A concentration inequality on Riemannian path space

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Abstract. The covariance representations method to obtain concentration inequalities is applied to functionals of Riemannian Brownian motion. This recovers, in particular, tail estimates for Brownian motion on a manifold.

1. Introduction and Background

Concentration and deviation inequalities have been obtained on the Wiener and Poisson spaces and for discrete random walks in [7] using the method of covariance representations ([1], [6]). In this note we present an application of this method to concentration inequalities on path space. The concentration results for the laws of general random variables presented here on Riemannian path space are slightly weaker than the ones obtained from logarithmic Sobolev inequalities (see [8]). Nevertheless they allow us to recover some classical bounds such as the one for the law of the supremum of the distance of Riemannian Brownian motion to the origin. The covariance representation method relies on the Clark formula on path space, a short proof of which is obtained by showing that the damped and flat gradients have the same adapted projections, as a consequence of an intertwining formula using Skorohod integrals.

In [7] we showed that the use of semi-groups for covariance representations allows us to recover the concentration and deviation inequalities obtained from logarithmic Sobolev inequalities and the Herbst method [8]. In particular, it turned out that covariance representations written in terms of the Clark formula generally yield weaker results than covariance representations written in terms of semi-groups. In the path case, however, covariance representations using Ornstein-Uhlenbeck type semi-groups are unknown to the authors. If available, they would allow us to recover the concentration results that follow from the Herbst method and the logarithmic Sobolev inequalities of [2].

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2. Preliminaries and notation

Let $(x(t))_{t\in[0,T]}$ denote the $\mathbb{R}^d$-valued Brownian motion on the Wiener space W with Wiener measure μ, generating the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Let M be a Riemannian manifold of dimension d whose Ricci curvature is uniformly bounded from below, and let $O(M)$ denote the bundle of orthonormal frames over M. The Levi-Civita parallel transport defines d canonical horizontal vector fields $A_1,\ldots,A_d$ on $O(M)$, and the Stratonovich stochastic differential equation

$$\begin{cases} dr(t) = \sum_{i=1}^{i=d} A_i(r(t)) \circ dx^i(t), & t\in[0,T], \\ r(0) = (m_0, r_0) \in O(M), \end{cases}$$

defines an $O(M)$-valued process $(r(t))_{t\in[0,T]}$. Let $\pi : O(M) \longrightarrow M$ be the canonical projection, let $\gamma(t) = \pi(r(t))$, $t\in[0,T]$, be the Brownian motion on M and let the Itô parallel transport along $(\gamma(t))_{t\in[0,T]}$ be defined as

$$t_{t\leftarrow 0} = r(t) r_0^{-1} : T_{m_0}M \simeq \mathbb{R}^d \longrightarrow T_{\gamma(t)}M, \quad t\in[0,T].$$

Let $\mathcal{C}_0([0,T];\mathbb{R}^d)$ denote the space of continuous $\mathbb{R}^d$-valued functions on $[0,T]$ vanishing at the origin. Let also $\mathbf{P}(M)$ denote the set of continuous paths on M starting at m_0, let

$$\begin{aligned} I \quad : \quad & \mathcal{C}_0([0,T];\mathbb{R}^d) \longrightarrow \mathbf{P}(M) \\ & (\omega(t))_{t\in[0,T]} \mapsto I(\omega) = (\gamma(t))_{t\in[0,T]} \end{aligned}$$

be the Itô map, and let ν denote the image measure on $\mathbf{P}(M)$ of the Wiener measure μ by I. Let Ω_r denote the curvature tensor of M, and let $\mathrm{ric}_r : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be the Ricci tensor at the frame $r\in O(M)$. Given an adapted process $(z_t)_{t\in[0,T]}$ with absolutely continuous trajectories, we let $(\hat{z}(t))_{t\in[0,T]}$ be defined by

$$\dot{\hat{z}}(t) = \dot{z}(t) + \frac{1}{2}\mathrm{ric}_{r(t)} z(t), \quad t\in[0,T], \quad \hat{z}(0) = 0. \tag{2.1}$$

We recall that $z \mapsto \hat{z}$ can be inverted, i.e. there exists a process $(\tilde{z}_t)_{t\in[0,T]}$ such that $\hat{\tilde{z}} = z$, cf. Sect. 3.7 of [5]. Finally, let $Q_{t,s} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, be defined as

$$\frac{dQ_{t,s}}{dt} = -\frac{1}{2}\mathrm{ric}_{r(t)} Q_{t,s}, \quad Q_{s,s} = \mathrm{Id}_{T_{m_0}}, \quad 0\le s\le t,$$

and let

$$q(t,z) = -\int_0^t \Omega_{r(s)}(\circ dx(s), z(s)), \quad t\in[0,T],$$

where $\circ dx(s)$ denotes the Stratonovich differential. Let $Q_{t,s}^*$ be the adjoint of $Q_{t,s}$, let $\mathbb{H} = L^2([0,T],\mathbb{R}^d)$, and let $\mathbf{H} = L^\infty(\mathbf{P}(M), \mathbb{H}; d\nu)$. Let finally $\mathcal{C}_c^\infty(M^n)$ denote the space of infinitely differentiable functions with compact support in M^n.

We now recall various notions of gradient developed for the analysis on path space. The following definitions are found in [5]. The notation ∇_i^M denotes the gradient on M applied to the i-th variable of f.

: i) (flat gradient) Given $F = f(x(t_1), \ldots, x(t_n))$, $f \in \mathcal{C}_c^\infty((\mathbb{R}^d)^n)$, let

$$D_t F = \sum_{i=1}^{i=n} 1_{[0,t_i]}(t) \nabla_i^{\mathbb{R}^d} f(x(t_1), , \ldots, x(t_n)), \quad t \in [0,T].$$

: ii) (intrinsic gradient) Given $F = f(\gamma(t_1), \ldots, \gamma(t_n))$, $f \in \mathcal{C}_c^\infty(M^n)$, let:

$$\hat{D}_t F = \sum_{i=1}^{i=n} t_{0 \leftarrow t_i} \nabla_i^M f(\gamma(t_1), \ldots, \gamma(t_n)) 1_{[0,t_i]}(t), \quad t \in [0,T].$$

: iii) (damped gradient) Given $f \in \mathcal{C}_c^\infty(M^n)$, let:

$$\tilde{D}_t F = \sum_{i=1}^{i=n} 1_{[0,t_i]}(t) Q^*_{t_i,t} t_{0 \leftarrow t_i} \nabla_i^M f(\gamma(t_1), \ldots, \gamma(t_n)), \quad t \in [0,T].$$

Let $\mathrm{Dom}(D)$, $\mathrm{Dom}(\hat{D})$ and $\mathrm{Dom}(\tilde{D})$ denote the respective domains of D, $\hat{D}$ and $\tilde{D}$ as closable operators.

Still developing our background, let us recall the intertwining formula between $\hat{D}$ and D (Th. 2.6 of [3]). This can be written as

$$\int_0^T \dot{z}(t) \cdot (\hat{D}_t F) \circ I \; dt = \int_0^T \dot{z}(t) \cdot D_t(F \circ I) \; dt - \int_0^T (\hat{D}_t F) \circ I \cdot q(t,z) \odot dx(t),$$

where $\odot dx(t)$ is the Stratonovich differential. We use a different intertwining formula which is stated in terms of the Itô-Skorohod differential $dx(t)$ instead of the Stratonovich integral $\odot dx(t)$, cf. Cor. 5.2.1 and Cor. 5.4.1 of [9].

Proposition 2.1. *([9]) Let $z : W \times [0,T] \to \mathbb{R}^d$ be adapted with absolutely continuous trajectories and $\dot{z} \in L^2(W \times [0,T])$. For $F = f(\gamma(t_1), \ldots, \gamma(t_n))$, $f \in \mathcal{C}_c^\infty(M^n)$ we have:*

: *i) (intrinsic gradient)*

$$\int_0^T \dot{z}(t) \cdot (\hat{D}_t F) \circ I \; dt = \int_0^T \dot{\hat{z}}(t) \cdot D_t(F \circ I) \; dt + \int_0^T q(t,z) D_t(F \circ I) \cdot dx(t), \tag{2.2}$$

: *ii) (damped gradient):*

$$\int_0^T \dot{z}(t) \cdot (\tilde{D}_t F) \circ I \; dt = \int_0^T \dot{z}(t) \cdot D_t(F \circ I) dt + \int_0^T q(t,\tilde{z}) D_t(F \circ I) \cdot dx(t). \tag{2.3}$$

The intertwining formula will now lead us to a Clark type formula. Indeed, since $E_\mu \left[\int_0^T q(t,\tilde{z}) D_t(F \circ I) \cdot dx(t) \right] = 0$, it follows directly from (2.2) and (2.3) that the processes DF and $\tilde{D}F$ have the same adapted projections:

$$E_\mu[D_t(F \circ I) \mid \mathcal{F}_t] = E_\mu[(\tilde{D}_t F) \circ I \mid \mathcal{F}_t], \quad t \in [0,T], \tag{2.4}$$

$F = f(\gamma(t_1), \ldots, \gamma(t_n))$. Using this relation and the classical Clark formula on Wiener space

$$F \circ I \;\; = \;\; E_\mu[F \circ I] + \int_0^T E_\mu[D_t(F \circ I) \mid \mathcal{F}_t] \cdot dx(t),$$

we obtain the expression of the Clark formula on path space:

Proposition 2.2. *([5]) Let $F \in \mathrm{Dom}(\tilde{D})$, then*

$$F \circ I \quad = \quad E_\nu[F] + \int_0^T E_\mu[(\tilde{D}_t F) \circ I \mid \mathcal{F}_t] \cdot dx(t).$$

3. Concentration inequalities on Riemannian path space

Let $\mathrm{Cov}_\nu(F,G) = E_\nu[FG] - E_\nu[F]E_\nu[G]$ for $F, G \in L^2(\mathbf{P}(M), \nu)$. The following covariance identity is an immediate consequence of the Clark formula.

Proposition 3.1. *Let $F, G \in \mathrm{Dom}(\tilde{D})$, then*

$$\mathrm{Cov}_\nu(F,G) = E_\mu \left[\int_0^T (\tilde{D}_t F) \circ I \cdot E_\mu[(\tilde{D}_t G) \circ I \mid \mathcal{F}_t] \, dt \right]. \tag{3.1}$$

Next, we apply this covariance representation to the proof of a concentration inequality on path space.

Lemma 3.2. *Let $F \in \mathrm{Dom}(\tilde{D})$. If $\|\tilde{D}F\|_{L^2([0,T], L^\infty(\mathbf{P}(M)))} \leq C$, for some $C > 0$, then*

$$\nu(F - E_\nu[F] \geq x) \leq \exp\left(-\frac{x^2}{2C\|\tilde{D}F\|_{\mathbf{H}}} \right), \quad x \geq 0. \tag{3.2}$$

In particular, $E[e^{\lambda F^2}] < \infty$, for $\lambda < (2C\|\tilde{D}F\|_{\mathbf{H}})^{-1}$.

Proof. We first consider a bounded random variable $F \in \mathrm{Dom}(\tilde{D})$. The general case follows by approximating $F \in \mathrm{Dom}(\tilde{D})$ by the sequence $(\max(-n, \min(F,n)))_{n \geq 1}$. Let

$$\eta_F(t) = E_\mu[(\tilde{D}_t F) \circ I \mid \mathcal{F}_t], \quad t \in [0,T].$$

Assuming first that $E_\nu[F] = 0$, we have

$$\begin{aligned}
E_\nu[F e^{sF}] &= E_\mu \left[\int_0^T (\tilde{D}_u e^{sF}) \circ I \cdot \eta_F(u) du \right] \\
&= sE_\mu \left[e^{sF \circ I} \int_0^T (\tilde{D}_u F) \circ I \cdot \eta_F(u) du \right] \\
&\leq sE_\mu \left[e^{sF \circ I} \|\tilde{D}F\|_{\mathbb{H}} \circ I \; \|\eta_F\|_{\mathbb{H}} \right] \\
&\leq sE_\nu \left[e^{sF} \right] \|\tilde{D}F\|_{\mathbf{H}} \|\eta_F\|_{L^\infty(W, \mathbb{H})} \\
&\leq sCE_\nu \left[e^{sF} \right] \|\tilde{D}F\|_{\mathbf{H}}.
\end{aligned}$$

In the general case, letting $L(s) = E_\nu[e^{s(F - E_\nu[F])}]$ we obtain:

$$\log(E_\nu[e^{t(F-E_\nu[F])}]) \quad = \quad \int_0^t \frac{L'(s)}{L(s)} ds \leq \int_0^t \frac{E_\nu[(F - E_\nu[F]) e^{s(F-E_\nu[F])}]}{E_\nu[e^{s(F-E_\nu[F])}]} ds$$

$$= \frac{1}{2}t^2 C\|\tilde{D}F\|_{\mathbf{H}}, \quad 0 \le t \le T.$$

We now have for all $x \in \mathbb{R}_+$ and $t \in [0,T]$:

$$\nu(F - E_\nu[F] \ge x) \le e^{-tx} E_\nu[e^{t(F-E_\nu[F])}] \le \exp\left(\frac{1}{2}t^2 C\|\tilde{D}F\|_{\mathbf{H}} - tx\right),$$

which yields (3.2) after minimization in $t \in [0,T]$. □

Since $\|\tilde{D}F\|_{\mathbf{H}} \le \|\tilde{D}F\|_{L^2([0,T],L^\infty(\mathbf{P}(M)))}$, the inequality (3.2) is weaker than:

$$\nu(F - E_\nu[F] \ge x) \le \exp\left(-\frac{x^2}{2\|\tilde{D}F\|^2_{\mathbf{H}}}\right), \quad x \ge 0, \tag{3.3}$$

which follows (via the Herbst method [8]) from the logarithmic Sobolev inequalities of [2], whose proof relies on the Itô formula and the Clark formula (see also [4]). The method presented in this paper is, somehow, simpler, and moreover (3.2) is sufficient to recover some classical tail estimates for functionals of $(\gamma(t))_{t\in\mathbb{R}_+}$, see [8] and references therein. Below, let ρ denote the distance on M.

Corollary 3.3. *Let the Ricci tensor be bounded from below by a constant $-K \in \mathbb{R}$. Then,*

(3.4)

$$\nu\left(\sup_{0\le t\le T}\rho(\gamma(t),m_0) \ge E_\nu\left[\sup_{0\le t\le T}\rho(\gamma(t),m_0)\right] + x\right) \le \exp\left(-\frac{x^2}{2Te^{KT}}\right), \quad x \ge 0,$$

Proof. Let $f(m_1,\dots,m_n) = \max_{1\le i\le n}\rho(m_i,m_0)$ and

$$F = f(\gamma(t_1),\dots,\gamma(t_n)) = \max_{1\le i\le n}\rho(\gamma(t_i),m_0),$$

where $0 < t_1 < \cdots < t_n < T$ determines a partition of $[0,T]$. Following the argument of [8], p. 196, we obtain for a certain partition $(A_j)_{1\le j\le n}$ of $\mathbf{P}(M)$:

$$|\tilde{D}_t F| \le \sum_{i=1}^{i=n} 1_{(t_{i-1},t_i]}(t) \sum_{j=i}^{j=n} e^{K(t_j-t)/2}|\nabla^M_j f| \le e^{KT/2}\sum_{i=1}^{i=n} 1_{(t_{i-1},t_i]}(t)\sum_{j=i}^{j=n} 1_{A_j} \le e^{KT/2},$$

hence $\|\tilde{D}F\|_{L^2([0,T],L^\infty(\mathbf{P}(M)))} \le Te^{KT}$, which gives (3.4) by monotone convergence as the mesh of the partition goes to zero. □

The Herbst method applied to the logarithmic Sobolev inequalities of [2] would yield the same estimate as (3.4), since here the bound on $\|\tilde{D}_t F\|_\infty$ is uniform in t.

When $K = 0$ this leads to the known estimate (cf. [8] and references therein):

$$\nu\left(\sup_{0\le t\le T}\rho(\gamma(t),m_0) \ge E_\nu\left[\sup_{0\le t\le T}\rho(\gamma(t),m_0)\right] + x\right) \le \exp\left(-\frac{x^2}{2T}\right), \quad x \ge 0.$$

In the following we assume that the Ricci tensor of M is bounded from above and below.

Theorem 3.4. *Assume that the Ricci tensor of M is bounded by $K \geq 0$, i.e. $\|\mathrm{ric}_u\| \leq K$, $u \in O(M)$, and let $F \in \mathrm{Dom}(\hat{D})$ be such that $\|\hat{D}F\|_{L^2([0,T],L^\infty(\mathrm{P}(M)))} \leq C$, for some $C > 0$. Then*

$$\nu(F - E_\nu[F] \geq x) \leq \exp\left(-\frac{x^2}{2e^{K/2}(C + (e^{K/2} - 1)\|\hat{D}F\|_{\mathbf{H}})\|\hat{D}F\|_{\mathbf{H}}}\right), \quad x \geq 0.$$

Proof. Again, we may start by assuming that F is bounded. We have the bound

$$\begin{aligned}
|\tilde{D}_t F|^2 &\leq \left(|\hat{D}_t F| + \int_t^T \left\|\frac{d}{ds} Q^*_{t,s}\right\| |\hat{D}_s F| ds\right)^2 \\
&\leq \left(|\hat{D}_t F| + \frac{1}{2} K \int_t^T e^{K(s-t)/2} |\hat{D}_s F| ds\right)^2 \\
&\leq \left(|\hat{D}_t F| + \frac{1}{2}\sqrt{K(e^{K(T-t)} - 1)}\|\hat{D}F\|_{\mathbb{H}}\right)^2 \\
&\leq |\hat{D}_t F|^2 + \sqrt{K(e^{K(T-t)} - 1)}|\hat{D}_t F|\|\hat{D}F\|_{\mathbb{H}} + \frac{1}{4}K(e^{K(T-t)} - 1)\|\hat{D}F\|^2_{\mathbb{H}}
\end{aligned}$$

(see p. 75 of [2]), which implies

$$\|\tilde{D}F\|^2_{\mathbb{H}} \leq \left(1 + \frac{1}{2}\sqrt{e^{KT} - KT - 1}\right)^2 \|\hat{D}F\|^2_{\mathbb{H}} \leq e^{KT}\|\hat{D}F\|^2_{\mathbb{H}},$$

and

$$\begin{aligned}
\|\tilde{D}F\|_{L^2([0,T],L^\infty(\mathrm{P}(M)))} &\leq \|\hat{D}F\|_{L^2([0,T],L^\infty(\mathrm{P}(M)))} + \frac{1}{2}\sqrt{e^{KT} - KT - 1}\|\hat{D}F\|_{\mathbf{H}} \\
&\leq C + (e^{KT/2} - 1)\|\hat{D}F\|_{\mathbf{H}}.
\end{aligned}$$

It remains to apply Lemma 3.2. □

As a corollary we obtain

$$\nu(F - E_\nu[F] \geq x) \leq \exp\left(-\frac{x^2}{2e^{KT}C\|\hat{D}F\|_{\mathbf{H}}}\right), \quad x \geq 0. \tag{3.5}$$

A damped gradient satisfying a Clark formula can also be defined for diffusion processes on Riemannian manifolds, and in this case similar bounds hold between $\tilde{D}_t F$ and $\hat{D}_t F$, cf. Sect. 3 of [2] and the references therein. Hence the argument of Th. 3.4 will also be valid on the space of a diffusion process on a manifold.

For the finite dimensional random variable $F = f(\gamma(t))$ we have $\hat{D}F = 1_{[0,t]} t_{0\leftarrow t} \nabla^M f(\gamma(t))$ and

$$\|\hat{D}F\|^2_{L^2([0,T],L^\infty(\mathrm{P}(M)))} = t|\nabla^M f|^2_\infty = \|\hat{D}F\|^2_{\mathbf{H}},$$

hence both norms coincide and (3.5) becomes:

$$\nu(f(\gamma(t)) - E_\nu[f(\gamma(t))] \geq x) \leq \exp\left(-\frac{x^2}{2e^{KT}t|\nabla^M f|^2_\infty}\right), \quad x \geq 0, \tag{3.6}$$

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