Conditional Stein approximation for Itô and Skorohod integrals

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Abstract

We derive conditional Edgeworth-type expansions for Skorohod and Itô integrals with respect to Brownian motion, based on cumulant operators defined by the Malliavin calculus. As a consequence we obtain conditional Stein approximation bounds for multiple stochastic integrals and quadratic Brownian functionals.

Key words: Stein method; Malliavin calculus; Edgeworth expansions; stochastic integral; conditioning; quadratic Brownian functionals.
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1 Introduction

Let $(B_t)_{t\in[0,T]}$ be a d-dimensional Brownian motion generating the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ on the Wiener space Ω. It has been shown in Theorem 2.1 of [4], that given B_T, the stochastic integral $\int_0^T AB_s dB_s$ is Gaussian $\mathcal{N}\left(0, \int_0^T |AB_s|^2\, ds\right)$-distributed given $\int_0^T |AB_s|^2\, ds$ when the $d\times d$ matrix A is skew-symmetrix, as an extension of results of [14] in the case of Lévy's stochastic area with $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. On the other hand, it has recently been shown in [12] that the distribution of $\int_0^T AB_s dB_s$ given $\int_0^T |AB_s|^2 \, ds$ is also Gaussian $\mathcal{N}\left(0, \int_0^T |AB_s|^2 \, ds\right)$ when A is a 2-nilpotent $d \times d$ matrix, in connection with results of [13] showing that the filtration $(\mathcal{F}_t^k)_{t\in[0,T]}$ of $t \mapsto \int_0^t AB_s dB_s$ is generated by k independent Brownian motions, where k is the number of distinct eigenvalues of $A^\top A$.

More generally, this type of result has been shown to hold in [12] for stochastic integrals of the form $\int_0^T u_t dB_t$ where $(u_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t)$-adapted process, under conditions formulated in terms of the Malliavin calculus, based on the cumulant-moment formulas of [9], [10]. Namely, sufficient conditions on the process $(u_t)_{t\in[0,T]}$ have been given for $\int_0^T u_t dB_t$ to be Gaussian $\mathcal{N}\left(0, \int_0^T |u_t|^2 dt\right)$-distributed given $\int_0^T |u_t|^2 dt$, cf. Theorem 2 therein.

In this paper, using the Malliavin-Stein method on the Wiener space we derive conditional estimates on the distance between the law of $\int_0^T u_t dB_t$ and the Gaussian $\mathcal{N}\left(0, \int_0^T |u_t|^2 dt\right)$ distribution given $\int_0^T |u_t|^2 dt$. For this, we rely on conditional Edgeworth type expansions for random variables represented as the Itô stochastic integral of $(u_t)_{t\in[0,T]}$ with respect to $(B_t)_{t\in[0,T]}$, extending results of [11] to a conditional setting. This approach relies on properties of the Skorohod integral operator δ, which coincides with the Itô stochastic integral with respect to d-dimensional Brownian motion on the square-integrable adapted processes.

Letting $H = L^2(\mathbb{R}_+; \mathbb{R}^d)$, we consider the standard Sobolev spaces of real-valued, resp. H-valued, functionals $\mathbb{D}_{p,k}$, resp. $\mathbb{D}_{p,k}(H)$, $p, k \geq 1$, for the Malliavin gradient D on the Wiener space, cf. § 1.2 of [7] for definitions. Recall that the Skorohod operator δ is the adjoint of the gradient D through the duality relation

$$E[F\delta(v)] = E[\langle DF, v\rangle_H], \quad F \in \mathbb{D}_{2,1}, \quad v \in \mathbb{D}_{2,1}(H), \tag{1.1}$$

and we have the commutation relation

$$D_t \delta(u) = u(t) + \delta(D_t u), \qquad t \in \mathbb{R}_+, \tag{1.2}$$

provided that $u \in \mathbb{D}_{2,1}(H)$ and $D_t u \in \mathbb{D}_{2,1}(H)$, dt-$a.e.$, cf. Proposition 1.3.2 of [7]. In the sequel we let $\langle h, h\rangle := \langle h, h\rangle_H$ and $\|h\| := \|h\|_H$, $h \in H$.

First order conditional duality and expansion

The duality relation (1.1) shows that we have

$$E\left[F\delta(u)f(\delta(u))\right] = E\left[F\langle u,u\rangle f'(\delta(u))\right] + E\left[\langle u, DF\rangle f(\delta(u))\right] + E\left[Ff'(\delta(u))\langle u, \delta(Du)\rangle\right],$$

for $f \in \mathcal{C}_b^1(\mathbb{R})$, provided that $u \in \mathbb{D}_{2,2}(H)$. Applying this relation to $F = g(\langle u,u\rangle)$ where $g : (0,\infty) \to (0,\infty)$ is in $\mathcal{C}_b^1((0,\infty))$, under the condition $\langle u, (Du)u\rangle = 0$ we have

$$\begin{aligned}
&E\left[F\delta(u)f(\delta(u))\right] \\
&= E\left[F\langle u,u\rangle f'(\delta(u))\right] + E\left[\langle u, D\langle u,u\rangle\rangle f(\delta(u))g'(\langle u,u\rangle)\right] + E\left[Ff'(\delta(u))\langle u,\delta(Du)\rangle\right] \\
&= E\left[F\langle u,u\rangle f'(\delta(u))\right] + 2E\left[\langle u,(Du)u\rangle\rangle f(\delta(u))g'(\langle u,u\rangle)\right] + E\left[F\langle u,\delta(Du)\rangle f'(\delta(u))\right] \\
&= E\left[F\langle u,u\rangle f'(\delta(u))\right] + E\left[F\langle u,\delta(Du)\rangle f'(\delta(u))\right],
\end{aligned}$$

which yields

$$E_{|u|}[\delta(u)f(\delta(u)) - \langle u,u\rangle f'(\delta(u))] = E_{|u|}[\langle u,\delta(Du)\rangle f'(\delta(u))], \qquad (1.3)$$

for $u \in \mathbb{D}_{2,1}(H)$, $F \in \mathbb{D}_{2,1}$ and $f \in \mathcal{C}_b^1(\mathbb{R})$, where

$$E_{|u|}[F] := E[F \mid \langle u,u\rangle]$$

denotes the conditional expectation given $\langle u,u\rangle$.

Let now $\mathcal{N}(0, g(\|u\|))$ denote a centered Gaussian random variable with variance $g(\|u\|)$, where $g : (0,\infty) \to (0,\infty)$ is a measurable function. Applying the above relation (1.3) to the solution f_x of the Stein equation

$$\mathbf{1}_{(-\infty,x]}(z) - \Phi_{g(\|u\|)}(x) = g(\|u\|)f_x'(z) - zf_x(z), \qquad z \in \mathbb{R}, \qquad (1.4)$$

satisfying the bounds $\|f_x\|_\infty \le \sqrt{2\pi}/4$ and $\|f_x'\|_\infty \le 1/\sqrt{g(\|u\|)}$, cf. Lemma 2.2-($v$) of [3], yields the conditional expansion

$$P(\delta(u) \le x \mid \|u\|) = \Phi_{g(\|u\|)}(x) + E_{|u|}[(g(\|u\|) - \langle u,u\rangle)f_x'(\delta(u))] - E\left[\langle u,\delta(Du)\rangle f_x'(\delta(u))\right],$$

$x \in \mathbb{R}$, around the Gaussian cumulative distribution function $\Phi_{g(\|u\|)}(x)$, with $u \in \mathbb{D}_{2,1}(H)$.

In Section 2 we will expand this approach to Edgeworth type expansion of all orders, based on a family of cumulant operators that are associated to the process $(u_t)_{t\in[0,T]}$. We refer to [1], [5], [2] for other approaches to Edgeworth expansions via the Stein method and the Malliavin calculus.

In Section 3, we derive conditional Stein approximation bounds for the distance between $\delta(u)$ and the Gaussian distribution with variance $g(\|u\|)$ Section 4 treats the case of double stochastic integrals, which includes the quadratic functional $\int_0^T AB_s dB_s$ as a particular case.

2 Conditional Edgeworth type expansions

Given $u \in \mathbb{D}_{2,1}(H)$ and $k \geq 1$, we define the operator composition $(Du)^k$ in the sense of matrix powers with continuous indices, namely, $(Du)^k$ denotes the random operator on H almost surely defined by

$$(Du)^k h_s = \int_0^\infty \cdots \int_0^\infty (D_{t_k} u_s D_{t_{k-1}} u_{t_k} \cdots D_{t_1} u_{t_2}) h_{t_1} dt_1 \cdots dt_k, \quad s \in \mathbb{R}_+, \quad h \in H, \tag{2.1}$$

cf. e.g. § 7 of [10], [9], [8] for details. The adjoint D^*u of Du on H satisfies

$$\langle (Du)v, h\rangle = \langle v, (D^*u)h\rangle, \qquad h, v \in H,$$

with $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_H$, and is given by

$$(D^*u)v_s = \int_0^\infty (D_s u_t) v_t dt, \qquad s \in \mathbb{R}_+, \quad v \in L^2(W; H).$$

The next proposition reformulates Proposition 2.1 of [11] in a conditional setting given the random value of $\langle u, u\rangle$, whereas the case where $\langle u, u\rangle$ is deterministic (random isometries) has been treated in [11] based on the relation $\langle u, (Du)^k u\rangle = \langle (Du)^{k-1} u, D\langle u, u\rangle\rangle/2$, $u \in \mathbb{D}_{k+2,1}(H)$, $k \geq 1$.

Proposition 2.1 *Let $n \geq 1$ and assume that $u \in \mathbb{D}_{k,2}(H)$ for all $k = 1, \ldots, n+2$ and $\langle u, (Du)^k u\rangle = 0$ for $k = 1, \ldots, n+1$. Then for all $f \in \mathcal{C}_b^{n+1}(\mathbb{R})$ we have*

$$E_{|u|}[\delta(u) f(\delta(u))] = E_{|u|}\left[\left(\langle u, u\rangle + \mathrm{trace}\,(Du)^2\right) f'(\delta(u))\right] \tag{2.2}$$
$$+ \sum_{k=2}^{n} E_{|u|}\left[\langle D^* u, D((Du)^{k-1}u)\rangle f^{(k)}(\delta(u))\right] + E_{|u|}\left[\langle (Du)^n u, \delta(Du)\rangle f^{(n+1)}(\delta(u))\right].$$

Proof. From Proposition 2.1 of [11] we have

$$E\left[F\delta(u) f(\delta(u))\right] = \sum_{k=0}^{n} E\left[f^{(k)}(\delta(u))\Gamma_{k+1}^u F\right] + E\left[F\langle (Du)^n u, \delta(Du)\rangle f^{(n+1)}(\delta(u))\right], \tag{2.3}$$

where

$$\Gamma_k^u : \mathbb{D}_{2,1} \longrightarrow L^2(\Omega), \qquad k \geq 1,$$

is defined for $u \in \mathbb{D}_{k,2}(H)$, by $\Gamma_1^u F := \langle u, DF\rangle$ and

$$\Gamma_k^u F := F\langle (Du)^{k-2}u, u\rangle + F\langle D^* u, D((Du)^{k-2}u)\rangle + \langle (Du)^{k-1}u, DF\rangle, \qquad k \geq 2.$$

Next, for F of the form $F = g\left(\int_0^T |u_t|^2 dt\right)$, $g \in \mathcal{C}_b^1(\mathbb{R})$ and $k \geq 1$ we have:

$$\begin{aligned}
&\Gamma_k^u F \\
&= \mathbf{1}_{\{k=2\}}\langle u, u\rangle g\left(\int_0^T |u_t|^2 dt\right) + g'\left(\int_0^T |u_t|^2 dt\right)\int_0^T \langle D_t \int_0^T |u_s|^2 ds, (Du)^{k-1}u_t\rangle_{\mathbb{R}^d} dt \\
&\quad + g\left(\int_0^T |u_t|^2 dt\right)\langle D^* u, D((Du)^{k-2}u)\rangle \\
&= \mathbf{1}_{\{k=2\}}\langle u, u\rangle g\left(\int_0^T |u_t|^2 dt\right) + 2g'\left(\int_0^T |u_t|^2 dt\right)\int_0^T \langle u_s, (Du)^k u_s\rangle_{\mathbb{R}^d} ds \\
&\quad + \mathbf{1}_{\{k\geq 2\}} g\left(\int_0^T |u_t|^2 dt\right)\langle D^* u, D((Du)^{k-2}u)\rangle \\
&= \mathbf{1}_{\{k=2\}}\langle u, u\rangle F + \mathbf{1}_{\{k\geq 2\}}\langle D^* u, D((Du)^{k-2}u)\rangle F.
\end{aligned} \tag{2.4}$$

Hence from (2.3) we find

$$E\left[F\delta(u) f(\delta(u))\right] = E\left[F\left(\langle u, u\rangle + \mathrm{trace}\,(Du)^2\right) f'(\delta(u))\right]$$
$$+ \sum_{k=2}^{n} E\left[F\langle D^* u, D((Du)^{k-1}u)\rangle f^{(k)}(\delta(u))\right] + E\left[F\langle (Du)^n u, \delta(Du)\rangle f^{(n+1)}(\delta(u))\right],$$

where we used the relation

$$\Gamma_2^u \mathbf{1} = \langle u, u\rangle + \langle D^* u, Du\rangle_{H\otimes H} = \langle u, u\rangle + \mathrm{trace}(Du)^2. \tag{2.5}$$

$\square$

Proposition 2.1 also covers the following particular settings.

(i) Quasi-nilpotent processes. Given $n \geq 0$, when $\mathrm{trace}(Du)^k = 0$ for all $k = 2, \ldots, n+1$, under the conditions of Proposition 2.1 we have

$$E_{|u|}[\delta(u) f(\delta(u))] = \langle u, u\rangle E_{|u|}[f'(\delta(u))] + E_{|u|}\left[f^{(n+1)}(\delta(u))\langle (Du)^n u, \delta(Du)\rangle\right].$$

When $\mathrm{trace}(Du)^k = 0$ for all $k \geq 2$, this recovers the conditional Gaussian integration by parts formula

$$E_{|u|}[\delta(u) f(\delta(u))] = \langle u, u\rangle E_{|u|}[f'(\delta(u))],$$

showing that $\delta(u)$ has the distribution $\mathcal{N}\left(0, \int_0^T |u_t|^2 dt\right)$ given $\int_0^T |u_t|^2 dt$. This setting includes the particular cases where $(u_t)_{t\in\mathbb{R}_+}$ is an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted process, cf. e.g. Lemma 3.5 of [8] and references therein, in which case $\delta(u)$ coincides with the Itô integral of u, cf. Proposition 1.3.11 of [7].

(ii) Multiple stochastic integrals. Taking $u_t := I_{m-1}(f_m(*, t))$, where $m \geq 1$ and f_m is a symmetric square-integrable function on $\mathbb{R}_+^m$, we have $\delta(u) = I_m(f_m)$ and

$$\delta(D_t u) = (m-1) I_{m-1}(f_m(*, t)) = (m-1) u_t, \qquad t \in \mathbb{R}_+. \tag{2.6}$$

Hence, under the conditions of Proposition 2.1 applied to $u_t = I_{m-1}(f_m(*, t))$, we get

$$\begin{aligned} & E_{|u|}[I_m(f_m) f(I_m(f_m))] \\ = \ & \langle u, u\rangle E_{|u|}[f'(I_m(f_m))] + \sum_{k=1}^n E_{|u|}\left[\langle D^* u, D((Du)^{k-1} u)\rangle f^{(k)}(I_m(f_m))\right]. \end{aligned}$$

Remark 2.2 *By replacing F in Equation* (2.4) *of the proof of Proposition 2.1 with F of the form*

$$F = g\left(\int_{a_i}^{b_i} |u_t|^2 dt, \ldots, \int_{a_d}^{b_d} |u_t|^2 dt\right), \quad 0 \leq a_i \leq b_i \leq T, \quad i = 1, \ldots, d,$$

where $g \in \mathcal{C}_b^1(\mathbb{R}^d)$, we can rewrite (2.2) by conditioning with respect to $(u_t)_{t\in[0,T]}$, under the stronger condition $\langle u_t, (Du)^k u_t\rangle_{\mathbb{R}^d} = 0$, $t \in [0,T]$, $k = 1,\ldots,n+1$.

3 Conditional Stein approximation

From now on we work with $d = 1$ and a one-dimensional Brownian motion $(B_t)_{t\in\mathbb{R}_+}$. Given $h : \mathbb{R} \to \mathbb{R}$ an absolutely continuous function with bounded derivative, the functional equation

$$h(z) - E_{|u|}\big[h\left(\mathcal{N}_{g(\|u\|)}\right)\big] = g(\|u\|)f'(z) - zf(z), \qquad z \in \mathbb{R}, \tag{3.1}$$

has a solution $f_h \in \mathcal{C}_b^1(\mathbb{R})$ which is twice differentiable and satisfies the bounds

$$\|f_h'\|_\infty \le \|h'\|_\infty/\sqrt{g(\|u\|)} \quad \text{and} \quad \|f_h''\|_\infty \le 2\|h'\|_\infty/g(\|u\|), \qquad x \in \mathbb{R},$$

cf. Lemma 1.2-(v) of [6]. Let

$$d_{|u|}(F,G) = \sup_{h\in\mathcal{L}} |E_{|u|}[h(F)] - E_{|u|}[h(G)]\,|$$

denote the Wasserstein distance between the conditional laws of F and G given $\|u\|$, where $\mathcal{L}$ denotes the class of 1-Lipschitz functions.

Proposition 3.1 *Let $u \in \bigcap_{k=1}^{3} I\!D_{k,2}(H)$, such that $\langle u, (Du)u\rangle = \langle u, (Du)^2 u\rangle = 0$. We have*

$$d_{|u|}(\delta(u), \mathcal{N}_{g(\|u\|)}) \le \frac{1}{\sqrt{g(\|u\|)}}\left|g(\|u\|) - \mathrm{Var}_{|u|}[\delta(u)]\right| + \frac{2}{g(\|u\|)} E_{|u|}[|\langle (Du)u, \delta(Du)\rangle|]\,. \tag{3.2}$$

Proof. Applying Proposition 2.1 with $n = 1$ shows that

$$E_{|u|}[\delta(u)f(\delta(u))] = E_{|u|}\big[\left(\langle u,u\rangle + \mathrm{trace}\,(Du)^2\right) f'(\delta(u))\big] + E_{|u|}[\langle (Du)u, \delta(Du)\rangle f''(\delta(u))]\,,$$

hence for any continuous function $h : \mathbb{R} \to [0,1]$, denoting by f_h the solution to (3.1) we have

$$|E_{|u|}[h(\delta(u))] - E_{|u|}\big[h(\mathcal{N}_{g(\|u\|)})\big]\,|$$

$$= |E_{|u|}[\delta(u) f_h(\delta(u)) - g(\|u\|) f_h'(\delta(u))] |$$
$$= |E_{|u|}\big[(\langle u,u\rangle - g(\|u\|))\, f_h'(\delta(u)) + \mathrm{trace}\,(Du)^2 f_h'(\delta(u)) + \langle (Du)u, \delta(Du)\rangle f_h''(\delta(u))\big] |$$
$$\leq \frac{\|h'\|_\infty}{\sqrt{g(\|u\|)}} \left|\langle u,u\rangle - g(\|u\|) + E_{|u|}[|\mathrm{trace}\,(Du)^2|]\right| + 2\frac{\|h'\|_\infty}{g(\|u\|)} E_{|u|}[|\langle (Du)u, \delta(Du)\rangle|]\,,$$

which yields (3.2) by (2.5) and the conditional Skorohod isometry

$$\mathrm{Var}_{|u|}[\delta(u)] = E_{|u|}\big[\delta(u)^2\big] = \langle u,u\rangle + E_{|u|}\big[\mathrm{trace}\,(Du)^2\big]\,, \tag{3.3}$$

that follows from (2.2) with $f(x) = x$ and $n = 1$. □

When $(u_t)_{t\in\mathbb{R}_+}$ is a quasi-nilpotent processes, and in particular when $(u_t)_{t\in\mathbb{R}_+}$ is $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted process, we obtain the following corollary.

Corollary 3.2 *Let* $u \in \bigcap_{k=1}^{3} \mathbb{D}_{k,2}(H)$*, such that* $\langle u, (Du)^2 u\rangle = \langle u, (Du)u\rangle = 0$ *and* $\mathrm{trace}\,(Du)^2 = 0$*. We have*

$$d_{|u|}(\delta(u), \mathcal{N}_{g(\|u\|)}) \leq \frac{1}{\sqrt{g(\|u\|)}} |\langle u,u\rangle - g(\|u\|)| + \frac{2}{g(\|u\|)} E_{|u|}[|\langle (Du)u, \delta(Du)\rangle|]\,.$$

Proof. This follows from Proposition 3.1 and the isometry (3.3). □

Note that from Proposition 2.1 we have

$$d_{|u|}(\delta(u), \mathcal{N}_{g(\|u\|)}) = |\sqrt{\langle u,u\rangle} - \sqrt{g(\|u\|)}| \leq \frac{1}{\sqrt{g(\|u\|)}} |\langle u,u\rangle - g(\|u\|)|$$

if, in addition to $\mathrm{trace}\,(Du)^2 = 0$, the condition $\langle u, (Du)^k u\rangle = 0$ holds for all $k \geq 1$.

Regarding multiple stochastic integrals, we have the following corollary.

Corollary 3.3 *Given* $f_m \in L^2(\mathbb{R}_+^m)$ *a symmetric function in* $m \geq 1$ *variables, let* $u_t := I_{m-1}(f_m(*, t))$*,* $t \in \mathbb{R}_+$*, and assume that* $\langle u, (Du)^2 u\rangle = \langle u, (Du)u\rangle = 0$*. Then we have*

$$d_{|u|}(I_m(f_m), \mathcal{N}_{g(\|u\|)}) \leq \frac{1}{\sqrt{g(\|u\|)}} \left|g(\|u\|) - \mathrm{Var}_{|u|}[I_m(f_m)]\right|. \tag{3.4}$$

Proof. We have $\langle (Du)u, \delta(Du)\rangle = (m-1)\langle (Du)u, u\rangle = 0$, hence the conclusion follows from Proposition 3.1. □

In particular, Corollary 3.3 shows that, given $\|u\|$, the multiple stochastic integral $I_m(f_m)$ is Gaussian distributed with mean 0 and variance $g(\|u\|) := \mathrm{Var}_{|u|}[I_m(f_m)]$.

4 Double stochastic integrals

In this section we take $(u_t)_{t\in\mathbb{R}_+}$ of the form $u_t = I_1(f_2(*,t))$, $t \in \mathbb{R}_+$, where the function $f_2 \in L^2(\mathbb{R}_+^2)$ may not be symmetric in its two variables. In this case we have $\delta(u) = I_2(\tilde{f}_2)$, where $\tilde{f}_2$ is the symmetrization of f_2, with $D_s u_t = f_2(s,t)$ and $\delta(D_s u) = I_1(f_2(s,*))$, $s,t \in \mathbb{R}_+$.

Proposition 4.1 *The condition $\langle u,(Du)u\rangle = 0$ implies $\langle u,(Du)^2 u\rangle = 0$ and is equivalent to $(Du)^2 = 0$, i.e. to the vanishing of the contraction*

$$(f_2 \otimes_1 f_2)(t_1,t_2) := \int_0^\infty f_2(t_1,s) f_2(s,t_2) ds = 0, \qquad t_1,t_2 \in \mathbb{R}_+, \tag{4.1}$$

and in this case, we have

$$d_{|u|}(I_2(\tilde{f}_2), \mathcal{N}_{g(\|u\|)}) \leq \frac{1}{\sqrt{g(\|u\|)}} \left| g(\|u\|) - \langle u,u\rangle \right|.$$

In particular, $I_2(\tilde{f}_2)$ is Gaussian $\mathcal{N}_{\|u\|^2}$-distributed given $\langle u,u\rangle$.

Proof. By Itô calculus we have

$$\begin{aligned}\langle u,(Du)u\rangle &= \int_0^\infty \int_0^\infty I_1(f_2(*,s)) f_2(s,t) I_1(f_2(*,t)) ds dt \qquad (4.2)\\ &= \int_0^\infty \int_0^\infty \int_0^\infty f_2(r,s) f_2(s,t) f_2(r,t) dr ds dt + \int_0^\infty \int_0^\infty f_2(s,t) I_2(f_2(*,s) \otimes f_2(*,t)) ds dt,\end{aligned}$$

hence the condition $\langle u,(Du)u\rangle = 0$ is equivalent to the vanishing

$$\int_0^\infty \int_0^\infty f_2(t_1,s) f_2(s,t) f_2(t_2,t) ds dt = 0, \qquad t_1,t_2 \in \mathbb{R}_+, \tag{4.3}$$

i.e.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty h(t_1) f_2(t_1,s) f_2(s,t) f_2(t_2,t) g(t_2) ds dt dt_1 dt_2 = 0,$$

and taking

$$g(t_2) := \int_0^\infty f_2(r,t_2) h(r) dr,$$

we get

$$\int_0^\infty \int_0^\infty \int_0^\infty \left(\int_0^\infty f_2(t_1,s) f_2(s,t) h(t_1) dt_1 \right) \left(\int_0^\infty f_2(r,t_2) f_2(t_2,t) h(r) dr \right) ds dt dt_2$$

$$= \int_0^\infty \left(\int_0^\infty \int_0^\infty f_2(r,s) f_2(s,t) h(r) dr ds \right)^2 dt = 0,$$

hence the condition

$$\int_0^\infty f_2(r,s) f_2(s,t) ds = 0, \qquad r, t \in \mathbb{R}_+,$$

which becomes equivalent to (4.3). Consequently we have $\text{trace}(f_2)^2 = 0$, hence by (3.3) we get $\text{Var}_{|u|}\left[I_2(\tilde{f}_2)\right] = \langle u, u \rangle$. Moreover, we note that

$$\begin{aligned}
\langle (Du)u, \delta(Du) \rangle &= \int_0^\infty \int_0^\infty u_s D_s u_t \delta(D_t u) ds dt \\
&= \int_0^\infty \int_0^\infty I_1(f_2(*, s)) f_2(s,t) I_1(f_2(t, *)) ds dt \\
&= \int_0^\infty \int_0^\infty \int_0^\infty f_2(r,s) f_2(s,t) f_2(t,r) dr ds dt \\
&\quad + \frac{1}{2} I_2 \left(\int_0^\infty \int_0^\infty f_2(s,t)(f_2(*, s) f_2(t, \cdot) + f_2(\cdot, s) f_2(t, *)) ds dt \right) \\
&= 0,
\end{aligned}$$

and we conclude by Proposition 3.1. □

Examples of functions satisfying (4.1) can be given by choosing $(e_k)_{k \geq 1}$ an orthonormal system in $L^2([0,T])$ and letting

$$f_2(s,t) := \sum_{i,j=1}^{d} a_{i,j} e_i(s) e_j(t), \qquad s, t \in [0,T],$$

where $A = (a_{i,j})_{1 \leq i,j \leq d}$ is a 2-nilpotent $d \times d$ matrix. Note that the vanishing (4.1) of the contraction $f_2 \otimes_2 f_2$ implies $f_2 = 0$ if the function f_2 is symmetric in its two variables, similarly we have $A = 0$ if A is symmetric and 2-nilpotent.

Next, given $A = (a_{i,j})_{1 \leq i,j \leq d}$ an $d \times d$ matrix, we note that the quadratic functional $\int_0^T A W_s dW_s$ of the n-dimensional Brownian motion

$$W_t := (B_t, B_{T+t} - B_T, \ldots, B_{(n-1)T+t} - B_{(n-1)T}), \qquad t \in [0,T],$$

can be represented as

$$I_2(\tilde{f}_2) = \int_0^T A W_s dW_s, \tag{4.4}$$

where f_2 is the function

$$f_2(s,t) := \sum_{i,j=1}^{d} a_{i,j} \mathbf{1}_{[(j-1)T,(j-1)T+t-(i-1)T]}(s) \mathbf{1}_{[(i-1)T,iT]}(t), \quad s,t \in [0,nT],$$

as we have

$$\begin{aligned}
I_2(\tilde{f}_2) &= \sum_{i,j=1}^{d} a_{i,j} \int_0^T \int_0^{t+(i-1)T} \mathbf{1}_{[(j-1)T,(j-i)T+t+(i-1)T]}(s) dB_s dW_t^i \\
&\quad + \sum_{i,j=1}^{d} a_{i,j} \int_{(j-1)T}^{nT} \int_{(i-1)T}^{t\wedge(iT)} \mathbf{1}_{[(j-1)T,(j-i)T+s]}(t) dB_s dB_t \\
&= \sum_{1\leq j\leq i\leq n} a_{i,j} \int_0^T \int_{(j-1)T}^{t+(j-1)T} dB_s dW_t^i + \sum_{1\leq i<j\leq n} a_{i,j} \int_{(j-1)T}^{jT} \int_{(i-1)T}^{t} dB_s dB_t \\
&= \sum_{1\leq j\leq i\leq n} a_{i,j} \int_0^T \int_0^t dW_s^j dW_t^i + \sum_{1\leq i<j\leq n} a_{i,j} \int_0^T \int_0^t dW_s^j dW_t^i \\
&= \int_0^T AW_s dW_s,
\end{aligned}$$

with

$$\begin{aligned}
\int_0^{nT} (I_1(f_2(*,t))^2 dt &= \sum_{i=1}^{d} \int_{(i-1)T}^{iT} \left(\sum_{j=1}^{d} a_{i,j} (B_{(j-1)T+t-(i-1)T} - B_{(j-1)T}) \right)^2 dt \\
&= \int_0^T |AW_t|^2 \, dt.
\end{aligned}$$

Note that here, the process $(u_t)_{t\in[0,T]} = (I_1(f_2(*,t)))_{t\in[0,T]}$ is not $(\mathcal{F}_t)_{t\in[0,T]}$-adapted. As a consequence of Proposition 4.1 we have the following.

Corollary 4.2 *The condition $\langle u, (Du)u \rangle = 0$ is satisfied if and only if the $d\times d$ matrix A is 2-nilpotent, and in this case we have*

$$d_{|u|}\left(\int_0^T AW_s dW_s, \mathcal{N}_{g(\|u\|)} \right) \leq \frac{1}{\sqrt{g(\|u\|)}} \left| g(\|u\|) - \int_0^T |AW_t|^2 \, dt \right|.$$

Proof. We have

$$\begin{aligned}
&\int_0^{nT} \int_0^{nT} f_2(t_1,s) f_2(s,t) f_2(t_2,t) ds dt \\
&= \sum_{i,j=1}^{d} \sum_{k,l=1}^{d} \sum_{p,q=1}^{d} \int_0^{nT} \int_0^{nT} \mathbf{1}_{[(j-1)T,(j-1)T+t-(i-1)T]}(t_2) \mathbf{1}_{[(q-1)T,(q-1)T+s-(p-1)T]}(t_1) a_{i,j} a_{k,l} a_{p,q}
\end{aligned}$$

$$\mathbf{1}_{[(i-1)T,iT]}(t)\mathbf{1}_{[(k-1)T,kT]}(t)\mathbf{1}_{[(l-1)T,(l-1)T+t-(i-1)T]}(s)\mathbf{1}_{[(p-1)T,pT]}(s)dsdt$$

$$= \sum_{i,j=1}^d \sum_{l=1}^d \sum_{q=1}^d \int_{(l-1)T}^{lT} \int_{(i-1)T}^{iT} \mathbf{1}_{[(j-1)T,(j-1)T+t-(i-1)T]}(t_2)\mathbf{1}_{[(q-1)T,(q-1)T+s-(p-1)T]}(t_1)a_{i,j}a_{i,l}a_{l,q}$$

$$\mathbf{1}_{[(l-1)T,(l-1)T+t-(i-1)T]}(s)dsdt$$

$$= \sum_{j,q=1}^d \int_0^T \int_0^T \mathbf{1}_{[(j-1)T,(j-1)T+t]}(t_2)\mathbf{1}_{[(q-1)T,(q-1)T+s]}(t_1)\mathbf{1}_{[0,t]}(s)dsdt \sum_{i,l=1}^d a_{i,j}a_{i,l}a_{l,q}$$

$$= \sum_{j,q=1}^d (A^\top A^2)_{j,q} \int_0^T \int_0^T \mathbf{1}_{[(j-1)T,(j-1)T+t]}(t_2)\mathbf{1}_{[(q-1)T,(q-1)T+s]}(t_1)\mathbf{1}_{[0,t]}(s)dsdt,$$

$t_1, t_2 \in [0,T]$, hence from (4.2), $\langle u, (Du)u\rangle = 0$ implies $A^\top A^2 = 0$, which in turn implies $A^2 = 0$ by the relation $\langle A^2x, A^2x\rangle_{\mathbb{R}^d} = \langle Ax, A^\top A^2 x\rangle_{\mathbb{R}^d}$, $x \in \mathbb{R}^d$. □

In particular, taking $g(x) = x^2$, Corollary 4.2 shows that, given $\int_0^T |AW_t|^2\, dt$, the quadratic functional $\int_0^T AW_t dW_t$ is Gaussian with variance $\int_0^T |AW_t|^2\, dt$ when $A^2 = 0$, which recovers Corollary 3 in [12] under the condition of [13].

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