Equivalence of gradients on configuration spaces

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Abstract

The gradient on a Riemannian manifold X is lifted to the configuration space Υ^X on X via a pointwise identity. This entails a norm equivalence that either holds under any probability measure or characterizes the Poisson measures, depending on the tangent space chosen on Υ^X. More generally, this approach links carré du champ operators on X to their counterparts on Υ^X, and also includes structures that do not admit a gradient.

Key words: Configuration spaces, Poisson measures, Stochastic analysis.
Mathematics Subject Classification (1991): 58G32, 60H07, 60J45, 60J75.

1 Introduction

Stochastic analysis under Poisson measures, cf. [5], [6], has been developed in several different directions. This is mainly due to the fact that, unlike on the Wiener space, the gradient on Fock space and the infinitesimal Poisson gradient do not coincide under the identification of the Fock space to the L^2 space of the Poisson process.

- The gradient on Fock space is interpreted as a difference operator and has been used in e.g. [13], [15], [18].

- On the other hand, infinitesimal perturbations of configurations, cf. [6], [8], [10], [19] lead to a gradient that defines a local Dirichlet form. The generator of its diffusion process is constructed by second quantization, cf. [3], [5], [21], [25].

In this paper we work in the framework of analysis on configuration spaces of [1], [2], [3], [4], [11], [20], which unifies and links the two main approaches described above. We lift the gradient on a Riemannian manifold X to the configuration space Υ^X of X, independently of the measure chosen on Υ^X. The key result of [3] is retrieved

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via a simple argument, by taking expectations under a Poisson measure on Υ^X. We also obtain an integration by parts characterization of the Poisson measures on Υ^X, stating that the creation and annihilation operators on Υ^X are mutually adjoint only under such measures. More generally, we compute the carré du champ operator Γ^Υ of the second quantization of structures on X that are not necessarily associated to a gradient.

We proceed as follows. In Sect. 2 we work on the configuration space Υ^X and state a unitary equivalence of gradients on Υ^X that holds pointwise. Sect. 3 is a simplified exposition of basic results on configuration spaces, cf. [3], [16], [17]. We show that these results can be proved without using a measure on Υ^X. In Sect. 4 we obtain an integration by parts characterization of Poisson measures on configuration spaces. In Sect. 5 we compute the carré du champ of second quantized operators in a more general setting, for structures that do not necessarily admit a gradient. In Sect. 6 we study the consequences of this relation on the stochastic integration of adapted processes with respect to a Poisson random measure. We work for processes indexed by X without referring to a particular notion of time. Sect. 7 is devoted to the gaussian case, in which both notions of gradient (finite difference and infinitesimal) coincide.

2 Differential calculus on configuration space

Let X be a metric space with Borel σ-algebra $\mathcal{B}$ and a diffuse Radon measure σ. Let ϵ_x denote the Dirac measure at $x \in X$. The results of this section hold without referring to a particular probability measure.

Definition 1 *Let Υ^X denote the configuration space on X, that is the set of Radon measures on $(X, \mathcal{B})$ of the form $\sum_{i=1}^{i=n} \epsilon_{x_i}$ with $(x_i)_{i=1}^{i=n} \subset X$, $x_i \neq x_j$ $\forall i \neq j$, $n \in \mathbb{N} \cup \{\infty\}$.*

The configuration space Υ^X is endowed with the vague topology and its associated σ-algebra, cf. [3]. The assumption that X is a metric space is needed in the above definition in order to restrict Υ^X to configurations that are finite on bounded sets. By a convenient abuse of notation we will identify $\gamma \in \Upsilon^X$ to its support $\{x \in X \ : \ \gamma(\{x\}) = 1\}$ and denote $\gamma \backslash \{x\}$ by $\gamma \backslash x$, resp. $\gamma \cup \{x\}$ by $\gamma \cup x$, $x \in X$. A process $u : X \times \Upsilon^X \longrightarrow \mathbb{R}$ will be denoted by $\{u_x(\gamma) \ : \ (x, \gamma) \in X \times \Upsilon^X\}$. The following operators have been introduced in [17].

Definition 2 *For any $x \in X$ and any mapping $F : \Upsilon^X \longrightarrow \mathbb{R}$ let $\varepsilon_x^+ F : \Upsilon^X \longrightarrow \mathbb{R}$ and $\varepsilon_x^- F : \Upsilon^X \longrightarrow \mathbb{R}$ be defined by*

$$\left[\varepsilon_x^- F\right](\gamma) = F(\gamma \backslash x), \quad \textit{and} \quad \left[\varepsilon_x^+ F\right](\gamma) = F(\gamma \cup x), \qquad \gamma \in \Upsilon^X.$$

If $u : X \times \Upsilon^X \longrightarrow \mathbb{R}$ is a given mapping we let

$$\varepsilon^- u(\gamma) = (\varepsilon_x^- u_x(\gamma))_{x \in X} = (u_x(\gamma \backslash x))_{x \in X} \;\text{ and }\; \varepsilon^+ u(\gamma) = (\varepsilon_x^+ u_x(\gamma))_{x \in X} = (u_x(\gamma \cup x))_{x \in X}.$$

Remark 1 *We have $\varepsilon^- \varepsilon^+ = \varepsilon^+$ and $\varepsilon^+ \varepsilon^- = \varepsilon^-$.*

Proof. Let $x \in X$. We have

$$\varepsilon_x^- \varepsilon_x^+ F(\gamma) = F((\gamma \backslash x) \cup x) = F(\gamma \cup x) = \varepsilon_x^+ F(\gamma),$$

and

$$\varepsilon_x^+ \varepsilon_x^- F(\gamma) = F((\gamma \cup x) \backslash x) = F(\gamma \backslash x) = \varepsilon_x^- F(\gamma), \quad \gamma \in \Upsilon^X. \quad \square$$

Consequently, $\varepsilon_x^- \varepsilon_x^+ F(\gamma) = F(\gamma)$ on $\{(x,\gamma) \ : \ x \in \gamma\}$, $\varepsilon_x^+ \varepsilon_x^- F(\gamma) = F(\gamma)$ on $\{(x,\gamma) \ : \ x \notin \gamma\}$, however ε_x^+ is not inverse of ε_x^-.

We now further assume that X is a Riemannian manifold. Let $T_x X$ denote the tangent space at $x \in X$, let $TX = \cup_{x \in X} T_x X$, let $L^2_\gamma(TX) = L^2(X, TX, \gamma)$ denote the "tangent space" to Υ^X at $\gamma \in \Upsilon^X$, cf. [3], and let $L^2_\sigma(TX) = L^2(X, TX, \sigma)$.

Definition 3 *Let*

$$\mathcal{FC}_b^\infty = \left\{ f\left(\int_X \varphi_1 d\gamma, \ldots, \int_X \varphi_n d\gamma \right), \quad \varphi_1, \ldots, \varphi_n \in \mathcal{C}_c^\infty(X), \ f \in \mathcal{C}_b^\infty(\mathbb{R}^n), \ n \in \mathbb{N} \right\},$$

$$\mathcal{FP} = \left\{ f\left(\int_X \varphi_1 d\gamma, \ldots, \int_X \varphi_n d\gamma \right), \quad \varphi_1, \ldots, \varphi_n \in \mathcal{C}_c^\infty(X), \ f \in \mathcal{P}(\mathbb{R}^n), \ n \in \mathbb{N} \right\},$$

and

$$\mathcal{UC}_b^\infty = \left\{ \sum_{i=1}^{i=n} F_i u_i \ : \ u_1, \ldots, u_n \in \mathcal{C}_c^\infty(X), \ F_1, \ldots, F_n \in \mathcal{FC}_b^\infty, \ n \geq 1 \right\},$$

$$\mathcal{UP} = \left\{ \sum_{i=1}^{i=n} F_i u_i \ : \ u_1, \ldots, u_n \in \mathcal{C}_c^\infty(X), \ F_1, \ldots, F_n \in \mathcal{FP}, \ n \geq 1 \right\}.$$

Let ∇^X denote the gradient on X. The following gradient has been defined in [3], [5].

Definition 4 *For $F \in \mathcal{F}\mathcal{C}_b^\infty \cup \mathcal{F}\mathcal{P}$ of the form*

$$F = f\left(\int_X \varphi_1 d\gamma, \ldots, \int_X \varphi_n d\gamma\right), \quad \varphi_1, \ldots, \varphi_n \in \mathcal{C}_c^\infty(X),$$

let

$$\nabla_x^\Upsilon F(\gamma) = \sum_{i=1}^{i=n} \partial_i f\left(\int_X \varphi_1 d\gamma, \ldots, \int_X \varphi_n d\gamma\right) \nabla^X \varphi_i(x), \quad x \in X.$$

Each vector field v with compact support on X defines a curve $(\phi_t^v(x))_{t\in\mathbb{R}_+}$ in X starting at $x \in X$, and a curve $(\phi_t^v(\gamma))_{t\in\mathbb{R}_+}$ in Υ^X starting at $\gamma \in \Upsilon^X$, with $\phi_t^v(\gamma) = \{\phi_t^v(x) \ : \ x \in \gamma\}$. With this notation, ∇^Υ on Υ^X can also be defined by

$$< \nabla^\Upsilon F(\gamma), v >_{L^2_\gamma(TX)} = \lim_{\varepsilon\to 0} \frac{F(\phi_\varepsilon^v(\gamma)) - F(\gamma)}{\varepsilon}, \quad F \in \mathcal{F}\mathcal{C}_b^\infty,$$

cf. Sect. 2.3 of [3]. The following result shows a unitary equivalence between ∇^Υ and the gradient ∇^X on X via ε^+ and ε^-.

Proposition 1 *Let $F \in \mathcal{F}\mathcal{C}_b^\infty \cup \mathcal{F}\mathcal{P}$ and $x \in X$. We have*

$$\nabla_x^\Upsilon F(\gamma) = \varepsilon_x^- \nabla^X \varepsilon_x^+ F(\gamma) \quad \text{on} \ \ \{\gamma \in \Upsilon^X \ : \ x \in \gamma\}, \tag{1}$$

and

$$\varepsilon_x^+ \nabla_x^\Upsilon F(\gamma) = \nabla^X \varepsilon_x^+ F(\gamma) \quad \text{on} \ \ \{\gamma \in \Upsilon^X \ : \ x \notin \gamma\}. \tag{2}$$

Proof. Let $F = f\left(\int_X \varphi_1 d\gamma, \ldots, \int_X \varphi_n d\gamma\right)$, $x \in X$, $\gamma \in \Upsilon^X$, and assume that $x \in \gamma$. We have

$$\begin{aligned}
\nabla_x^\Upsilon F(\gamma) &= \sum_{i=1}^{i=n} \partial_i f\left(\int_X \varphi_1 d\gamma, \ldots, \int_X \varphi_n d\gamma\right) \nabla^X \varphi_i(x) \\
&= \sum_{i=1}^{i=n} \partial_i f\left(\varphi_1(x) + \int_X \varphi_1 d(\gamma\backslash x), \ldots, \varphi_n(x) + \int_X \varphi_n d(\gamma\backslash x)\right) \nabla^X \varphi_i(x) \\
&= \nabla^X f\left(\varphi_1(x) + \int_X \varphi_1 d(\gamma\backslash x), \ldots, \varphi_n(x) + \int_X \varphi_n d(\gamma\backslash x)\right) \\
&= \nabla^X \varepsilon_x^+ f\left(\int_X \varphi_1 d(\gamma\backslash x), \ldots, \int_X \varphi_n d(\gamma\backslash x)\right) \\
&= \left(\nabla^X \varepsilon_x^+ F\right)(\gamma\backslash x).
\end{aligned}$$

If $x \notin \gamma$, then

$$\begin{aligned}
\varepsilon_x^+ \nabla_x^\Upsilon F(\gamma) &= \sum_{i=1}^{i=n} \partial_i f\left(\varphi_1(x) + \int_X \varphi_1 d\gamma, \ldots, \varphi_n(x) + \int_X \varphi_n d\gamma\right) \nabla^X \varphi_i(x) \\
&= \nabla^X f\left(\varphi_1(x) + \int_X \varphi_1 d\gamma, \ldots, \varphi_n(x) + \int_X \varphi_n d\gamma\right) \\
&= \nabla^X \varepsilon_x^+ f\left(\int_X \varphi_1 d\gamma, \ldots, \int_X \varphi_n d\gamma\right) = \nabla^X \varepsilon_x^+ F(\gamma). \quad \square
\end{aligned}$$

Let us mention the consequences of this identity.

Remark 2 *For $F, G \in \mathcal{FC}_b^\infty \cup \mathcal{FP}$ we have the isometries*

$$< \nabla^\Upsilon F, \nabla^\Upsilon G >_{L^2_\gamma(TX)} \quad = \quad < \varepsilon^- \nabla^X \varepsilon^+ F, \varepsilon^- \nabla^X \varepsilon^+ G >_{L^2_\gamma(TX)}, \quad \gamma \in \Upsilon^X, \qquad (3)$$

and

$$< \varepsilon^+ \nabla^\Upsilon F, \varepsilon^+ \nabla^\Upsilon G >_{L^2_\sigma(TX)} \quad = \quad < \nabla^X \varepsilon^+ F, \nabla^X \varepsilon^+ G >_{L^2_\sigma(TX)} . \qquad (4)$$

Proof. This follows from the fact that for fixed $\gamma \in \Upsilon^X$, (1) and (2) hold respectively $\gamma(dx)$-a.e. and $\sigma(dx)$-a.e.

□

This extends the relation

$$< \nabla^\Upsilon \int_X u d\gamma, \nabla^\Upsilon \int_X v d\gamma >_{L^2_\gamma(TX)} = < \nabla^X u, \nabla^X v >_{L^2_\gamma(TX)}$$

for deterministic $u, v \in \mathcal{C}_c^\infty(X)$, which is interpreted in [3], Sect. 3.4, Relation (3.30), as a lifting to Υ^X of the geometry on X. In the next sections we will show that taking expectations in (3) under a Poisson measure with intensity σ yields Th. 5-2 of [3], cf. Cor. 2 below.

3 Multiple integrals

This section is a self-contained and simplified exposition of basic results in analysis on Poisson space, cf. [16], [17]. The difference is that they are stated on configuration spaces and that here their proofs do not make use of a (Poisson) measure on Υ^X.

Definition 5 *For $x \in X$ we define the difference operator D_x on $F : \Upsilon^X \longrightarrow \mathbb{R}$ as*

$$D_x F(\gamma) = \varepsilon_x^+ F(\gamma) - F(\gamma) = F(\gamma \cup x) - F(\gamma), \quad x \in X, \ \gamma \in \Upsilon_X^X,$$

and the operator δ on $u : X \times \Upsilon^X \longrightarrow \mathbb{R}$ with $u(\cdot, \gamma) \in L^1(X, \sigma)$, $\gamma \in \Upsilon^X$, as

$$\delta(u) = \int_X \varepsilon_x^- u_x(\gamma)(\gamma(dx) - \sigma(dx)) = \int_X u_x(\gamma \backslash x)(\gamma(dx) - \sigma(dx)),$$

provided the series converges.

The operator D satisfies the finite difference identity

$$D_x(FG) = F D_x G + G D_x F + D_x F D_x G, \quad x \in X, \ F, G : \Upsilon^X \longrightarrow \mathbb{R}. \qquad (5)$$

One has

$$\delta(u) = \int_X u d\gamma - \int_X u d\sigma = \sum_{x\in\gamma} u(x) - \int_X u d\sigma, \quad u \in \mathcal{C}_c(X).$$

Since σ is diffuse, for $u : X \times \Upsilon^X \longrightarrow \mathbb{R}$ we have $u_x(\gamma\backslash x) = u_x(\gamma)$, $\sigma(dx)$-a.e., $\gamma \in \Upsilon^X$, hence

$$\int_X u_x(\gamma\backslash x)\sigma(dx) = \int_X u d\sigma, \quad \gamma \in \Upsilon^X, \tag{6}$$

and

$$\delta(u) = \int_X \varepsilon_x^- u_x(\gamma)\gamma(dx) - \int_X u_x(\gamma)\sigma(dx), \quad \gamma \in \Upsilon^X.$$

For the same reason the action of D coincides in $L^2(X,\sigma)$ with that of the symmetric difference operator defined in [17], i.e.

$$D_x F(\gamma) = F(\gamma \cup x) - F(\gamma\backslash x), \quad \sigma(dx) - a.e., \ \gamma \in \Upsilon_X^X.$$

Remark 3 *With this notation the isometry (3) of Remark 2 becomes*

$$< \nabla^\Upsilon F, \nabla^\Upsilon G >_{L^2_\gamma(TX)} \ = \ \delta\Big(< \nabla^X DF, \nabla^X DG >_{TX}\Big) + < \nabla^X DF, \nabla^X DG >_{L^2_\sigma(TX)},$$

$\gamma \in \Upsilon^X$, *$F, G \in \mathcal{FC}_b^\infty \cup \mathcal{FP}$.*

Relation (4), however, can not be formulated using the operator δ. The following proposition is a classical result in Poissonian analysis. Its proof is usually stated under a Poisson measure, via the Fock space or the Kabanov multiplication formula. Here we use a simple trajectorial argument.

Proposition 2 *Let $u : X \times \Upsilon^X \longrightarrow \mathbb{R}$ and $F : \Upsilon^X \longrightarrow \mathbb{R}$ such that $u(\cdot,\gamma)$, $D.F(\gamma)$, $u(\cdot,\gamma)D.F(\gamma) \in L^1(X,\sigma)$, $\forall \gamma \in \Upsilon^X$. We have*

$$F\delta(u) = \delta(uF) + (u, DF)_{L^2(X,\sigma)} + \delta(uDF),$$

provided the series converge.

Proof. If $x \in \gamma$ we have

$$\varepsilon_x^- D_x F(\gamma) = \varepsilon_x^- \varepsilon_x^+ F(\gamma) - \varepsilon_x^- F(\gamma) = F(\gamma) - \varepsilon_x^- F(\gamma) = F(\gamma) - F(\gamma\backslash x),$$

hence

$$\begin{aligned}
\delta(uDF)(\gamma) &= \int_X u_x(\gamma\backslash x) D_x F(\gamma\backslash x)\gamma(dx) - \int_X u_x(\gamma\backslash x) D_x F(\gamma\backslash x)\sigma(dx) \\
&= \int_X u_x(\gamma\backslash x) F(\gamma)\gamma(dx) - \int_X u_x(\gamma\backslash x) F(\gamma\backslash x)\gamma(dx) - (DF(\gamma), u(\gamma))_{L^2(X,\sigma)} \\
&= F(\gamma)\delta(u)(\gamma) - \delta(uF)(\gamma) - (DF(\gamma), u(\gamma))_{L^2(X,\sigma)},
\end{aligned}$$

since from (6),

$$F(\gamma)\int_X u_x(\gamma\backslash x)\sigma(dx) = F(\gamma)\int_X u_x(\gamma)\sigma(dx) = \int_X F(\gamma\backslash x)u_x(\gamma\backslash x)\sigma(dx). \quad \square$$

For $f_n \in \mathcal{C}_c(X^n)$, the multiple integral

$$I_n(f_n) = \int_{\Delta_n} f_n(x_1,\dots,x_n)(\gamma(dx_1)-\sigma(dx_1))\cdots(\gamma(dx_n)-\sigma(dx_n))$$

can be defined without probability, with

$$\Delta_n = \{(x_1,\dots,x_n)\in X^n \ : \ x_i \neq x_j, \ \forall i\neq j\},$$

cf. [15]. We denote by $f_n\otimes g_m$ the tensor product of two functions $f_n \in L^2(X,\sigma)^{\otimes n}$, $g_m \in L^2(X,\sigma)^{\otimes m}$, defined as

$$f_n\otimes g_m(x_1,\dots,x_n,y_1,\dots,y_m) = f_n(x_1,\dots,x_n)g_m(y_1,\dots,y_m),$$

$(x_1,\dots,x_n,y_1,\dots,y_m)\in X^{n+m}$. The action of D and δ on multiple integrals is given by the following proposition.

Proposition 3 *We have for $f_n \in \mathcal{C}_c(X^n)$:*

$$D_xI_n(f_n) = 1_{\{x\notin\gamma\}}\sum_{i=1}^{i=n} I_{n-1}(f_n(\underbrace{\cdots}_{i-1},x,\underbrace{\cdots}_{n-i})), \quad x\in X,$$

and for $g_n \in \mathcal{C}_c(X^{n+1})$:

$$\delta(I_n(g_n(*,\cdot))) = I_{n+1}(g_n). \tag{7}$$

Proof. Let $\tilde\gamma = \gamma-\sigma$. We have, using the relation $1_{\Delta_n}(x_1,\dots,x_n)\epsilon_x(dx_i)\epsilon_x(dx_j)=0$:

$$\begin{aligned}
D_xI_n(f_n) &= D_x\int_{\Delta_n} f_n(x_1,\dots x_n)\tilde\gamma(dx_1)\cdots\tilde\gamma(dx_n) \\
&= \int_{\Delta_n} f_n(x_1,\dots,x_n)\prod_{i=1}^{i=n}(\tilde\gamma(dx_i)+(1-\gamma(\{x_i\}))\epsilon_x(dx_i)) \\
&\quad -\int_{\Delta_n} f(x_1)\cdots f(x_n)\tilde\gamma(dx_1)\cdots\tilde\gamma(dx_n) \\
&= (1-\gamma(\{x\}))\sum_{i=1}^{i=n}\int_{\Delta_{n-1}} f_n(x_1,\dots,\hat x_i,\dots,x_n)\tilde\gamma(dx_1)\cdots\hat\gamma(dx_i)\cdots\tilde\gamma(dx_n) \\
&= (1-\gamma(\{x\}))\sum_{i=1}^{i=n} I_{n-1}(f_n(\underbrace{\cdots}_{i-1},x,\underbrace{\cdots}_{n-i})), \quad x\in X,
\end{aligned}$$

where $\hat{x}_i$ and $\hat{\gamma}(dx_i)$ respectively denote the omission of x_i, and $\tilde{\gamma}(dx_i)$. On the other hand,

$$\begin{aligned}
\delta(I_n(g_n(*,\cdot))) &= \int_X I_n(g_n(x_1,\ldots,x_n,x))(\gamma\backslash x)\tilde{\gamma}(dx) \\
&= \int_X \int_{\Delta_n} 1_{\{x\notin\{x_1,\ldots,x_n\}\}} g_n(x_1,\ldots,x_n,x)\tilde{\gamma}(dx_1)\cdots\tilde{\gamma}(dx_n)\tilde{\gamma}(dx) \\
&= \int_{\Delta_{n+1}} g_n(x_1,\ldots,x_n,x)\tilde{\gamma}(dx)\tilde{\gamma}(dx_1)\cdots\tilde{\gamma}(dx_n) = I_{n+1}(g_n). \quad \square
\end{aligned}$$

In (7), the integral I_n operates on any $(n-1)$-subset of the n variables $(x_1,\ldots,x_n)$. If f_n is symmetric then

$$D_x I_n(f_n) = 1_{\{x\notin\gamma\}} n I_{n-1}(f_n(*,x)), \quad x \in X.$$

Moreover, since σ is diffuse we have

$$D_x I_n(f_n) = n I_{n-1}(f_n(*,x)), \quad \sigma(dx) - a.e.$$

Prop. 2 implies the Kabanov multiplication formula, i.e. taking $f,u \in \mathcal{C}_c(X)$ and $F = I_n(f^{\otimes n})$ we have

$$I_1(u)I_n(f^{\otimes n}) = I_{n+1}(u\otimes f^{\otimes n}) + n(u,f)_{L^2(X,\sigma)} I_{n-1}(f^{\otimes(n-1)}) + nI_n((uf)\otimes f^{\otimes(n-1)}). \quad (8)$$

We use the convention $I_0(f_0) = f_0$, $f_0 \in \mathbb{R}$.

Remark 4 *Relation (8) implies that $\mathcal{FP}$ coincides with the vector space generated by*

$$\{I_n(f_1\otimes\cdots\otimes f_n) \ : \ f_1,\ldots,f_n \in \mathcal{C}_c^\infty(X), \ n\in\mathbb{N}\}.$$

This remark shows in particular that a linear relation on $\mathcal{FP}$ needs only be proved on elements of the form $I_n(f_1\otimes\cdots\otimes f_n)$, $f_1,\ldots,f_n \in \mathcal{C}_c^\infty(X)$, $n\in\mathbb{N}$. We end this section with a definition that usually requires the notion of Fock space, but can also be stated on configuration spaces.

Definition 6 *The differential second quantization of an operator $H : \mathcal{C}_c^\infty(X) \longrightarrow \mathcal{C}_c^\infty(X)$ is the operator $d\Gamma(H)$ defined as*

$$d\Gamma(H)F = \delta(HDF), \quad F \in \mathcal{FC}_b^\infty \cup \mathcal{FP}.$$

4 Integration by parts characterization

Let π_σ denote the Poisson measure with intensity σ on Υ^X, such that $\gamma \mapsto \gamma(A)$ is Poisson distributed with mean $\sigma(A)$, $A \in \mathcal{B}$, and such that

$$\left\{ \int_X \varphi_1 d\gamma, \ldots, \int_X \varphi_n d\gamma \right\}$$

are independent random variables if $\varphi_1, \ldots, \varphi_n \in \mathcal{C}_c(X)$ have disjoint supports. For $f_n \in L^2(X,\sigma)^{\otimes n}$ and $g_m \in L^2(X,\sigma)^{\otimes m}$, let $f_n \odot g_m$ denote the symmetrization of $f_n \otimes g_m$ in $n+m$ variables. The multiple integral $I_n(f_n)$ satisfies under π_σ the well-known isometry property

$$(I_n(f_n), I_m(g_m))_{L^2(\Upsilon^X,\pi_\sigma)} = 1_{\{n=m\}} n! (f_n, g_m)_{L^2(X,\sigma)^{\otimes n}}$$

for f_n, g_m in the symmetric tensor products $L^2(X,\sigma)^{\odot n}$, $L^2(X,\sigma)^{\odot m}$, which allows to extend the mapping I_n to a continuous operator from $L^2(X,\sigma)^{\odot n}$ into $L^2(\Upsilon^X,\pi_\sigma)$. A linear isometric isomorphism is constructed between the symmetric Fock space

$$\Gamma(L^2(X,\sigma)) = \bigoplus_{n=0}^{\infty} L^2(X,\sigma)^{\odot n}$$

and $L^2(\Upsilon^X,\pi_\sigma)$, associating $f_n \in L^2(X,\sigma)^{\odot n}$ to $I_n(f_n)$. The operators

$$D : \Gamma(L^2(X,\sigma)) \longrightarrow \Gamma(L^2(X,\sigma)) \otimes L^2(X,\sigma)$$

and

$$\delta : \Gamma(L^2(X,\sigma)) \otimes L^2(X,\sigma) \longrightarrow \Gamma(L^2(X,\sigma))$$

are identified from Prop. 3 to the gradient and divergence operators on $\Gamma(L^2(X,\sigma))$, i.e.

$$Df^{\odot n} = n f^{\odot(n-1)} \otimes f, \quad \delta\left(f^{\odot n} \otimes g\right) = f^{\odot n} \odot g, \quad f, g \in L^2(X,\sigma). \tag{9}$$

The following well-known result follows from the identification of $L^2(\Upsilon^X,\pi_\sigma)$ with $\Gamma(L^2(X,\sigma))$.

Proposition 4 *Under π_σ the operators $D : L^2(\Upsilon^X,\pi_\sigma) \longrightarrow L^2(\Upsilon^X,\pi_\sigma) \otimes L^2(X,\sigma)$ and $\delta : L^2(\Upsilon^X,\pi_\sigma) \otimes L^2(X,\sigma) \longrightarrow L^2(\Upsilon^X,\pi_\sigma)$ are closable and mutually adjoint, with*

$$E_{\pi_\sigma}[(DF,u)_{L^2(X,\sigma)}] = E[F\delta(u)], \qquad F \in \mathcal{FC}_b^\infty, \ u \in \mathcal{UC}_b^\infty.$$

We now show D and δ are mutually adjoint only under $\{\pi_\sigma \otimes \sigma, \pi_\sigma\}$. This converse to Prop. 4 completes the analog result of [3] concerning ∇^Υ and its adjoint $\mathrm{div}_\pi^\Upsilon$.

Proposition 5 *Let π be a probability measure on Υ^X such that $\forall h \in \mathcal{C}_c^\infty(X)$, $I_1(h)(=\delta(h))$ has finite moments of all orders under π. Assume that*

$$E_\pi\left[\delta(u)\right] = 0, \qquad \forall u \in \mathcal{UP}, \tag{10}$$

or equivalently

$$E_\pi\left[(DF, h)_{L^2(X,\sigma)}\right] = E_\pi\left[F\delta(h)\right], \quad \forall F \in \mathcal{FP},\ h \in \mathcal{C}_c^\infty(X). \tag{11}$$

Then π is the Poisson measure π_σ with intensity σ.

Proof. First, we note that from Remark 4, if $I_1(h)$ has finite moments of all orders under π, $\forall h \in \mathcal{C}_c^\infty(X)$, then $\delta(u)$ is integrable under π, $\forall u \in \mathcal{UP}$. Relation (10) implies (11) from Prop. 2. Conversely, if

$$E_\pi\left[\delta(gI_n(f^{\otimes n}))\right] = 0, \quad f, g \in \mathcal{C}_c^\infty(X),$$

then $E_\pi\left[\delta(gI_{n+1}(f^{\otimes n+1}))\right] = 0$, since

$$E\left[\delta(g)I_{n+1}(f^{\otimes(n+1)}) - (g, DI_{n+1}(f^{\otimes(n+1)}))_{L^2(X,\sigma)}\right] = 0,$$

and from the Kabanov multiplication formula (Prop. 2):

$$\begin{aligned}
&\delta(gI_{n+1}(f^{\otimes(n+1)})) \\
&\quad = \delta(g)I_{n+1}(f^{\otimes(n+1)}) - (g, DI_{n+1}(f^{\otimes(n+1)}))_{L^2(X,\sigma)} - (n+1)\delta((gf)I_n(f^{\otimes n})).
\end{aligned}$$

Hence by induction, (11) implies (10), given Remark 4. Let us now show that $\pi = \pi_\sigma$. We have for $h \in \mathcal{C}_c^\infty(X)$ and $n \geq 1$, using (11):

$$\begin{aligned}
E_\pi\left[\left(\int_X hd\gamma\right)^n\right] &= E_\pi\left[\delta(h)\left(\int_X hd\gamma\right)^{n-1}\right] + \left(\int_X hd\sigma\right)E_\pi\left[\left(\int_X hd\gamma\right)^{n-1}\right] \\
&= E_\pi\left[\left(h, D\left(\int_X hd\gamma\right)^{n-1}\right)_{L^2(X,\sigma)}\right] + \left(\int_X hd\sigma\right)E_\pi\left[\left(\int_X hd\gamma\right)^{n-1}\right] \\
&= E_\pi\left[\int_X h(x)\left(h(x) + \int_X hd(\gamma\backslash x)\right)^{n-1}\sigma(dx)\right] \\
&= E_\pi\left[\int_X h(x)\left(h(x) + \int_X hd\gamma\right)^{n-1}\sigma(dx)\right] \\
&= \sum_{k=0}^{k=n-1}\binom{n-1}{k}\left(\int_X h^{n-k}d\sigma\right)E_\pi\left[\left(\int_X hd\gamma\right)^k\right].
\end{aligned}$$

This induction relation defines the moments of $\gamma \mapsto \int_X hd\gamma$ under π. Moreover it holds under π_σ from Prop. 4, hence the moments of $\gamma \mapsto \int_X hd\gamma$ under π are that of a Poisson random variable with intensity $\int_X h d\sigma$. By dominated convergence this implies

$$E_\pi\left[\exp\left(iz\int_X h d\gamma\right)\right] = \exp\int_X (e^{izh}-1)d\sigma, \quad z \in \mathbb{R}, \quad h \in \mathcal{C}_c^\infty(X),$$

hence $\pi = \pi_\sigma$.

□

This proposition can be modified as follows.

Proposition 6 *Let π be a probability measure on Υ^X such that $\delta(u)$ is integrable, $\forall u \in \mathcal{UC}_b^\infty$. Assume that*

$$E_\pi\left[\delta(u)\right] = 0, \quad u \in \mathcal{UC}_b^\infty, \tag{12}$$

or equivalently

$$E_\pi\left[(DF,u)_{L^2(X,\sigma)}\right] = E_\pi\left[F\delta(u)\right], \quad F \in \mathcal{FC}_b^\infty, \; u \in \mathcal{UC}_b^\infty. \tag{13}$$

Then π is the Poisson measure π_σ with intensity σ.

Proof. Clearly, (12) implies (13) as in the proof of Prop. 5. The implication (13) $\Rightarrow$ (12) follows in this case by taking $F = 1$. Denoting the characteristic function of $\gamma \mapsto \int_X h d\gamma$ by $\psi(z) = E_\pi[\exp(iz\int_X h d\gamma)]$, $z \in \mathbb{R}$, we have:

$$\begin{aligned}
\frac{d}{dz}\psi(z) &= iE_\pi\left[\int_X h d\gamma \exp\left(iz\int_X h d\gamma\right)\right] \\
&= iE_\pi\left[\delta(h)\exp\left(iz\int_X h d\gamma\right)\right] + iE_\pi\left[\int_X h d\sigma \exp\left(iz\int_X h d\gamma\right)\right] \\
&= iE_\pi\left[\left(h, D\exp\left(iz\int_X h d\gamma\right)\right)_{L^2(X,\sigma)}\right] + i\psi(z)\int_X h d\sigma \\
&= i(h, e^{izh}-1)_{L^2(X,\sigma)}E_\pi\left[\exp\left(iz\int_X h d\gamma\right)\right] + i\psi(z)\int_X h d\sigma \\
&= i\psi(z)(h, e^{izh})_{L^2(X,\sigma)}, \quad z \in \mathbb{R}.
\end{aligned}$$

We used the relation

$$D_x \exp\left(iz\int_X h d\gamma\right) = (e^{izh(x)}-1)\exp\left(iz\int_X h d\gamma\right), \quad x \in X,$$

which holds from the definition of D_x as a finite difference operator, cf. Def. 5. With the initial condition $\psi(0) = 1$ we obtain

$$\psi(z) = \exp\int_X (e^{izh}-1)d\sigma, \quad z \in \mathbb{R}. \quad \square$$

Corollary 1 *Let π be a probability measure on Υ^X such that $I_n(f^{\otimes n})$ is integrable under π, $\forall f \in \mathcal{C}_c^\infty(X)$. The relation*

$$E_\pi\left[I_n(f^{\otimes n})\right] = 0, \quad \forall f \in \mathcal{C}_c^\infty(X), \quad n \geq 1, \tag{14}$$

holds if and only if π is the Poisson measure π_σ with intensity σ.

Proof. If (14) holds then by polarisation,

$$E_\pi\left[\delta(g \otimes I_n(f_1 \otimes \cdots \otimes f_n))\right] = 0, \quad g, f_1, \ldots, f_n \in \mathcal{C}_c^\infty(X), \quad n \geq 0,$$

and from Remark 4, $E_\pi[\delta(u)] = 0$, $u \in \mathcal{UP}$, hence $\pi = \pi_\sigma$ from Prop. 5.

□

In the sequel we make the following hypothesis.

Definition 7 *We now assume that*

(i) X is a Riemannian manifold with volume element m,

(ii) $\sigma(dx) = \rho(x)m(dx)$ with $\rho \in L^2_{loc}(X, m)$,

(iii) div_σ^X is defined on every $\nabla^X f$, $f \in \mathcal{C}_c^\infty(X)$, with

$$\int_X g(x)\mathrm{div}_\sigma^X \nabla^X f(x)\sigma(dx) = \int_X < \nabla^X g(x), \nabla^X f(x) >_{T_xX} \sigma(dx), \quad f, g \in \mathcal{C}_c^1(X).$$

Let also H_σ^X denote the Laplace-Beltrami operator on X, defined as $H_\sigma^X = \mathrm{div}_\sigma^X \nabla^X$. As a corollary of our pointwise lifting of gradients we obtain Th. 5-2 of [3], page 489, by taking expectations in Remark 3, and a characterization of the Poisson measure.

Corollary 2 *(i) The isometry relation*

$$E_\pi\left[< \nabla^\Upsilon F, \nabla^\Upsilon G >_{L^2_\gamma(TX)}\right] = E_\pi\left[< \nabla^X DF, \nabla^X DG >_{L^2_\sigma(TX)}\right], \quad F, G \in \mathcal{FC}_b^\infty, \tag{15}$$

holds if π is the Poisson measure π_σ with intensity σ.

(ii) Under the hypothesis $\{H_\sigma^X f \ : \ f \in \mathcal{C}_c^\infty(X)\} = \mathcal{C}_c^\infty(X)$, Relation (15) implies that $\pi = \pi_\sigma$.

Proof. (i) Remark 3, Prop. 4 and Prop. 5 show that (15) holds if $\pi = \pi_\sigma$. (ii) if (15) is satisfied, then taking $F = I_n(u^{\odot n})$ and $G = I_1(h)$, $h, u \in \mathcal{C}_c^\infty(X)$, gives $E_\pi\left[\delta\left(< \nabla^X DF, \nabla^X h >_{TX}\right)\right] = 0$, i.e. $E_\pi\left[\delta((H_\sigma^X h)uI_{n-1}(u^{\odot(n-1)}))\right] = 0$, $n \geq 1$, hence $\pi = \pi_\sigma$ from Cor. 1.

$\square$

Remark 5 *Instead of δ, the proof of Relation (15) in [3] consists in using (2) and the relation*

$$E_{\pi_\sigma}\left[\int_X \varepsilon_x^+ < \nabla^\Upsilon F, \nabla^\Upsilon G >_{T_xX} \sigma(dx)\right] = E_{\pi_\sigma}\left[\int_X < \nabla^\Upsilon F, \nabla^\Upsilon G >_{T_xX} \gamma(dx)\right],$$

$F, G \in \mathcal{FC}_b^\infty$, cf. e.g. [14].

Let $\operatorname{div}_{\pi_\sigma}^\Upsilon$ denote the adjoint of ∇^Υ under π_σ, defined as

$$E_{\pi_\sigma}\left[F \operatorname{div}_{\pi_\sigma}^\Upsilon G\right] = E_{\pi_\sigma}\left[< \nabla^\Upsilon F, \nabla^\Upsilon G >_{L^2_\gamma(TX)}\right],$$

on $G \in \mathcal{FC}_b^\infty$ such that $\mathcal{FC}_b^\infty \ni F \mapsto E_{\pi_\sigma}\left[< \nabla^\Upsilon F, \nabla^\Upsilon G >_{L^2_\gamma(TX)}\right]$ extends to a bounded operator on $L^2(\Upsilon^X, \pi_\sigma)$. By density of $\mathcal{FC}_b^\infty$ in $L^2(\Upsilon^X, \pi_\sigma)$ it follows from Cor. 2 that under π_σ, $\operatorname{div}_{\pi_\sigma}^\Upsilon \nabla^\Upsilon$ coincides with $d\Gamma(H_\sigma^X)$. The following remark is connected to Th. 1 of [23].

Remark 6 *Assume that there exists a unique probability measure σ on $(X, \mathcal{B})$ such that*

$$\int_X f \operatorname{div}_\sigma^X \nabla^X \varphi d\sigma = < \nabla^X f, \nabla^X \varphi >_{L^2_\sigma(TX)}, \quad \varphi, f \in \mathcal{C}_c^\infty(X). \tag{16}$$

(i) Let ρ be a probability measure on $(X, \mathcal{B})$. The relation

$$E_\pi\left[\int_X u_x(\gamma\backslash x) \operatorname{div}_\sigma^X \nabla^X \varphi(x)\gamma(dx)\right] = E_\pi\left[< \nabla^X u, \nabla^X \varphi >_{L^2_\rho(TX)}\right], \tag{17}$$

$u \in \mathcal{UC}_b^\infty, \varphi \in \mathcal{C}_c^\infty(X)$, holds if $\rho = \sigma$ and $\pi = \pi_\sigma$.

(ii) If $\delta(u)$ is integrable under π, $\forall u \in \mathcal{UC}_b^\infty$, then under the hypothesis

$$\{H_\sigma^X f \ : \ f \in \mathcal{C}_c^\infty(X)\} = \mathcal{C}_c^\infty(X), \tag{18}$$

Relation (17) implies $\rho = \sigma$ and $\pi = \pi_\sigma$.

Proof. (i) From the above discussion, (17) is satisfied if $\pi = \pi_\sigma$. (ii) Taking $u = f \in \mathcal{C}_c^\infty(X)$ in (17) implies $\rho = \sigma$ from (16). Moreover from (17), Relation (16) becomes

$$E_\pi\left[\int_X u_x(\gamma\backslash x) H_\sigma^X \varphi(x)\gamma(dx)\right] = E_\pi\left[< \nabla^X u, \nabla^X \varphi >_{L^2_\sigma(TX)}\right] = E_\pi\left[< u, H_\sigma^X \varphi >_{L^2(X,\sigma)}\right].$$

i.e.

$$E_\pi\left[\delta\left(u H_\sigma^X \varphi\right)\right] = 0, \quad u \in \mathcal{UC}_b^\infty, \ \varphi \in \mathcal{C}_c^\infty(X),$$

hence π is the Poisson measure π_σ with intensity σ from Prop. 5, under the hypothesis (18).

$\square$

In particular one can choose X to be the Wiener space as in [23]. In this case, σ is the Wiener measure from the integration by parts characterizations (Th. 1.2) of [22]. Finally we mention a result also known as the Skorohod isometry, which can be proved from Prop. 2 and will be used in the sequel.

Proposition 7 *For $u \in \mathcal{UP} \cup \mathcal{UC}_b^\infty$ we have*

$$E_{\pi_\sigma}\left[\delta(u)^2\right] = E\left[\|u\|^2_{L^2(X,\sigma)}\right] + E\left[\int_X\int_X D_x u(y) D_y u(x)\sigma(dx)\sigma(dy)\right]. \tag{19}$$

Proof. We have, applying Prop. 2 and the relation $D_x\delta(u) = u(x) + \delta(D_x u)$:

$$\begin{aligned}
E_{\pi_\sigma}\left[\delta(u)^2\right] &= E_{\pi_\sigma}\left[\delta(u\delta(u)) + (u, D\delta(u))_{L^2(X,\sigma)} + \delta(uD\delta(u))\right] \\
&= E_{\pi_\sigma}\left[(u, D\delta(u))_{L^2(X,\sigma)}\right] \\
&= E_{\pi_\sigma}\left[\|u\|^2_{L^2(X,\sigma)} + \int_X u(x)\delta(D_x u)\sigma(dx)\right] \\
&= E_{\pi_\sigma}\left[\|u\|^2_{L^2(X,\sigma)} + \int_X D_y u(x) D_x u_y \sigma(dx)\sigma(dy)\right]. \quad \square
\end{aligned}$$

5 Second quantization and carré du champ operators

In this section we compute the carré du champ operator associated to second quantized operators. The difference with the previous sections is that we are not restricted to the structure given by ∇^X. The following definition is adapted from [24], cf. also [9].

Definition 8 *Let M be a metric space with Borel measure ν. The carré du champ associated to an operator H defined on a domain $\mathcal{D} \subset L^2(M,\nu)$ stable by pointwise multiplication and by H is the bilinear operator $\Gamma^M : \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{D}$ defined as*

$$\Gamma^M(\varphi,\psi) = \frac{1}{2}(H^M(\varphi\psi) - \varphi H^M\psi - \psi H^M\varphi), \quad \varphi,\psi \in \mathcal{D}.$$

- *The operator Γ^M is local if it satisfies the property*

$$\Gamma^M(\varphi\psi,\phi) = \psi\Gamma^M(\varphi,\phi) + \varphi\Gamma^M(\psi,\phi), \quad \varphi,\psi,\phi \in \mathcal{D}.$$

- *The operator H^M is said to be conservative on $\mathcal{D}$ under ν if*

$$\int_M (H^M u) d\nu = 0, \quad \forall u \in \mathcal{D}.$$

Only the conservativity assumption makes use of the measure ν. If Γ^M is local, then

$$\begin{aligned} H^M f(\varphi_1, \ldots, \varphi_n) &= \sum_{i,j=1}^{n} \partial_i \partial_j f(\varphi_1, \ldots, \varphi_n) \Gamma^M(\varphi_i, \varphi_j) \\ &\quad + \sum_{i=1}^{i=n} \partial_i f(\varphi_1, \ldots, \varphi_n) H^M \varphi_i, \quad \varphi_1, \ldots, \varphi_n \in \mathcal{D}, \end{aligned}$$

$f \in \mathcal{P}(\mathbb{R}^n)$, and H^M is called a diffusion operator. The following lemma shows how to compute the carré du champ of a second quantized (or "lifted") operator.

Lemma 1 *Let H^X be an operator on $\mathcal{C}_c^\infty(X)$ with carré du champ Γ^X.*

(i) The carré du champ of $d\Gamma(H^X)$ is

$$\Gamma^\Upsilon(F,G) = \delta\left(\Gamma^X(DF,DG)\right) - \frac{1}{2}\left((DF, H^X DG)_{L^2(X,\sigma)} + (H^X DF, DG)_{L^2(X,\sigma)}\right), \tag{20}$$

$F, G \in \mathcal{FC}_b^\infty$.

(ii) If H^X is conservative on $\mathcal{C}_c^\infty(X)$ under σ, then

$$\Gamma^\Upsilon(F,G) = \delta\left(\Gamma^X(DF,DG)\right) + \int_X \Gamma^X(DF,DG) d\sigma, \quad F, G \in \mathcal{FC}_b^\infty. \tag{21}$$

Proof. We have from Prop. 3 and (5):

$$\begin{aligned} \frac{1}{2} d\Gamma(H^X)(F^2) &= \frac{1}{2}\delta\left(H^X D(F^2)\right) = \delta\left(F H^X DF + H^X(DF DF)\right) \\ &= F\delta\left(H^X DF\right) - (DF, H^X DF)_{L^2(X,\sigma)} - \delta\left(DF H^X DF\right) \\ &\quad + \delta\left(DF H^X DF + \Gamma^X(DF,DF)\right) \\ &= F d\Gamma(H^X) F - (DF, H^X DF)_{L^2(X,\sigma)} + \delta\left(\Gamma^X(DF,DF)\right), \end{aligned}$$

hence (20) holds. If H^X is conservative on $\mathcal{C}_c^\infty(X)$ under σ,

$$-\frac{1}{2}\left((DF, H^X DG)_{L^2(X,\sigma)} + (H^X DF, DG)_{L^2(X,\sigma)}\right) = \int_X \Gamma^X(DF,DG) d\sigma,$$

hence (21).

□

If H^X is conservative on $\mathcal{C}_c^\infty(X)$, then we can also write

$$\Gamma^\Upsilon(F,G)(\gamma) = \int_X \Gamma^X(D_x F, D_x G)(\gamma \backslash x)\gamma(dx), \quad F, G \in \mathcal{FC}_b^\infty.$$

Proposition 8 *Let H^X be a conservative diffusion operator on $\mathcal{C}_c^\infty(X)$ under σ with local carré du champ Γ^X. Then* *(i) $d\Gamma(H^X)$ is a diffusion operator with carré du champ*

$$\Gamma^{\Upsilon}(F,G) = \delta\left(\Gamma^X(DF,DG)\right) + \int_X \Gamma^X(DF,DG)d\sigma, \quad F,G \in \mathcal{FC}_b^\infty. \tag{22}$$

(ii) $d\Gamma(H^X)$ is conservative on $\mathcal{FC}_b^\infty$ under π_σ,

(iii) the Dirichlet form $(F,G) \mapsto \mathcal{E}(F,G)$ associated to $d\Gamma(H^X)$ and defined as $\mathcal{E}(F,G) = E_{\pi_\sigma}\left[\Gamma^{\Upsilon}(F,G)\right]$ satisfies

$$\mathcal{E}(F,G) = E_{\pi_\sigma}\left[\int_X \Gamma^X(DF,DG)d\sigma\right], \quad F,G \in \mathcal{FC}_b^\infty.$$

Proof. (i) Given Lemma 1 it suffices to prove that the locality property is satisfied. We have $D(F^2) = 2FDF + DFDF$ and

$$\begin{aligned}
&\delta\left(\Gamma^X(D(F^2),DG)\right) + \int_X \Gamma^X(D(F^2),DG)d\sigma \\
&= 2\delta\left(F\Gamma^X(DF,DG)\right) + \delta\left(\Gamma^X(DFDF,DG)\right) \\
&\quad +2F\int_X \Gamma^X(DF,DG)d\sigma + \int_X \Gamma^X(DFDF,DG)d\sigma \\
&= 2\delta\left(F\Gamma^X(DF,DG)\right) + 2\delta\left(DF\Gamma^X(DF,DG)\right) + 2(DF,\Gamma^X(DF,DG)))_{L^2(X,\sigma)} \\
&\quad +2F\int_X \Gamma^X(DF,DG)d\sigma \\
&= 2F\delta\left(\Gamma^X(DF,DG)\right) + 2F\int_X \Gamma^X(DF,DG)d\sigma = 2F\Gamma^X(DF,DG).
\end{aligned}$$

The statements (ii) and (iii) are a consequence of Cor. 2, Prop. 5 and (22): the conservativity of $d\Gamma(H^X)$ states that $E_{\pi_\sigma}[\delta(H^X DF)] = 0$, $F \in \mathcal{FP}$, and (iii) holds because $E_{\pi_\sigma}[\delta(\Gamma^X(DF,DG))] = 0$, $F,G \in \mathcal{FC}_b^\infty$.

□

In particular if H^X is the generator of a local Dirichlet form $\mathcal{E}^X$ on X with carré du champ Γ^X, then $d\Gamma(H^X)$ is the generator of a local Dirichlet form on Υ^X with carré du champ

$$\Gamma^{\Upsilon}(F,G) = \delta(\Gamma^X(DF,DG)) + \mathcal{E}^X(F,G), \quad F,G \in \mathcal{FC}_b^\infty.$$

As a particular case, for $H_\sigma^X = \operatorname{div}_\sigma^X \nabla^X$ we retrieve Remark 3 which holds independently of the measure π chosen on Υ^X. In the following corollary we use the assumptions of Def. 7.

Corollary 3 *The differential second quantization $d\Gamma(H_\sigma^X)$ of the Laplace-Beltrami operator $H_\sigma^X = \operatorname{div}_\sigma^X \nabla^X$ on X is a diffusion operator with local carré du champ*

$$\Gamma^{\Upsilon}(F,G)(\gamma) = < \nabla^{\Upsilon} F, \nabla^{\Upsilon} G >_{L^2_\gamma(TX)}, \quad F,G \in \mathcal{FP}. \tag{23}$$

Proof. From Prop. 8 it suffices to prove (23). Given Prop. 8, the mapping

$$(F,G) \mapsto < \nabla^\Upsilon F, \nabla^\Upsilon G >_{L^2_\gamma(TX)}, \quad F,G \in \mathcal{FP},$$

satisfies the locality property since ∇^Υ is a derivation. Since Γ^Υ is also local from Prop. 8, it suffices to prove that (23) holds for $G(\gamma) = \int_X u d\gamma$ and $F(\gamma) = \int_X v d\gamma$, $u, v \in \mathcal{C}_c^\infty(X)$. Since u and v are deterministic, we have from Prop. 1 and (22):

$$\begin{aligned}(\nabla^\Upsilon F, \nabla^\Upsilon G)_{L^2_\sigma(TX)} &= \int_X < \nabla^X u, \nabla^X v >_{T_xX} \gamma(dx) = \int_X \Gamma^X(u,v) d\gamma \\ &= \delta\left(\Gamma^X(u,v)\right) + \int_X \Gamma^X(u,v) d\sigma = \Gamma^\Upsilon(F,G),\end{aligned}$$

Γ^X being defined as $\Gamma^X(u,v) =< \nabla^X u, \nabla^X v >_{L^2_\sigma(TX)}$, $u, v \in \mathcal{C}_c^\infty(X)$.

□

Relation (23) shows that Γ^Υ is the carré du champ associated to the Dirichlet form of Th. 4-1 of [3]. If $F = I_1(u)$ and $G = I_1(v)$, $u, v \in \mathcal{C}_c^\infty(X)$, are first chaos random variables we retrieve as a particular case the identity

$$\Gamma^\Upsilon(F,G)(\gamma) = \int_X \Gamma^X(u,v) d\gamma,$$

which is apparent in e.g. Relation (4.7) of [3], page 476. This also shows that the expression of $d\Gamma(H^X_\sigma)$ for first chaos random variables:

$$\begin{aligned}& d\Gamma(H^X_\sigma) f\left(\int_X \varphi_1 d\gamma, \ldots, \int_X \varphi_n d\gamma\right) \\ &= \sum_{i=1}^{i=n} \partial_i f\left(\int_X \varphi_1 d\gamma, \ldots, \int_X \varphi_n d\gamma\right) \int_X H^X_\sigma \varphi_i d\gamma \\ &\quad + \sum_{i,j=1}^{n} \partial_i \partial_j f\left(\int_X \varphi_1 d\gamma, \ldots, \int_X \varphi_n d\gamma\right) \int_X < \nabla^X \varphi_i, \nabla^X \varphi_j >_{TX} d\gamma,\end{aligned}$$

cf. (4.3) and (4.7), pp. 474 and 476 of [3], can be extended to $F_1, \ldots, F_n \in \mathcal{FC}_b^\infty$:

$$\begin{aligned}d\Gamma(H^X_\sigma) f(F_1, \ldots, F_n) &= \sum_{i=1}^{i=n} \partial_i f(F_1, \ldots, F_n)\, d\Gamma(H^X_\sigma) F_i \\ &\quad + \sum_{i,j=1}^{n} \partial_i \partial_j f(F_1, \ldots, F_n)\, \Gamma^\Upsilon(F_i, F_j).\end{aligned}$$

Th. 9-18 of [5] also follows from the application of (23) to the special case of adapted functionals.

6 Stochastic integration

In this section we work with the triple $(H_\sigma^X, \mathcal{E}_\sigma^X, \Gamma_\sigma^X)$ of conservative generator, Dirichlet form and carré du champ given by ∇^X. If $X = \mathbb{R}_+$ then it is well-known that $\mathrm{div}_{\pi_\sigma}^\Upsilon$ and $\delta \circ \mathrm{div}_\sigma^X$ both coincide with the compensated Poisson stochastic integral on adapted processes, cf. [10], [15], [19]. We show that this property is a consequence of Prop. 8 and can be extended to a Riemannian manifold X using a definition of adaptedness that does not require an ordering on X, extending the Wiener space construction of [26].

Definition 9 *Let $\mathcal{V} = \{\nabla^X h \ : \ h \in \mathcal{UC}_b^\infty\}$. Let $u : X \times \Upsilon^X \longrightarrow TX$ be a mapping in $\mathcal{V}$ written as*

$$u = \sum_{i=1}^{i=n} F_i \nabla^X h_i, \quad F_1, \ldots, F_n \in \mathcal{FC}_b^\infty, \ h_1, \ldots, h_n \in \mathcal{C}_c^\infty(X).$$

(i) $v \in \mathcal{V}$ is said to be a ∇^Υ-adapted vector if $< \nabla^X h_i(x), \nabla_x^\Upsilon F_i(\gamma) >_{T_xX} = 0$ $\forall x \in X$, $\gamma \in \Upsilon^X$, $i = 1, \ldots, n$.

(ii) $v \in \mathcal{V}$ is said to be a D-adapted vector if for all $i, j \in \{1, \ldots, n\}$, $x \in X$, and $\gamma \in \Upsilon^X$:

$$< \nabla^X h_i(x), \nabla^X D_x F_j(\gamma) >_{T_xX} = 0 \quad \text{or} \quad < \nabla^X h_j(x), \nabla^X D_x F_i(\gamma) >_{T_xX} = 0, \tag{24}$$

for at least one such representation of v.

(iii) Let $L^2_{ad}(\Upsilon^X \times X, TX)$ denote the completion under the norm

$$v \mapsto \|v\|_{L^2(\Upsilon^X \times X, TX; \pi_\sigma)} + E_{\pi\sigma}\left[\int_X \left(\mathrm{div}_\sigma^X v\right)^2 d\sigma\right]^{1/2}$$

of the subset of $\mathcal{V}$ made of vectors that are both ∇^Υ-adapted and D-adapted.

If $X = \mathbb{R}_+$, then processes which are adapted in the usual sense with respect to the canonical Poisson filtration are identified to ∇^Υ-adapted vectors. We define the bilinear form trace_x on $T_xX \otimes T_xX$ by

$$\mathrm{trace}_x u \otimes v = < u, v >_{T_xX}, \quad u \otimes v \in T_xX \otimes T_xX, \ x \in X.$$

The following proposition shows that the operators $\mathrm{div}_{\pi_\sigma}^\Upsilon$ and $\delta \circ \mathrm{div}_\sigma^X$ coincide on $L^2_{ad}(\Upsilon^X \times X, TX)$ with the Poisson stochastic integral.

Proposition 9 *(i) Let π be a probability under which $\mathrm{div}_\pi \nabla^X h$ is defined, $h \in \mathcal{C}_c^\infty(X)$, and such that $\mathcal{FC}_b^\infty$ is dense in $L^2(\Upsilon^X, \pi)$. We have*

$$\mathrm{div}_\pi^\Upsilon v = \sum_{i=1}^{i=n} G_i \mathrm{div}_\pi^\Upsilon \nabla^X h_i - \int_X \mathrm{trace}_x \nabla_x^\Upsilon v(x) \gamma(dx), \tag{25}$$

where $v \in \mathcal{V}$ is written as $v(x,\gamma) = \sum_{i=1}^{i=n} \nabla^X h_i(x) G_i(\gamma)$, $x \in X$, $\gamma \in \Upsilon^X$.

(ii) If π is the Poisson measure π_σ with intensity σ, then

$$\mathrm{div}_{\pi_\sigma}^\Upsilon v = \int_X \mathrm{div}_\sigma^X v d\gamma - \int_X \mathrm{trace}_x \nabla_x^\Upsilon v(x) \gamma(dx), \quad v \in \mathcal{V}. \tag{26}$$

(iii) For all simple ∇^Υ-adapted vector $v \in \mathcal{V}$ we have

$$\mathrm{div}_{\pi_\sigma}^\Upsilon (v) = \delta \circ \mathrm{div}_\sigma^X (v) = \int_X \mathrm{div}_\sigma^X v d\gamma. \tag{27}$$

(iv) The Poisson integral extends to $L_{ad}^2(\Upsilon^X \times X, TX)$ as a continuous operator with the relation

$$\mathrm{div}_{\pi_\sigma}^\Upsilon (v) = \delta \circ \mathrm{div}_\sigma^X (v) = \int_X \mathrm{div}_\sigma^X v d\gamma, \qquad v \in L_{ad}^2(\Upsilon^X \times X, TX), \tag{28}$$

and the equality

$$E_{\pi_\sigma}\left[\left(\delta \circ \mathrm{div}_\sigma^X (v)\right)^2\right] = E_{\pi_\sigma}\left[\|\mathrm{div}_\sigma^X v\|_{L^2(X,\sigma)}^2\right], \quad v \in L_{ad}^2(\Upsilon^X \times X, TX). \tag{29}$$

Proof. (i) We assume that v is of the form $v = G \nabla^X h$ with $h \in \mathcal{C}_c^\infty(X)$ and $G \in \mathcal{FC}_b^\infty$. We have $G \nabla^\Upsilon F = \nabla^\Upsilon (FG) - F \nabla^\Upsilon G$, hence

$$\begin{aligned} & E_\pi\left[G < \nabla^\Upsilon F, \nabla^X h >_{L_\gamma^2(TX)}\right] \\ = \ & E_\pi\left[< \nabla^\Upsilon (FG), \nabla^X h >_{L_\gamma^2(TX)} - F < \nabla^\Upsilon G, \nabla^X h >_{L_\gamma^2(TX)}\right], \end{aligned}$$

i.e.

$$E_\pi\left[F \mathrm{div}_\pi^\Upsilon v\right] = E_\pi\left[F\left(G \mathrm{div}_\pi^\Upsilon \nabla^X h - < \nabla^\Upsilon G, \nabla^X h >_{L_\gamma^2(TX)}\right)\right], \quad F \in \mathcal{FC}_b^\infty,$$

hence (25).

(ii) Cor. 2 shows that under π_σ,

$$E_{\pi_\sigma}\left[(\nabla^\Upsilon F, \nabla^X h)_{L_\gamma^2(TX)}\right] \ = \ E_{\pi_\sigma}\left[< \nabla^X DF, \nabla^X h >_{L_\sigma^2(TX)}\right], \quad h \in \mathcal{C}_c^\infty(X),$$

i.e. since $\int_X \mathrm{div}_\sigma^X \nabla^X h d\sigma = 0$:

$$E_{\pi_\sigma}\left[F \mathrm{div}_{\pi_\sigma}^\Upsilon (v)\right] = E_{\pi_\sigma}\left[F \delta\left(\mathrm{div}_\sigma^X \nabla^X h\right)\right] = E_{\pi_\sigma}\left[F \int_X \mathrm{div}_\sigma^X \nabla^X h d\gamma\right], \quad F \in \mathcal{FC}_b^\infty,$$

hence

$$\operatorname{div}_{\pi_\sigma}^{\Upsilon}(v) = \int_X \operatorname{div}_\sigma^X \nabla^X h d\gamma,$$

which implies (26) from (25).

(iii) If $v \in \mathcal{V}$ is ∇^Υ-adapted then $< \nabla^X h(x), \nabla_x^\Upsilon G >_{T_x X} = 0$, $x \in X$, hence $\operatorname{trace}_x \nabla_x^\Upsilon v_x = 0$, which proves (27) from (25) by linearity.

(iv) For all D-adapted vector $v \in \mathcal{V}$ we have

$$\begin{aligned}
&\int_X \int_X D_x \operatorname{div}_\sigma^X v(y) D_y \operatorname{div}_\sigma^X v(x) \sigma(dx)\sigma(dy) \\
&= \sum_{i,j=1}^n < \operatorname{div}_\sigma^X \nabla^X h_i, DG_j >_{L^2(X,\sigma)} < \operatorname{div}_\sigma^X \nabla^X h_j, DG_i >_{L^2(X,\sigma)} \\
&= \sum_{i,j=1}^n < \nabla^X h_i, \nabla^X DG_j >_{L^2_\sigma(TX)} < \nabla^X h_j, \nabla^X DG_i >_{L^2_\sigma(TX)} = 0,
\end{aligned}$$

hence the Skorohod isometry (19) shows that (29) holds, and $\delta \circ \operatorname{div}_\sigma^X$ extends to a continuous operator on $L^2_{ad}(\Upsilon^X \times X, TX)$, which proves (28) by density.

□

If $X = \mathbb{R}$, σ being the Lebesgue measure, we find the classical result

$$\operatorname{div}_{\pi_\sigma}^{\Upsilon}(v) = \delta \circ \operatorname{div}_\sigma^X(v) = -\delta(\nabla^X v) = -\int_{\mathbb{R}_+} \nabla^X v(t) dN_t,$$

for $\nabla^X v \in \mathcal{V}$, $v = 0$ on $\mathbb{R}_-$, adapted to the canonical filtration of the standard Poisson process $(N_t)_{t \in \mathbb{R}_+}$ on $\mathbb{R}_+$, cf. [10], [15], [19].

Remark 7 *By duality, (28) shows the projection identity*

$$E[(\nabla^\Upsilon F, v)_{L^2_\gamma(TX)}] = E[(\nabla^X DF, v)_{L^2(X,\sigma)}], \qquad F \in \mathcal{FC}_b^\infty, \tag{30}$$

for all ∇^Υ-adapted vector $v \in \mathcal{V}$.

In case $X = \mathbb{R}_+$, (30) can be interpreted as an equality between the adapted projections of $\nabla^\Upsilon F$ and $\nabla^X DF$ with the respect to the canonical Poisson filtration, cf. [19].

7 The Gaussian case

The aim of this section is to examine the Gaussian counterpart of the above construction. We consider a centered random Gaussian measure W with variance σ on the Riemannian manifold X, on a probability space (Ω, μ). For $h \in L^2(X, \sigma)$, the first order stochastic integral of h is denoted by

$$I_1(h) = \int_X h(x) W(dx).$$

If $\{h_1, \ldots, h_d\}$ is orthonormal in $L^2(X, \sigma)$ and $n_1 + \cdots + n_d = n$, let

$$I_n(h_1^{\odot n_1} \odot \cdots \odot h_d^{\odot n_d}) = H_{n_1}(I_1(h_1)) \cdots H_{n_d}(I_1(h_d)),$$

where $(H_n)_{n \in \mathbb{N}}$ denotes the family of Hermite polynomials. The isometry property

$$(I_n(f_n), I_m(g_m))_{L^2(\Omega,\mu)} = 1_{\{n=m\}} n! (f_n, g_m)_{L^2(X,\sigma)^{\odot n}}$$

allows to define $I_n(f_n)$ for any f_n in the symmetric tensor product $L^2(X, \sigma)^{\odot n}$, and to identify $L^2(\Omega, \mu)$ to the Fock space $\Gamma(L^2(X, \sigma))$. Under this identification the annihilation operator D becomes a derivation operator. Let δ denote the adjoint of D, also called the Skorohod integral, and defined as in (9). The construction presented here does not rely on Brownian motion as in [7], [12]. It is in fact the direct analogue of the construction presented above on configuration spaces. Let ∇^W be the gradient defined on

$$\mathcal{S} = \{ f(I_1(h_1), \ldots I_1(h_n)) \; : \; f \in \mathcal{C}_b^\infty(\mathbb{R}^n), \quad h_1, \ldots, h_n \in \mathcal{C}_c^\infty(X) \}$$

by

$$\nabla_x^W F = \sum_{i=1}^{i=n} \partial_i f(I_1(h_1), \ldots, I_1(h_n)) \nabla^X h_i(x), \quad x \in X,$$

for $F \in \mathcal{S}$ of the form $F = f(I_1(h_1), \ldots, I_1(h_n))$, i.e. for $u = \nabla^X h$, $h \in \mathcal{C}_c^\infty(X)$:

$$\begin{aligned}
(\nabla^W F, u)_{L^2_\sigma(TX)} &= \sum_{i=1}^{i=n} \partial_i f(I_1(h_1), \ldots, I_1(h_n)) (\nabla^X h_i, u)_{L^2_\sigma(TX)} \\
&= \sum_{i=1}^{i=n} \partial_i f(I_1(h_1), \ldots, I_1(h_n)) (h_i, \operatorname{div}_\sigma^X u)_{L^2(X,\sigma)},
\end{aligned}$$

or more formally:

$$(\nabla^W F, u)_{L^2_\sigma(TX)} = \lim_{\varepsilon \to 0} \frac{F(dW + \operatorname{div}_\sigma^X u d\sigma) - F(dW)}{\varepsilon}.$$

Given the relation

$$D_x f(I_1(h_1), \ldots, I_1(h_n)) = \sum_{i=1}^{i=n} \partial_i f(I_1(h_1), \ldots, I_1(h_n)) h_i(x), , \quad x \in X,$$

in the Gaussian case the relation between ∇^W, ∇^X and D becomes simply

$$\nabla^W_x F = \nabla^X D_x F, \qquad x \in X, \ F \in \mathcal{S},$$

in other terms, according to the terminology of Sect. 3.1 of [3], on the Wiener space D coincides with the flat gradient, which is not the case on the Poisson space. The following result applies in particular to the Laplace-Beltrami operator $H^X_\sigma = \mathrm{div}^X_\sigma \nabla^X$.

Theorem 1 *If H^X is an operator on $\mathcal{C}^\infty_c(X)$ with carré du champ Γ^X, then $d\Gamma(H^X)$ is a diffusion operator with carré du champ*

$$\Gamma^W(F,G) = -\frac{1}{2}((DF, H^X DG) + (DG, H^X DF)), \quad F, G \in \mathcal{S}.$$

If H^X is conservative on $\mathcal{C}^\infty_c(X)$ under σ, then $\Gamma^W(F,G) = \Gamma^X(DF,DG)$, $F, G \in \mathcal{S}$.

Proof. We have

$$\begin{aligned}
d\Gamma(H^X)(FG) &= \delta\left(H^X D(FG)\right) = \delta\left(FH^X DG + GH^X DF\right) \\
&= F\delta\left(H^X DG\right) + G\delta\left(H^X DF\right) - (DF, H^X DG)_{L^2(X,\sigma)} - (DG, H^X DF)_{L^2(X,\sigma)} \\
&= F d\Gamma(H^X)G + G d\Gamma(H^X)F - (DF, H^X DG)_{L^2(X,\sigma)} - (DG, H^X DF)_{L^2(X,\sigma)},
\end{aligned}$$

hence

$$\begin{aligned}
\Gamma^W(F,G) &= \frac{1}{2}\left(d\Gamma(H^X)(FG) - F d\Gamma(H^X)G - G d\Gamma(H^X)F\right) \\
&= -\frac{1}{2}\left((DF, H^X DG)_{L^2(X,\sigma)} + (H^X DF, DG)_{L^2(X,\sigma)}\right).
\end{aligned}$$

The locality property is satisfied since D is a derivation operator:

$$\begin{aligned}
\Gamma^W(F^2,G) &= (D(F^2), H^X DG)_{L^2(X,\sigma)} + (H^X D(F^2), DG)_{L^2(X,\sigma)} \\
&= 2F\left((DF, H^X DG)_{L^2(X,\sigma)} + (H^X DF, DG)_{L^2(X,\sigma)}\right) = 2F\Gamma^W(F,G).
\end{aligned}$$

If H^X is conservative, then $(DF, H^X DG) + (H^X DF, DG) = \Gamma^X(DF,DG)$, i.e. $\Gamma^W(F,G) = \Gamma^X(DF,DG)$, $F, G \in \mathcal{S}$.

$\square$

We note that $d\Gamma(H^X)$ is a derivation (i.e. $\Gamma^W = 0$) if H^X is antisymmetric.
The relationship between div^W_μ and the stochastic integral can be described as follows. The space $L^2_{ad}(\Omega \times X, TX)$ of adapted vectors is defined as in the Poisson case, by completion of simple adapted vectors $u : X \times \Omega \longrightarrow TX$ in $\mathcal{V}$ of the form

$$u = \sum_{i=1}^{i=n} F_i \nabla^X h_i, \quad F_1, \ldots, F_n \in \mathcal{S}, \ h_1, \ldots, h_n \in \mathcal{C}^\infty_c(X),$$

with $\forall x \in X$, $\omega \in \Omega$, and $i, j \in \{1, \ldots, n\}$:

$$< \nabla^X h_i(x), \nabla_x^W F_j(\omega) >_{T_x X} = 0 \quad \text{or} \quad < \nabla^X h_j(x), \nabla_x^W F_i(\omega) >_{T_x X} = 0.$$

Note that in the Gaussian case, the ∇^W-adapted vectors are D-adapted since $\nabla^W = \nabla^X D$. As in the Poisson case we have the following result.

Proposition 10 *(i) We have for $v \in \mathcal{V}$:*

$$\operatorname{div}_\mu^W v = \int_X \operatorname{div}_\sigma^X v dW - \int_X \operatorname{trace}_x \nabla_x^W v(x) \sigma(dx). \tag{31}$$

(ii) For $v \in L^2_{ad}(\Omega \times X, TX)$ we have the isometry

$$E_\mu \left[\left(\delta \circ \operatorname{div}_\sigma^X (v) \right)^2 \right] = E_\mu \left[\| \operatorname{div}_\sigma^X v \|^2_{L^2(X,\sigma)} \right], \quad v \in L^2_{ad}(\Upsilon^X \times X, TX),$$

and the relation

$$\operatorname{div}_\mu^W v = \delta \circ \operatorname{div}_\sigma^X v = \int_X (\operatorname{div}_\sigma^X v) dW. \tag{32}$$

Proof. This proof is close to that of Prop. 9.
(i) We choose v of the form $v = G \nabla^X f$ with $f \in \mathcal{C}_c^\infty(X)$ and $G \in \mathcal{S}$. We have

$$\begin{aligned} & E_\mu \left[G < \nabla^W F, \nabla^X f >_{L^2_\sigma(TX)} \right] \\ = \; & E_\mu \left[< \nabla^W (FG), \nabla^X f >_{L^2_\sigma(TX)} - F < \nabla^W G, \nabla^X f >_{L^2_\sigma(TX)} \right], \end{aligned}$$

i.e.

$$E_\mu \left[F \operatorname{div}_\mu^W v \right] = E_\mu \left[FG \operatorname{div}_\mu^W \nabla^X f \right] - E_\mu \left[F < \nabla^W G, \nabla^X f >_{L^2_\sigma(TX)} \right], \quad F \in \mathcal{S},$$

hence (25). By duality we have

$$\operatorname{div}_\mu^W \left(\nabla^X f \right) = \delta(\operatorname{div}_\sigma^X \nabla^X f) = I_1(\operatorname{div}_\sigma^X \nabla^X f) = \int_X \operatorname{div}_\sigma^X \nabla^X f dW,$$

hence (31).
(ii) If $v \in \mathcal{V}$ is adapted then $\operatorname{trace}_x \nabla_x^W v_x = 0$, hence (32) holds for simple adapted processes. Its extension by density to $L^2_{ad}(\Omega \times X, TX)$ follows as in the Poisson case from the Skorohod isometry (19) which is well-known to hold also on the Wiener space.

□

If $X = \mathbb{R}_+$ this corresponds to the relation

$$\operatorname{div}_\mu^W (v) = \delta(\operatorname{div}_\sigma^X v) = -\delta(\nabla^X v) = -\int_{\mathbb{R}_+} \nabla^X v(t) dW_t,$$

where $(W_t)_{t \in \mathbb{R}_+}$ denotes the standard Wiener process on $\mathbb{R}_+$ and $\nabla^X v \in L^2(\Omega, \mu) \otimes L^2(\mathbb{R}_+, \sigma)$ is adapted.

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