on the Fock structure, this identity makes sense on flat Wiener space via the Wiener-Itô isomorphism and can be rewritten as a Weitzenböck identity, cf. [26]:

$$\|\delta(u)\|_\Phi^2 + \frac{1}{2} \int_M \int_M \|D_x u(y) - D_y u(x)\|_\Phi^2 \sigma(dx) \sigma(dy) = \|u\|_{\Phi \otimes L^2(M)}^2 + \|Du\|_{\Phi \otimes L^2(M)^{\otimes 2}}^2.$$
$$(2.1)$$

Using the Wiener-Itô isomorphism, this identity applies on Poisson space as well as on the flat Wiener space of $\mathbb{R}^d$-valued Brownian motion, but in the latter case it is not directly relevant to Riemannian Brownian motion for which a special geometry has to be developed via intrinsic differential operators, cf. [9]. The situation in the Poisson case is similar. Let Γ denote the configuration space on a metric space M, that is the set of Radon measures on M of the form

$$\Gamma = \left\{ \gamma = \sum_{i=1}^{i=n} \epsilon_{x_i} \quad : \quad (x_i)_{i=1}^{i=n} \subset M, \quad x_i \neq x_j \; \forall i \neq j, \; n \in \mathbb{N} \cup \{\infty\} \right\},$$

where ϵ_x denotes the Dirac measure at $x \in M$, with the vague topology and associated σ-algebra, cf. [4]. Let σ be a diffuse Radon measure on M, and let P denote the Poisson measure with intensity σ on Γ. Under the Wiener-Itô identification of Poisson space and Fock space, D is a finite difference operator, cf. e.g. [15]:

$$D_x F(\gamma) = F(\gamma + (1 - \gamma(\{x\}))\epsilon_x) - F(\gamma), \quad x \in M, \; \gamma \in \Gamma,$$

for measurable $F : \Gamma \longrightarrow \mathbb{R}$, and δ acts as a compensated Poisson stochastic integral, in particular if $h \in \mathcal{C}_c^\infty(M)$ is deterministic,

$$\delta(h)(\gamma) = \int_M h \, d\gamma - \int_M h \, d\sigma = \sum_{\{x \in M \; : \; \gamma(\{x\})=1\}} h(x) - \int_M h(x)\sigma(dx), \quad h \in \mathcal{C}_c^\infty(M).$$

3

Definition 2.1 *Let $\mathcal{S}$ denote the space of cylindrical functionals of the form*

$$F(\gamma) = f\left(\int_M u_1 d\gamma, \ldots, \int_M u_n d\gamma\right), \quad u_1, \ldots, u_n \in \mathcal{C}_c^\infty(M), \quad f \in \mathcal{C}_b^\infty(\mathbb{R}^n). \quad (2.2)$$

Definition 2.2 *Let $\mathcal{U}_c^\infty(M)$, resp. $\mathcal{U}_b^\infty(M)$, denote the space of smooth vector fields of the form*

$$v(\gamma, x) = \sum_{i=1}^{i=n} F_i(\gamma) h_i(x), \quad (\gamma, x) \in \Gamma \times M, \quad F_i \in \mathcal{S}, \quad (2.3)$$

$h_i \in \mathcal{C}_c^\infty(M)$, *resp.* $h_i \in \mathcal{C}_b^\infty(M)$, $i = 1, \ldots, n$.

In general for $h \in \mathcal{U}_c^\infty(M)$ we have

$$\delta(h)(\gamma) = \int_M h(\gamma \setminus \{x\}, x) \gamma(dx) - \int_M h d\sigma,$$

cf. e.g. [19]. As in the Brownian case, we are interested in isometry formulas that directly involve intrinsic differential operators on Γ. For this we will need a differentiable structure on M.

3　The one-dimensional case

In this section we introduce the construction of connection, covariant derivative and Weitzenböck type identities for a Poisson random measure on $M = \mathbb{R}_+$, and we make a complete use of the particularities of the one-dimensional case, where the curvature of Γ vanishes. Some results of this section will be consequences of the more general framework developed in the next sections for Riemannian manifolds. Here, every configuration $\gamma \in \Gamma$ can be viewed as the ordered sequence $\gamma = (T_n)_{n \geq 1}$ of jump times of a standard Poisson process $(N_t)_{t \in \mathbb{R}_+}$ on $\mathbb{R}_+$. Let $\mathcal{S}$ denote the space of cylindrical functionals of the form

$$F = f(T_1, \ldots, T_n), \quad f \in \mathcal{C}_b^\infty(\mathbb{R}^n). \quad (3.1)$$

Let $\hat{D}$ be the intrinsic gradient operator defined as

$$\hat{D}_t F = \sum_{i=1}^{i=n} 1_{\{t = T_i\}} \partial_i f(T_1, \ldots, T_n), \quad dN_t - a.e.,$$

i.e.

$$\begin{aligned} \hat{D}_u F = \langle \hat{D}F, u \rangle_{L^2(\mathbb{R}_+, dN)} &= \int_0^\infty u(t) \hat{D}_t F dN_t = \sum_{i=1}^{i=n} u(T_i) \, \partial_i f(T_1, \ldots, T_n) \\ &= \frac{d}{d\varepsilon} f(T_1 + \varepsilon u(T_1), \ldots, T_n + \varepsilon u(T_n))_{|\varepsilon=0}, \end{aligned}$$

$u \in \mathcal{C}_c^\infty(\mathbb{R}_+)$. We have if $t = T_k$:

$$\hat{D}_t F = \partial_k f(T_1, \ldots, T_n) = \frac{\partial}{\partial s} f(T_1, \ldots, T_{k-1}, s, T_{k+1}, \ldots, T_n)_{|s=T_k} = \frac{\partial}{\partial s} D_s F(\gamma \backslash \{s\})_{|s=T_k},$$

hence for $v \in \mathcal{C}_0^\infty(\mathbb{R}_+) := \{f \in \mathcal{C}_c^\infty(\mathbb{R}_+) \ : \ f(0) = 0\}$:

$$\langle \hat{D} F, v \rangle_{L^2(\mathbb{R}_+, dN)} = \delta(v \partial DF) + \int_0^\infty \partial_t D_t F v(t) dt = \delta(v \partial DF) - \int_0^\infty \dot{v}(t) D_t F dt, \quad (3.2)$$

which implies the integration by parts formula by taking expectations:

$$E\left[\langle \hat{D} F, v \rangle_{L^2(\mathbb{R}_+, dN)}\right] = -E\left[\int_0^\infty \dot{v}(t) D_t F dt\right] = -E[F\delta(\dot{v})], \quad v \in \mathcal{C}_0^\infty(\mathbb{R}_+),$$

$F \in \mathcal{S}$. The damped gradient $\tilde{D}$ is defined as

$$\tilde{D}_t F = -\sum_{i=1}^{i=n} 1_{[0, T_i]}(t) \partial_i f(T_1, \ldots, T_n), \quad dt - a.e., \quad (3.3)$$

i.e.

$$\tilde{D}_u F = \langle u, \tilde{D} F \rangle_{L^2(\mathbb{R}_+)} = \langle \tilde{u}, \hat{D} F \rangle_{L^2(\mathbb{R}_+, dN_t)} = \hat{D}_{\tilde{u}} F,$$

with $\tilde{u}(t) = -\int_0^t u(s) ds$, $t \in \mathbb{R}_+$. We denote by $\tilde{\delta} : L^2(\Gamma \times \mathbb{R}_+) \longrightarrow L^2(\Gamma)$ the closable adjoint of $\tilde{D}$, which satisfies

$$E[F\tilde{\delta}(u)] = E[\langle \tilde{D} F, u \rangle_{L^2(\mathbb{R}_+, dt)}], \quad F \in \mathcal{S}, \ \ u \in \mathcal{U}_c^\infty(\mathbb{R}_+).$$

If $u \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ then

$$E[F\tilde{\delta}(u)] = E[\langle \tilde{D} F, u \rangle_{L^2(\mathbb{R}_+, dt)}] = E[\langle \hat{D} F, \tilde{u} \rangle_{L^2(\mathbb{R}_+, dN)}] = -E[F\delta(\partial \tilde{u})] = E[F\delta(u)].$$

In particular, $\tilde{\delta}$ coincides with the compensated Poisson stochastic integral on the adapted square-integrable processes. Given $u \in \mathcal{U}_c^\infty(\mathbb{R}_+)$ we define the covariant derivative $\nabla_u^\Gamma v \in \mathcal{U}_b^\infty(\mathbb{R}_+)$ of the vector field $v = \sum_{i=1}^{i=n} F_i h_i \in \mathcal{U}_c^\infty(\mathbb{R}_+)$ as

$$\nabla_u^\Gamma v(t) = \sum_{i=1}^{i=n} h_i(t) \tilde{D}_u F_i - F_i h_i'(t) \int_0^t u(s) ds, \quad t \in \mathbb{R}_+. \quad (3.4)$$

In particular,

$$\nabla_u^\Gamma v(t) = \dot{v}(t) \tilde{u}(t), \quad t \in \mathbb{R}_+, \ \ u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+),$$

and

$$\nabla_{uF}^\Gamma (vG) = Fv\tilde{D}_u G + FG\nabla_u^\Gamma v, \quad u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+), \ \ F, G \in \mathcal{S}.$$

Letting

$$\nabla_s^\Gamma v(t) = \sum_{i=1}^{i=n} h_i(t)\tilde{D}_s F_i - F_i h_i'(t)1_{[0,t]}(s), \quad s, t \in \mathbb{R}_+,$$

we have

$$\nabla_u^\Gamma v(t) = \int_0^\infty u(s)\nabla_s^\Gamma v(t)ds, \quad t \in \mathbb{R}_+, \quad u, v \in \mathcal{U}_c^\infty(\mathbb{R}_+).$$

Lemma 3.1 *We have*

$$\tilde{D}_u\tilde{D}_v - \tilde{D}_v\tilde{D}_u = \tilde{D}_{\nabla_u^\Gamma v - \nabla_v^\Gamma u}, \quad u, v \in \mathcal{U}_c^\infty(\mathbb{R}_+).$$

Proof. Since $\tilde{D}$ is a derivation it suffices to consider $F = T_n$.

i) We have for $u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$:

$$
\begin{aligned}
(\tilde{D}_u\tilde{D}_v - \tilde{D}_v\tilde{D}_u)T_n &= -\tilde{D}_u \int_0^{T_n} v(s)ds + \tilde{D}_v \int_0^{T_n} u(s)ds \\
&= v(T_n)\int_0^{T_n} u(s)ds - u(T_n)\int_0^{T_n} v(s)ds \\
&= \int_0^{T_n}\left(v'(t)\int_0^t u(s)ds - u'(t)\int_0^t v(s)ds\right)dt \\
&= \tilde{D}_{\nabla_u^\Gamma v - \nabla_v^\Gamma u}T_n.
\end{aligned}
$$

ii) If $u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ and $F, G \in \mathcal{S}$,

$$
\begin{aligned}
(\tilde{D}_{uF}\tilde{D}_{vG} - \tilde{D}_{vG}\tilde{D}_{uF})T_n &= F\tilde{D}_u G\tilde{D}_v T_n - G\tilde{D}_v F\tilde{D}_u T_n + FG(\tilde{D}_u\tilde{D}_v - \tilde{D}_v\tilde{D}_u)T_n \\
&= \tilde{D}_w T_n,
\end{aligned}
$$

with

$$w = Fv\tilde{D}_u G - Gu\tilde{D}_v F + FG(\dot{v}\tilde{u} - \dot{u}\tilde{v}) = \nabla_{uF}^\Gamma(vG) - \nabla_{vG}^\Gamma(uF).$$

$\square$

Definition 3.1 *The Lie bracket $\{u, v\}$ of $u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$, is defined to be the unique element of $\mathcal{C}_c^\infty(\mathbb{R}_+)$ satisfying $(\tilde{D}_u\tilde{D}_v - \tilde{D}_v\tilde{D}_u)F = \tilde{D}_w F$, $F \in \mathcal{S}$.*

The bracket $\{u, v\}$ is defined for $u, v \in \mathcal{U}_c^\infty(\mathbb{R}_+)$ with

$$\{Fu, Gv\}(x) = FG\{u, v\}(x) + v(x)F\tilde{D}_u G - u(x)G\tilde{D}_v F, \quad x \in M,$$

$u, v \in \mathcal{C}_c^\infty(M)$, $F, G \in \mathcal{S}$. The next proposition is a consequence of Lemma 3.1 and Def. 3.1.

Proposition 3.1 *The Lie bracket $\{u, v\}$ of $u, v \in \mathcal{U}_c^\infty(\mathbb{R}_+)$ satisfies*

$$\{u, v\} = \nabla_u^\Gamma v - \nabla_v^\Gamma u,$$

i.e. the connection defined by ∇^Γ has a vanishing torsion.

Proposition 3.2 *The curvature tensor $\Omega^\Gamma : \mathcal{U}_c^\infty(\mathbb{R}_+) \times \mathcal{U}_c^\infty(\mathbb{R}_+) \times \mathcal{U}_c^\infty(\mathbb{R}_+) \longrightarrow \mathcal{U}_c^\infty(\mathbb{R}_+)$, of the connection ∇^Γ vanishes on $\mathcal{U}_c^\infty(\mathbb{R}_+)$, i.e.*

$$\Omega^\Gamma(u, v)h = [\nabla_u^\Gamma, \nabla_v^\Gamma]h - \nabla_{\{u,v\}}^\Gamma h = 0, \quad u, v, h \in \mathcal{U}_c^\infty(\mathbb{R}_+),$$

and $\mathcal{U}_c^\infty(\mathbb{R}_+)$ is a Lie algebra under the bracket $\{\cdot, \cdot\}$.

Proof. We have

$$\Omega^\Gamma(uF, vG)(hH) = FGh([\tilde{D}_u, \tilde{D}_v] - \tilde{D}_{\{u,v\}})H + FGH\Omega^\Gamma(u, v)h = FGH\Omega^\Gamma(u, v)h,$$

$h, u, v, \in \mathcal{C}_c^\infty(\mathbb{R}_+)$, $F, G, H \in \mathcal{S}$. Hence it suffices to show that $\Omega^\Gamma(u, v)h = 0$, $h, u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$. We have

$$[\nabla_u^\Gamma, \nabla_v^\Gamma]h = \tilde{u}\overset{\cdot}{\widetilde{\nabla_v^\Gamma h}} - \tilde{v}\overset{\cdot}{\widetilde{\nabla_u^\Gamma h}} = \tilde{u}\ \overset{\cdot}{\widetilde{\dot{v}h}} - \tilde{v}\ \overset{\cdot}{\widetilde{\dot{u}h}} = -\tilde{u}v\dot{h} + \tilde{u}\tilde{v}\dot{h} + \tilde{v}u\dot{h} - \tilde{v}\tilde{u}\dot{h}, = -\tilde{u}v\dot{h} + \tilde{v}u\dot{h},$$

and

$$\nabla_{\{u,v\}}^\Gamma h = \nabla_{\tilde{u}\dot{v} - \tilde{v}\dot{u}}^\Gamma h = (\widetilde{\tilde{u}\dot{v} - \tilde{v}\dot{u}})\dot{h} = (u\tilde{v} - v\tilde{u})\dot{h}.$$

The Lie algebra property follows from the vanishing of Ω^Γ. $\qquad\qquad\square$

The exterior derivative $d^\Gamma u$ of a smooth vector field $u \in \mathcal{U}_c^\infty(\mathbb{R}_+)$ is defined from

$$\langle d^\Gamma u, h_1 \wedge h_2 \rangle_{L^2(\mathbb{R}_+) \wedge L^2(\mathbb{R}_+)} = \langle \nabla_{h_1}^\Gamma u, h_2 \rangle_{L^2(\mathbb{R}_+)} - \langle \nabla_{h_2}^\Gamma u, h_1 \rangle_{L^2(\mathbb{R}_+)},$$

$h_1, h_2 \in \mathcal{U}_c^\infty(\mathbb{R}_+)$. We have

$$d^\Gamma u(s, t) = \frac{1}{2}(\nabla_s^\Gamma u(t) - \nabla_t^\Gamma u(s)), \quad s, t \in \mathbb{R}_+,$$

and the relation

$$\|d^\Gamma u\|_{L^2(\mathbb{R}_+) \wedge L^2(\mathbb{R}_+)}^2 = \frac{1}{2} \int_0^\infty \int_0^\infty (\nabla_s^\Gamma u(t) - \nabla_t^\Gamma u(s))^2 ds dt, \tag{3.5}$$

$u \in \mathcal{U}_c^\infty(\mathbb{R}_+)$. We now state a Weitzenböck type identity on configuration space. For this we will use the commutation relation satisfied by the damped gradient $\tilde{D}$:

$$\tilde{D}_u \tilde{\delta}(v) = \tilde{\delta}(\nabla_u^\Gamma v) + \langle u, v \rangle_{L^2(\mathbb{R}_+)}, \quad u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+), \tag{3.6}$$

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which can be proved as follows:

$$
\begin{aligned}
\tilde{D}_u \tilde{\delta}(v) &= \tilde{D}_u \sum_{k=1}^{\infty} v(T_k) = -\sum_{k=1}^{\infty} v'(T_k) \int_0^{T_k} u(s)ds \\
&= -\tilde{\delta}\left(v(\cdot) \int_0^{\cdot} u(s)ds \right) - \int_0^{\infty} v'(t) \int_0^t u(s)ds dt \\
&= -\tilde{\delta}\left(v(\cdot) \int_0^{\cdot} u(s)ds \right) + \int_0^{\infty} u(t)v(t)dt \\
&= \tilde{\delta}(\nabla_u^{\Gamma} v) + \langle u, v \rangle_{L^2(\mathbb{R}_+)}.
\end{aligned}
$$

Proposition 3.3 *We have for $u \in \mathcal{U}_c^{\infty}(\mathbb{R}_+)$:*

$$
E[\tilde{\delta}(u)^2] + E\left[\|d^{\Gamma} u\|_{L^2(\mathbb{R}_+) \wedge L^2(\mathbb{R}_+)}^2 \right] = E[\|u\|_{L^2(\mathbb{R}_+)}^2] + E\left[\|\nabla^{\Gamma} u\|_{L^2(\mathbb{R}_+) \otimes L^2(\mathbb{R}_+)}^2 \right]. \quad (3.7)
$$

Proof. We have

$$
\begin{aligned}
E[\tilde{\delta}(u_i F_i)\tilde{\delta}(u_j F_j)] &= E[F_i \tilde{D}_{u_i}\tilde{\delta}(u_j F_j)] \\
&= E[F_i \tilde{D}_{u_i}(F_j \tilde{\delta}(u_j) - \tilde{D}_{u_j} F_j)] \\
&= E[F_i F_j \tilde{D}_{u_i}\tilde{\delta}(u_j) + F_i \tilde{\delta}(u_j)\tilde{D}_{u_i} F_j - F_i \tilde{D}_{u_i}\tilde{D}_{u_j} F_j] \\
&= E[F_i F_j \langle u_i, u_j \rangle_{L^2(\mathbb{R}_+)} + F_i F_j \tilde{\delta}(\nabla_{u_i}^{\Gamma} u_j) + F_i \tilde{\delta}(u_j)\tilde{D}_{u_i} F_j - F_i \tilde{D}_{u_i}\tilde{D}_{u_j} F_j] \\
&= E[F_i F_j \langle u_i, u_j \rangle_{L^2(\mathbb{R}_+)} + \tilde{D}_{\nabla_{u_i}^{\Gamma} u_j}(F_i F_j) + \tilde{D}_{u_j}(F_i \tilde{D}_{u_i} F_j) - F_i \tilde{D}_{u_i}\tilde{D}_{u_j} F_j] \\
&= E[F_i F_j \langle u_i, u_j \rangle_{L^2(\mathbb{R}_+)} + \tilde{D}_{\nabla_{u_i}^{\Gamma} u_j}(F_i F_j) + \tilde{D}_{u_j} F_i \tilde{D}_{u_i} F_j + F_i(\tilde{D}_{u_j}\tilde{D}_{u_i} F_j - \tilde{D}_{u_i}\tilde{D}_{u_j} F_j)] \\
&= E[F_i F_j \langle u_i, u_j \rangle_{L^2(\mathbb{R}_+)} + \tilde{D}_{\nabla_{u_i}^{\Gamma} u_j}(F_i F_j) + \tilde{D}_{u_j} F_i \tilde{D}_{u_i} F_j + F_i \tilde{D}_{\nabla_{u_j}^{\Gamma} u_i - \nabla_{u_i}^{\Gamma} u_j} F_j] \\
&= E[F_i F_j \langle u_i, u_j \rangle_{L^2(\mathbb{R}_+)} + F_j \tilde{D}_{\nabla_{u_i}^{\Gamma} u_j} F_i + F_i \tilde{D}_{\nabla_{u_j}^{\Gamma} u_i} F_j + \tilde{D}_{u_j} F_i \tilde{D}_{u_i} F_j] \\
&= E\left[F_i F_j \langle u_i, u_j \rangle_{L^2(\mathbb{R}_+)} + F_j \int_0^{\infty} \tilde{D}_s F_i \int_0^{\infty} \nabla_t^{\Gamma} u_j(s) u_i(t) dt ds \right. \\
&\quad \left. + F_i \int_0^{\infty} \tilde{D}_t F_j \int_0^{\infty} \nabla_s^{\Gamma} u_i(t) u_j(s) ds dt + \int_0^{\infty} u_i(t)\tilde{D}_t F_j \int_0^{\infty} u_j(s)\tilde{D}_s F_i ds dt \right],
\end{aligned}
$$

which implies

$$
E[\tilde{\delta}(u)^2] = E[\|u\|_{L^2(\mathbb{R}_+)}^2] + E\left[\int_0^{\infty}\int_0^{\infty} \nabla_s^{\Gamma} u(t)\nabla_t^{\Gamma} u(s) ds dt \right],
$$

and (3.7) for $u = \sum_{i=1}^n u_i F_i \in \mathcal{U}_c^{\infty}(\mathbb{R}_+)$. $\qquad \square$

4 Intrinsic differential structure on configuration space

In this section we work in the general case where M is a Riemannian manifold. We start by recalling the definition of the intrinsic gradient of [4], Sect. 3, see also [5], p.

152, and state a short proof of the integration by parts formula. If K is a compact set such that $u_1, \ldots, u_n \in \mathcal{C}_c^\infty(K)$ and $\mathrm{card}(\gamma \cap K) = n$, then $F(\gamma)$ can be represented as

$$F(\gamma) = f_n(x_1, \ldots, x_n), \quad \text{if } \gamma = \{x_1, \ldots, x_n\} \in \Gamma, \quad n \geq 1,$$

where $f_n \in \mathcal{C}_c^\infty(M^n)$ is symmetric. Let ∇^M and div^M denote the gradient and divergence on M, let $T_x M$ denote the tangent space at $x \in M$, and assume that σ is the volume element of M, under which div^M and ∇^M are adjoint:

$$\langle \nabla^M u, U \rangle_{L^2(M, d\sigma; TM,)} = \langle u, \mathrm{div}^M U \rangle_{L^2(M, \sigma)}, \quad U \in \mathcal{C}_0^\infty(M; TM), \quad u \in \mathcal{C}_c^\infty(M),$$

where $\mathcal{C}_0^\infty(M; TM)$ is a space of $\mathcal{C}^\infty$ vector fields satisfying suitable boundary conditions for integration by parts (see examples below). Given $U \in T_x M$ and $f \in \mathcal{C}_c^\infty(M)$, we adopt the notation $U f(x) = \langle U, f(x) \rangle_{T_x M}$, $x \in M$. Let $\mathcal{C}^\infty(M; TM)$ be the Lie algebra of $\mathcal{C}^\infty$ vector fields on M, let $\mathrm{Diff}(M)$ denote the group of diffeomorphisms of M, let $(\phi_t^U)_{t \in \mathbb{R}_+}$ denote the flow generated by the vector field $U \in \mathcal{C}^\infty(M; TM)$, let $\phi_t^U(\gamma)$ denote the image measure of γ by ϕ_t^U, $U \in \mathcal{C}^\infty(M; TM)$ and let $\hat{D}$ be the gradient operator defined in [4] as

$$
\begin{aligned}
\langle \hat{D} F(\gamma), U \rangle_{L^2(M, d\gamma; TM)} &= \lim_{\varepsilon \to 0} \frac{F(\phi_\varepsilon^U(\gamma)) - F(\gamma)}{\varepsilon} \\
&= \sum_{i=1}^{i=n} \int_M U u_i \, d\gamma \, \partial_i f \left(\int_M u_1 \, d\gamma, \ldots, \int_M u_n \, d\gamma \right),
\end{aligned}
$$

$U \in \mathcal{C}^\infty(M; TM)$, i.e.

$$\hat{D}_x F(\gamma) = \sum_{i=1}^{i=n} \nabla^M u_i(x) \, \partial_i f \left(\int_M u_1 \, d\gamma, \ldots, \int_M u_n \, d\gamma \right), \quad x \in M.$$

We can also formulate this definition as

$$\hat{D}_x F(\gamma) = \sum_{i=1}^{i=n} 1_{\{x_i\}}(x) \nabla_i^M f_n(x_1, \ldots, x_n), \quad \gamma(dx) - a.e.,$$

with $F(\gamma) = f_n(x_1, \ldots, x_n)$, $\gamma = \{x_1, \ldots, x_n\} \in \Gamma$, $n \geq 1$, and ∇_i^M is the gradient of f with respect to its i-th variable. We recall the following explicit expression of $\hat{D}$ in terms of the flat gradient D and flat divergence δ, cf. Remark 3 of [24] and Th. 8.2.1 of [20].

Proposition 4.1 *We have for $V \in \mathcal{C}_0^\infty(M; TM)$ and $F \in \mathcal{S}$:*

$$\langle \hat{D} F(\gamma), V \rangle_{L^2(M, d\gamma; TM)} = \langle \nabla^M D F(\gamma), V \rangle_{L^2(M, d\sigma; TM)} + \delta(\langle \nabla^M D F, V \rangle_{TM})(\gamma). \quad (4.1)$$

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Proof. This identity follows from the relations $\hat{D}_x F(\gamma) = (\nabla_x^M D_x F)(\gamma \setminus x)$ and

$$\delta(u) = \int_M u(x, \gamma \setminus x) \gamma(dx) - \int_M u(x, \gamma) \sigma(dx).$$

$\square$

Taking expectations on both sides in (4.1), we obtain the integration by parts formula for $\hat{D}$, cf. [4]:

$$E[\langle \hat{D} F(\gamma), V \rangle_{L^2(M, d\gamma; TM)}] = E[\langle \nabla^M D F, V \rangle_{L^2(M, d\sigma; TM)}] = E[F \delta(\operatorname{div}^M V)], \qquad (4.2)$$

$V \in \mathcal{C}_0^\infty(M; TM)$, $F \in \mathcal{D}$. However the gradient $\hat{D}$ is not satisfactory here, due to the presence of $d\gamma$ in Relation (4.1), see Sect. 6 and Prop. 9.1 below. For this reason we need to define a damped gradient.

5 Damped gradient and tangent bundle

This section recalls the definition and properties of the damped gradient $\tilde{D}$ on configuration space, cf. [20], and introduces the corresponding tangent bundle. We assume that the Laplacian $\mathcal{L} = \operatorname{div}^M \nabla^M$ is invertible on $\mathcal{C}_c^\infty(M)$, and that its inverse $\mathcal{L}^{-1}$ is given by a Green kernel $g : M \times M \longrightarrow \mathbb{R}$:

$$\mathcal{L}^{-1} u(x) = \int_M g(x, y) u(y) \sigma(dy), \quad x \in M, \ u \in \mathcal{C}_c^\infty(M).$$

In general, $g(\cdot, y)$ belongs to the Sobolev space $W^{1,1}(M; TM)$, and if M is of dimension one then $g(\cdot, y) \in W^{1,p}(M; TM)$ for all $p \geq 1$, $y \in M$. We define $\partial_x(y) \in T_x M$ as

$$\partial_x(y) = \nabla_x^M g(x, y), \quad \sigma \otimes \sigma(dx, dy) - a.e.$$

We have $\partial_x(\cdot) \in L^1(M; TM)$ and define $\tilde{u} \in \mathcal{C}_0^\infty(M; TM)$ as

$$\tilde{u}(x) = \nabla^M \mathcal{L}^{-1} u(x) = \int_M u(y) \partial_x(y) \sigma(dy) \in T_x M, \quad x \in M, \ u \in \mathcal{C}_c^\infty(M).$$

Moreover $\tilde{u} \in \mathcal{C}_0^\infty(M)$ and satisfies

P-i) $\operatorname{div}^M \tilde{u} = u, \quad u \in \mathcal{C}_c^\infty(M)$, and

P-ii) $\langle v, u \rangle_{L^2(M, d\sigma)} = \langle v, \operatorname{div}^M \tilde{u} \rangle_{L^2(M, d\sigma)} = \langle \nabla^M v, \tilde{u} \rangle_{L^2(M, d\sigma; TM)}, \ u, v \in \mathcal{C}_c^\infty(M),$

Examples:

If $M = \mathbb{R}_+$ we set $\mathcal{C}_0^\infty(\mathbb{R}_+; \mathbb{R}) = \{u \in \mathcal{C}_b^\infty(\mathbb{R}_+; \mathbb{R}) \ : \ u(0) = 0\}$, $g(x, y) = -x \vee y$, $\partial_x(y) = -1_{[0,x]}(y)$, $x, y \in \mathbb{R}_+$, and $\tilde{u}(x) = -\int_0^x u(y)dy$ (cf. Sects. 3 and 10).

If $M = \mathbb{R}^d$ we let $\mathcal{C}_0^\infty(\mathbb{R}^d; \mathbb{R}^d) = \{u \in \mathcal{C}^\infty(\mathbb{R}^d; \mathbb{R}^d) \ : \ \lim_{x \to \infty} u(x) = 0\}$, with $g(x, y) = \frac{1}{2\pi} \log |x - y|$ if $d = 2$, and $g(x, y) = \frac{1}{(d-2)c_d}|x - y|^{1-d}$ if $d \geq 3$, where c_d is the volume of the unit ball, i.e.

$$\partial_x(y) = \frac{1}{2\pi}|x - y|^{-2}(x_1 - y_1, x_2 - y_2) \quad \text{if } d = 2,$$

and

$$\partial_x(y) = \frac{1}{(d-2)c_d}|x - y|^{-d}(x_1 - y_1, \ldots, x_d - y_d) \quad \text{if } d \geq 3.$$

Definition 5.1 *The damped gradient of $F \in \mathcal{S}$ in the direction $h \in \mathcal{C}_c^\infty(M)$ is $\tilde{D}_h F \in L^2(\Gamma, P)$, defined as*

$$\tilde{D}_h F(\gamma) = \hat{D}_{\tilde{h}} F(\gamma) = \sum_{i=1}^{i=n} \int_M \tilde{h} u_i d\gamma \ \partial_i f \left(\int_M u_1 d\gamma, \ldots, \int_M u_n d\gamma \right), \qquad (5.1)$$

with F as in (2.2).

This description of the gradient would be incomplete without a description of the tangent bundle of Γ. The space $L^2(M, d\gamma)$, which explicitly depends on the random element $\gamma \in \Gamma$, is a natural candidate as a tangent space to Γ at γ, cf. [4]. However this choice is compatible with the gradient $\hat{D}$ which is not damped in the sense of [14] and is not appropriate to our context, in particular it can not be used to state the commutation relation of Prop. 9.1 below, and does not seem to lead to Weitzenböck type identities. Instead of $L^2(M, d\gamma)$ we will choose $\mathcal{C}_c^\infty(M)$ as tangent space to Γ. We choose the trivial tangent bundle to Γ with group $\text{Diff}(M)$ and fiber $\mathcal{C}_c^\infty(M)$ which is defined as $T\Gamma = \Gamma \times \mathcal{C}_c^\infty(M)$, with group action

$$T\Gamma \times \text{Diff}(M) \longrightarrow T\Gamma$$

$$((\gamma, u), \phi) \mapsto (\gamma, u \circ \phi).$$

Each stochastic process $u \in \mathcal{U}_c^\infty(M)$ is identified to a smooth vector field $\gamma \mapsto (\gamma, u(\gamma, \cdot)) \in T\Gamma$. Let

$$\tilde{D} : L^2(\Gamma, P) \longrightarrow L^2(\Gamma; L^1(M, \sigma), P)$$

be defined on $\mathcal{S}$ as

$$\tilde{D}_y F(\gamma) = \int_M \langle \partial_x(y), \hat{D}_x F \rangle_{T_x M} \gamma(dx) = \langle \partial_.(y), \nabla \hat{D}_. F \rangle_{L^2(M, \gamma; TM)}, \quad y \in M,$$

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or

$$\tilde{D}_y F(\gamma) = \sum_{i=1}^{i=n} \int_M \partial_x(y) u_i(x) \gamma(dx) \; \partial_i f\left(\int_M u_1 d\gamma, \ldots, \int_M u_n d\gamma\right), \quad y \in M. \quad (5.2)$$

We have

$$\tilde{D}_u F = \int_M u(y) \tilde{D}_y F \sigma(dy), \quad u \in \mathcal{C}_c^\infty(M), \quad F \in \mathcal{S},$$

and more generally we will let $\tilde{D}_u F = \langle \tilde{D} F, u \rangle_{L^2(M, d\sigma)}$, $u \in \mathcal{U}_c^\infty(M)$. We may also write

$$\tilde{D}_y F(\gamma) = \sum_{i=1}^{i=n} \partial_{x_i}(y) f_n(x_1, \ldots, x_n), \quad y \in M, \quad \gamma = \{x_1, \ldots, x_n\} \in \Gamma,$$

with $F(\gamma) = f_n(x_1, \ldots, x_n)$, where the notation $\partial_{x_i}(y) f_n(x_1, \ldots, x_n)$ denotes the application of the derivation $\partial_{x_i}(y)$ to the i-th variable of f_n.

Definition 5.2 *We define the anticipating integral of $u \in \mathcal{U}_b^\infty(M)$ as*

$$\tilde{\delta}(u) = \sum_{i=1}^{i=n} F_i \delta(h_i) - \langle \tilde{D} F_i, h_i \rangle_{L^2(M, d\sigma)},$$

if $u \in \mathcal{U}_c^\infty(M)$ is of the form (2.3).

In particular we have

$$\tilde{\delta}(h)(\gamma) = \delta(h)(\gamma) = \int_M h \, d\gamma - \int_M h \, d\sigma, \quad h \in \mathcal{C}_c^\infty(M).$$

Proposition 5.1 *The operators $\tilde{D} : L^\infty(\Gamma) \longrightarrow L^2(\Gamma; L^1(M))$ and $\tilde{\delta} : L^2(\Gamma; L^\infty(M)) \longrightarrow L^1(\Gamma)$ are mutually adjoint:*

$$E[F \tilde{\delta}(u)] = E[\langle \tilde{D} F, u \rangle_{L^2(M, d\sigma)}], \quad F \in \mathcal{S}, \; u \in \mathcal{U}_c^\infty(M). \quad (5.3)$$

Proof. For $u \in \mathcal{C}_c^\infty(M)$ we apply (4.1) to $V = \tilde{u} \in \mathcal{C}_0^\infty(M; TM)$ and property P-ii) to obtain the identity

$$\tilde{D}_u F = \langle DF, u \rangle_{L^2(M, d\sigma)} + \delta(\tilde{u} DF), \quad F \in \mathcal{S}. \quad (5.4)$$

see Prop. 8 of [23] when $M = \mathbb{R}_+$. Taking the expectation and using the duality between D and δ provides (5.3) for $u \in \mathcal{C}_c^\infty(M)$. If $u \in \mathcal{U}_c^\infty(M)$ is of the form (2.3) then $\tilde{\delta}(u)$ also satisfies (5.3) due to the derivation property of $\tilde{D}$. The closability of $\tilde{\delta}$ follows from the integration by parts formula and the density of $\mathcal{S}$ and $\mathcal{U}_c^\infty(M)$ in $L^2(\Gamma, P)$ and $L^2(\Gamma \times M, P \otimes \sigma)$. $\qquad \square$

As a consequence, $\tilde{D}$ is closable in the sense that if $(F_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ is bounded in $L^\infty(\Gamma)$ and converges a.s. to 0 and $(DF_n)_{n\in\mathbb{N}}$ converges to U in $L^2(\Gamma; L^1(M))$, then $U = 0$. If $U = \nabla f$ is a gradient field, $f \in \mathcal{C}_c^\infty(M)$, then we have

$$\tilde{D}_{\mathrm{div}^M U}F = \hat{D}_U F, \quad F \in \mathcal{S}, \tag{5.5}$$

since

$$\widetilde{\mathrm{div}^M U} = \widetilde{\mathrm{div}^M \nabla}f = \widetilde{\mathcal{L}f} = \nabla^M \mathcal{L}^{-1}\mathcal{L}f = \nabla^M f = U.$$

In general for $U \in \mathcal{C}_0^\infty(M; TM)$, Relation (5.5) does not hold but we have

$$E[\tilde{D}_{\mathrm{div}^M U}F] = E[\hat{D}_U F], \quad F \in \mathcal{S}, \quad U \in \mathcal{C}_0^\infty(M; TM), \tag{5.6}$$

since from (4.1), and (5.4),

$$E[\hat{D}_U F] = E[\langle \nabla^M DF, U \rangle_{L^2(M,d\sigma;TM)}] = E[\langle DF, \mathrm{div}^M U \rangle_{L^2(M,d\sigma;TM)}] = E[\tilde{D}_{\mathrm{div}^M U}F].$$

6   Covariant derivative

We define the covariant derivative ∇_u^Γ, $u \in \mathcal{C}_c^\infty(M)$, as

$$\nabla_u^\Gamma v(x) = \langle \tilde{u}(x), \nabla^M v(x) \rangle_{T_x M} = \tilde{u}v(x), \quad x \in M, \quad v \in \mathcal{C}_c^\infty(M),$$

hence $\nabla_u^\Gamma v \in \mathcal{C}_b(M)$. Since $\nabla_u^\Gamma v$ depends only on u and $\nabla^M v$, and given its commutation properties with stochastic integrals, cf. Prop. 9.1, this connection will be called the Markovian connection in reference to [9]. Its definition extends to vector fields.

Definition 6.1 *Given $u \in \mathcal{U}_c^\infty(M)$ we define the covariant derivative $\nabla_u^\Gamma v \in \mathcal{U}_b^\infty(M)$ of the vector field $v \in \mathcal{U}_c^\infty(M)$ as*

$$\nabla_u^\Gamma v(x) = \sum_{i=1}^{i=n} h_i(x)\tilde{D}_u F_i + F_i \nabla_u^\Gamma h_i(x) = \tilde{D}_u v(x) + \tilde{u}v(x), \quad x \in M, \tag{6.1}$$

with v as in (2.3).

This definition has an interpretation in a decomposition of the tangent space to $T\Gamma$ at $u \in T\Gamma$ in horizontal and vertical subspaces is $Q \oplus \mathcal{C}_0^\infty(M; TM)$, where $Q = \{(u, -\tilde{u}) : u \in \mathcal{C}_c^\infty(M)\}$, i.e. $(v, V) = (v, -\tilde{v}) \oplus (0, V + \tilde{v})$, $v \in \mathcal{C}_c^\infty(M)$, $V \in \mathcal{C}_0^\infty(M; TM)$. The horizontal lift starting from $(\gamma, v) \in T\Gamma$ of the curve $t \mapsto \phi_t^{\tilde{u}}(\gamma)$ is $t \mapsto (\phi_t^{\tilde{u}}(\gamma), v \circ \phi_{-t}^{\tilde{u}})$, the parallel transport $\tau_t^u v : T_{\phi_t^{\tilde{u}}(\gamma)}\Gamma \to T_\gamma\Gamma$ along $t \mapsto \phi_t^{\tilde{u}}(\gamma)$ is

given by $\tau_t^u v = v \circ \phi_t^{\tilde{u}}$. Given a vector field $u \in \mathcal{U}_c^\infty(M)$ the covariant derivative $\nabla_u^\Gamma v$ of $v \in \mathcal{U}_c^\infty(M)$ is

$$\nabla_u^\Gamma v(\gamma, x) = \lim_{\varepsilon \to 0} \frac{\tau_\varepsilon^u v(\phi_\varepsilon^{\tilde{u}}(\gamma), x) - v(\gamma, x)}{\varepsilon}, \qquad x \in M, \ \gamma \in \Gamma.$$

Relation (6.1) is similar to (2.2) of [12] which uses the Lie bracket of deterministic vector fields on the space of Brownian paths in a Lie group instead of $\tilde{u}v$. We have the relation

$$\nabla_u^\Gamma(Fv)(x) = v(x)\tilde{D}_u F + F\nabla_u^\Gamma v(x), \quad x \in M, \ u, v \in \mathcal{U}_c^\infty(M), \ F \in \mathcal{S}.$$

Definition 6.2 *We extend naturally $\partial_x(y)$ from a derivation on $\mathcal{C}_c^\infty(M)$ to a derivation on $\mathcal{U}_c^\infty(M)$, with $\partial_x(y)(Fh) = F\partial_x(y)h$, $F \in \mathcal{S}$, $h \in \mathcal{C}_c^\infty(M)$, and let*

$$\nabla_y^\Gamma u(x) = \sum_{i=1}^{i=n} h_i(x)\tilde{D}_y F_i + F_i \langle \partial_x(y), \nabla^M h_i(x)\rangle_{T_x M} = \tilde{D}_y v(x) + \partial_x(y)v(x), \quad x, y \in M,$$

$F \in \mathcal{S}$, for $u \in \mathcal{U}_c^\infty(M)$ of the form (2.3), i.e.

$$\nabla_v^\Gamma u(x) = \int_M v(y)\nabla_y^\Gamma u(x)\sigma(dy), \quad x \in M, \quad u, v \in \mathcal{U}_c^\infty(M).$$

7 Torsion, curvature and Lie-Poisson bracket

The Lie bracket $\{u, v\}$ of vector fields is normally defined from the commutator $[\tilde{D}_u, \tilde{D}_v]$. We have $\tilde{u}v \in \mathcal{C}_c^\infty(M)$, $u, v \in \mathcal{C}_c^\infty(M)$, and from (4.2) and (5.3),

$$
\begin{aligned}
(\tilde{D}_u\tilde{D}_v - \tilde{D}_v\tilde{D}_u)\int_M h\,d\gamma &= \tilde{D}_u \int_M \tilde{v}h\,d\gamma - \tilde{D}_v \int_M \tilde{u}h\,d\gamma \\
&= \int_M \tilde{u}\tilde{v}h\,d\gamma - \int_M \tilde{v}\tilde{u}h\,d\gamma = \int_M [\tilde{u}, \tilde{v}]h\,d\gamma \\
&= \langle [\tilde{u}, \tilde{v}], \nabla^M h\rangle_{L^2(M, d\gamma)} = \hat{D}_{[\tilde{u}, \tilde{v}]} \int_M h\,d\gamma.
\end{aligned}
$$

If $\dim M > 1$, $[\tilde{u}, \tilde{v}]$ may not be a gradient field, in particular it can not be written as $[\tilde{u}, \tilde{v}] = \tilde{w}$, hence in general there may not exist $w \in \mathcal{C}_c^\infty(M)$ such that

$$\tilde{D}_u\tilde{D}_v - \tilde{D}_v\tilde{D}_u = \tilde{D}_w. \tag{7.2}$$

A definition of the Lie bracket $\{u, v\}$ is nevertheless possible via an equality between expectations, due to the following Lemma.

Lemma 7.1 *Let $u, v, w \in \mathcal{C}_c^\infty(M)$. The relation*

$$E[\tilde{D}_u \tilde{D}_v F - \tilde{D}_v \tilde{D}_u F] = E[\tilde{D}_w F], \quad F \in \mathcal{S}, \tag{7.3}$$

holds if and only if $w = \nabla_u^\Gamma v - \nabla_v^\Gamma u$.

Proof. We have $(\tilde{D}_u \tilde{D}_v - \tilde{D}_v \tilde{D}_u)F = \hat{D}_{[\tilde{u},\tilde{v}]}F$, and

$$E\left[(\tilde{D}_u \tilde{D}_v - \tilde{D}_v \tilde{D}_u)F\right] = E\left[\hat{D}_{[\tilde{u},\tilde{v}]}F,\right] = E\left[F\delta(\mathrm{div}^M[\tilde{u},\tilde{v}])\right] = E\left[\tilde{D}_{\mathrm{div}^M([\tilde{u},\tilde{v}])}F\right], \quad F \in \mathcal{S}.$$

Since the connection on M has no torsion we have:

$$\mathrm{div}^M([\tilde{u},\tilde{v}]) = \mathrm{div}^M(\nabla_{\tilde{u}}^M \tilde{v} - \nabla_{\tilde{v}}^M \tilde{u}).$$

Let $(X_1, \ldots, X_d)$ denote a set of normal coordinates at $x \in M$ with $\tilde{u} = \sum_{j=1}^{j=d} \tilde{u}^j X_j$, $\tilde{v} = \sum_{i=1}^{i=d} \tilde{v}^i X_i$. We have at $x \in M$: $\nabla_{X_j}^M X_i = 0$, $[X_i, X_j] = [\nabla_{X_i}^M, \nabla_{X_j}^M] = 0$, $i, j = 1, \ldots, d$, and from [16], p. 282,

$$
\begin{aligned}
\mathrm{div}^M(\nabla_{\tilde{u}}^M \tilde{v}) &= \sum_{l=1}^{l=d} \langle \nabla_{X_l}^M \nabla_{\tilde{u}}^M \tilde{v}, X_l \rangle_{T_x M} = \langle \sum_{l=1}^{l=d} \nabla_{X_l}^M \sum_{i,j=1}^{d} \tilde{u}^j \nabla_{X_j}^M (\tilde{v}^i X_i), X_l \rangle_{T_x M} \\
&= \sum_{i,l=1}^{d} (X_l \tilde{u}^i)(X_i \tilde{v}^l) + \sum_{l=1}^{l=d} \sum_{i,j=1}^{d} \tilde{u}^j \langle (X_l X_j \tilde{v}^i) X_i, X_l \rangle_{T_x M} \\
&= \sum_{i,l=1}^{d} (X_l \tilde{u}^i)(X_i \tilde{v}^l) + \sum_{i,j=1}^{d} \tilde{u}^j X_i X_j \tilde{v}^i \\
&= \sum_{i,l=1}^{d} (X_l \tilde{u}^i)(X_i \tilde{v}^l) + \sum_{i,j=1}^{d} \tilde{u}^j X_j X_i \tilde{v}^i \\
&= \sum_{i,l=1}^{d} (X_l \tilde{u}^i)(X_i \tilde{v}^l) + \sum_{i,j=1}^{d} \tilde{u}^j X_j \langle \nabla_{X_i}^M \tilde{v}, X_i \rangle_{T_x M}, \\
&= \sum_{i,l=1}^{d} (X_l \tilde{u}^i)(X_i \tilde{v}^l) + \sum_{j=1}^{j=d} \tilde{u}^j X_j \mathrm{div}(\tilde{v}) \\
&= \sum_{i,l=1}^{d} (X_l \tilde{u}^i)(X_i \tilde{v}^l) + \tilde{u} \mathrm{div}(\tilde{v}).
\end{aligned}
$$

Hence

$$\mathrm{div}^M([\tilde{u},\tilde{v}]) = \mathrm{div}^M(\nabla_{\tilde{u}}^M \tilde{v} - \nabla_{\tilde{v}}^M \tilde{u}) = \tilde{u}\mathrm{div}(\tilde{v}) - \tilde{v}\mathrm{div}(\tilde{u}) = \tilde{u}v - \tilde{v}u = \nabla_u^\Gamma v - \nabla_v^\Gamma u,$$

and

$$E\left[(\tilde{D}_u \tilde{D}_v - \tilde{D}_v \tilde{D}_u)F\right] = E\left[\tilde{D}_{\nabla_u^\Gamma v - \nabla_v^\Gamma u}F\right].$$

On the other hand, given $w_1, w_2 \in \mathcal{C}_c^\infty(M)$, we note that $E[\tilde{D}_{w_1} F] = E[\tilde{D}_{w_2} F]$, $F \in \mathcal{S}$, implies in particular for $F = \int_M h \, d\gamma$:

$$\int_M \langle \tilde{w}_1(x), \nabla^M h(x) \rangle_{T_x M} \sigma(dx) = \int_M \langle \tilde{w}_2(x), \nabla^M h(x) \rangle_{T_x M} \sigma(dx),$$

hence

$$\int_M h(x) \mathrm{div}^M \tilde{w}_1(x) \sigma(dx) = \int_M h(x) \mathrm{div}^M \tilde{w}_2(x) \sigma(dx),$$

i.e. $\langle h, w_1 \rangle_{L^2(M)} = \langle h, w_2 \rangle_{L^2(M)}$ for all $h \in \mathcal{C}_c^\infty(M)$ and $x \in M$, and $w_1 = w_2$. $\qquad \square$

This allows to state the following definition.

Definition 7.3 *The Lie bracket $\{u, v\}$ of $u, v \in \mathcal{C}_c^\infty(M)$ is defined to be the unique element of $\mathcal{C}_c^\infty(M)$ satisfying $E[\tilde{D}_u \tilde{D}_v F - \tilde{D}_v \tilde{D}_u F] = E[\tilde{D}_w F]$, $F \in \mathcal{S}$. The bracket $\{u, v\}$ is extended to $u, v \in \mathcal{U}_c^\infty(M)$ by*

$$\{Fu, Gv\}(x) = FG\{u, v\}(x) + v(x)F\tilde{D}_u G - u(x)G\tilde{D}_v F, \quad x \in M, \qquad (7.4)$$

$u, v \in \mathcal{C}_c^\infty(M)$, $F, G \in \mathcal{S}$.

The following is an immediate consequence of Lemma 7.1 and Def. 7.3.

Proposition 7.1 *The connection defined by ∇^Γ has a vanishing torsion:*

$$\{u, v\} = \nabla_u^\Gamma v - \nabla_v^\Gamma u, \quad u, v \in \mathcal{U}_c^\infty(M).$$

Proof. We have

$$
\begin{aligned}
\{Fu, Gv\}(x) &= FG\{u, v\}(x) - Gu(x)\tilde{D}_v F + Fv(x)\tilde{D}_u G \\
&= FG(\nabla_u^\Gamma v(x) - \nabla_v^\Gamma u(x)) - Gu(x)\tilde{D}_v F + Fv(x)\tilde{D}_u G \\
&= \nabla_{Fu}^\Gamma (Gv)(x) - \nabla_{Gv}^\Gamma (Fu)(x), \quad x \in M, \ u, v \in \mathcal{C}_c^\infty(M), \ F, G \in \mathcal{S}.
\end{aligned}
$$

$\qquad \square$

The curvature is defined as a trilinear mapping on smooth processes.

Definition 7.4 *Let $\Omega^\Gamma : \mathcal{U}_c^\infty(M) \times \mathcal{U}_c^\infty(M) \times \mathcal{U}_c^\infty(M) \longrightarrow \mathcal{U}_c^\infty(M)$, defined as*

$$\Omega^\Gamma(u, v)h = [\nabla_u^\Gamma, \nabla_v^\Gamma]h - \nabla_{\{u,v\}}^\Gamma h, \quad u, v, h \in \mathcal{U}_c^\infty(M),$$

denote the curvature tensor of the connection ∇^Γ.

We let $[\cdot, \cdot]$ denotes the commutator of operators.

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Proposition 7.2 *Let $u, v \in \mathcal{U}_c^\infty(M)$. We have*

i) $[\tilde{D}_{uF}, \tilde{D}_{vG}] - \tilde{D}_{\{uF,vG\}} = FG([\tilde{D}_u, \tilde{D}_v] - \tilde{D}_{\{u,v\}})$, $F, G \in \mathcal{S}$,

ii) $\Omega^\Gamma(uF, vG)(hH) = hFG([\tilde{D}_u, \tilde{D}_v] - \tilde{D}_{\{u,v\}})H + HFG\Omega^\Gamma(u, v)h$, $h \in \mathcal{C}_c^\infty(M)$, $u, v \in \mathcal{U}_c^\infty(M)$, $F, G, H \in \mathcal{S}$,

iii) $\int_M \Omega^\Gamma(u, v)h(x)\sigma(dx) = 0$ if $h \in \mathcal{C}_c^\infty(M)$, or if $h \in \mathcal{U}_c^\infty(M)$ and $\int_M h d\sigma = 0$.

Proof. *i*) We have

$$\tilde{D}_{uF}(G\tilde{D}_v) - \tilde{D}_{vG}(F\tilde{D}_u) = G\tilde{D}_{uF}\tilde{D}_v + (\tilde{D}_{uF}G)\tilde{D}_v - (F\tilde{D}_{vG}\tilde{D}_u + (\tilde{D}_{vG}F)\tilde{D}_u),$$

and

$$\tilde{D}_{\{uF,vG\}} = FG\tilde{D}_{\{u,v\}} + F(\tilde{D}_uG)\tilde{D}_v - G(\tilde{D}_vF)\tilde{D}_u.$$

ii) We have

$$
\begin{aligned}
\nabla_{uF}^\Gamma \nabla_{vG}^\Gamma(Hh) &= \nabla_{uF}^\Gamma(GH\nabla_v^\Gamma h + Gh\tilde{D}_vH) \\
&= hF\tilde{D}_u(G\tilde{D}_vH) + FG\tilde{D}_vH\nabla_u^\Gamma h + F\nabla_v^\Gamma h\tilde{D}_u(HG) + FGH\nabla_u^\Gamma\nabla_v^\Gamma h \\
&= hFG\tilde{D}_u\tilde{D}_vH + hF\tilde{D}_uG\tilde{D}_vH + FG\tilde{D}_vH\nabla_u^\Gamma h + FH\nabla_v^\Gamma h\tilde{D}_uG \\
&\quad + FG\nabla_v^\Gamma h\tilde{D}_uH + FGH\nabla_u^\Gamma\nabla_v^\Gamma h,
\end{aligned}
$$

hence

$$
\begin{aligned}
&\nabla_{uF}^\Gamma \nabla_{vG}^\Gamma(Hh) - \nabla_{vG}^\Gamma\nabla_{uF}^\Gamma(Hh) \\
&= hFG[\tilde{D}_u, \tilde{D}_v]H + hF\tilde{D}_uG\tilde{D}_vH + FG\tilde{D}_vH\nabla_u^\Gamma h + FH\nabla_v^\Gamma h\tilde{D}_uG + FG\nabla_v^\Gamma h\tilde{D}_uH \\
&\quad + FGH\nabla_u^\Gamma\nabla_v^\Gamma h - hG\tilde{D}_vF\tilde{D}_uH - FG\tilde{D}_uH\nabla_v^\Gamma h - GH\nabla_u^\Gamma h\tilde{D}_vF - FG\nabla_u^\Gamma h\tilde{D}_vH \\
&\quad - FGH\nabla_v^\Gamma\nabla_u^\Gamma h \\
&= hFG[\tilde{D}_u, \tilde{D}_v]H + hF\tilde{D}_uG\tilde{D}_vH + FG\tilde{D}_vH\nabla_u^\Gamma h + FH\nabla_v^\Gamma h\tilde{D}_uG + FG\nabla_v^\Gamma h\tilde{D}_uH \\
&\quad + FGH[\nabla_u^\Gamma, \nabla_v^\Gamma]h - hG\tilde{D}_vF\tilde{D}_uH - FG\tilde{D}_uH\nabla_v^\Gamma h - GH\nabla_u^\Gamma h\tilde{D}_vF - FG\nabla_u^\Gamma h\tilde{D}_vH \\
&= hFG[\tilde{D}_u, \tilde{D}_v]H + hF\tilde{D}_uG\tilde{D}_vH + FH\nabla_v^\Gamma h\tilde{D}_uG + FGH[\nabla_u^\Gamma, \nabla_v^\Gamma]h - hG\tilde{D}_vF\tilde{D}_uH \\
&\quad - GH\nabla_u^\Gamma h\tilde{D}_vF,
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_{\{uF,vG\}}^\Gamma(Hh) &= FG\nabla_{\{u,v\}}^\Gamma(Hh) + F\tilde{D}_uG\nabla_v^\Gamma(Hh) - G\tilde{D}_vF\nabla_u^\Gamma(hH) \\
&= FGH\nabla_{\{u,v\}}^\Gamma h + FH\tilde{D}_uG\nabla_v^\Gamma h - GH\tilde{D}_vF\nabla_u^\Gamma h \\
&\quad + FGh\tilde{D}_{\{u,v\}}H + hF\tilde{D}_uG\tilde{D}_vH - hG\tilde{D}_vF\tilde{D}_uH \\
&= FGH\nabla_{\{u,v\}}^\Gamma h + FH\tilde{D}_uG\nabla_v^\Gamma h - GH\tilde{D}_vF\nabla_u^\Gamma h \\
&\quad + FGh\tilde{D}_{\{u,v\}}H + hF\tilde{D}_uG\tilde{D}_vH - hG\tilde{D}_vF\tilde{D}_uH.
\end{aligned}
$$

Hence

$$[\nabla^\Gamma_{uF}, \nabla^\Gamma_{vG}](Hh) - \nabla^\Gamma_{\{uF,vG\}}(Hh) = hFG([\tilde{D}_u, \tilde{D}_v] - \tilde{D}_{\{u,v\}})H$$
$$+ FGH\left([\nabla^\Gamma_u, \nabla^\Gamma_v]h - \nabla^\Gamma_{\{u,v\}}h\right),$$

which implies $ii)$.

$iii)$ We have $[\nabla^\Gamma_u, \nabla^\Gamma_v]h(x) = [\tilde{u}, \tilde{v}]h(x)$ $x \in M$, $u, v, h \in \mathcal{C}^\infty_c(M)$, and

$$\int_M [\nabla^\Gamma_u, \nabla^\Gamma_v]h d\sigma = \int_M [\tilde{u}, \tilde{v}]h d\sigma = \int_M h\mathrm{div}^M[\tilde{u}, \tilde{v}]d\sigma$$
$$= \int_M h\{u, v\}d\sigma = \int_M h\mathrm{div}^M\widetilde{\{u, v\}}d\sigma$$
$$= \int_M \widetilde{\{u, v\}}h d\sigma = \int_M \nabla^\Gamma_{\{u,v\}}h d\sigma, \quad u, v, h \in \mathcal{C}^\infty_c(M),$$

hence from $ii)$, $\int_M \Omega^\Gamma(u, v)h d\sigma = 0$, $h \in \mathcal{C}^\infty_c(M)$, $u, v \in \mathcal{U}^\infty_c(M)$, and

$$\int_M \Omega^\Gamma(u, v)(hH)d\sigma = \int_M h d\sigma([\tilde{D}_u, \tilde{D}_v] - \tilde{D}_{\{u,v\}})H, \quad h \in \mathcal{C}^\infty_c(M), \quad H \in \mathcal{S},$$

hence (iii). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

It follows in particular that for $h \in \mathcal{C}^\infty_c(M)$ we have $\Omega^\Gamma(uF, vG)h = FG\Omega^\Gamma(u, v)h$. The relation

$$\int_M [\nabla^\Gamma_u, \nabla^\Gamma_v]h d\sigma = \int_M \nabla^\Gamma_w h d\sigma, \quad h \in \mathcal{C}^\infty_c(M),$$

holds if and only if $w = \{u, v\}$.

Lemma 7.2 Let $u, v \in \mathcal{U}^\infty_c(M)$.

$i)$ If $\Omega^\Gamma(u, v) = 0$ on $\mathcal{C}^\infty_c(M)$ then $\Omega^\Gamma(u, v) = 0$ on $\mathcal{U}^\infty_c(M)$ and

$$(\tilde{D}_u\tilde{D}_v - \tilde{D}_v\tilde{D}_u)F = \tilde{D}_{\{u,v\}}F, \quad F \in \mathcal{S}.$$

$ii)$ If $M = \mathbb{R}^+$ then $\Omega^\Gamma(u, v) = 0$ on $\mathcal{U}^\infty_c(\mathbb{R}_+)$, $\tilde{D}_u\tilde{D}_v - \tilde{D}_v\tilde{D}_u = \tilde{D}_{\{u,v\}}$ on $\mathcal{S}$, and $\mathcal{U}^\infty_c(\mathbb{R}_+)$ is a Lie algebra under the bracket $\{\cdot, \cdot\}$.

$iii)$ If $u, v \in \mathcal{C}^\infty_c(M)$ and F is $\sigma(G)$-measurable, then

$$E[F(\tilde{D}_u\tilde{D}_v - \tilde{D}_v\tilde{D}_u)G] = E[F\tilde{D}_{\{u,v\}}G], \quad F, G \in \mathcal{S}.$$

Proof. *i*) We have

$$
\begin{aligned}
\left(\tilde{D}_u\tilde{D}_v - \tilde{D}_v\tilde{D}_u\right)\int_M hd\gamma &= \int_M [\tilde{u},\tilde{v}]hd\gamma \\
&= \int_M \widetilde{\{u,v\}}hd\gamma + \int_M [\tilde{u},\tilde{v}]hd\gamma - \int_M \widetilde{\{u,v\}}hd\gamma \\
&= \int_M \widetilde{\{u,v\}}hd\gamma + \int_M \Omega^\Gamma(u,v)hd\gamma \\
&= \tilde{D}_{\{u,v\}}\int_M hd\gamma + \int_M \Omega^\Gamma(u,v)hd\gamma, \quad u,v \in \mathcal{U}_c^\infty(M),
\end{aligned}
$$

hence

$$
\begin{aligned}
\left(\tilde{D}_u\tilde{D}_v - \tilde{D}_v\tilde{D}_u\right)F(\gamma) &= \tilde{D}_{\{u,v\}}F(\gamma) \\
&\quad + \sum_{i=1}^{i=n}\partial_i f\left(\int_M h_1 d\gamma,\ldots,\int_M h_1 d\gamma\right)\int_M \Omega^\Gamma(u,v)h_i d\gamma,
\end{aligned}
$$

$u,v \in \mathcal{U}_c^\infty(M)$, $F \in \mathcal{S}$. Consequently, $\Omega^\Gamma(u,v) = 0$ on $\mathcal{C}_c^\infty(M)$ implies $[\tilde{D}_u,\tilde{D}_v] = \tilde{D}_{\{u,v\}}$ on $\mathcal{S}$, which from Prop. 7.2-ii) implies $\Omega^\Gamma(u,v) = 0$ on $\mathcal{U}_c^\infty(M)$.

ii) If $M = \mathbb{R}_+$ the relation $\widetilde{\operatorname{div}^M U} = U$, $U \in \mathcal{C}_0^\infty(M;TM)$, implies

$$
\begin{aligned}
\Omega^\Gamma(uF,vG)h &= FG\Omega^\Gamma(u,v)h = FG\left([\nabla_u^\Gamma,\nabla_v^\Gamma] - \nabla_{\{u,v\}}^\Gamma\right)h \\
&= FG\left([\tilde{u},\tilde{v}] - \nabla_{\operatorname{div}^M[\tilde{u},\tilde{v}]}^\Gamma\right)h = 0, \quad h \in \mathcal{U}_c^\infty(M),
\end{aligned}
$$

hence $\Omega^\Gamma(u,v)$ vanishes on $\mathcal{C}_c^\infty(M)$ and it remains to apply *i*). We also have

$$
\begin{aligned}
\{\{u,v\},w\} &= \nabla_{\{u,v\}}^\Gamma w - \nabla_w^\Gamma\{u,v\} \\
&= \nabla_{\{u,v\}}^\Gamma w - \nabla_w^\Gamma(\nabla_u^\Gamma v - \nabla_v^\Gamma u) \\
&= (\nabla_u^\Gamma\nabla_v^\Gamma - \nabla_v^\Gamma\nabla_u^\Gamma)w - \nabla_w^\Gamma(\nabla_u^\Gamma v - \nabla_v^\Gamma u), \quad u,v,w \in \mathcal{U}_c^\infty(\mathbb{R}_+),
\end{aligned}
$$

which implies the Jacobi identity: $\{\{u,v\},w\} + \{\{v,w\},u\} + \{\{w,u\},v\} = 0$.

iii) If F is $\sigma(G)$-measurable then there is a Borel measurable function f such that $F = f(G)$, hence

$$
\begin{aligned}
E[F(\tilde{D}_u\tilde{D}_v - \tilde{D}_v\tilde{D}_u - \tilde{D}_{\{u,v\}})G] &= E[f(G)(\tilde{D}_u\tilde{D}_v - \tilde{D}_v\tilde{D}_u - \tilde{D}_{\{u,v\}})G] \\
&= E\left[(\tilde{D}_u\tilde{D}_v - \tilde{D}_v\tilde{D}_u - \tilde{D}_{\{u,v\}})\int_0^G f(t)dt\right] = 0,
\end{aligned}
$$

from (7.3), since $\int_0^G f(t)dt \in \mathcal{S}$. $\qquad\square$

Proposition 7.3 *We have for* $u, v, h \in \mathcal{U}_c^\infty(M)$:

$$\tilde{D}_h \langle u, v \rangle_{L^2(M,d\sigma)} = \langle u, \nabla_h^\Gamma v \rangle_{L^2(M,d\sigma)} + \langle \nabla_h^\Gamma u, v \rangle_{L^2(M,d\sigma)} + \langle u, v \rangle_{L^2(M,hd\sigma)}, \qquad (7.5)$$

i.e. the connection ∇^Γ *is not Riemannian.*

Proof. Integrating by parts on M and using property P-ii), we have

$$
\begin{aligned}
\int_M uvhd\sigma &= \int_M uv\mathrm{div}^M \tilde{h}d\sigma = \int_M \tilde{h}(uv)d\sigma \\
&= \langle u, \tilde{h}v \rangle_{L^2(M,d\sigma)} + \langle v, \tilde{h}u \rangle_{L^2(M,d\sigma)} \\
&= \langle u, \nabla_h^\Gamma v \rangle_{L^2(M,d\sigma)} + \langle v, \nabla_h^\Gamma u \rangle_{L^2(M,d\sigma)},
\end{aligned}
$$

hence for $u, v \in \mathcal{C}_c^\infty(M)$, $F, G \in \mathcal{S}$ and $h \in \mathcal{U}_c^\infty(M)$,

$$
\begin{aligned}
\tilde{D}_h \langle Fu, Gv \rangle_{L^2(M,d\sigma)} &= \langle u\tilde{D}_h F, Gv \rangle_{L^2(M,d\sigma)} + \langle uF, v\tilde{D}_h G \rangle_{L^2(M,d\sigma)} \\
&= \langle \nabla_h^\Gamma(uF), vG \rangle_{L^2(M,d\sigma)} + \langle uF, \nabla_h^\Gamma(vG) \rangle_{L^2(M,d\sigma)} \\
&\quad + FG \int_M uvhd\sigma.
\end{aligned}
$$

$\square$

The identity that links ∇^Γ to the metric σ is

$$
\begin{aligned}
2\langle \nabla_h^\Gamma u, v \rangle_{L^2(M,d\sigma)} &= \tilde{D}_h \langle u, v \rangle_{L^2(M,d\sigma)} + \tilde{D}_u \langle h, v \rangle_{L^2(M,d\sigma)} - \tilde{D}_v \langle h, u \rangle_{L^2(M,d\sigma)} \\
&\quad + \langle \{h, u\}, v \rangle_{L^2(M,d\sigma)} + \langle \{v, h\}, u \rangle_{L^2(M,d\sigma)} - \langle \{u, v\}, h \rangle_{L^2(M,d\sigma)} \\
&\quad + \int_M uvhd\sigma, \quad u, v, h \in \mathcal{U}_c^\infty(M),
\end{aligned}
$$

and the connection can be expressed in terms of the trilinear form

$$G(u, v, h) = \langle \{u, v\}, h \rangle_{L^2(M,d\sigma)} + \int_M uvhd\sigma, \quad u, v, h \in \mathcal{U}_c^\infty(M),$$

as

$$\langle \nabla_h^\Gamma u, v \rangle_{L^2(M,d\sigma)} = \frac{1}{2}(G(h, u, v) - G(u, v, h) - G(v, h, u)), \quad u, v, h \in \mathcal{U}_c^\infty(M).$$

The bracket $\{\cdot, \cdot\}$ maps couples of $\mathcal{C}_c^\infty$ functions on M to $\mathcal{C}_c^\infty$ functions on M, in this sense it is similar to the Poisson bracket, to the exception that the Leibniz rule is not satisfied. Also, this bracket is not local and it is non-vanishing even if $\dim M = 1$. One may prefer to write

$$H(u, v, h) = \langle \{u, v\}, h \rangle_{L^2(M,d\sigma)}, \quad K(u, v, h) = \int_M uvhd\sigma, \quad u, v, h \in \mathcal{C}_c^\infty(M),$$

and

$$\langle \nabla^\Gamma_h u, v\rangle_{L^2(M,d\sigma)} \;=\; \frac{1}{2}(H(h,u,v) - H(u,v,h) - H(v,h,u) + K(u,v,h)),$$

$u, v, h \in \mathcal{U}^\infty_c(M)$. The Markovian connection ∇^Γ is not Riemannian, for this reason we may define a Levi-Civita connection $\tilde{\nabla}^\Gamma$ as

$$\tilde{\nabla}^\Gamma_u v(x) = \nabla^\Gamma_u v(x) - \frac{1}{2}u(x)v(x), \qquad x \in M, \ u, v \in \mathcal{U}^\infty_c(M).$$

Proposition 7.4 *The connection $\tilde{\nabla}^\Gamma$ has Ω^Γ for curvature tensor, its torsion vanishes, and it is Riemannian:*

$$\tilde{D}_h\langle u,v\rangle_{L^2(M,d\sigma)} = \langle u, \tilde{\nabla}^\Gamma_h v\rangle_{L^2(M,d\sigma)} + \langle \tilde{\nabla}^\Gamma_h u, v\rangle_{L^2(M,d\sigma)}, \qquad u,v,h \in \mathcal{U}^1_c(M). \qquad (7.6)$$

Proof. The vanishing of torsion and (7.6) are obvious from Prop. 7.1 and (7.5). Concerning the curvature we have for $u,v,h \in \mathcal{U}^\infty_c(M)$:

$$\begin{aligned}
\tilde{\nabla}^\Gamma_u \tilde{\nabla}^\Gamma_v h(x) &= \tilde{\nabla}^\Gamma_u(\nabla^\Gamma_v h(x) - \frac{1}{2}h(x)v(x)) \\
&= \nabla^\Gamma_u \nabla^\Gamma_v h(x) - \frac{1}{2}h(x)\nabla^\Gamma_u v(x) - \frac{1}{2}v(x)\nabla^\Gamma_u h(x) \\
&\quad - \frac{1}{2}u(x)\nabla^\Gamma_v h(x) - \frac{1}{4}h(x)u(x)v(x),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\nabla}^\Gamma_v \tilde{\nabla}^\Gamma_u h(x) &= \nabla^\Gamma_v \nabla^\Gamma_u h(x) - \frac{1}{2}h(x)\nabla^\Gamma_v u(x) - \frac{1}{2}u(x)\nabla^\Gamma_v h(x) \\
&\quad - \frac{1}{2}v(x)\nabla^\Gamma_u h(x) - \frac{1}{4}h(x)u(x)v(x),
\end{aligned}$$

hence

$$\begin{aligned}
\tilde{\nabla}^\Gamma_v \tilde{\nabla}^\Gamma_u h(x) - \tilde{\nabla}^\Gamma_u \tilde{\nabla}^\Gamma_v h(x) &= (\Omega^\Gamma(u,v)h)(x) + \nabla^\Gamma_{\{u,v\}}h(x) - \frac{1}{2}h(x)(\nabla^\Gamma_u v(x) - \nabla^\Gamma_v u(x)) \\
&= (\Omega^\Gamma(u,v)h)(x) + \nabla^\Gamma_{\{u,v\}}h(x) - \frac{1}{2}h(x)\{u,v\}(x) \\
&= (\Omega^\Gamma(u,v)h)(x) + \tilde{\nabla}^\Gamma_{\{u,v\}}h(x), \qquad x \in M.
\end{aligned}$$

$\square$

We have

$$\begin{aligned}
2\langle \tilde{\nabla}^\Gamma_h u, v\rangle_{L^2(M,d\sigma)} &= \tilde{D}_h\langle u,v\rangle_{L^2(M,d\sigma)} + \tilde{D}_u\langle h,v\rangle_{L^2(M,d\sigma)} - \tilde{D}_v\langle h,u\rangle_{L^2(M,d\sigma)} \\
&\quad + \langle \{h,u\}, v\rangle_{L^2(M,d\sigma)} + \langle \{v,h\}, u\rangle_{L^2(M,d\sigma)} - \langle \{u,v\}, h\rangle_{L^2(M,d\sigma)},
\end{aligned}$$

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$u, v, h \in \mathcal{U}_c^\infty(M)$, and

$$\langle \tilde{\nabla}_h^\Gamma u, v \rangle_{L^2(M,d\sigma)} \;\; = \;\; \frac{1}{2}(H(h, u, v) - H(u, v, h) - H(v, h, u)), \quad u, v, h \in \mathcal{U}_c^\infty(M).$$

However, unlike the Markovian connection ∇^Γ, the Levi-Civita connection $\tilde{\nabla}^\Gamma$ will not be used in the sequel because it does not possess suitable commutation properties with the stochastic integral, cf. Prop. 9.1.

8   Exterior derivative

The exterior derivative $d^\Gamma F$ of $F \in \mathcal{S}$ is the 1-differential form defined as

$$\langle d^\Gamma F, h \rangle_{L^2(M,d\sigma)} = \tilde{D}_h F, \quad h \in \mathcal{C}_c^\infty(M).$$

We identify $L^2(M)$ to its dual $L^2(M)^*$ via the scalar product, let

$$h_1 \wedge h_2 = (h_1 \otimes h_2 - h_2 \otimes h_1), \quad h_1, h_2 \in L^2(M, d\sigma),$$

and let $L^2(M) \wedge L^2(M)$ denote the space of continuous antisymmetric bilinear forms on $L^2(M) \otimes L^2(M)$. The above Lemma allows to set the following definition.

Definition 8.1 *Let $u \in \mathcal{U}_c^\infty(M)$ be a smooth vector field.*

i) The exterior product $d^\Gamma F \wedge u$, $F \in \mathcal{S}$, is defined from

$$\langle d^\Gamma F \wedge u, h_1 \wedge h_2 \rangle_{L^2(M) \wedge L^2(M)} = (\tilde{D}_{h_1} F)\langle u, h_2 \rangle_{L^2(M)} - (\tilde{D}_{h_2} F)\langle u, h_1 \rangle_{L^2(M)},$$

$h_1, h_2 \in \mathcal{U}_c^\infty(M)$.

ii) The exterior derivative $d^\Gamma u$ of $u \in \mathcal{U}_c^\infty(M)$ is defined as

$$\langle d^\Gamma u, h_1 \wedge h_2 \rangle_{L^2(M) \wedge L^2(M)} = \langle \nabla_{h_1}^\Gamma u, h_2 \rangle_{L^2(M)} - \langle \nabla_{h_2}^\Gamma u, h_1 \rangle_{L^2(M)},$$

$h_1, h_2 \in \mathcal{U}_c^\infty(M)$.

If $u, h_1, h_2 \in \mathcal{C}_c^\infty(M)$ do not depend on the random element γ, then

$$\langle d^\Gamma u, h_1 \wedge h_2 \rangle_{L^2(M) \wedge L^2(M)} = -\langle u, \{h_1, h_2\} \rangle_{L^2(M)}.$$

We also have the relations

$$\langle d^\Gamma(Fu), h_1 \wedge h_2 \rangle_{L^2(M) \wedge L^2(M)} = F\langle d^\Gamma u, h_1 \wedge h_2 \rangle_{L^2(M) \wedge L^2(M)} + \langle d^\Gamma F \wedge u, h_1 \wedge h_2 \rangle_{L^2(M) \wedge L^2(M)},$$

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$F \in \mathcal{S}$, $h_1, h_2, u \in \mathcal{U}_c^\infty(M)$, and

$$\langle d^\Gamma u, (F h_1) \wedge h_2 \rangle_{L^2(M) \wedge L^2(M)} = F \langle d^\Gamma u, F h_1 \wedge h_2 \rangle_{L^2(M) \wedge L^2(M)},$$

$F \in \mathcal{S}$, $h_1, h_2, u \in \mathcal{U}_c^\infty(M)$. The following relation relies on the symmetry of the trilinear form $(u, v, h) \mapsto K(u, v, h) = \int_M uvhd\sigma$, $u, v, h \in \mathcal{U}_c^\infty(M)$.

Proposition 8.1 *We have for $h_1, h_2, u \in \mathcal{U}_c^\infty(M)$:*

$$\langle d^\Gamma u, h_1 \wedge h_2 \rangle_{L^2(M) \wedge L^2(M)} = \tilde{D}_{h_1} \langle u, h_2 \rangle_{L^2(M)} - \tilde{D}_{h_2} \langle u, h_1 \rangle_{L^2(M)} - \langle u, \{h_1, h_2\} \rangle_{L^2(M)}.$$

Proof. We apply Relation (7.5):

$$\tilde{D}_{h_1} \langle u, h_2 \rangle_{L^2(M)} = \langle \nabla_{h_1}^\Gamma u, h_2 \rangle_{L^2(M)} + \langle u, \nabla_{h_1}^\Gamma h_2 \rangle_{L^2(M)} + \int_M h_1 h_2 u d\sigma, \quad h_1, h_2, u \in \mathcal{U}_c^\infty(M),$$

and the fact that ∇^Γ has no torsion. $\qquad\square$

The exterior derivative $d^\Gamma u$ can be written in terms of the kernel $\nabla_x^\Gamma u(y)$ of Def. 7.3 as

$$
\begin{aligned}
\langle d^\Gamma u, h_1 \wedge h_2 \rangle_{L^2(M) \wedge L^2(M)} &= \frac{1}{2} \int_M \int_M (\nabla_x^\Gamma u(y) - \nabla_y^\Gamma u(x))(h_1 \wedge h_2)(x, y) \sigma(dx)\sigma(dy). \\
&= \int_M \int_M (\nabla_x^\Gamma u(y) - \nabla_y^\Gamma u(x)) h_1(x) h_2(y) \sigma(dx)\sigma(dy),
\end{aligned}
$$

$u, h_1, h_2 \in \mathcal{U}_c^\infty(M)$. The following Lemma is valid only in dimension one due to the integrability property of the gradient of the Green kernel.

Lemma 8.1 *We have*

$$\|d^\Gamma u\|_{L^2(M) \wedge L^2(M)}^2 = \frac{1}{2} \int_M \int_M (\nabla_y^\Gamma u(x) - \nabla_x^\Gamma u(y))^2 \sigma(dx)\sigma(dy), \qquad (8.1)$$

$u \in \mathcal{U}_c^\infty(M)$, *the right hand side being finite if* $\dim M = 1$.

Proof. Using the relation

$$\|h_1 \wedge h_2\|_{L^2(M) \wedge L^2(M)} = \|h_1\|_{L^2(M)} \|h_2\|_{L^2(M)} = \frac{1}{\sqrt{2}} \|h_1 \wedge h_2\|_{L^2(M^2)},$$

we have

$$
\begin{aligned}
\|d^\Gamma u\|_{L^2(M) \wedge L^2(M)}^2 &= \sup_{\|h_1 \wedge h_2\|_{L^2(M) \wedge L^2(M)} \le 1} \left| \langle d^\Gamma u, h_1 \wedge h_2 \rangle_{L^2(M) \wedge L^2(M)} \right| \\
&= \sup_{\|h_1 \wedge h_2\|_{L^2(M^2)} \le \sqrt{2}} \left| \langle d^\Gamma u, h_1 \wedge h_2 \rangle_{L^2(M) \wedge L^2(M)} \right| \\
&= \frac{1}{2} \int_M \int_M (\nabla_y^\Gamma u(x) - \nabla_x^\Gamma u(y))^2 \sigma(dx)\sigma(dy).
\end{aligned}
$$

$\qquad\square$

As a consequence of this Lemma we obtain for $u \in \mathcal{U}_c^\infty(M)$:

$$\|d^\Gamma u\|_{L^2(M) \wedge L^2(M)}^2 = \int_M \int_M (\nabla_y^\Gamma u(x))^2 \sigma(dx)\sigma(dy) - \int_M \int_M \nabla_y^\Gamma u(x) \nabla_x^\Gamma u(y) \sigma(dx)\sigma(dy).$$

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9 Commutation relation and energy identity

In this section we obtain energy identities on configuration space and in particular a bound for the damped anticipating stochastic integral operator $\tilde{\delta}$. The following result follows from Relation (5.4) applied to the first chaos random variable $F = \tilde{\delta}(h)$, $h \in \mathcal{C}_c^\infty(M)$. A direct proof is available in this particular case.

Proposition 9.1 *We have the commutation relation*

$$\tilde{D}_u \tilde{\delta}(v) = \tilde{\delta}(\nabla_u^\Gamma v) + \langle u, v \rangle_{L^2(M, d\sigma)}, \quad u, v \in \mathcal{C}_c^\infty(M). \tag{9.1}$$

Proof. We have

$$
\begin{aligned}
\tilde{D}_u \tilde{\delta}(v) &= \tilde{D}_u \int_0^\infty v(x) \gamma(dx) = \int_M \tilde{u} v \, d\gamma = \tilde{\delta}(\tilde{u}v) + \int_M \tilde{u} v(x) \sigma(dx) \\
&= \tilde{\delta}(\tilde{u}v) + \int_M v(x) \operatorname{div}^M \tilde{u}(x) \sigma(dx) \\
&= \tilde{\delta}(\tilde{u}v) + \int_M u(x) v(x) \sigma(dx) = \tilde{\delta}(\nabla_u^\Gamma v) + \langle u, v \rangle_{L^2(M, d\sigma)}.
\end{aligned}
$$

$\square$

Given two vector fields $U, V \in \mathcal{C}_0^\infty(M; TM)$, the usual bracket $[U, V]$ satisfies

$$\hat{D}_{[U,V]} = \hat{D}_U \hat{D}_V - \hat{D}_V \hat{D}_U,$$

however this bracket can not be used to state a commutation relation such as (9.1), since here, Poisson integrals are naturally defined as integrals of real-valued functions on M, not of vector fields on M, cf. also Prop. 9.1 above.

Proposition 9.2 *Let u be a process of the form $u = \sum_{i=1}^{i=n} u_i F_i$ and assume that for all $i, j = 1, \ldots, n$, either i) F_i is $\sigma(F_j)$-measurable, or ii) $\Omega^\Gamma(u_i, u_j) = 0$. Then*

$$
\begin{aligned}
& E[\tilde{\delta}(u)^2] + E\left[\int_M \int_M \tilde{D}_y u(x) \tilde{D}_x u(y) \sigma(dy) \sigma(dx) \right] \\
&= E[\|u\|_{L^2(M, d\sigma)}^2] + 2E\left[\int_M \int_M \nabla_y^\Gamma u(x) \tilde{D}_x u(y) \sigma(dy) \sigma(dx) \right]. \tag{9.2}
\end{aligned}
$$

Proof. We will show that as in the proof of Th. 4.3 of [7] or Th. 3.3. of [12],

$$E[\tilde{\delta}(u)^2] = E[\|u\|_{L^2(M, d\sigma)}^2] + 2E\left[\sum_{i,j=1}^n F_i \tilde{D}_{\nabla_{u_j}^\Gamma u_i} F_j \right] + E\left[\sum_{i,j=1}^n \tilde{D}_{u_j} F_i \tilde{D}_{u_i} F_j \right].$$

The proof of this identity relies on the use of the damped gradient $\tilde{D}$ and on the relation

$$E[F_i(\tilde{D}_{u_j} \tilde{D}_{u_i} - \tilde{D}_{u_i} \tilde{D}_{u_j}) F_j] = E[F_i \tilde{D}_{\{u_j, u_i\}} F_j], \tag{9.3}$$

if $\sigma(F_i) = \sigma(F_j)$, or $\tilde{D}_{u_j}\tilde{D}_{u_i} - \tilde{D}_{u_i}\tilde{D}_{u_j} = \tilde{D}_{\{u_j,u_i\}}$, if $\Omega^\Gamma(u_j, u_i) = 0$, cf. Prop. 7.2. We have

$$
E[\tilde{\delta}(u_i F_i)\tilde{\delta}(u_j F_j)] = E[F_i \tilde{D}_{u_i}\tilde{\delta}(u_j F_j)]
$$

$$
= E[F_i \tilde{D}_{u_i}(F_j \tilde{\delta}(u_j) - \tilde{D}_{u_j}F_j)]
$$

$$
= E[F_i F_j \tilde{D}_{u_i}\tilde{\delta}(u_j) + F_i \tilde{\delta}(u_j)\tilde{D}_{u_i}F_j - F_i \tilde{D}_{u_i}\tilde{D}_{u_j}F_j]
$$

$$
= E[F_i F_j \langle u_i, u_j \rangle_{L^2(M,d\sigma)} + F_i F_j \tilde{\delta}(\nabla^\Gamma_{u_i}u_j) + F_i \tilde{\delta}(u_j)\tilde{D}_{u_i}F_j - F_i \tilde{D}_{u_i}\tilde{D}_{u_j}F_j]
$$

$$
= E[F_i F_j \langle u_i, u_j \rangle_{L^2(M,d\sigma)} + \tilde{D}_{\nabla^\Gamma_{u_i}u_j}(F_i F_j) + F_i \tilde{\delta}(u_j)\tilde{D}_{u_i}F_j - F_i \tilde{D}_{u_i}\tilde{D}_{u_j}F_j]
$$

$$
= E[F_i F_j \langle u_i, u_j \rangle_{L^2(M,d\sigma)} + \tilde{D}_{\nabla^\Gamma_{u_i}u_j}(F_i F_j) + \tilde{D}_{u_j}(F_i \tilde{D}_{u_i}F_j) - F_i \tilde{D}_{u_i}\tilde{D}_{u_j}F_j]
$$

$$
= E[F_i F_j \langle u_i, u_j \rangle_{L^2(M,d\sigma)} + \tilde{D}_{\nabla^\Gamma_{u_i}u_j}(F_i F_j) + \tilde{D}_{u_j}F_i \tilde{D}_{u_i}F_j + F_i(\tilde{D}_{u_j}\tilde{D}_{u_i}F_j - \tilde{D}_{u_i}\tilde{D}_{u_j}F_j)]
$$

$$
= E[F_i F_j \langle u_i, u_j \rangle_{L^2(M,d\sigma)} + \tilde{D}_{\nabla^\Gamma_{u_i}u_j}(F_i F_j) + \tilde{D}_{u_j}F_i \tilde{D}_{u_i}F_j + F_i \tilde{D}_{\{u_j,u_i\}}F_j]
$$

$$
= E[F_i F_j \langle u_i, u_j \rangle_{L^2(M,d\sigma)} + \tilde{D}_{\nabla^\Gamma_{u_i}u_j}(F_i F_j) + \tilde{D}_{u_j}F_i \tilde{D}_{u_i}F_j + F_i \tilde{D}_{\nabla^\Gamma_{u_j}u_i - \nabla^\Gamma_{u_i}u_j}F_j]
$$

$$
= E[F_i F_j \langle u_i, u_j \rangle_{L^2(M,d\sigma)} + F_j \tilde{D}_{\nabla^\Gamma_{u_i}u_j}F_i + F_i \tilde{D}_{\nabla^\Gamma_{u_j}u_i}F_j + \tilde{D}_{u_j}F_i \tilde{D}_{u_i}F_j].
$$

On the other hand we have

$$
\sum_{i,j=1}^n \left(\tilde{D}_{u_j}F_i \tilde{D}_{u_i}F_j + F_j \tilde{D}_{\nabla^\Gamma_{u_i}u_j}F_i + F_i \tilde{D}_{\nabla^\Gamma_{u_j}u_i}F_j \right)
$$

$$
= \sum_{i,j=1}^n \tilde{D}_{u_j}F_i \tilde{D}_{u_i}F_j + 2F_i \int_M \tilde{D}_x F_j \nabla^\Gamma_{u_j}u_i(x)\sigma(dx)
$$

$$
= \sum_{i,j=1}^n \int_M \int_M \tilde{D}_y F_i u_i(x) \tilde{D}_x F_j u_j(y)\sigma(dy)\sigma(dx)
$$

$$
+ 2\sum_{i,j=1}^n \int_M \int_M F_i \tilde{D}_x F_j u_j(y)\partial_x(y)u_i(x)\sigma(dx)\sigma(dy)
$$

$$
= 2\int_M \int_M \nabla^\Gamma_y u(x)\tilde{D}_x u(y)\sigma(dy)\sigma(dx) - \int_M \int_M \tilde{D}_y u(x)\tilde{D}_x u(y)\sigma(dy)\sigma(dx).
$$

$\square$

The following proposition is valid in particular if $\dim M = 1$.

Proposition 9.3 *Under the assumptions of Prop. 9.2 we have*

$$
E[\tilde{\delta}(u)^2] + E\left[\int_M \int_M \partial_x(y)u(x)\partial_y(x)u(y)\sigma(dy)\sigma(dx)\right]
$$

$$
= E[\|u\|^2_{L^2(M,d\sigma)}] + E\left[\int_M \int_M \nabla^\Gamma_y u(x)\nabla^\Gamma_x u(y)\sigma(dy)\sigma(dx)\right],
$$

and

$$E[\tilde{\delta}(u)^2] + E\left[\int_M \int_M \partial_x(y)u(x)\partial_y(x)u(y)\sigma(dy)\sigma(dx)\right] + E\left[\|d^\Gamma u\|^2_{L^2(M)\wedge L^2(M)}\right]$$
$$= E[\|u\|^2_{L^2(M)}] + E\left[\|\nabla^\Gamma u\|^2_{L^2(M)\otimes L^2(M)}\right], \quad u \in \mathcal{U}_c^\infty(M).$$

Proof. We use the relation

$$2\int_M \int_M \nabla^\Gamma_y u(x)\tilde{D}_x u(y)\sigma(dy)\sigma(dx) - \int_M \int_M \tilde{D}_y u(x)\tilde{D}_x u(y)\sigma(dy)\sigma(dx) =$$
$$\int_M \int_M \nabla^\Gamma_y u(x)\nabla^\Gamma_x u(y)\sigma(dy)\sigma(dx) - \int_M \int_M \partial_x(y)u(x)\partial_y(x)u(y)\sigma(dy)\sigma(dx), \quad (9.4)$$

and apply Lemma 8.1 and Prop. 9.2. $\qquad\square$

If dim $M > 1$ however, (9.4) becomes an equality between finite terms that are cancellations of infinite terms since we only have $g(x, \cdot) \in W^{1,1}(M; TM)$, and Prop. 9.3 should be interpreted accordingly. Similarly we have

$$E[\tilde{\delta}(u)^2] + E\left[\int_M \int_M (\nabla^\Gamma_y u(x) - \tilde{D}_y u(x))(\nabla^\Gamma_x u(y) - \tilde{D}_x u(y))\sigma(dy)\sigma(dx)\right]$$
$$= E[\|u\|^2_{L^2(M,d\sigma)}] + E\left[\int_M \int_M \nabla^\Gamma_y u(x)\nabla^\Gamma_x u(y)\sigma(dy)\sigma(dx)\right].$$

If dim $M = 1$ we obtain a bound on the anticipating stochastic integral operator $\tilde{\delta}$:

$$E[\tilde{\delta}(u)^2] \leq E[\|u\|^2_{L^2(M)}] + 3E\left[\|\nabla^\Gamma u\|^2_{L^2(M)\otimes L^2(M)}\right] + 2E\left[\|\tilde{D}u\|^2_{L^2(M)\otimes L^2(M)}\right], \quad u \in \mathcal{U}_c^\infty(M).$$

There is a possible formal axiomatization of this construction in order to include simultaneously the path group and configuration space cases. For this one needs a damped gradient written as

$$\tilde{D}_u F = D_u F + \delta(q(u)DF),$$

where $q(u)$ is a deterministic differential operator. In the configuration space case, $q(u)f$ will be the application of the vector field u to the function f on M. In the path or loop group case, $q(u)v$ will be the bracket $[u, v]$ of two vector fields $u, v \in L^2(\mathbb{R}_+, \mathcal{G})$, see [1], [12], [13], where $\mathcal{G}$ denotes the Lie algebra of the Lie group G. In both cases the determinism of $q(u)$ is linked to the triviality of the tangent bundle. On the path space on a Riemannian manifold, $q(u)$ involves a stochastic integral of the curvature tensor on M, cf. [9], [20], hence a different and more complex framework is needed.

10 De Rham-Hodge-Kodaira operator and Weitzenböck type identity

In this section we give again a particular attention to the one-dimensional case which appears to be more closely related to the setting of Lie group valued Brownian paths of [12]. We consider $M = \mathbb{R}_+$ with σ the Lebesgue measure and the integration by parts $\int_0^\infty u(t)\tilde{v}(t)dt = -\int_0^\infty \tilde{u}(s)v(s)ds$, $u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$. The Green function associated to the problem $-\frac{d^2}{dt^2}u = f$, $u(0) = 0$, is $g(t,s) = -t \vee s$. We have $\partial_t(s) = -1_{[0,t]}(s)$, and $\tilde{u}$ is the function defined as $\tilde{u}(t) = -\int_0^t u(s)ds$, $t \in \mathbb{R}_+$, $u \in \mathcal{C}_c^\infty(\mathbb{R}_+)$. From (5.2) we have

$$\tilde{D}_t F = -\sum_{i=1}^{i=n}\sum_{k=1}^\infty \dot{u}_i(T_k)1_{[0,T_k]}(t)\partial_i f\left(\int_0^\infty u_1 d\gamma, \ldots, \int_0^\infty u_n d\gamma\right), \quad t \in \mathbb{R}_+,$$

where T_k, $k \geq 1$, is the position of the k-th particle in the Poisson sample γ. In other terms, on cylindrical functionals of the form $f(T_1, \ldots, T_n)$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$, we have

$$\tilde{D}_t f(T_1, \ldots, T_n) = -\sum_{i=1}^{i=n} 1_{[0,T_i]}(t)\partial_i f(T_1, \ldots, T_n), \quad t \in \mathbb{R}_+,$$

In the one-dimensional case, $\tilde{D}$ is a modification of the gradient of [6], see also [11], and Sobolev spaces $\mathbb{D}_{2,1}$ of real-valued functionals are defined by completion of $\mathcal{S}$ under the norm

$$\|F\|_{2,1}^2 = \|\tilde{D}F\|_{L^2(\Gamma \times \mathbb{R}_+)}^2 + \|F\|_{L^2(\Gamma)}^2 = E\left[F(\tilde{\delta}\tilde{D} + I_d)F\right], \quad F \in \mathcal{S}.$$

The Sobolev spaces $\mathbb{D}_{2,1}(l^2(\mathbb{N}))$ of $l^2(\mathbb{N})$-valued functionals of [21] rely on a discrete parameter. In order to define Sobolev spaces of $L^2(\mathbb{R}_+)$-valued functionals, we need the notion of covariant derivative. We have

$$\nabla_u^\Gamma v(t) = \sum_{i=1}^{i=n} h_i(t)\tilde{D}_u F_i - F_i \dot{h}_i(t)\int_0^t u(s)ds = \tilde{D}_u v(t) - \dot{v}(t)\int_0^t u(s)ds, \quad t \in \mathbb{R}_+,$$

and

$$\nabla_s^\Gamma v(t) = \sum_{i=1}^{i=n} h_i(t)\tilde{D}_s F_i - F_i \dot{h}_i(t)1_{[0,t]}(s) = \tilde{D}_s v(t) - \dot{v}(t)1_{[0,t]}(s), \quad s, t \in \mathbb{R}_+,$$

where $v \in \mathcal{U}_c^\infty(\mathbb{R}_+)$ is a smooth vector field as in (2.3): in one dimension, the Poisson case and the Brownian path group case are formally very close, see [12], although their geometrical aspects are quite different.

Corollary 10.1 *We have the energy identity*

$$E[\tilde{\delta}(u)^2] = E[\|u\|^2_{L^2(\mathbb{R}_+)}] + E\left[\int_0^\infty \int_0^\infty \nabla_s^\Gamma u(t) \nabla_t^\Gamma u(s) ds dt\right], \quad u \in \mathcal{U}_c^\infty(\mathbb{R}_+),$$

and the Weitzenböck type identity

$$E\left[\tilde{\delta}(u)^2\right] + E\left[\|d^\Gamma u\|^2_{L^2(\mathbb{R}_+)\wedge L^2(\mathbb{R}_+)}\right] = E\left[\|\nabla^\Gamma u\|^2_{L^2(\mathbb{R}_+)\otimes L^2(\mathbb{R}_+)}\right] + E\left[\|u\|^2_{L^2(\mathbb{R}_+)}\right].$$

Proof. These formulas follow from Props. 9.2, 9.3 and Lemma 7.2, since the relation $\partial_t(s) = -1_{[0,t]}(s)$ implies $\partial_t(s)\partial_s(t) = 0$, $\sigma \otimes \sigma(ds, dt)$-a.e. □

Consider the quadratic form

$$q(u) = E\left[\tilde{\delta}(u)^2\right] + E\left[\|d^\Gamma u\|^2_{L^2(\mathbb{R}_+)\wedge L^2(\mathbb{R}_+)}\right], \quad u \in \mathcal{U}_c^\infty(\mathbb{R}_+).$$

We have

$$q(u) - E\left[\|\nabla^\Gamma u\|^2_{L^2(\mathbb{R}_+)\otimes L^2(\mathbb{R}_+)}\right] = E\left[\|u\|^2_{L^2(\mathbb{R}_+)}\right].$$

The interpretation of this relation is that the Ricci tensor $R^\Gamma : \mathcal{U}_c^\infty(\mathbb{R}_+) \longrightarrow \mathcal{U}_c^\infty(\mathbb{R}_+)$ of Γ under the Poisson measure is identity. We define the Sobolev space $\mathbb{D}_{2,1}(L^2(\mathbb{R}_+))$ of Hilbert-valued functionals to be the completion of $\mathcal{U}_c^\infty(\mathbb{R}_+)$ under the norm

$$\|u\|^2_{2,1} = q(u) = E\left[\|u\|^2_{L^2(\mathbb{R}_+)}\right] + E\left[\|\nabla^\Gamma u\|^2_{L^2(\mathbb{R}_+)\otimes L^2(\mathbb{R}_+)}\right], \quad u \in \mathcal{U}_c^\infty(\mathbb{R}_+),$$

and deduce the following bound for the anticipating stochastic integral operator $\tilde{\delta}$:

$$E\left[\tilde{\delta}(u)^2\right] \leq \|u\|^2_{2,1}, \quad u \in \mathcal{U}_c^\infty(\mathbb{R}_+).$$

Proposition 10.1 *The de Rham-Hodge-Kodaira operator $\square = dd^* + d^*d$ is self-adjoint on $\mathcal{U}_c^\infty(\mathbb{R}_+)$, it can be written as $\square = \nabla^*\nabla + I_d$, with*

$$\begin{aligned}(\square u)(t) &= \sum_{i=1}^{i=n} h_i(t)\tilde{\delta}\tilde{D}F_i + \dot{h}_i(t)\int_0^t \tilde{D}_s F_i ds - \dot{h}_i(t)\tilde{\delta}(F_i 1_{[0,t]}) \\ &\quad - F_i t\ddot{h}_i(t) - F_i \dot{h}_i(t) + h_i(t)\tilde{D}_t F_i + F_i h_i(t), \quad t \in \mathbb{R}_+,\end{aligned}$$

if $u \in \mathcal{U}_c^\infty(\mathbb{R}_+)$ is of the form (2.3). The eigenvalues of $\square$ are greater than one.

Proof. We have

$$\begin{aligned}&E[\langle \nabla(Fh), \nabla(Gv)\rangle_{L^2(\mathbb{R}_+^2)}] \\ &= E\left[\int_0^\infty \int_0^\infty (h(t)\tilde{D}_s F - F\dot{h}(t)1_{[0,t]}(s))(v(t)\tilde{D}_s G - G\dot{v}(t)1_{[0,t]}(s)) ds dt\right]\end{aligned}$$

$$
\begin{aligned}
= \ & E\left[G\tilde{\delta}\tilde{D}F\int_0^\infty h(t)v(t)dt - G\int_0^\infty h(t)\dot{v}(t)\int_0^t \tilde{D}_sFdsdt\right.\\
& \left. -F\int_0^\infty v(t)\dot{h}(t)\int_0^t \tilde{D}_sGdsdt + FG\int_0^\infty t\dot{h}(t)\dot{v}(t)dt\right]\\
= \ & E\left[G\tilde{\delta}\tilde{D}F\int_0^\infty h(t)v(t)dt + G\int_0^\infty h(t)v(t)\tilde{D}_tFdt\right.\\
& +G\int_0^\infty \dot{h}(t)v(t)\int_0^t \tilde{D}_sFdsdt - G\int_0^\infty v(t)\dot{h}(t)\tilde{\delta}(1_{[0,t]}F)dt\\
& \left. -FG\int_0^\infty \ddot{h}(t)v(t)tdt - FG\int_0^\infty t\dot{h}(t)v(t)dt\right], \quad F,G\in\mathcal{S},\ h,v\in\mathcal{C}_c^\infty(\mathbb{R}_+).
\end{aligned}
$$

The relation

$$
E[\langle \Box u, u\rangle_{L^2(\mathbb{R}_+)}] = q(u) \geq E[\langle u, u\rangle_{L^2(\mathbb{R}_+)}], \quad u\in\mathcal{U}_c^\infty(\mathbb{R}_+),
$$

shows that the eigenvalues of $\Box$ are greater than one. $\qquad\Box$

We also have

$$
\|u\|_{2,1}^2 = q(u) = E[\langle u, (\mathrm{Id} + \nabla^*\nabla)u\rangle_{L^2(\mathbb{R}_+)}], \quad u\in\mathcal{U}_c^\infty(\mathbb{R}_+).
$$

11 Vanishing of the Ricci tensor in Poisson numerical models

A configuration on $\mathbb{R}_+ \times [-1,1]^d$ under the Lebesgue intensity measure can be constructed as a collection $(\tau_0^k, \tau_1^k, \ldots, \tau_d^k)$, $k\in\mathbb{N}$, of $\mathbb{R}_+\times[-1,1]^d$-valued random variables with $(\tau_0^k)_{k\in\mathbb{N}}$ being a family of i.i.d. exponentially distributed random variables representing distances between first coordinates of configuration points, and $\tau_1^k, \ldots, \tau_d^k$, $k\in\mathbb{N}$, denoting i.i.d. uniformly distributed random variables (the marks, or heights of configuration points). In this manner, the Poisson space can be constructed as the linear space of sequences $\mathbb{R}^{\mathbb{N}}$ with σ-field $\bigotimes_{k=0}^\infty \mathcal{B}(\mathbb{R})$ and infinite product measure $\bigotimes_{k=0}^\infty \mu_k$ with either

$$
\mu_k(dt) = 1_{[0,\infty[}(t)e^{-t}dt \quad\text{or}\quad \mu_k(dt) = \frac{1}{2}1_{[-1,1]}(t)dt,
$$

depending on the values of $k\in\mathbb{N}$. The k-th coordinate functional e_k^* on $\mathbb{R}^{\mathbb{N}}$ is either exponentially distributed on $\mathbb{R}_+$ or uniformly distributed on $[-1,1]$, depending whether e_k^* represents an interjump time or a "mark". This space is a (non-Gaussian) numerical model in the sense of [17], Ch. 1, Sects. 3 and 4. Let $\mathcal{V}$ a space of finite random sequences $(u_k)_{k\in\mathbb{N}}$ satisfying suitable boundary conditions for integration by parts, i.e. $u_k(x) = 0$

for $x_k = 0$ or $x_k \in \{-1, 1\}$, $x \in \mathbb{R}^{\mathbb{N}}$, according to whether μ_k is exponential or uniform. We let ∂ denote the gradient on $\mathbb{R}^{\mathbb{N}}$.

Proposition 11.1 *We have the Weitzenböck type identity*

$$E[(d^*u)^2] + E\left[\langle du, du \rangle_{l^2(\mathbb{N}) \wedge l^2(\mathbb{N})}\right] = E\left[\langle \partial u, \partial u \rangle_{l^2(\mathbb{N}) \otimes l^2(\mathbb{N})}\right], \quad u \in \mathcal{V},$$

i.e. the Ricci tensor vanishes under the Poisson measure.

Proof. We are in the setting of Ch. 1 of [17], integrating with respect to densities of the form $e^{\varphi(t_1, \ldots, t_l)} = \prod_{i=1}^{i=l} e^{\varphi_i(t_i)}$ with $\varphi_i(t) = -t$ or $\varphi_i(t) = 0$, $i = 1, \ldots, l$. Let d_0 denote the exterior derivative on $\mathbb{R}^l$ and let $d_\varphi^* = e^{-\varphi}(d_0)^* e^\varphi$, $\Box_0 = d_0^* d_0 + d_0 d_0^*$, and $\Box_\varphi = d_\varphi^* d_\varphi + d_\varphi d_\varphi^*$. Then, both in the exponential and uniform cases we have $\mathrm{Hess}(\varphi) = 0$ and

$$\Box_\varphi = \Box_0 + \mathrm{Hess}(\varphi) = \Box_0 = \Delta.$$

Hence from Lemma 6.7.7 of [17],

$$\langle d_\varphi^*(u), d_\varphi^*(u) \rangle_{L^2(\mathbb{R}^l, d\mu)} + \langle d_\varphi u, d_\varphi u \rangle_{L^2(\mathbb{R}^l, d\mu; \mathbb{R}^l \wedge \mathbb{R}^l)}$$

$$= \langle \Box_0 u, u \rangle_{L^2(\mathbb{R}^l, d\mu)} = \langle \Delta u, u \rangle_{L^2(\mathbb{R}^l, d\mu)} = \langle \partial u, \partial u \rangle_{L^2(\mathbb{R}^l, d\mu; \mathbb{R}^l \otimes \mathbb{R}^l)}.$$

$\Box$

We deduce the bound

$$E[(d^*u)^2] \leq E[\|\partial u\|^2_{l^2(\mathbb{N}) \otimes l^2(\mathbb{N})}], \quad u \in \mathcal{V}. \tag{11.1}$$

Remark. The vanishing of the Ricci tensor in Prop. 11.1 is due to the vanishing of the second derivative of φ, and can also be linked to the linearity of $\mathbb{R}^\infty$ as a space of sequences. On the other hand, the existence of curvature on Γ is due to the nonlinearity of φ.

These identities can be rewritten as

$$E[(d^*u)^2] + \frac{1}{2}E\left[\sum_{k,l=0}^\infty (\partial_k u_l - \partial_l u_k)^2\right] = E\left[\sum_{k,l=0}^\infty \partial_k u_l \partial_k u_l\right],$$

and

$$E[(d^*u)^2] = E\left[\langle \partial u, (\partial u)^* \rangle_{l^2(\mathbb{N}) \otimes l^2(\mathbb{N})}\right] = E\left[\sum_{k,l=0}^\infty (-\varphi_k' u_l - \partial_k u_l)(-\varphi_l' u_k - \partial_l u_k)\right],$$

$u \in \mathcal{U}_c^\infty(\mathbb{R}_+)$. The bound (11.1) has been used in the anticipating Girsanov theorem on Poisson space, cf. Prop. 3 of [22].

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Note added in proof:

A different construction of differential forms on configuration spaces has been recently introduced in S. Albeverio, A. Daletskii and E. Lytvynov, "Laplace operators on differential forms over configuration spaces", J. Geom. Phys. **37** (2001), no. 1-2, 15–46.