Conditional calculus on Poisson space and enlargement of filtration

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Abstract

We obtain sufficient conditions for the existence of conditional densities of functionals of the Poisson process, and expressions of these densities via the Malliavin calculus on Poisson space. These results are applied to enlargement of filtrations on Poisson space with explicit examples of computations via the Clark formula which is presented as a consequence of the Itô formula and the martingale property. The gradient operators of stochastic analysis on Poisson space are classified into three families, and each type of gradient is considered separately.

Key words: Poisson process, conditional densities, enlargement of filtration. *Mathematics Subject Classification.* 60H07, 60J75.

1 Introduction

The Malliavin calculus provides sufficient conditions for the existence of densities of random variables. It has also been used to study conditional densities, see [14] and [8] on the Wiener space, [3] in the general case of Dirichlet forms, and [6] on the Wiener-Poisson space.

In this paper we obtain sufficient conditions for the existence of conditional densities on Poisson space. The main differences with the Wiener case are the existence of two different approaches to the stochastic calculus of variations, and the fact that on Poisson space, even elementary smooth functionals such as jump times do not have conditional densities with respect to the past of the Poisson process. In the stochastic calculus of variations on Poisson space, one can distinguish three families of operators, cf. [16]: intrinsic gradients defined by infinitesimal shifts of configuration

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points, finite difference gradients defined through Fock space decompositions, and damped gradients whose adjoints coincide with the compensated Poisson stochastic integral. In this paper we mainly use the damped gradient for the standard Poisson process because of its particular properties that make it closer to the Wiener space derivative. The properties of the two other families of gradients with respect to conditional calculus are also studied.

We proceed as follows. In Sect. 2 we deal with a transformation of Poisson paths and a simple procedure to compute conditional laws. In Sect. 3 we recall the definition of the damped gradient operator which is used Sect. 4 to obtain sufficient conditions for the existence of conditional densities. The method of [8] (transformation of trajectories) is used, cf. Th. 1, as well as the general results of [3], cf. Th. 2. In Sect. 5 we state the Clark formula with an elementary proof that does not rely on chaos expansions. On Poisson space, the method of transformations of trajectories can be easier to apply to jump times functionals than the product spaces method, cf. [6]. Sect. 6 deals with the enlargement of filtrations, cf. [10], with an example of explicit computation. In general, simple random variables on Poisson space such as jump times or increments of the process do not have conditional densities. We choose to consider the interjump time τ_{N_T}, which has a conditional density given $\mathcal{F}_t$, $t \leq T$. An application to insider trading in mathematical finance as in [7] is also mentioned. In Sects. 7 and 8 we review the properties of the two main other types of gradient operators (finite difference and intrinsic gradients) with respect to the conditional calculus on Poisson space.

2 Transformation of Poisson paths

This section introduces the notation and transformations of trajectories that allow to compute conditional laws. We consider the Poisson space $(B, \mathcal{F}, P)$ where B is a space of sequences and P is such that the canonical projections $\tau_k : B \to \mathbb{R}$ form a sequence of i.i.d. exponentially distributed random variables. We denote by $T_n = \sum_{k=0}^{k=n-1} \tau_k$ the n-th jump time of the Poisson process on $\mathbb{R}_+$ constructed as

$$N_t(\omega) = \sum_{k=1}^{\infty} 1_{[T_k(\omega), +\infty[}(t), \quad t \in \mathbb{R},$$

and denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by $(N_t)_{t\geq 0}$. Let

$$p_n^t = P(N_t = n) = 1_{\{n\geq 0\}} \frac{(t \vee 0)^n}{n!} e^{-t\vee 0}, \quad n \in \mathbb{Z}, \ t \in \mathbb{R},$$

with the convention $0^0 = 1$. For fixed $n \geq 1$, $p_{n-1} : \mathbb{R}_+ \to \mathbb{R}_+$ is also the density of the n-th jump time T_n of $(N_t)_{t\in\mathbb{R}_+}$:

$$P(T_n \geq x) = \int_x^\infty p_{n-1}(t)dt, \quad x \in \mathbb{R}.$$

For fixed $t > 0$, we let for $\omega, \omega' \in B$:

$$N_s^t(\omega, \omega') = \begin{cases} N_s(\omega), & \text{if } 0 \leq s \leq t, \\ N_t(\omega) + N_{s-t}(\omega'), & \text{if } 0 \leq t < s, \end{cases}$$

which defines a Poisson process indexed by $s \geq 0$, with n-th jump time

$$T_n^t(\omega, \omega') = \begin{cases} T_n(\omega), & \text{if } 1 \leq n \leq N_t(\omega) \ (\text{i.e. } T_n \leq t), \\ t + T_{n-N_t(\omega)}(\omega'), & \text{if } 1 \leq N_t(\omega) < n \ (\text{i.e. } T_n > t), \end{cases}$$

and k-th interjump time

$$\tau_k^t(\omega, \omega') = \begin{cases} \tau_k(\omega), & \text{if } 0 \leq k < N_t(\omega) \ (\text{i.e. } T_{k+1} < t), \\ t - T_k(\omega) + \tau_0(\omega'), & \text{if } k = N_t(\omega) \ (\text{i.e. } T_k \leq t < T_{k+1}), \\ \tau_{k-N_t(\omega)}(\omega'), & \text{if } 1 \leq N_t(\omega) < k \ (\text{i.e. } t < T_k). \end{cases}$$

Let $\mathcal{T}^t : B \times B \to B$ be defined as $\mathcal{T}^t(\omega, \omega') = (\tau_k^t(\omega, \omega'))_{k\in\mathbb{N}}$. The following properties are easily shown:

i) $\mathcal{T}^t$ is $(\mathcal{F}_t \otimes \mathcal{F}) - \mathcal{F}$ measurable,

ii) we have $[P \otimes P] \circ (\mathcal{T}^t)^{-1} = P$,

iii) given $F \in L^1(B)$ we have for P-a.e $w \in B$:

$$E[F \mid \mathcal{F}_t](\omega) = E[F \circ \mathcal{T}^t(\omega, \cdot)],$$

iv) for all $A \in \mathcal{F}_t$ we have $(\mathcal{T}^t)^{-1}(A) = A \times B$ and $\mathcal{T}^t(A \times B) = A$.

Property (iii) is helpful to compute the conditional law of functionals of the Poisson process, e.g.:

$$
\begin{aligned}
P(T_n \geq x \mid \mathcal{F}_t)(\omega) &= E[1_{\{T_n \geq x\}} \mid \mathcal{F}_t](\omega) = E[1_{\{T_n^t(\omega, \cdot) \geq x\}}] \\
&= 1_{\{x \leq T_n(\omega) \leq t\}} + 1_{\{T_n(\omega) > t\}} E[1_{\{t + T_{n-N_t(\omega)} \geq x\}}] \\
&= 1_{\{x \leq T_n(\omega) \leq t\}} + 1_{\{T_n(\omega) > t\}} \int_{x-t}^\infty p_{n-1-N_t(\omega)}^u du,
\end{aligned}
$$

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which shows that T_n does not have a conditional density given $\mathcal{F}_t$, except locally on the set $\{T_n > t\}$.

3 Commutation relation for the damped gradient

Let

$$\mathcal{S} = \{F = f(T_1, \ldots, T_n) \; : \; f \in \mathcal{C}_c^\infty(\mathbb{R}^n), \; n \geq 1\},$$

and let $\tilde{D} : L^2(B) \to L^2(B \times \mathbb{R}_+)$ be the closable linear operator defined on $\mathcal{S}$ as

$$\tilde{D}_t F = -\sum_{k=1}^{k=n} 1_{[0, T_k]}(t) \partial_k f(T_1, \ldots, T_n), \quad F \in \mathcal{S}, \quad t \in \mathbb{R}_+,$$

cf. [4], [17], and extended to the completion Dom $\tilde{D}$ of $\mathcal{S}$ in $L^2(B)$, with respect to the norm

$$\|F\|_{L^2(B)} + \|\tilde{D}F\|_{L^2(B \times \mathbb{R}_+)}, \quad F \in \mathcal{S}.$$

The operator $\tilde{D}$ has an adjoint $\tilde{\delta} : L^2(B \times \mathbb{R}_+) \to L^2(B)$ which coincides with the stochastic integral on the adapted processes, with the duality relation

$$E[\langle \tilde{D}F, u \rangle_{L^2(\mathbb{R}_+)}] = E[F\tilde{\delta}(u)], \quad F \in \text{Dom } \tilde{D}, \quad u \in \text{Dom } \tilde{\delta}.$$

If $G : B \times B \to \mathbb{R}$ is such that $G(\omega, \cdot) \in \text{Dom } \tilde{D}$, $P(d\omega)$-a.s., we denote by $\tilde{D}^2$ the action of $\tilde{D}$ on the second variable, and define $\tilde{\delta}^2$ similarly. The following Lemma states the commutation relations between gradient operators and transformations of trajectories. The commutation relations of $\tilde{D}$ and $\tilde{\delta}$ with $\mathcal{T}^t$ are similar to the ones found in the Wiener case in e.g. [8].

Lemma 1 *Let $t > 0$. If $F \in \text{Dom } \tilde{D}$ then $F \circ \mathcal{T}^t(\omega, \cdot) \in \text{Dom } \tilde{D}^2$, $P(d\omega)$-a.s., and*

$$[\tilde{D}_{s+t}F] \circ \mathcal{T}^t = \tilde{D}_s^2(F \circ \mathcal{T}^t), \quad ds \otimes P \otimes P - a.e. \tag{1}$$

Proof. Since $\tilde{D}$ has the derivation property it suffices to consider $F = f(T_n)$, $f \in \mathcal{C}_c^\infty(\mathbb{R})$. We have

$$F \circ \mathcal{T}^t(\omega, \omega') = 1_{\{T_n(\omega) < t\}} f(T_n(\omega)) + 1_{\{T_n(\omega) > t\}} f(t + T_{n-N_t(\omega)}(\omega')),$$

and

$$\tilde{D}_s^2(F \circ \mathcal{T}^t) = 1_{\{T_n(\omega) > t\}} 1_{[0, T_{n-N_t(\omega)}(\omega')]}(s) f'(t + T_{n-N_t(\omega)}(\omega')).$$

Moreover,

$$\tilde{D}_{s+t} F = f'(T_n) 1_{[0, T_n]}(s + t),$$

and

$$
\begin{aligned}
[\tilde{D}_{s+t} F] \circ \mathcal{T}^t(\omega, \omega') &= f'(T_n^t(\omega, \omega')) 1_{[0, T_n^t(\omega, \omega')]}(s + t) \\
&= 1_{\{T_n(\omega) < t\}} f'(T_n(\omega)) 1_{[0, T_n(\omega)]}(s + t) \\
&\quad + 1_{\{T_n(\omega) > t\}} f'(t + T_{n-N_t(\omega)}(\omega')) 1_{[0, t + T_{n-N_t(\omega)}(\omega')]}(s + t) \\
&= 1_{\{T_n(\omega) > t\}} f'(t + T_{n-N_t(\omega)}(\omega')) 1_{[0, T_{n-N_t(\omega)}(\omega')]}(s) \\
&= \tilde{D}_s^2(F \circ \mathcal{T}^t(\omega, \omega')),
\end{aligned}
$$

which implies for $s, t \in \mathbb{R}_+$ and all $F \in \mathcal{S}$:

$$[\tilde{D}_{s+t} F] \circ \mathcal{T}^t(\omega, \cdot) = \tilde{D}_s^2(F \circ \mathcal{T}^t(\omega, \cdot)).$$

Given $F \in \text{Dom } \tilde{D}$ there exists a sequence $(F_n)_{n \in \mathbb{N}} \subset \mathcal{S}$ such that $(\tilde{D}_{\cdot+t} F_n)_{n \in \mathbb{N}}$ converges to $\tilde{D}_{\cdot+t} F$ in $L^2(B \times \mathbb{R}_+)$. This implies that $(\tilde{D}_\cdot^2(F_n \circ \mathcal{T}^t))_{n \in \mathbb{N}} = (\tilde{D}_{\cdot+t} F_n \circ \mathcal{T}^t)_{n \in \mathbb{N}}$, converges to $\tilde{D}_{\cdot+t} F \circ \mathcal{T}^t$ in $L^2(B \times B \times \mathbb{R}_+)$, hence $F \circ \mathcal{T}^t(\omega, \cdot) \in \text{Dom } \tilde{D}$ and (1) holds. $\square$

By duality we obtain a similar result for $\tilde{\delta}$. Let $\pi_{[t}$ denote the multiplication by $1_{[t, \infty[}$.

Lemma 2 *Let $t > 0$. Let $u \in \text{Dom } \tilde{\delta}$. Then $u_{\cdot+t} \circ \mathcal{T}^t(\omega, \cdot) \in \text{Dom } \tilde{\delta}^2$, $P(d\omega) - a.s$, and*

$$\tilde{\delta}^2(u_{\cdot+t} \circ \mathcal{T}^t) = \tilde{\delta}(\pi_{[t} u) \circ \mathcal{T}^t, \qquad P \otimes P - a.s. \tag{2}$$

Proof. Let $\mathcal{F}_{[t}$ denote the σ-algebra generated by $\{N_s - N_t : s > t\}$, $t \in \mathbb{R}_+$. Let $F, G \in \mathcal{S}$ be respectively $\mathcal{F}_t$ and $\mathcal{F}_{[t}$ measurable. We have

$$E \otimes E[\tilde{\delta}^2(u_{\cdot+t} \circ \mathcal{T}^t) F \circ \mathcal{T}^t G \circ \mathcal{T}^t]$$

$$
\begin{aligned}
&= E \otimes E[F \circ \mathcal{T}^t \langle u_{\cdot+t} \circ \mathcal{T}^t, \tilde{D}_\cdot^2(G \circ \mathcal{T}^t) \rangle] = E \otimes E[F \circ \mathcal{T}^t \langle u_\cdot \circ \mathcal{T}^t, \pi_{[t} \tilde{D}_{\cdot-t}^2(G \circ \mathcal{T}^t) \rangle] \\
&= E \otimes E[F \circ \mathcal{T}^t \langle u_\cdot \circ \mathcal{T}^t, \pi_{[t} [\tilde{D}_\cdot G] \circ \mathcal{T}^t \rangle] = E[F \langle u, \pi_{[t} \tilde{D} G \rangle] \\
&= E[\langle u, \pi_{[t} \tilde{D}(FG) \rangle] = E[\tilde{\delta}(\pi_{[t} u) FG] = E \otimes E[\tilde{\delta}(\pi_{[t} u) \circ \mathcal{T}^t F \circ \mathcal{T}^t G \circ \mathcal{T}^t].
\end{aligned}
$$

Hence $\pi_{[t} u \circ \mathcal{T}^t \in \text{Dom } \tilde{\delta}^2$ and (2) holds. $\square$

4 Existence of conditional densities

The following result shows in particular that if

$$\int_t^{+\infty} (\tilde{D}_s F)^2 ds > 0, \quad P - a.s.,$$

then the conditional law of F given $\mathcal{F}_t$ is absolutely continuous with respect to the Lebesgue measure. For $t = 0$, i.e. in the unconditional case, this result can be found in [18], Th. 6, cf. also Th. 4.1. of [4]. Its analog on Wiener space is Th. 2.1.3 in [12].

Theorem 1 *Let $t > 0$, $A \in \mathcal{F}_t$ and $F \in \text{Dom } \tilde{D}$. If*

$$\int_t^{+\infty} (\tilde{D}_s F)^2 ds > 0, \quad 1_A P - a.e.,$$

then for P-a.e $w \in A$, F has a conditional density given $\mathcal{F}_t$.

Proof. Since $A \in \mathcal{F}_t$, we have $\mathcal{T}^t(A \times B) = A$, hence for P-a.e $w \in A$:

$$\int_0^{+\infty} [\tilde{D}_s^2(F \circ \mathcal{T}^t(\omega, \omega'))]^2 ds = \int_t^{+\infty} [\tilde{D}_{s-t}^2(F \circ \mathcal{T}^t(\omega, \omega'))]^2 ds = \int_t^{+\infty} (\tilde{D}_s F)^2 \circ \mathcal{T}^t(\omega, \omega') ds > 0,$$

$P(d\omega')$-a.s, which implies by Th. 6 of [18] that for P-a.e $w \in A$ the law of $F \circ \mathcal{T}^t(\omega, \cdot)$ admits a density which is the conditional density of F given $\mathcal{F}_t$. $\square$

The jump time T_n satisfies the hypothesis of this theorem with $A = \{T_n > t\}$, since

$$\int_t^\infty (\tilde{D}_s T_n)^2 ds = (T_n - t)^+,$$

hence T_n has a conditional density on the set $\{T_n > t\}$, in fact:

$$dP(T_n = x \mid \mathcal{F}_t)(\omega) = 1_{\{T_n(\omega) \leq t\}} \delta_{T_n(\omega)}(dx) + 1_{\{T_n(\omega) > t\}} p_{n-1-N_t(\omega)}^{x-t} dx, \quad n \geq 1.$$

If $A = B$, Th. 1 can be obtained from the general results on conditional Dirichlet forms of [3], see also Th. 4.2 of [14] in the Wiener case.

Theorem 2 *Let $t > 0$ and $F \in \text{Dom } \tilde{D}$. If*

$$\int_{T_{N_t}}^{+\infty} (\tilde{D}_s F)^2 ds > 0, \quad P - a.s,$$

then F has a conditional density given $\mathcal{F}_t$.

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Proof. Let

$$\mathcal{K}_t = \{G\tilde{D}u(T_k) \ : \ k \geq 1, \ u \in \mathcal{C}_c^\infty([0,t]), \ G \in L^\infty(B)\},$$

and let

$$\mathcal{H}_t = \left\{v \in L^2(B \times \mathbb{R}_+) \ : \ \int_0^{T_k} v(s)ds = 0, \ \forall \ T_k < t\right\}$$

denote the orthogonal of $\mathcal{K}_t$ in $L^2(B \times \mathbb{R}_+)$. For $F \in \text{Dom } \tilde{D}$, the orthogonal projection $\pi_{\mathcal{H}_t}\tilde{D}F$ of $\tilde{D}F$ on $\mathcal{H}_t$ is

$$\pi_{\mathcal{H}_t}\tilde{D}F = 1_{[T_{N_t},+\infty[}\tilde{D}F$$

since $1_{[T_{N_t},+\infty[}\tilde{D}F \in \mathcal{H}_t$ and $\tilde{D}F - 1_{[T_{N_t},+\infty[}\tilde{D}F = 1_{[0,T_{N_t}[}\tilde{D}F \in \mathcal{H}_t^\perp$. Th. 5-2-7-a of [3], page 228, implies that if

$$\int_{T_{N_t}}^{+\infty}(\tilde{D}_sF)^2ds = \langle\pi_{\mathcal{H}_t}\tilde{D}F, \pi_{\mathcal{H}_t}\tilde{D}F\rangle > 0, \quad P-a.s.,$$

then F has a conditional density given

$$\mathcal{F}_t = \sigma(u(T_k) \ : \ k \in \mathbb{N}, \ u \in \mathcal{C}_c^\infty([0,t])).$$

$\square$

The relation

$$\int_t^\infty (\tilde{D}_sF)^2ds \leq \int_{T_{N_t}}^{+\infty}(\tilde{D}_sF)^2ds \ = \ \frac{t - T_{N_t}}{T_{N_t+1} - t}\int_t^{T_{N_t+1}}(\tilde{D}_sF)^2ds + \int_t^{+\infty}(\tilde{D}_sF)^2ds$$

$$\leq \ \frac{T_{N_t+1} - T_{N_t}}{T_{N_t+1} - t}\int_t^\infty(\tilde{D}_sF)^2ds,$$

shows that for $A = B$, Th. 1 and Th. 2 are equivalent.

We now turn to representation formulas for the conditional densities.

Theorem 3 *Let $t > 0$. We assume that $F \in \text{Dom } \tilde{D}$ and that*

$$\left(\int_t^\infty (\tilde{D}_sF)^2ds\right)^{-1}\pi_{[t}\tilde{D}F \in \text{Dom } \tilde{\delta}.$$

Then, conditionally to $\mathcal{F}_t$, F has a bounded continuous density given by:

$$q^t(x,\omega) = E\left[1_{\{F>x\}}\tilde{\delta}\left(\frac{\pi_{[t}\tilde{D}F}{\|\pi_{[t}\tilde{D}F\|_{L^2(\mathbb{R}_+)}^2}\right) \mid \mathcal{F}_t\right].$$

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Proof. The duality between $\tilde{D}$ and $\tilde{\delta}$ and the derivation property of $\tilde{D}$ show that the proof of Prop. 2.1.1 in [12] applies here, hence if

$$\frac{\tilde{D}^2(F \circ \mathcal{T}^t)}{\|\tilde{D}^2(F \circ \mathcal{T}^t)\|^2_{L^2(\mathbb{R}_+)}} \in \text{Dom } \tilde{\delta}^2,$$

then the law of $F \circ \mathcal{T}^t(\omega, \cdot)$ admits a bounded continuous density given by

$$
\begin{aligned}
q^t(x, \omega) &= E\left[1_{\{F \circ \mathcal{T}^t(\omega, \cdot) > x\}} \tilde{\delta}^2 \left(\frac{\tilde{D}^2(F \circ \mathcal{T}^t(\omega, \cdot))}{\|\tilde{D}^2(F \circ \mathcal{T}^t)\|^2_{L^2(\mathbb{R}_+)}} \right) \right] \\
&= E\left[1_{\{F \circ \mathcal{T}^t(\omega, \cdot) > x\}} \tilde{\delta}^2 \left(\frac{\tilde{D}_{\cdot + t} F}{\|\tilde{D}_{\cdot + t} F\|^2_{L^2(\mathbb{R}_+)}} \circ \mathcal{T}^t(\omega, \cdot) \right) \right] \\
&= E\left[1_{\{F \circ \mathcal{T}^t(\omega, \cdot) > x\}} \tilde{\delta} \left(\frac{\pi_{[t} \tilde{D} F}{\|\pi_{[t} \tilde{D} F\|^2_{L^2(\mathbb{R}_+)}} \right) \circ \mathcal{T}^t(\omega, \cdot) \right],
\end{aligned}
$$

where we applied Lemma 1 and Lemma 2. $\qquad\square$

5 The Clark formula

The Clark formula yields the predictable representation of a random variable using the operator D. On Poisson space, the first statements of this formula have been obtained in [19], [20] and [21]. An application of the Itô formula

$$
\begin{aligned}
f(N_v, v) &= f(N_u, u) + \int_u^v (f(1 + N_{s^-}, s) - f(N_{s^-}, s)) d\tilde{N}_s \\
&\quad + \int_u^v (f(1 + N_s, s) - f(N_s, s)) ds + \int_u^v \partial_2 f(N_s, s) ds,
\end{aligned}
$$

to the martingale in t

$$
\begin{aligned}
f(N_t(\omega), t) &= E[1_{\{N_T - N_u = n\}} \mid \mathcal{F}_t] = E[1_{\{N_T - N_u = n\}} \circ \mathcal{T}^t(\omega, \cdot)] \\
&= E[1_{\{N_T^t(\omega, \cdot) - N_u^t(\omega, \cdot) = n\}}] = E[1_{\{N_{T-t}(\cdot) + N_t(\omega) - N_u(\omega) = n\}}] \\
&= p_{n - (N_t - N_u)(\omega)}^{T-t}, \quad u < t < T,
\end{aligned}
$$

gives the predictable representation

$$
p_{n - (N_v - N_u)}^{T-v} = p_n^{T-u} + \int_u^v (p_{n-1 - N_t + N_u}^{T-t} - p_{n - N_t + N_u}^{T-t}) d\tilde{N}_t, \quad 0 \le u \le v \le T,
$$

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and in particular for $T = v$,

$$1_{\{N_v - N_u = n\}} = p_n^{v-u} + \int_u^v (p_{n-1-N_t+N_u}^{v-t} - p_{n-N_t+N_u}^{v-t})d\tilde{N}_t, \quad n \in \mathbb{Z}. \tag{3}$$

For $F = 1_{\{N_v - N_u = n\}}$ and $u \le t \le v$ we have

$$\begin{aligned} p_{n-1-N_t+N_u}^{v-t} - p_{n-N_t+N_u}^{v-t} &= E[1_{\{N_v - N_u + 1 = n\}} - 1_{\{N_v - N_u = n\}} \mid \mathcal{F}_t] \\ &= E[D_t 1_{\{N_v - N_u = n\}} \mid \mathcal{F}_t], \end{aligned}$$

where $D : L^2(B) \to L^2(B) \otimes L^2(\mathbb{R}_+)$ is the finite difference (linear) operator defined as

$$D_t f(N_v - N_u) = 1_{[u,v]}(t)(f(N_v - N_u + 1) - f(N_v - N_u)),$$

$f : \mathbb{N} \to \mathbb{R}$, and extended to functionals of the form

$$F = f(N_{u_1} - N_{v_1}, \ldots, N_{u_n} - N_{v_n}), \quad 0 < u_1 < v_1 < \cdots < u_n < v_n, \tag{4}$$

with the product rule

$$D_t(FG) = FD_tG + GD_tF + D_tFD_tG, \quad t \in \mathbb{R}_+. \tag{5}$$

Proposition 1 *Let F be as in (4), where f has finite support on $\mathbb{N}^n$. We have the Clark formula*

$$F = E[F] + \int_0^\infty E[D_tF \mid \mathcal{F}_t]d\tilde{N}_t.$$

Proof. For $F = 1_{\{N_v - N_u = n\}}$, the formula follows from (3). Using the product rule (5), we check that if the formula holds for $\mathcal{F}_u$-measurable F and for G independent of $\mathcal{F}_u$, then

$$\begin{aligned} FG &= E[G]F + F(G - E[G]) \\ &= E[G]\left(E[F] + \int_0^u E[D_tF \mid \mathcal{F}_t]d\tilde{N}_t\right) + F\int_u^\infty E[D_tG \mid \mathcal{F}_t]d\tilde{N}_t \\ &= E[FG] + \int_0^u E[GD_tF \mid \mathcal{F}_t]d\tilde{N}_t + \int_u^\infty E[FD_tG \mid \mathcal{F}_t]d\tilde{N}_t \\ &= E[FG] + \int_0^\infty E[D_t(FG) \mid \mathcal{F}_t]d\tilde{N}_t. \end{aligned}$$

$\square$

Let

$$F = E[F] + \sum_{n=1}^{\infty} I_n(f_n), \quad f_n \in L^2(\mathbb{R}_+)^{\circ n},$$

denote the chaos expansion of $F \in L^2(B, \mathcal{F}, P)$, where $I_n(f_n)$ denotes the Poisson compensated multiple stochastic integral. The operator D can be extended as a closed operator with domain Dom D, such that $I_n(f_n) \in$ Dom D, $f_n \in L^2(\mathbb{R}_+)^{\circ n}$, and

$$D_t I_n(f_n) = n I_{n-1}(f_n(t, \cdot)), \quad dP \times dt - a.e.,$$

cf. [13]. The Clark formula can then be extended to $F \in$ Dom D and to $F \in$ Dom $\tilde{D}$ since functionals of the form (4) are dense in $L^2(B)$ and $F \mapsto (E[D_t F \mid \mathcal{F}_t])_{t \in \mathbb{R}_+}$ is in fact continuous from $L^2(B)$ into $L^2(B \times \mathbb{R}_+)$, with $E[D_t F \mid \mathcal{F}_t] = E[\tilde{D}_t F \mid \mathcal{F}_t]$, $t \in \mathbb{R}_+$, cf. Prop. 20 and Prop. 21 of [17]. In particular,

Proposition 2 *([17], Th. 1) We have for $F \in L^2(B)$:*

$$F = E[F] + \int_0^{\infty} E[D_t F \mid \mathcal{F}_t] d\tilde{N}_t = E[F] + \int_0^{\infty} E[\tilde{D}_t F \mid \mathcal{F}_t] d\tilde{N}_t.$$

We stress that predictable representations can not always be obtained from the combination of Markovian and martingale methods described above. In such cases the Clark formula can be the only possible way to compute explicitly a predictable representation, as in the example given in the next section.

6 Enlargement of filtration

As noted above, many absolutely continuous random variables on Poisson space do not have conditional densities given $\mathcal{F}_t$. A simple example is the conditional law of T_n given $\mathcal{F}_t$:

$$dP(T_n = x \mid \mathcal{F}_t)(\omega) = 1_{\{T_n(\omega) \le t\}} \delta_{T_n(\omega)}(dx) + 1_{\{T_n(\omega) > t\}} dP(t + T_{n - N_t} = x).$$

Thus the conditional law of a random variable such as T_n does not satisfy the absolute continuity condition imposed in [9]. Such restrictions are not directly imposed in [23], which deals with continuous processes. Our first step is to check that Th. 1 of [23], see also [22], p. 82, also holds in the discontinuous case, and in particular for the Poisson

process. However, Prop. 3 does not apply to the jump time T_n, since its conditional law does not satisfy Relation (6) below.

Proposition 3 *Given a random variable F, let $\lambda_t(dx)$ denote a version of the conditional law of F given $\mathcal{F}_t$, and assume that its predictable representation is given as*

$$\lambda_t(g) = \lambda_0(g) + \int_0^t \dot{\lambda}_u(g) d\tilde{N}_u, \qquad g \in \mathcal{C}_b(\mathbb{R}),$$

and that there exists $l : \mathbb{R}_+ \times B \times \mathbb{R}_+ \to \mathbb{R}$ measurable and a stopping time S, such that

$$1_{\{u \leq S\}} \dot{\lambda}_u(dx) = 1_{\{u \leq S\}} \lambda_u(dx) l(x, u). \tag{6}$$

Then

$$\tilde{N}_t - \int_0^t l(F, u) du$$

is a $\mathcal{G}_t$-local martingale up to the time S, where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(F)$, $t \in \mathbb{R}_+$.

Proof. We note that the argument of Th. 1 in [23], applies here without continuity assumption on the trajectories. Let F_s be an $\mathcal{F}_s$-measurable set. We have $g(F) = E[g(F)] + \int_0^\infty \dot{\lambda}_u(g) d\tilde{N}_u$, and

$$
\begin{aligned}
E[1_{F_s} g(F)(\tilde{N}_{t \wedge S} - \tilde{N}_{s \wedge S})] &= E\left[1_{F_s} \left(E[g(F)] + \int_0^\infty \dot{\lambda}_u(g) d\tilde{N}_u\right)(\tilde{N}_{t \wedge S} - \tilde{N}_{s \wedge S})\right] \\
&= E\left[1_{F_s} \int_{s \wedge S}^{t \wedge S} \dot{\lambda}_u(g) du\right] \\
&= E\left[1_{F_s} \int_{s \wedge S}^{t \wedge S} \int_{-\infty}^\infty l(x, u) g(x) \lambda_u(dx) du\right] \\
&= E\left[1_{F_s} \int_{s \wedge S}^{t \wedge S} E[g(F) l(F, u) \mid \mathcal{F}_u] du\right] \\
&= E\left[1_{F_s} g(F) \int_{s \wedge S}^{t \wedge S} l(F, u) du\right].
\end{aligned}
$$

$\square$

Next we perform an explicit enlargement of filtration, using the Clark formula. We consider the absolutely continuous random variable $\tau_{N_T} = T_{N_T+1} - T_{N_T}$ which represents the length of the jump interval $[T_{N_T}, T_{N_T+1}]$ straddling over T. Let $\mathcal{G}_t =$

$\mathcal{F}_t \vee \sigma(\tau_{N_T})$. We have for $0 < t < T$:

$$T_{N_T} \circ \mathcal{T}^t(\omega, \omega') = \begin{cases} t + T_{N_{T-t}}(\omega'), & T_1(\omega') < T - t, \\ T_{N_t}(\omega), & T_1(\omega') > T - t, \end{cases}$$

hence

$$\begin{aligned}
P(T_{N_T} \geq x \mid \mathcal{F}_t)(\omega) &= E[1_{\{T_{N_T} \circ \mathcal{T}^t(\omega, \cdot) \geq x\}}] \\
&= E[1_{\{T_{N_{T-t}} \geq x-t\}} 1_{\{T_1 < T-t\}}] + E[1_{\{T_{N_t}(\omega) \geq x\}} 1_{\{T_1 > T-t\}}] \\
&= E[1_{\{N_{T-t} - N_{x-t} > 0\}}] + P(T_1 > T - t) 1_{\{T_{N_t}(\omega) \geq x\}} \\
&= 1 - e^{-(T - t \vee x)} + e^{-(T-t)} 1_{\{T_{N_t}(\omega) \geq x\}},
\end{aligned}$$

and

$$dP(T - T_{N_T} = x \mid \mathcal{F}_t) = 1_{[0,T-t]}(x) e^{-x} dx + e^{-(T-t)} \delta_{T - T_{N_t}}(dx). \tag{7}$$

It is well-known that $T_{N_T+1} - T$ is exponentially distributed and independent of $T - T_{N_T}$, conditionally to $\mathcal{F}_t$, $0 \leq t \leq T$, since $T_{N_T+1} \circ \mathcal{T}^t(\omega, \omega') = T_1(\omega')$ and:

$$E[1_{\{T_{N_T+1} - T \geq x\}} \mid \mathcal{F}_t] = E[1_{\{T_{N_T+1} \circ \mathcal{T}^t(\omega, \cdot) - T \geq x\}}] = E[1_{\{T_1 \geq x\}}] = 1 - e^{-x}, \quad x \geq 0,$$

hence by convolution the conditional density of τ_{N_T} given $\mathcal{F}_t$ is

$$\begin{aligned}
\frac{\lambda_t(dx)}{dx} &= \frac{dP}{dx}(\tau_{N_T} = x \mid \mathcal{F}_t) = \int_0^{x \wedge (T-t)} e^{-(x-y)} e^{-y} dy + e^{-(T-t)} \int_0^x e^{-(x-y)} \delta_{T - T_{N_t}}(dy) \\
&= 1_{[0,\infty[}(x) e^{-x} (T - t) \wedge x + e^{t - T_{N_t} - x} 1_{[T - T_{N_t}, \infty[}(x).
\end{aligned}$$

The predictable representation of $\lambda_t(dx)$ is obtained from the Clark formula, and does not follow directly from the Itô formula. From (7), we get

$$\begin{aligned}
E[D_s f(T_{N_t}) \mid \mathcal{F}_s] &= E[1_{\{T_{N_t} < s\}}(f(s) - f(T_{N_t})) \mid \mathcal{F}_s] \\
&= \int_0^s e^{-(t-s)}(f(s) - f(y)) \delta_{T_{N_s}}(dy) = e^{-(t-s)}(f(s) - f(T_{N_s})),
\end{aligned}$$

hence with $f(u) = e^{-u} 1_{[T-x,\infty[}(u)$ we have

$$E[D_s(e^{-T_{N_t}} 1_{[T-x,\infty[}(T_{N_t})) \mid \mathcal{F}_s] = e^{-t}(1_{[T-x,\infty[}(s) - e^{s - T_{N_s}} 1_{[T-x,\infty[}(T_{N_s})),$$

and the predictable representation of the conditional density $\lambda_t(dx)/dx$ is

$$\frac{\lambda_0(dx)}{dx} + e^{t-x}\int_0^t E[D_s(e^{-T_{N_t}}1_{[T-x,\infty[}(T_{N_t})) \mid \mathcal{F}_s]d\tilde{N}_s$$

$$= 1_{[0,\infty[}(x)e^{-x}(T\wedge x) + e^{-x}1_{[T,\infty[}(x) + e^{-x}\int_0^t (1_{\{T-x\leq s\}} - e^{-(T_{N_s}-s)}1_{\{T-x\leq T_{N_s}\}})d\tilde{N}_s$$

$$= 1_{[0,\infty[}(x)e^{-x}(T\wedge x) + e^{-x}1_{[T,\infty[}(x)$$

$$+ e^{-x}\int_0^t (1_{[T-s,T-T_{N_s}]}(x) + 1_{[T-T_{N_s},\infty[}(x)(1 - e^{-(T_{N_s}-s)}))d\tilde{N}_s.$$

Hence

$$\dot{\lambda}_t(dx) = e^{-x}1_{[T-t,T-T_{N_t}[}(x)dx + e^{-x}1_{[T-T_{N_t},\infty[}(x)(1 - e^{-(T_{N_t}-t)})dx$$

$$= e^{-x}(1_{[T-t,\infty[}(x) - 1_{[T-T_{N_t},\infty[}(x)e^{t-T_{N_t}})dx,$$

and the process $l(x,t)$ defined in (6) is equal to

$$l(x,t) = \frac{1}{T-t}1_{[T-t,T-T_{N_t}[}(x) + 1_{[T-T_{N_t},\infty[}(x)\frac{1 - e^{t-T_{N_t}}}{(T-t) + e^{t-T_{N_t}}}$$

$$= 1_{[T-t,\infty[}(x)\frac{1 - e^{t-T_{N_t}}1_{[T-T_{N_t},\infty[}(x)}{T - t + e^{t-T_{N_t}}1_{[T-T_{N_t},\infty[}(x)}.$$

Consequently,

$$N_t - t - \int_0^t \Lambda_s ds, \quad t \in \mathbb{R}_+,$$

is a compensated $\mathcal{G}_t$-Poisson process, with

$$\Lambda_s = 1_{[T-\tau_{N_T},\infty[}(s)\frac{1 - e^{s-T_{N_s}}1_{[T-\tau_{N_T},\infty[}(T_{N_s})}{T - s + e^{s-T_{N_s}}1_{[T-\tau_{N_T},\infty[}(T_{N_s})}, \quad s \in [0,T].$$

For an application in mathematical finance, consider a price process $(S_t)_{t\in\mathbb{R}_+}$ given by

$$\frac{dS_t}{S_t} = \nu dt + \phi d\tilde{N}_t,$$

where ν and ϕ are positive constants. Assume that an informed agent knows the random value of τ_{N_T}. Since there is always at least one jump at T_{N_T} in the time interval $[T - \tau_{N_T}, T]$, the agent can buy at time $T - \tau_{N_T}$ and sell at a higher price after the next jump, inducing a possibility of arbitrage. We have shown that the dynamics of prices for the informed agent is

$$\frac{dS_t}{S_t} = (\nu + \Lambda_t)dt + \phi(d\tilde{N}_t - \Lambda_t dt).$$

Note that $\Lambda_s = 0$ if $s \leq T - \tau_{N_T}$.

7 The finite difference gradient

In this section we investigate the properties of the finite difference gradient with respect to the existence of conditional densities. Instead of a Poisson process on $\mathbb{R}_+$ we may consider a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$. Let

$$\Omega = \left\{ \omega = \sum_{i=1}^{i=n} \epsilon_{t_i, x_i} \; : \; (t_i, x_i) \in \mathbb{R}_+ \times \mathbb{R}^d, \;\; n \in \mathbb{N} \cup \{+\infty\} \right\},$$

where $\epsilon_{t,x}$ denotes the Dirac measure at $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. The space Ω is equipped with the Poisson random measure with intensity $d\mu(t, x) = dt \otimes \sigma(dx)$, where σ is a σ-finite measure on $\mathbb{R}^d$, by letting

$$P(\{\omega \in \Omega \;\; : \;\; \omega(A_1) = k_1, \ldots, \omega(A_n) = k_n\}) = \frac{\mu(A_1)^{k_1}}{k_1!} e^{-\mu(A_1)} \cdots \frac{\mu(A_n)^{k_n}}{k_n!} e^{-\mu(A_n)},$$

where $A_1, \ldots, A_n$ are disjoint compact subsets of $\mathbb{R}_+ \times \mathbb{R}^d$. This measure is characterized by its Fourier transform

$$\int_\Omega e^{i \int_{\mathbb{R}_+ \times \mathbb{R}^d} f(t,x) d\omega(t,x)} dP(\omega) = \exp\left(\int_{\mathbb{R}_+ \times \mathbb{R}^d} (e^{if(t,x)} - 1) d\mu(t,x) \right), \quad f \in \mathcal{C}_c(\mathbb{R}_+ \times \mathbb{R}^d).$$

Let $D : L^2(\Omega) \to L^2(\Omega \times \mathbb{R}_+ \times \mathbb{R}^d)$ be the linear operator defined as

$$D_{t,x} F = F(\omega + \epsilon_{t,x}) - F(\omega), \quad \omega \in \Omega.$$

The domain of $D : L^2(\Omega) \to L^2(\Omega \times \mathbb{R}_+ \times \mathbb{R}^d)$ is the space Dom D of random variables $F : \Omega \to \mathbb{R}$ such that

$$\|F\|_{L^2(\Omega)} + \|DF\|_{L^2(\Omega \times \mathbb{R}_+ \times \mathbb{R}^d)} < \infty, \quad u \in \mathbb{R}^d.$$

The operator D has an adjoint $\delta : L^2(\Omega \times \mathbb{R}_+ \times \mathbb{R}^d) \to L^2(\Omega)$, with the duality relation

$$E[\langle DF, u \rangle_{L^2(\mathbb{R}_+ \times \mathbb{R}^d)}] = E[F\delta(u)], \quad F \in \text{Dom } D, \; u \in \text{Dom } \delta.$$

If $G : \Omega \times \Omega \to \mathbb{R}$ we will denote by D^2 the action of D on the second variable. The transformation $\mathcal{T}^t : \Omega \times \Omega \to \Omega$ is here defined as

$$\mathcal{T}^t(\omega, \omega') = \mathcal{T}^t \left(\sum_{i=1}^{i=n} \epsilon_{t_i, x_i}, \sum_{i=1}^{i=n'} \epsilon_{t'_i, x'_i} \right) = \sum_{t_i < t} \epsilon_{t_i, x_i} + \sum_{i=1}^{i=n'} \epsilon_{t+t'_i, x'_i}.$$

The following Lemma states the commutation relations between gradient operators and transformations of trajectories. The commutation relation of D_s with $\mathcal{T}^t$ coincides also with that of $\tilde{D}$, or [8] on the Wiener space.

Lemma 3 *Let $t > 0$. If $F \in \mathrm{Dom}\ D$ then $F \circ \mathcal{T}^t \in \mathrm{Dom}\ D^2$ and :*

$$[D_{r+t,x}F] \circ \mathcal{T}^t = D_{r,x}^2(F \circ \mathcal{T}^t), \qquad dr \otimes \sigma(dx) \otimes P \otimes P - a.e.$$

Proof. We have for $r > t$ and $\omega = \sum_{i=1}^{i=n} \epsilon_{t_i, x_i}$, $\omega' = \sum_{i=1}^{i=n'} \epsilon_{t'_i, x'_i}$:

$$
\begin{aligned}
& [D_{r,x}F] \circ \mathcal{T}^t(\omega, \omega') \\
&= \mathbf{1}_{[t,\infty[}(r)\left(F\left(\sum_{t_i<t} \epsilon_{t_i,x_i} + \epsilon_{r,x} + \sum_{i=1}^{i=n'} \epsilon_{t+t'_i,x'_i} \right) - F\left(\sum_{t_i<t} \epsilon_{t_i,x_i} + \sum_{i=1}^{i=n'} \epsilon_{t+t'_i,x'_i} \right) \right) \\
&= D_{r-t,x}^2(F \circ \mathcal{T}^t(\omega, \omega')).
\end{aligned}
$$

$\square$

Let $\pi_{[t}$ denote the multiplication by $\mathbf{1}_{[t,\infty[} \otimes 1$. By duality we obtain similar results for the adjoint δ^2 of D^2, with the same proof as in Lemma 2, if we consider the σ-algebra $\mathcal{F}_{[t}$ generated by $\{\omega \mapsto \omega([s,r] \times A) : t < s < r, \ A \in \mathcal{B}(\mathbb{R}^d)\}$, $t \in \mathbb{R}_+$.

Lemma 4 *Let $t > 0$. Let $u \in \mathrm{Dom}\ \delta$. Then $u \circ \mathcal{T}^t(\omega, \cdot) \in \mathrm{Dom}\ \delta^2$, $P(d\omega) - a.s$, and*

$$\delta^2(u_{.+t} \circ \mathcal{T}^t) = \delta(\pi_{[t}u) \circ \mathcal{T}^t, \qquad P \otimes P - a.s. \tag{8}$$

We note that existence results for conditional densities, using the finite difference gradient D can also be stated from the result of [11], see also [15]. Assume that for some $\alpha \in]0, 2[$,

$$\liminf_{\rho \to 0} \frac{1}{\rho^\alpha} \int_{|z| \le \rho} |z|^2 \sigma(dz) > 0$$

(condition (1.9) in [11]). Let M be a symmetric positive definite $d \times d$ matrix such that

$$(u, Mu)_{\mathbb{R}^d} \le \liminf_{\rho \to 0} \left(u, \frac{\int_{|z| \le \rho} zz^T \sigma(dz)}{\int_{|z| \le \rho} |z|^2 \sigma(dz)} u \right),$$

(condition (1.4) in [11]). Let $\mathcal{F}_t$ be the σ-algebra generated by $\{\omega \mapsto \omega([s,r] \times A) : s < r < t, \ A \in \mathcal{B}(\mathbb{R}^d)\}$, $t \in \mathbb{R}_+$.

Proposition 4 *Let $F = (F_1, \ldots, F_d)$ such that $F_i \in \text{Dom } D$, $i = 1, \ldots, d$, and assume that the covariance matrix*

$$(< DF_i, DF_j >_{L^2([t,+\infty[\times\mathbb{R}^d)})_{1 \leq i,j \leq d} + \int_t^\infty (\partial_x D_{s,0} F) M (\partial_x D_{s,0} F)^T ds$$

is positive definite. Then $F : \Omega \to \mathbb{R}^d$ has a conditional density given $\mathcal{F}_t$.

For $t = 0$ this is Th. 2.1 of [11]. The proof of this proposition is then a straightforward application to $F \circ \mathcal{T}^t$ of the result of [11], combined with Lemma 3.

8 Conditional calculus with the intrinsic gradient

The intrinsic gradient $\hat{D}$ is defined as

$$\hat{D}_{t,x} F = \sum_{i=1}^{i=n} \partial_i f \left(\int_{\mathbb{R}_+\times\mathbb{R}^d} u_1 d\omega, \ldots \int_{\mathbb{R}_+\times\mathbb{R}^d} u_n d\omega \right) \nabla_{t,x} u_i, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

where $\nabla_{t,x}$ is the gradient on $\mathbb{R}_+ \times \mathbb{R}^d$ and

$$F = f \left(\int_{\mathbb{R}_+\times\mathbb{R}^d} u_1 d\omega, \ldots \int_{\mathbb{R}_+\times\mathbb{R}^d} u_n d\omega \right), \tag{9}$$

$u_1, \ldots, u_n \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$, $f \in \mathcal{C}_c^1(\mathbb{R}^n)$, cf. [2], [1]. The transformation $\mathcal{T}^t$ is defined as in the previous section, i.e.

$$F \circ \mathcal{T}^t(\omega, \omega') = f \left(\int_{[0,t]\times\mathbb{R}^d} u d\omega + \int_{\mathbb{R}_+\times\mathbb{R}^d} u(s+t,x) d\omega'(s,x) \right),$$

for $F = f \left(\int_{\mathbb{R}_+\times\mathbb{R}^d} u d\omega \right)$, $u \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$.

Lemma 5 *Let $t > 0$. If $F \in \text{Dom } \hat{D}$ then $F \circ \mathcal{T}^t \in \text{Dom } \hat{D}^2$ and :*

$$[\hat{D}_{r+t,x} F] \circ \mathcal{T}^t = \hat{D}_{r,x}^2 (F \circ \mathcal{T}^t), \quad dr \otimes dx \otimes P \otimes P - a.e.$$

Proof. Let $F = f \left(\int_{\mathbb{R}_+\times\mathbb{R}^d} u d\omega \right)$. We have

$$\begin{aligned}
[\hat{D}_{r+t,x} F] \circ \mathcal{T}^t(\omega, \omega') &= f' \left(\int_{[0,t]\times\mathbb{R}^d} u d\omega + \int_{\mathbb{R}_+\times\mathbb{R}^d} u(s+t,x) d\omega'(s,x) \right) \nabla u(r+t,x) \\
&= \hat{D}_{r,x}^2 (F \circ \mathcal{T}^t(\omega, \omega')).
\end{aligned}$$

$\square$

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Although the elements ω of Ω are not invariant by translation, a similar commutation relation holds for the norm of $\hat{D}$.

Proposition 5 *Let F be a smooth functional of the form* (9). *We have the relation*

$$\|\hat{D}F\|^2_{L^2([t,\infty[\times\mathbb{R}^d;\mathbb{R}^{d+1},d\omega)} \circ \mathcal{T}^t(\omega,\omega') = \|\hat{D}^2(F \circ \mathcal{T}^t(\omega,\omega'))\|^2_{L^2(\mathbb{R}_+\times\mathbb{R}^d;\mathbb{R}^{d+1},d\omega')},$$

$\omega, \omega' \in \Omega.$

Proof. We have, using the notation $\omega' = \sum_k \epsilon_{t'_k,x'_k}$:

$$\|\hat{D}F\|^2_{L^2([t,\infty[\times\mathbb{R}^d;\mathbb{R}^{d+1},d\omega)} \circ \mathcal{T}^t(\omega,\omega')$$

$$= \int_{[t,\infty[\times\mathbb{R}^d} \sum_{k=1}^{k=n} \|\hat{D}_{r,x}F\|^2_{\mathbb{R}^{d+1}} \circ \mathcal{T}^t(\omega,\omega')\epsilon_{t+t'_k,x'_k}(dr,dx)$$

$$= \int_{[0,\infty[\times\mathbb{R}^d} \sum_{k=1}^{k=n} \|\hat{D}_{r+t,x}F\|^2_{\mathbb{R}^{d+1}} \circ \mathcal{T}^t(\omega,\omega')\epsilon_{t'_k,x'_k}(dr,dx)$$

$$= \int_{[0,\infty[\times\mathbb{R}^d} \sum_{k=1}^{k=n} \|\hat{D}_{r,x}(F \circ \mathcal{T}^t(\omega,\omega'))\|^2_{\mathbb{R}^{d+1}}\epsilon_{t'_k,x'_k}(dr,dx)$$

$$= \int_{\mathbb{R}_+\times\mathbb{R}^d} \|\hat{D}^2_r(F \circ \mathcal{T}^t(\omega,\omega'))\|^2_{\mathbb{R}^{d+1}}d\omega'.$$

$\square$

This implies, using an argument similar to Th. 1 or Prop. 4 above, and e.g. Th. 1.3.1 in [5], that F has a conditional density given $\mathcal{F}_t$ whenever $F \in \text{Dom}(\hat{D})$ and

$$\int_{[t,\infty[\times\mathbb{R}^d} \|\hat{D}F(\omega)\|^2_{\mathbb{R}^{d+1}}d\omega > 0, \quad P(d\omega) - a.s.$$

However the integration by parts formula for $\hat{D}$ reads

$$E[\langle\hat{D}F(\omega),U\rangle_{L^2(\mathbb{R}_+\times\mathbb{R}^d;\mathbb{R}^{d+1},d\omega)}] = E[F\delta(\text{div }U)], \quad U \in \mathcal{C}^\infty_c(\mathbb{R}_+ \times \mathbb{R}^d;\mathbb{R}^{d+1}),$$

where div U is the divergence of U, cf. e.g. [1], hence there is no commutation relation between $\mathcal{T}^t$ and the adjoint $\hat{\delta}$ of $\hat{D}$ since $\pi_{[t}U$ does not belong to Dom $\hat{\delta}$. Thus Lemma 2, Lemma 4 and Th. 3 can not be stated with $\hat{D}$.

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