Chaotic and variational calculus in discrete and continuous time for the Poisson process

Nicolas Privault

Abstract

We study a new interpretation of the Poisson space as a triplet (H, B, P) where H is a Hilbert space, B is the completion of H and P is the extension to the Borel $\sigma-$algebra of B of a cylindrical measure on B. A discrete chaotic decomposition of $L^2(B, P)$ is defined, along with multiple stochastic integrals of elements of H^{on}. It turns out that the directional derivative of functionals in $L^2(B, P)$ in the direction of an element of H is an annihilation operator on the discrete chaotic decomposition. By composition with the Poisson process, we deduce continuous-time operators of derivation and divergence that form the number operator on the discrete chaotic decomposition. Those results are applied to the representation of random variables in the Wiener-Poisson chaotic decomposition.

KEY WORDS: Poisson process, Stochastic calculus of variations, Chaotic decompositions.

1 Introduction

The space of square-integrable functionals of the Poisson process is isomorphic to the Fock space on $L^2(\mathbb{R}_+)$, through the Wiener-Poisson decomposition, cf. Nualart-Vives[8]. It is also known that the annihilation operator on the Wiener-Poisson chaotic decomposition is a finite difference operator, whose adjoint coincides with the stochastic integral with respect to the compensated Poisson process on the square-integrable predictable processes. This finite difference operator, however, lacks a property which can be important in applications. Namely, it is not a derivation, and it can not be expressed as a directional derivative. Our aim is to introduce an interpretation of the Poisson space which is closely connected to the Wiener space approach, and to define an annihilation operator which is also a directional derivative. In this way, the methods that were applied on Wiener space work also on Poisson space, and can yield results in Malliavin calculus, anticipative stochastic differential equations and for the absolute continuity of the Poisson measure with respect to nonlinear anticipative transformations. The starting point of this work is the article [4] by Carlen and Pardoux who introduced a notion of differential calculus on Poisson space by shifting the jump times of a standard Poisson process, see also the book by Bouleau and Hirsch, [3] pp. 235-239 for another approach. For other existing approaches to the stochastic calculus of variations and the Malliavin calculus on Poisson space, we refer for instance to [1] and [2]. We proceed as follows. In section 2 we introduce a triplet (H, B, P) where $H = l^2(\mathbb{N})$ is the Hilbert space of square-summable sequences, B is the completion of H with respect to a certain norm, and P is the extension to the Borel $\sigma-$algebra of B of a cylindrical measure such that the canonical projections $(\tau_k)_{k\in\mathbb{N}}$ from B to $\mathbb{R}$ are independent exponentially distributed random variables under P, and represent the interjump times of the Poisson process. We show that there exists a discrete chaotic decomposition of $L^2(B, P)$ in which any element of $L^2(B, P)$ can be represented as a sum of multiple stochastic integrals of kernels in $H^{on} = l^2(\mathbb{N})^{\circ n}$, space of symmetric square-summable functions on $\mathbb{N}^n$. For any $F \in L^2(B, P)$, we have

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

with $f_n \in H^{on}$, $n \in \mathbb{N}$, and $I_n(f_n)$ is the discrete multiple stochastic integral of a symmetric function of discrete variable, defined with the Laguerre polynomials.
In section 3, we deduce from this decomposition an annihilation operator D and its adjoint δ, the divergence operator. We show that D is also a directional derivative. For F smooth and cylindrical, we have

$$(DF, h)_H(\omega) = -\lim_{\varepsilon\to 0} \frac{F(\omega + \varepsilon h) - F(w)}{\varepsilon} \quad h \in H.$$

The operators D and δ are closable and identified with their closed extensions. In order to obtain results that would be more closely related to the Poisson process, we need to define continuous-time analogues of D and δ, this is the object of section 4. Given $F \in Dom(D)$, $(D_nF)_{n\in\mathbb{N}}$ is a discrete-time stochastic process, and we want to construct from $(D_nF)_{n\in\mathbb{N}}$ a continuous-time stochastic process. For $\omega \in B$ fixed, we look at the composition of $DF(\omega) : \mathbb{N} \to \mathbb{R}$ with the Poisson process trajectory $(N_t(\omega))_{t\in\mathbb{R}_+}$, and define for $t \in \mathbb{R}_+$: $\tilde{D}_tF = D_{N_{t-}}F$. It turns out that $\tilde{D}$ is the derivation operator on Poisson space that was defined in [3, 4]. As expected, the adjoint $\tilde{\delta}$ of $\tilde{D}$ coincides with the stochastic integral with respect to the compensated Poisson process on the predictable processes in $L^2(B) \otimes L^2(\mathbb{R}_+)$. We denote by ∇ the finite difference operator which is the annihilation operator of the Wiener-Poisson chaotic decomposition, cf. [8]. The adjoint ∇^* of ∇ coincides with the compensated Poisson stochastic integral on the predictable processes in $L^2(B) \otimes L^2(\mathbb{R}_+)$. Hence $\tilde{\delta}$ coincides also with ∇^* on the predictable processes, and this gives two different expressions for the Clark-Hausmann-Üstünel formula. However, $\tilde{\delta}$ does not coincide with ∇^* for anticipative processes, and $\tilde{D}$ is totally different from ∇. Another property of the annihilation operator and its adjoint on the Wiener, cf. [13] and Wiener-Poisson, cf. [8], decompositions is that their composition is a number operator. The composition δD did not turn out to be a number operator on the discrete chaotic decomposition. We find in section 5 that the composition $L = \tilde{\delta}\tilde{D}$ is effectively a number operator on the discrete chaotic decomposition of $L^2(B, P)$. The operator L can be used to define spaces of test functions and distributions on B, and will help characterize the domain of $\tilde{D}$. Finally, in section 6 we use $\tilde{D}$ and ∇ to determine the developments in the Wiener-Poisson chaotic decomposition of the jump times of the Poisson process, and it appears that the combined use of $\tilde{D}$ and ∇ makes explicit calculations possible.

2 Exponential space

2.1 The triplet (H,B,P)

The aim of this section is to construct a probability measure P on a sequence space B such that the projections (i.e. the coordinate functionals)

$$\begin{aligned} \tau_n : \ & B \to \mathbb{R} \\ & x \to x_n \end{aligned}$$

where $x \in B$ is the sequence $x = (x_n)_{n\in\mathbb{N}}$, are independent exponentially distributed random variables. The projection τ_n represents the time between the n-th and n+1-th jumps of the Poisson process, and it is easy to reconstruct the Poisson process from the $\tau_n' s$. We let $H = l^2(\mathbb{N})$. The Hilbert space H will have the role played by the Cameron-Martin space on Wiener space, and H will be the domain of the basic isometry I_1 which corresponds to the stochastic integral in the Wiener and Wiener-Poisson cases. Consider the norm $\| \cdot \|_B$ on H defined by

$$\| x \|_B = \sup_{k\in\mathbb{N}} \frac{| x_k |}{k+1}.$$

Definition 1 *We denote by B the completion of H with respect to the norm $\| \,.\, \|$.*

Note that if $(\xi_k)_{k\in\mathbb{N}}$ is a sequence of independent exponential random variables on a probability space (Ω, P), then

$$\sup_{k\in\mathbb{N}} \frac{| \xi_k |}{k+1} < +\infty \quad P-a.s.$$

Indeed,

$$\begin{aligned} P(\sup_{k\in\mathbb{N}} \frac{| \xi_k |}{k+1} < a) &= \lim_{n\to+\infty} \prod_{k=0}^{k=n} (\int_0^{(k+1)a} e^{-x} dx) \\ &\geq \lim_{n\to+\infty} 1 - e^{-a} \sum_{k=0}^{k=n} e^{-ak} \geq 1 - \frac{e^{-a}}{1-e^{-a}}, \end{aligned}$$

which goes to 1 as a goes to infinity. As a consequence of this remark, we have:

Proposition 1 *There is a probability measure P on the Borel σ-algebra $\mathcal{F}$ of B such that the coordinate functionals $(\tau_k)_{k\in\mathbb{N}}$ are exponential independent random variables.*

See [9] for a detailed proof.
We have that $P(H) = 0$, since

$$\left(\frac{1}{n+1} \sum_{k=0}^{k=n} \tau_k^2 \right)_{n\in\mathbb{N}}$$

converges to $E[\tau_0^2]$ as n goes to infinity, from the law of large numbers.
To end this section, we relate (H, B, P) to the Poisson process. We let $L^2(B) =$

$L^2(B, \mathcal{F}, P)$. From the above construction, the family of projections $(\tau_n)_{n\in\mathbb{N}}$ is a sequence of independent exponentially distributed random variables. We can easily reconstruct a Poisson process, starting from this system of random variables. Let

$$T_n = \sum_{k=0}^{k=n-1} \tau_k$$

for $n \geq 1$ and $T_0 = 0$. The $T_n's$, $n \geq 1$, represent the jump times of a Poisson process. If we let

$$N_t = \sum_{n=1}^{\infty} 1_{[T_n, +\infty[}(t), \quad n \geq 1,$$

then $(N_t)_{t\in\mathbb{R}_+}$ is a Poisson process on $(B, \mathcal{F}, P)$.

2.2 Discrete multiple stochastic integrals

We start by defining the stochastic integral of a function $f \in l^2(\mathbb{N})$ as an isometry from $H = l^2(\mathbb{N})$ to $L^2(B, \mathcal{F}, P)$. We let

$$I_1(f) = \int_0^{+\infty} f(N_{t^-}) d(N_t - t).$$

This also means that $I_1(f) = \sum_{k=0}^{\infty}(1 - \tau_k).f_k$ if $f \in l^2(\mathbb{N})$ is the sequence $(f_k)_{k\in\mathbb{N}}$, cf. also Bouleau-Hirsch[3], p. 239. We could also say that we integrate f with respect to the process $(n - T_n)_{n\geq 0}$ with independent increments $(1 - \tau_n)_{n\geq 0}$ of zero expectation, which is a martingale. The application I_1 is clearly an isometry, and this can be shown in several ways, e.g.

$$E[I_1(f)^2] = \sum_{k,l\in\mathbb{N}} f_k f_l E[(1 - \tau_k).(1 - \tau_l)] = \sum_{k=0}^{\infty} f_k^2 = \| f \|^2_{l^2(\mathbb{N})} .$$

Let us now define multiple stochastic integrals of elements of the completed symmetric tensor product $H^{\circ n} = l^2(\mathbb{N})^{\circ n}$. In the Wiener and Wiener-Poisson cases, this definition is achieved by iterating the stochastic integral, or using Hermite or Charlier polynomials. We prefer to use polynomials here than to iterate the stochastic integral I_1. In the Wiener-Poisson case the iteration of the stochastic integral is performed on predictable processes, by avoiding the integration on the diagonals in $\mathbb{R}_+^n$, which are of null Lebesgue measure. But here diagonals in $\mathbb{N}^n$ play an important role, and we must also integrate on them, which means that the discrete multiple stochastic integrals can not be defined as iterated stochastic integrals of discrete-time predictable processes. Instead of the Hermite or Charlier polynomials, we need the Laguerre polynomials $(L_n)_{n\in\mathbb{N}}$, defined as

$$L_n(x) = \sum_{k=0}^{k=n} C_n^k (-1)^k \frac{x^k}{k!} \qquad x \in \mathbb{R}_+.$$

The Laguerre polynomials $(L_n)_{n\in\mathbb{N}}$ form an orthonormal sequence in $L^2(\mathbb{R}, 1_{\{x>0\}}e^{-x}dx)$. The main properties of this sequence are summarized below.

$$xL_n(x) = -(n + 1)L_{n+1}(x) + (2n + 1)L_n(x) - nL_{n-1}(x) \quad n \geq 1 \tag{1}$$

$$\frac{exp(\frac{-\alpha x}{1-\alpha})}{1-\alpha} = \sum_{n\geq 0} \alpha^n L_n(x) \quad \mid \alpha \mid < 1, x \geq 0 \tag{2}$$

$$\frac{d}{dx} L_n(x) = -\sum_{k=0}^{k=n-1} L_k(x). \tag{3}$$

Definition 2 *Let $\mathcal{P}$ be the set of functionals of the form $Q(\tau_0, ..., \tau_{n-1})$ where Q is a real polynomial and $n \in \mathbb{N}$.*

We now define a "Wick product" between elements of $\mathcal{P}$. Since $\mathcal{P}$ is an algebra generated by $\{L_n(\tau_k) \ : \ k, n \in \mathbb{N}\}$, it is sufficient to define this product between elements of $\{L_n(\tau_k) \ : \ k, n \in \mathbb{N}\}$. The Wick product allows to define the multiple stochastic integrals in several steps, instead of writing the combinatorial expression that defines $I_n(f_n)$. If $\alpha \in \mathbb{N}^d$, we let $\alpha! = \alpha_1! \cdots \alpha_d!$.

Definition 3 *The Wick product of two elements $F, G \in \mathcal{P}$ is denoted by $F{:}G$ and is defined by*

$$(L_{n_1}(\tau_{k_1}) \cdots L_{n_d}(\tau_{k_d})) : (L_{m_1}(\tau_{k_1}) \cdots L_{m_d}(\tau_{k_d})) = \frac{(n+m)!}{n!m!} L_{n_1+m_1}(\tau_{k_1}) \cdots L_{n_d+m_d}(\tau_{k_d})$$

if $n = (n_1, \ldots, n_d)$, $m = (m_1, \ldots, m_d)$ and $k_1 \neq \cdots \neq k_d$.

Now, the discrete multiple stochastic integral $I_n(f_n)$, f_n symmetric in $l^2(\mathbb{N}^n)$ with a finite support, is directly defined using the Wick product. If $f_1, \ldots, f_n \in H$, the symmetric tensor product $f_1 \circ \cdots \circ f_n$ is defined as

$$f_1 \circ \cdots \circ f_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}$$

where Σ_n is the set of all permutations of $\{1, \ldots, n\}$.

Definition 4 *For $g_1, ..., g_n \in l^2(\mathbb{N})$ with finite supports, we let*

$$I_n(g_1 \circ \cdots \circ g_n) = I_1(g_1) : \cdots : I_1(g_n)$$

where $I_1(g_i) = \sum_{k=0}^{\infty} g_i(k) L_1(k)$, $1 \leq i \leq n$.

The application I_n will later be extended to any element of $l^2(\mathbb{N})^{\circ n}$ by a density argument. So far, $I_n(f_n)$ is still an element of $\mathcal{P}$, i.e. a polynomial in the $\tau_k{}'s$. Next, we look in more details at explicit formulae for I_n, deduced from the above definitions.

Proposition 2 *Let $(e_n)_{n\in\mathbb{N}}$ be the canonical basis in $l^2(\mathbb{N})$.*

- *For $k_1 \neq \cdots \neq k_d$ and $n_1 + \cdots + n_d = n$,*

$$I_n(e_{k_1}^{\circ n_1} \circ \cdots \circ e_{k_d}^{\circ n_d}) = n_1! \cdots n_d! L_{n_1}(\tau_{k_1}) \cdots L_{n_d}(\tau_{k_d}).$$

- *If $f \in l^2(\mathbb{N})$ has a finite support,*

$$I_n(f^{\circ n}) = n! \sum_{\substack{k_1 \neq \cdots \neq k_d \\ n_1 + \cdots + n_d = n \\ n_1, ..., n_d > 0}} f_{k_1}^{n_1} \cdots f_{k_d}^{n_d} L_{n_1}(\tau_{k_1}) \cdots L_{n_d}(\tau_{k_d}).$$

- *If $f_n \in l^2(\mathbb{N}^n)$, $g_m \in l^2(\mathbb{N}^m)$ are symmetric with finite supports,*

$$I_n(f_n) : I_m(g_m) = I_{n+m}(f_n \circ g_m).$$

The last part of the proposition shows the utility of the Wick product.
Proof. The first part is obvious, and the third part is a consequence of the definition of I_n by the Wick product. Let us prove the second part. We have $f = \sum_{k=0}^{\infty} f_k.e_k$ and the combinatorial expression

$$f^{on} = \sum_{\substack{k_1 \neq \cdots \neq k_d \\ n_1 + \cdots + n_d = n}} \frac{n!}{n_1! \cdots n_d!} f_{k_1}^{n_1} \cdots f_{k_d}^{n_d}.e_{k_1}^{on_1} \circ \cdots \circ e_{k_d}^{on_d}.$$

Using the first part of the proposition yields the formula. □
The multiple stochastic integrals with respect to the Brownian motion or to the compensated Poisson process are isometries and such multiple integrals of different orders are orthogonal random variables. We now establish an isometry property for I_n that presents a small difference with the isometry properties existing in the Wiener and Wiener-Poisson cases. Define $D_n = \{(k_1, ..., k_n) \in \mathbb{N}^n \ : \ \exists i \neq j \ \textit{such that} \ k_i = k_j\}$, which represents the diagonals in $\mathbb{N}^n$, and let $X_n = \mathbb{N}^n \setminus D_n$. The application I_n satisfies to the following isometry property:

Proposition 3 *If $f_n \in l^2(\mathbb{N}^n)$ and $g_m \in l^2(\mathbb{N}^m)$ are symmetric with finite supports,*

$$< I_n(f_n), I_m(g_m) >_{L^2(B)} = n! < f_n, g_m >_{l^2(X_n)} + (n!)^2 < f_n, g_m >_{l^2(D_n)}$$

if n=m, and

$$< I_n(f_n), I_m(g_m) >_{L^2(B)} = 0$$

if $n \neq m$.

The supplementary term $(n!)^2 < f_n, g_m >_{l^2(D_n)}$ corresponds to the integration on the diagonals in $\mathbb{N}^n$.
Proof. If $n \neq m$, it is clear that the scalar product in $L^2(B)$ is zero, by looking at the exact expressions for $I_n(f_n)$ and $I_m(g_m)$, f_n and g_m being elementary functions, using the facts that Laguerre polynomials are orthonormal in $L^2(\mathbb{R}, 1_{\{x>0\}}e^{-x})$ and that $(\tau_k)_{k \in \mathbb{N}}$ are independent exponentially distributed random variables. If $n = m$, we only treat the case where $f_n = g_n = f^{on}$, because the formulae for general f_n and g_n can be obtained by use of the polarization identity

$$h_1 \circ \cdots \circ h_n = \frac{1}{n!} \sum_{k=1}^{k=n} (-1)^{n-k} \sum_{l_1 < \cdots < l_k} (h_{l_1} + \cdots + h_{l_k})^{\circ n}.$$

The RHS of the equality that we have to prove is equal to

$$n!(< f^{on}, f^{on} >_{l^2(X_n)} + n! < f^{on}, f^{on} >_{l^2(D_n)}) = (n!)^2 \sum_{\substack{k_1 \neq \cdots \neq k_d \\ n_1 + \cdots + n_d = n}} (f_{k_1}^2)^{n_1} \cdots (f_{k_d}^2)^{n_d}.$$

From the second part of the preceding proposition, this equals the LHS of our equality. □

The preceding definitions and statements are now extended to the completed symmetric tensor product $H^{\circ n} = l^2(\mathbb{N})^{\circ n}$, because the l^2-norm $\| \cdot \|_{l^2(\mathbb{N}^n)}$ on $l^2(\mathbb{N}^n)$ is equivalent to $f_n \to \sqrt{n!}(< f_n, f_n >_{l^2(X_n)} + n! < f_n, f_n >_{l^2(D_n)})^{\frac{1}{2}}$: we have

$$\begin{aligned} &< f^{on}, f^{on} >_{l^2(X_n)} + n! < f^{on}, f^{on} >_{l^2(D_n)} \\ &\leq n! \| f \|_{l^2(\mathbb{N})} \\ &\leq n!(< f^{on}, f^{on} >_{l^2(X_n)} + n! < f^{on}, f^{on} >_{l^2(D_n)}). \end{aligned}$$

2.3 The discrete chaotic decomposition of $L^2(B, \mathcal{F}, P)$

We define the chaos C_n of order $n \in \mathbb{N}$ in $L^2(B)$ by $C_n = \{I_n(f_n) \ : \ f_n \in l^2(\mathbb{N})^{\circ n} \}$. It contains the polynomials of degree n in the $\tau_k's$, and is orthogonal to $C_0, C_1, ..., C_{n-1}$.

Proposition 4 *$L^2(B)$ has a chaotic decomposition which is different from the Wiener-Poisson decomposition:*

$$L^2(B, \mathcal{F}, P) = \bigoplus_{n=0}^{\infty} C_n.$$

Proof. We know that the chaos $(C_n)_{n\geq 0}$ are orthogonal, from the orthogonality of multiple stochastic integrals of different orders. We only need to show that $\mathcal{P}$ is dense in $L^2(B)$. From [7], p. 540, we have that the polynomials are dense in $L^2(\mathbb{R}_+, e^{-x}dx)$, since the function $t \longrightarrow \Phi(tz) = \frac{1}{1-itz}$ is analytic in a neighborhood of 0 $\forall z \in \mathbb{R}$, where Φ is the characteristic function of the exponential law. Hence, if $F \in L^2(B)$ and

$$E[FP_0(\tau_0) \cdots P_n(\tau_n)] = 0$$

for any polynomials $P_0, \dots, P_n$, $n \in \mathbb{N}$, then $E[F \mid \tau_0, \dots \tau_n] = 0$, $n \in \mathbb{N}$. Moreover, $(E[F \mid \tau_0, \dots, \tau_n])_{n\in\mathbb{N}}$ is a discrete-time martingale, and from [6], Th.3.20,

$$\lim_{n\to\infty} E[F \mid \tau_0, \dots, \tau_n] = F \quad P - a.s.,$$

so that $F = 0$ P-a.s., and $\mathcal{P}$ is dense in $L^2(B)$. □

Remark 1 *If we define the Fock space $\mathcal{F}(H)$ in the following way, then it is isomorphic to $L^2(B)$. Let K_n be the tensor product H^{on}, endowed with the norm*

$$||| f_n |||_n^2 = n! < f_n, f_n >_{l^2(X_n)} + n!^2 < f_n, f_n >_{l^2(D_n)}$$

which is equivalent to $\| \cdot \|_{l^2(\mathbb{N})}$, and let $\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} K_n$. Taking $F = \sum_{n=0}^{\infty} I_n(f_n)$ in $L^2(B) = \bigoplus_{n=0}^{\infty} C_n$, we can associate $\sum_{n=0}^{\infty} f_n \in \mathcal{F}(H)$ to F with $\| F \|_{L^2(B)}^2 = \sum_{n=0}^{\infty} ||| f_n |||_n^2 = \| \sum_{n=0}^{\infty} f_n \|_{\mathcal{F}(H)}^2$, hence $\mathcal{F}(H)$ is isometrically isomorphic to $L^2(B)$.

3 The derivative and its adjoint

In this section, we define an annihilation operator on the discrete chaotic decomposition of $L^2(B)$, and we show that it coincides with a directional derivative defined by

perturbations of the jump times of the Poisson process. We also give an expression for its adjoint operator.
We define the directional derivative in the case where the functionals that we derive are Hilbert space-valued. Let X be a real separable Hilbert space with an orthonormal basis $(h_k)_{k\in\mathbb{N}}$, and let $\mathcal{P}(X)=\left\{\sum_{i=0}^{i=n} Q_i h_i \;:\; Q_1,\dots Q_n\in\mathcal{P},\;\; n\in\mathbb{N}\right\}$.

Definition 5 *We define an operator $D:\mathcal{P}(X)\longrightarrow L^2(B\times\mathbb{N};X)$ by*

$$(DF(\omega),h)_{l^2(\mathbb{N})}=-\lim_{t\to 0}\frac{F(\omega+th)-F(\omega)}{t}\qquad h\in l^2(\mathbb{N}).$$

This defines an X-valued discrete-time process $(D_kF)_{k\in\mathbb{N}}$. We omit writing X if $X=\mathbb{R}$. The difference with the definition of the annihilation operator Ψ_k which is defined below lies in the fact that we used here the vector space structure of B to define the derivation D.

Definition 6 *For $k\in\mathbb{N}$, we define an annihilation operator Ψ_k on $\mathcal{P}$ by*

$$\Psi_k(I_n(f_n))=\sum_{p=0}^{n-1}\frac{n!}{p!}I_p(f_n(*,\underbrace{k,...,k}_{n-p\ times})).$$

In terms of Fock space, this operator Ψ_k transforms a n-particle state into a mixed state with $0,1,2,...,n-2$ or $n-1$ particles. The next proposition shows that D_k is identical to Ψ_k, $k\in\mathbb{N}$ when $X=\mathbb{R}$.

Proposition 5 *If $F\in\mathcal{P}$, then $D_kF=\Psi_kF$, i.e.*

$$D_k(I_n(f_n))=\sum_{p=0}^{p=n-1}\frac{n!}{p!}I_p\left(f_n(*,\underbrace{k,...,k}_{n-p\ times})\right)\qquad k\in\mathbb{N}.$$

Proof. It suffices to prove this equality for any F of the form

$$F=I_n(e_{k_1}^{\circ n_1}\circ\cdots\circ e_{k_d}^{\circ n_d})=n_1!\cdots n_d!L_{n_1}(\tau_{k_1})\cdots L_{n_d}(\tau_{k_d}),$$

whith $n=n_1+\cdots+n_d$ and $k_1\neq\cdots\neq k_d$. We have

$$\begin{aligned}
D_kF &= n_1!\cdots n_d!\sum_{i=1}^{i=d}L_{n_1}(\tau_{k_1})\cdots\widehat{L_{n_i}(\tau_{k_i})}\cdots L_{n_d}(\tau_{k_d})\times\left(\sum_{j=1}^{j=n_i}L_{n_i-j}(\tau_{k_i})1_{\{k_i=k\}}\right)\\
&= \sum_{i=1}^{i=d}1_{\{k_i=k\}}\sum_{j=1}^{j=n_i}\frac{n_i!}{(n_i-j)!}I_{n-j}(e_{k_1}^{\circ n_1}\circ\cdots\circ e_{k_i}^{\circ(n_i-j)}\circ\cdots\circ e_{k_d}^{\circ n_d}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Psi_kF &= \sum_{p=0}^{p=n-1}\frac{n!}{p!}I_p(e_{k_1}^{\circ n_1}\circ\cdots\circ e_{k_d}^{n_d}(*,k,\dots k))\\
&= \sum_{i=1}^{i=d}\sum_{p=n-n_i}^{p=n-1}\frac{n_i!}{(n_i-(n-p))!}1_{\{k_i=k\}}I_p(e_{k_1}^{\circ n_1}\circ\cdots\circ e_{k_i}^{\circ(n_i-(n-p))}\circ\cdots\circ e_{k_d}^{\circ n_d})\\
&= \sum_{i=1}^{i=d}1_{\{k_i=k\}}\sum_{j=1}^{j=n_i}\frac{n_i!}{(n_i-j)!}I_{n-j}(e_{k_1}^{\circ n_1}\circ\cdots\circ e_{k_i}^{\circ(n_i-j)}\circ\cdots\circ e_{k_d}^{\circ n_d}).\quad\square
\end{aligned}$$

We now turn to the definition of the adjoint δ of D.

Proposition 6 *Let $\mathcal{U}$ denote the set of elements of $u \in L^2(B) \otimes l^2(\mathbb{N})$ such that $u_k = \tau_k h_k$ $k \in \mathbb{N}$, where $h : \mathbb{N} \longrightarrow \mathcal{P}$ has a finite support in $\mathbb{N}$. We have that $\mathcal{U}$ is dense in $L^2(B) \otimes l^2(\mathbb{N})$.*

Proof. It is sufficient to prove this in dimension one, since $\mathcal{P}$ is dense in $L^2(B)$. If $f(x) = \sum_{n=0}^{\infty} a_n L_n(x)$ with $\sum_{n=0}^{\infty} a_n^2 < \infty$, then using the relation $xL_n(x) = -(n+1)L_{n+1}(x) + (2n+1)L_n(x) - nL_{n-1}(x)$ gives

$$xf(x) = a_0 L_0 - a_0 L_1(x) + \sum_{n=1}^{\infty} -(n+1)a_n L_{n+1}(x) + (2n+1)a_n L_n(x) - na_n L_{n-1}(x).$$

If $\int_0^\infty xf(x)L_k(x)e^{-x}dx = 0$, $\forall k \in \mathbb{N}$, then $a_0 = a_1$ for $k = 0$, and $ka_{k-1} - (2k+1)a_k + (k+1)a_{k+1} = 0$ for $k \geq 1$. This implies that $a_{k+1} = a_k$, $\forall k \in \mathbb{N}$. Hence $a_k = 0$, $\forall k \in \mathbb{N}$, given that $(a_k)_{k\in\mathbb{N}}$ is square-summable. $\square$

Proposition 7 *We define an operator $\delta : \mathcal{U} \longrightarrow L^2(B)$ by*

$$\delta(u) = -\sum_{k=0}^{\infty}(u_k + D_k u_k).$$

Then D, δ are closable and adjoint operators. We identify D and δ with their closed extensions, and denote their domains by $Dom(D)$ and $Dom(\delta)$. For any $F \in Dom(D)$ and $u \in Dom(\delta)$, we have

$$E[(DF, u)_{l^2(\mathbb{N})}] = E[F\delta(u)]. \tag{4}$$

Proof. First we prove relation 4 in the cylindrical case, and then the closability of D and δ. Let $u \in \mathcal{U}$ and $F \in \mathcal{P}$ with $F = f(\tau_0, \ldots, \tau_n)$ and $u_k = g_k(\tau_0, \ldots, \tau_n)$ $k \in \mathbb{N}$. Note that $g_k(x_0, \ldots, x_n) = 0$ if $x_k = 0$. We have

$$\begin{aligned} E[(DF, u)_{l^2(\mathbb{N})}] &= -\sum_{k=0}^{k=n} \int_0^\infty \cdots \int_0^\infty g_k \partial_k f e^{-(x_0+\cdots+x_n)} dx_0 \cdots dx_n \\ &= \sum_{k=0}^{k=n} \int_0^\infty \cdots \int_0^\infty f(x_0, ..., x_n) \partial_k (g_k e^{-(x_0+\cdots+x_n)}) dx_0 \cdots dx_n \end{aligned}$$

$$+ \int_0^\infty \cdots \int_0^\infty f(x_0, ..., 0, ..., x_n) g_k(x_0, \ldots, 0, \ldots, x_n) e^{-(x_0+\cdots+\widehat{x_k}+\cdots+x_n)} dx_0 \cdots \widehat{dx_k} \cdots dx_n$$

$$= -\int_0^\infty \cdots \int_0^\infty f(x_0, ..., x_n) \sum_{k=0}^{k=n} (g_k(x_0, ..., x_n) - \partial_k g_k(x_0, ..., x_n)) e^{-(x_0+\cdots+x_n)} dx_0 \cdots dx_n$$

$$= E[F\delta(u)].$$

Now we prove the closability of D and δ. Let $u \in \mathcal{U}$ and $F \in \mathcal{P}$. We have

$$E[\delta(u)F] = E[(DF, u)_{l^2(\mathbb{N})}]$$

Let $(F_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}$ that tends to 0 in $L^2(B)$, and such that $(DF_n)_{n\in\mathbb{N}}$ tends to $\Phi \in L^2(B) \otimes l^2(\mathbb{N})$. We have

$$\begin{aligned} \mid E[(DF_n, u)_{l^2(\mathbb{N})}] - E[(\Phi, u)_{l^2(\mathbb{N})}] \mid &\leq E\left[\mid (DF_n - \Phi, u)_{l^2(\mathbb{N})} \mid\right] \\ &\leq \parallel DF_n - \Phi \parallel_{L^2(B)\otimes l^2(\mathbb{N})} . \parallel u \parallel_{L^2(B)\otimes l^2(\mathbb{N})} . \end{aligned}$$

Hence $\lim_{n\to\infty} E[(DF_n, u)_{l^2(\mathbb{N})}] = E[(\Phi, u)_{l^2(\mathbb{N})}]$.

Now $E[(\Phi,u)_{l^2(\mathbb{N})}] = \lim_{n\to\infty} E[F_n\delta(u)] = 0$ because $\lim_{n\to\infty} F_n \stackrel{L^2(B)}{=} 0$. Hence $E[(\Phi,u)_{l^2(\mathbb{N})}] = 0 \quad \forall u \in \mathcal{U}$. This implies that $\Phi = 0$ P.a.s. since $\mathcal{U}$ is dense in $L^2(B) \otimes l^2(\mathbb{N})$. Consequently, D is closable. The closability of δ follows since $Dom(D)$ is dense in $L^2(B)$. □

Proposition 8 *We have the following bound for $\| \delta(u) \|_{L^2(B)}$:*

$$E[\delta(u)^2] \leq E[\| Du \|^2_{H\otimes H}] \quad u \in \mathcal{U}.$$

Proof. Let $u \in \mathcal{U}$ and $n \in \mathbb{N}$ such that $support(u) \subset \{0,\dots,n\}$ and $u_k = g_k(\tau_0,\dots,\tau_n)$. We have

$$\left[\sum_{i=0}^{i=n} -g_i + \partial_i g_i\right]^2 = \sum_{i=0}^{i=n}(-g_i + \partial_i g_i)^2 + 2\sum_{0\leq i<j\leq n}(-g_i+\partial_i g_i)\times(-g_j+\partial_j g_j).$$

Note that $g_k(x_0,\dots,x_n) = 0$ if $x_k = 0$, and $\partial_k g_k(x_0,\dots,x_n) = 0$ if $x_k = 0$ and $k \neq l$. Fixing $x_0, ..., x_{i-1}, x_{i+1}, ..., x_n$ and integrating on x_i, we get if $x = (x_0,\dots,x_n)$:

$$\begin{aligned}\int_0^\infty (-g_i(x) + \partial_i g_i(x))^2 e^{-x_i}dx_i &= \int_0^\infty \left[(\partial_i g_i)^2 + (g_i(x))^2 - 2g_i(x)\partial_i g_i(x)\right] e^{-x_i}dx_i \\ &= \int_0^\infty (\partial_i g_i)^2 e^{-x_i}dx_i.\end{aligned}$$

On the other hand, integrating on x_i and x_j for $i \neq j$ gives:

$$2\int_{\mathbb{R}_+^2}(-g_i(x)+\partial_i g_i(x))(-g_j(x)+\partial_j g_j(x))e^{-(x_i+x_j)}dx_i dx_j.$$

$$\begin{aligned} &= 2\int_{\mathbb{R}_+^2}(\partial_i(g_i e^{-x_i}))\partial_j(g_j e^{-x_j})dx_i dx_j \\ &= 2\int_{\mathbb{R}_+^2}\partial_j g_i \partial_i g_j e^{-(x_i+x_j)}dx_i dx_j \\ &\leq \int_{\mathbb{R}_+^2}\left[(\partial_j g_i)^2 + (\partial_i g_j)^2\right] e^{-(x_i+x_j)}dx_i dx_j.\end{aligned}$$

We obtain the inequality we are looking for by integration with respect to the remaining variables and summation. □

4 Extending the derivative from $L^2(B)\otimes l^2(\mathbb{N})$ to $L^2(B)\otimes L^2(\mathbb{R}_+)$

We are going to define a continuous-time process $(\tilde{D}_t F)_{t\in\mathbb{R}_+}$ from the discrete-time process $(D_k F)_{k\in\mathbb{N}}$. More precisely, we define an injection

$$i : L^2(B)\otimes l^2(\mathbb{N}) \to L^2(B)\otimes L^2(\mathbb{R}_+)$$

by

$$i_t(f) = f(N_{t^-}).$$

And we study the dual operator of i. Namely, we have

$$j : L^2(B) \otimes L^2(\mathbb{R}_+) \to L^2(B) \otimes l^2(\mathbb{N})$$

defined by

$$j_k(u) = \int_{T_k}^{T_{k+1}} u(t)dt, \quad k \in \mathbb{N},$$

for $f \in L^2(B) \otimes l^2(\mathbb{N})$ and $u \in L^2(B) \otimes L^2(\mathbb{R}_+)$. Then

$$j(u) = \sum_{k=0}^{\infty} < u, i(e_k) >_{L^2(\mathbb{R}_+)} .e_k,$$

which means that i and j are dual in the following sense:

$$< i(f), u >_{L^2(\mathbb{R}_+)} = < f, j(u) >_{l^2(\mathbb{N})} \quad P.a.s. \tag{5}$$

The introduction of i and j allows to think of different pairs of unbounded operators. Namely, we will consider

- $\tilde{D} := ioD$ and $\tilde{\delta} := \delta oj$. Here, $\tilde{\delta}(v)$ will be the stochastic integral of a continuous-time predictable process v with respect to the compensated Poisson process.
- $jo\tilde{D}$ and $\tilde{\delta}oi$. In this case, $\tilde{\delta}oi(u)$ will be identical to the integral of u with respect to the discrete-time martingale $(n - T_n)_{n\in\mathbb{N}}$ if u is a discrete-time predictable process.
- $\Delta_k := (jo\tilde{D}\cdot)_k$ and $\Delta_k^* := \tilde{\delta}oi(\cdot e_k) = \delta(\tau_k \cdot)$, $\quad k \in \mathbb{N}$. The sum $\Delta_k + \Delta_k^*$ is the multiplication by the white noise $(1 - \tau_k)$.

4.1 Operators $\tilde{D}$ and $\tilde{\delta}$

From $(D_n F)_{n\geq 0}$ we define a continuous-time process $(\tilde{D}_t F)_{t\in\mathbb{R}_+}$ by

$$\tilde{D}F = i \circ DF.$$

This means that if $F \in \mathcal{P}$, $\tilde{D}_t F = D_{N_{t^-}} F$. If $F = f(\tau_0, ..., \tau_n)$, we have the following expression for $\tilde{D}_t F$:

$$\tilde{D}_t F = - \sum_{k=0}^{k=n} \partial_k f(\tau_0, ..., \tau_n) 1_{]T_k, T_{k+1}]}(t)$$

since for $l \in \mathbb{N}$, $D_l F = -\partial_l f(\tau_0, ..., \tau_n) = -\sum_{k=0}^{k=n} \partial_k f(\tau_0, ..., \tau_n) 1_{\{k=l\}}$. We notice that this is similar to the expression of the gradient operator given in Carlen-Pardoux[4] and Bouleau-Hirsch[3], pp. 235-239. We will show that $\tilde{D}$ is closable and that the dual operator of $\tilde{D}$ is $\delta \circ j$, so that we let $\tilde{\delta} = \delta \circ j$. This gives two commutative diagrams.

$$
\begin{array}{ccc}
L^2(B) & \xrightarrow{\ D\ } & L^2(B)\otimes l^2(\mathbb{N}) \\
\updownarrow & & \downarrow i \\
L^2(B) & \xrightarrow{\ \tilde{D}\ } & L^2(B)\otimes L^2(\mathbb{R}_+)
\end{array}
\qquad
\begin{array}{ccc}
L^2(B)\otimes l^2(\mathbb{N}) & \xrightarrow{\ \delta\ } & L^2(B) \\
j \uparrow & & \updownarrow \\
L^2(B)\otimes L^2(\mathbb{R}_+) & \xrightarrow{\ \tilde{\delta}\ } & L^2(B)
\end{array}
$$

The injection i can be interpreted as an injection from the tangent space associated to the discrete-time Fock space into the tangent space associated to the continuous-time Fock space. Let $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ be the filtration generated by the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$, and denote by $\mathcal{C}_c^\infty(\mathbb{R}^n)$ the space of $\mathcal{C}^\infty$ functions with compact support in $\mathbb{R}^n$.

Definition 7 *We denote by $\mathcal{V}$ the class of processes $v \in L^2(B)\otimes L^2(\mathbb{R}_+)$ of the following form:*

$$v(t) = f(t, \tau_0, \ldots, \tau_n)$$

where $n \in \mathbb{N}$, $f \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$ with

$$f(y, x_0, \ldots, x_n) = 0$$

if $y > x_0 + \cdots + x_n$.

Remark 2 *The space $\mathcal{V}$ is dense in $L^2(B)\otimes L^2(\mathbb{R}_+)$.*

The following proposition states the properties that we expect from $\tilde{D}$ and $\tilde{\delta}$. We call a simple cylindrical predictable process on $(B, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ a process v of the form $v(t) = 1_{]s,\infty[}(t)F$ with $F = f(\tau_0, \ldots, \tau_n)$, F $\mathcal{F}_s$-measurable, $f \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$, $n \in \mathbb{N}$, $s \geq 0$.

Proposition 9

1. *If $v \in \mathcal{V}$, then $\tilde{\delta}(v) \in L^2(B)$, with*

$$\tilde{\delta}(v) = \int_{\mathbb{R}_+} v(t)d(N_t - t) - trace(\tilde{D}v). \tag{6}$$

2. *For $F \in \mathcal{P}$ and $u \in \mathcal{V}$,*

$$E[(\tilde{D}F, u)_{L^2(\mathbb{R}_+)}] = E[F\tilde{\delta}(u)].$$

The operators $\tilde{D}$ and $\tilde{\delta}$ are closable and $\tilde{\delta}$ is adjoint of $\tilde{D}$. We identify $\tilde{D}$ (resp. $\tilde{\delta}$) with its closed extension and denote by $\mathbb{D}_{2,1}$ (resp. $Dom(\tilde{\delta})$) the domain of $\tilde{D}$ (resp. $\tilde{\delta}$).

3. *Let v be a simple cylindrical predictable process on $(B, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$. We have*

$$\tilde{D}_t v(u) = 0 \quad P.a.s. \quad t \geq u \geq 0 \ \ and \ \ trace(\tilde{D}v) = 0 \ \ P.a.s.$$

Consequently, $\tilde{\delta}$ coincides with the compensated Poisson stochastic integral on the predictable processes in $L^2(B) \otimes L^2(\mathbb{R}_+)$.

Proof.

- The proof of the closability of $\tilde{D}$ is the same as for D. To prove relation 6, we do the calculations for $v_t = f(t, \tau_1, ..., \tau_n)$. Since $j_k(v) = 0$ for $\tau_k = 0$, and $j_k(v) = 0$ for $k > n$, we have from Prop. 7:

$$\tilde{\delta}(v) = \delta o j(v) = \sum_{k=0}^{\infty} -j_k(v) - D_k j_k(v)$$

  with $j_k(v) = \int_{T_k}^{T_{k+1}} v(t)dt$. Now we use the fact that D_k is a partial derivative with respect to τ_k:

$$D_k j_k(v) = -\partial_k j_k(v) = -v(T_{k+1}) + \int_{T_k}^{T_k+1} D_k v(t)dt$$

  and we get

$$\tilde{\delta}(v) = \int_0^\infty v(t)d(N_t - t) - \int_0^\infty \tilde{D}_t v(t)dt.$$

- Using the duality property between i and j, we have

$$E[(\tilde{D}F, u)_{L^2(\mathbb{R}_+)}] = E[(ioDF, u)_{L^2(\mathbb{R}_+)}]$$
$$= E[(DF, j(u))_{l^2(\mathbb{N})}] = E[F\delta o j(u)] = E[F\tilde{\delta}(u)]$$

  for $F \in \mathcal{P}$ and $u \in \mathcal{V}$. Since $\mathcal{V}$ is dense in $L^2(B) \otimes L^2(\mathbb{R}_+)$ and $Dom(\tilde{D})$ is dense in $L^2(B)$, $\tilde{\delta}$ is closable and is the adjoint of $\tilde{D}$.

- The idea of this proof was found in Carlen-Pardoux[4]. We consider first the case where v is a cylindrical elementary predictable process $v = f(\tau_0, ..., \tau_n)1_{]s,\infty[}(t)$, i.e. $v = F1_{]s,\infty[}(t)$, $F = f(\tau_0, ..., \tau_n)$ and $f \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$. Since v is predictable, F is $\mathcal{F}_s$-measurable. This means that F does not depend on the future of the Poisson process after s, or that F does not depend on the k-th jump time T_k if $T_k > s$, i.e.

$$\partial_i f(\tau_0, ..., \tau_n) = 0 \quad for \ \ \tau_0 + \cdots + \tau_k > s, \ \ i \leq k.$$

  This implies $\partial_i f(\tau_0, ..., \tau_n)1_{]T_i,T_{i+1}]}(t) = 0 \ \ t \geq s \ \ i = 0, ..., n$ and

$$\tilde{D}_t F = \sum_{i=0}^{i=n} \partial_i f(\tau_0, ..., \tau_n)1_{]T_i,T_{i+1}]}(t) = 0 \ \ t \geq s.$$

  Hence $\tilde{D}_t v(u) = 0 \ \ \forall t \geq u$. Then we use the fact that $\tilde{D}$ is linear, to extend the property to the linear combinations of elementary predictable processes. The compensated Poisson stochastic integral coincides with $\tilde{\delta}$ on the predictable square-integrable processes from relation 6 and a density argument. □

We also remark that $trace(\tilde{D}v)$ can be expressed as:

$$trace(\tilde{D}v) = \int_0^\infty \tilde{D}_t v(t)dt = \sum_{k=0}^{\infty} \int_{T_k}^{T_{k+1}} D_k v(t)dt.$$

4.2 Operators $j \circ \tilde{D}$ and $\tilde{\delta} \circ i$

The next proposition gives the properties of $\tilde{\delta} \circ i$ as a stochastic integral operator with respect to the martingale $(n - T_n)_{n \geq 1}$, and shows that $\tilde{\delta} \circ i$ is a creation operator on $\mathcal{F}(H)$.

Proposition 10

1. *If $u : \mathbb{N} \longrightarrow \mathcal{P}$ has a finite support in $\mathbb{N}$, then $\tilde{\delta} \circ i(u) \in L^2(B)$ and*

$$\tilde{\delta} \circ i(u) = \sum_{k=0}^{\infty} (1 - \tau_k).u_k - trace(j \circ \tilde{D} u).$$

2. *For $F \in \mathcal{P}$ and $u : \mathbb{N} \longrightarrow \mathcal{P}$ with a finite support in $\mathbb{N}$, $u \in Dom(\tilde{\delta} \circ i)$ and*

$$E[(j \circ \tilde{D} F, u)_{l^2(\mathbb{N})}] = E[F \tilde{\delta} \circ i(u)].$$

   Hence $j \circ \tilde{D}$ and $\tilde{\delta} \circ i$ are closable. They are identified with their closed extensions, and $\tilde{\delta} \circ i$ is adjoint of $j \circ \tilde{D}$.

3. *$\tilde{\delta} \circ i$ has the following expression as a creation operator:*

$$\tilde{\delta} \circ i(I_n(f_{n+1}(*, \cdot))) = I_{n+1}(\hat{f}_{n+1}) - n I_n(\hat{f}^1_{n+1})$$

   where $f_{n+1} \in l^2(\mathbb{N})^{\circ n} \otimes l^2(\mathbb{N})$, $\hat{f}_{n+1}$ is the symmetrization of f_{n+1} in n+1 variables, and $\hat{f}^1_{n+1}$ denotes the symmetrization of the contraction f^1_{n+1} of $\hat{f}_{n+1}$ defined by

$$f^1_{n+1}(k_1, ..., k_n) = \hat{f}_{n+1}(k_1, ..., k_n, k_n).$$

Proof.

- We have $\tilde{\delta} o i(u) = \delta o j o i(u) = -\left(\sum_{k=0}^{\infty} (joi(u))_k + D_k(joi(u))_k\right)$. But joi is a random linear operator from $l^2(\mathbb{N})$ to $l^2(\mathbb{N})$ with a diagonal matrix, its eigenvalues being $(\tau_k)_{k \in \mathbb{N}}$: $(joi(u), e_k)_H = \int_{T_k}^{T_{k+1}} u_k dt = \tau_k u_k$. Hence

$$\tilde{\delta} o i(u) = -\left(\sum_{k=0}^{\infty} \tau_k u_k - u_k + \tau_k D_k u_k \right) = \sum_{k=0}^{\infty} (1 - \tau_k).u_k - trace(j \circ \tilde{D} u).$$

- Using again the duality between i and j, we have:

$$E[(u, jo\tilde{D} F)_{l^2(\mathbb{N})}] = E[(i(u), \tilde{D} F)_{L^2(\mathbb{R}_+)}] = E[\tilde{\delta} o i(u) F]$$

  where $F \in \mathcal{P}$, and $u : \mathbb{N} \longrightarrow \mathcal{P}$ has a finite support in $\mathbb{N}$. The proof of the closability of $j \circ \tilde{D}$ is the same as for D.

- It is sufficient to consider the case where $f_{n+1} = e_{k_1}^{\circ n_1} \circ \cdots \circ e_{k_d}^{\circ n_d} \otimes e_{k_l}$ with $n_1 + \cdots + n_d = n$, $k_1 < \cdots < k_d$ and either $l \in \{1, 2, \ldots, d\}$ or $k_l \notin \{k_1, ..., k_d\}$. Let $u_k = I_n(f_{n+1}(*, k))$, $k \in \mathbb{N}$. Then if $l \in \{1, 2, \ldots, n\}$:

$$\begin{aligned} \tilde{\delta} o i(u) &= -n_1! \cdots n_d! L_{n_1}(\tau_{k_1}) \cdots L_{n_d}(\tau_{k_d})(\tau_{k_l} - 1) \\ &\quad - \tau_{k_l} n_1! \cdots n_d! L_{n_1}(\tau_{k_1}) \cdots \widehat{L_{n_l}(\tau_{k_l})} \cdots L_{n_d}(\tau_{k_d}) \sum_{i=0}^{n_l - 1} L_i(\tau_{k_l}) \\ &= n_1! \cdots n_d! L_{n_1}(\tau_{k_1}) \cdots \widehat{L_{n_l}(\tau_{k_l})} \cdots L_{n_d}(\tau_{k_d}) \left[-\sum_{i=0}^{n_l} \tau_{k_l} L_i(\tau_{k_l}) + L_{n_l}(\tau_{k_l}) \right] \end{aligned}$$

where "$\widehat{L_{n_l}(\tau_{k_l})}$" means that this term is excluded of the product. Now,

$$\begin{aligned}\tau_{k_l}\sum_{i=1}^{n_l}L_i(\tau_{k_l}) &= \sum_{i=1}^{n_l}-(i+1)L_{i+1}(\tau_{k_l})+(2i+1)L_i(\tau_{k_l})-iL_{i-1}(\tau_{k_l})\\ &= \sum_{i=2}^{n_l+1}-iL_i(\tau_{k_l})+\sum_{i=1}^{n_l}(2i+1)L_i(\tau_{k_l})-\sum_{i=0}^{n_l-1}(i+1)L_i(\tau_{k_l})\\ &= -(n_l+1)L_{n_l+1}(\tau_{k_l})+(n_l+1)L_{n_l}(\tau_{k_l})+3L_1(\tau_{k_l})-2L_1(\tau_{k_l})-1.\end{aligned}$$

We used here the recurrence relation 1. Hence

$$\tilde{\delta}oi(u)=n_1!\cdots(n_l+1)!\cdots n_d!L_{n_1}(\tau_{k_1})\cdots L_{n_l+1}(\tau_{k_l})\cdots L_{n_d}(\tau_{k_d})$$

$$-n_ln_1!\cdots n_d!L_{n_1}(\tau_{k_1})\cdots L_{n_d}(\tau_{k_d})=I_{n+1}(\hat{f}_{n+1})-n\frac{n_l}{n}n_1!\cdots n_d!L_{n_1}(\tau_{k_1})\cdots L_{n_d}(\tau_{k_d}).$$

By definition of the symmetrized contraction $\hat{f}^1_{n+1}$, we have and $\hat{f}^1_{n+1}=\frac{n_l}{n}e_{k_1}^{on_1}\circ\cdots\circ e_{k_d}^{on_d}$. Hence $\tilde{\delta}\circ i(u)=I_{n+1}(\hat{f}_{n+1})-nI_n(\hat{f}^1_{n+1})$. In case $k_l\notin\{k_1,...,k_d\}$, $\tilde{\delta}oi(u)=L_1(\tau_{k_l})n_1!\cdots n_d!L_{n_1}(\tau_{k_1})\cdots L_{n_d}(\tau_{k_d})=I_{n+1}(\hat{f}_{n+1})$ and $\hat{f}^1_{n+1}=0$. Hence we also have $\tilde{\delta}\circ i(u)=I_{n+1}(\hat{f}_{n+1})-nI_n(\hat{f}^1_{n+1})$. By linear combinations, density and closedness of $\tilde{\delta}\circ i$, we obtain the result for $f_n\in l^2(\mathbb{N})^{\circ n}\otimes l^2(\mathbb{N})$.□

The domain of $\tilde{\delta}\circ i$ will be further determined in Prop. 17. The following statement comes from [4] without modification.

Proposition 11 *For $F\in\mathbb{D}_{2,1}$ and $v\in Dom(\tilde{\delta})$ such that $F\tilde{\delta}(v)-\int_0^\infty v(t)\tilde{D}_tFdt\in L^2(B)$ and $Fv\in L^2(B)\otimes L^2(\mathbb{R}_+)$, we have $Fv\in Dom(\tilde{\delta})$ and*

$$\tilde{\delta}(Fv)=F\tilde{\delta}(v)-\int_0^\infty v(t)\tilde{D}_tFdt.$$

Proof. This identity comes from the fact that $\tilde{D}$ is a derivation:

$$\begin{aligned}E[G\tilde{\delta}(Fu)] &= E[\int_0^\infty(\tilde{D}_t)GFu_tdt]\\ &= E[\int_0^\infty(\tilde{D}_t(FG)-G\tilde{D}_tF)u_tdt]\\ &= E[GF\tilde{\delta}(u)]-E[G\int_0^\infty u_t\tilde{D}_tFdt]\\ &= E[(F\tilde{\delta}(u)-\int_0^\infty u_t\tilde{D}_tFdt)G].\end{aligned}$$

This is true for $F\in\mathcal{P}$, $u\in\mathcal{V}$ and proves the proposition, since $\tilde{\delta}$, $\tilde{D}$ are closed, $\mathcal{P}$ is dense in $L^2(B)$ and $\mathcal{V}$ is dense in $L^2(B)\otimes L^2(\mathbb{R}_+)$. □

4.3 Operators Δ_k and Δ_k^*

In this section, we will see how $\tilde{D}$ and $\tilde{\delta}$ can be used to express the multiplication by the white noise $(1-\tau_k)_{k\geq 0}$, and study their commutation relations. We define the operators Δ_k and Δ_k^* $k\in\mathbb{N}$ as $\Delta_k^*F=\tilde{\delta}oi(F.e_k)=\delta(\tau_kF)$ and $\Delta_kF=(jo\tilde{D}F)_k=\tau_kD_kF$.

Proposition 12

1. *For* $F, G \in \mathcal{P}$*,* $E[F\Delta_k G] = E[G\Delta_k^* F]$*. Hence* Δ_k^**,* Δ_k *are closable and are adjoint operators on* $L^2(B)$*.*

2. $\Delta_k + \Delta_k^*$ *is the multiplication by the white noise* $(1-\tau_k)_{k\in\mathbb{N}}$*:* $(\Delta_k + \Delta_k^*)F = (1-\tau_k).F$ *for* $F \in \mathcal{P}$*.*

3. *We have the following commutation relations between* Δ_k *and* Δ_l^**:*

$$[\Delta_k, \Delta_l^*] = \tau_k 1_{\{k=l\}} \qquad [\Delta_k, \Delta_l] = [\Delta_k^*, \Delta_l^*] = 0.$$

4. *Writing* $a(f)F = (f, \Delta F)_H$ *and* $a^*(g)F = (g, \Delta^* F)_H = \tilde{\delta} oi(Fg)$ *for* $f, g \in H$*,* $F \in \mathcal{P}$*, we have*

$$[a(f), a^*(g)] = (i(f), i(g))_{L^2(\mathbb{R}_+)}$$

$$[a(f), a(g)] = [a^*(f), a^*(g)] = 0$$

and $a(f)$ *is adjoint of* $a^*(f)$ *on* $L^2(B)$*.*

Proof. Let $F \in \mathcal{P}$. We have $\Delta_k^* F = \tilde{\delta} oi(Fe_k) = (1-\tau_k)F - \Delta_k F$, by Prop 11. We only need to prove the commutation relation:

$$\begin{aligned} [\Delta_k, \Delta_l^*]F &= \Delta_k \Delta_l^* F - \Delta_l^* \Delta_k F \\ &= \Delta_k((1-\tau_l)F - \tau_l D_l F) - ((1-\tau_l)\tau_k D_k F - \tau_l D_l \tau_k D_k F) \\ &= [\Delta_k, \Delta_l]F + \tau_k 1_{\{k=l\}} F + (1-\tau_l)(D_k - D_k)F \\ &= [\Delta_k, \Delta_l]F + \tau_k 1_{\{k=l\}} F. \end{aligned}$$

It remains to check that $[\Delta_k, \Delta_l] = [\Delta_k^*, \Delta_l^*] = 0$, but this is evident. □

5 A number operator on the discrete chaotic decomposition

5.1 Definition

We call number operator on the discrete chaotic decomposition an operator L on $L^2(B, P)$ which has the set $\mathbb{N}$ for eigenvalues, the eigenspace associated to $n \in \mathbb{N}$ being the n-th chaos $C_n = \{I_n(f_n) \ : \ f_n \in H^{on}\}$. This means that if $f_n \in l^2(\mathbb{N})^{\circ n}$, then

$$L(I_n(f_n)) = nI_n(f_n).$$

On the Wiener and Wiener-Poisson decompositions, the derivatives D and ∇ are annihilation operators, and their adjoints are creation operators. For example, we have on the Wiener-Poisson decomposition, cf. [8]:

$$\nabla_t(\tilde{I}_{n+1}(f_{n+1})) = (n+1)\tilde{I}_n(f_{n+1}(*, t))$$

and

$$\nabla^*(\tilde{I}_n(f_{n+1}(*, t))) = \tilde{I}_{n+1}(\hat{f}_{n+1})$$

where ∇^* is the adjoint of ∇. Both ∇ and ∇^* are closed operators and $\mathcal{P} \subset Dom(\nabla)$. Recall that $\tilde{I}_n(f_n)$ is here the Wiener-Poisson multiple stochastic integral of a symmetric function $f_n \in L^2(\mathbb{R}_+)^{\circ n}$, and $\hat{f}_{n+1}$ is the symmetrization of f_{n+1} in $n+1$ variables. The composition $\nabla^*\nabla$ is a number operator on the Wiener-Poisson chaotic decomposition:

$$\nabla^*\nabla\tilde{I}_{n+1}(f_{n+1}) = (n+1)\tilde{I}_{n+1}(f_{n+1}).$$

As we already noticed, the composition δD is not a number operator on the discrete chaotic decomposition.

Proposition 13 *The operator $L = \tilde{\delta}\tilde{D}$ is a number operator on the discrete chaotic decomposition of $L^2(B)$. This means that if $f_n \in l^2(\mathbb{N})^{\circ n}$, then $I_n(f_n) \in Dom(L)$ and*

$$\tilde{\delta}\tilde{D}(I_n(f_n)) = L(I_n(f_n)) = nI_n(f_n).$$

Proof. The proof uses the fact that D and $\tilde{\delta}oi$ are annihilation and creation operators. Let $F = I_n(f_n)$, $f_n \in l^2(\mathbb{N})^{\circ n}$,

$$f_n^i(k_1, ..., k_{n-i}, l) = f_n(k_1, ..., k_{n-i}, \underbrace{l, ..., l}_{i \quad times}) \quad i \geq 1,$$

and denote by $\hat{f}_n^i$ the symmetrization of f_n^i in its $n-i+1$ variables. We have

$$\begin{aligned} LF = \tilde{\delta} \circ i \circ DF &= \tilde{\delta} \circ i(\sum_{k=0}^{n-1} \frac{n!}{k!} I_k(f_n(*, l, ..., l))) \\ &= \sum_{k=0}^{n-1} \frac{n!}{k!} \tilde{\delta} \circ i(I_k(f_n^{n-k}(*, l))) \\ &= \sum_{k=0}^{n-1} \frac{n!}{k!} (I_{k+1}(\hat{f}_n^{n-k}) - kI_k(\hat{f}_n^{n-k+1})) \\ &= nI_n(f_n^0) = nI_n(f_n) \end{aligned}$$

Here, $\tilde{\delta} \circ i$ operates on the variable l and we used the result of Prop. 10. □
The following result uses the number operator L to determine the domain of $\tilde{D}$, which is a test function space.

Proposition 14 $F = \sum_{n=0}^{\infty} I_n(f_n)$ *belongs to the domain of $\tilde{D}$ if and only if*

$$\| \tilde{D}F \|^2_{L^2(B)\otimes L^2(\mathbb{R}_+)} = \sum_{n=0}^{\infty} n \| I_n(f_n) \|^2_{L^2(B)} < +\infty.$$

Proof. We have

$$\begin{aligned} \| \tilde{D}F \|^2_{L^2(B)\otimes L^2(\mathbb{R}_+)} &= E[\int_0^{\infty} (\tilde{D}_t F)^2 dt] \\ &= E[FLF] = \sum_{m,n\geq 0} nE[I_n(f_n)I_m(f_m)] \\ &= \sum_{n\geq 0} n \| I_n(f_n) \|^2_{L^2(B)} . \quad \square \end{aligned}$$

We end this section with a remark concerning the number operator L:

Proposition 15 *We have on $\mathcal{P}$:*

$$L = \sum_{k=0}^{\infty} \Delta_k^* D_k$$

where Δ_k^* is defined in section 4.3.

Proof: This follows from the definitions:

$$LF = \tilde{\delta}\tilde{D}F = \sum_{k=0}^{\infty} \tilde{\delta} oi(D_k F e_k) = \sum_{k=0}^{\infty} \Delta_k^* D_k F. \quad \square$$

5.2 Commutation and composition relations

In this section we state commutation relations that can be useful to construct distributions on Poisson space.

Proposition 16 *We have the following commutation relations on $\mathcal{P}$:*

$$D_k \Delta_k^* = \Delta_k^* D_k + I + D_k \quad k \in \mathbb{N}.$$

$$D_k L = L D_k + D_k(I + D_k), \quad k \in \mathbb{N}$$

Proof. For $F = I_n(f_n)$, where $f_n \in l^2(\mathbb{N})^{\circ n}$ has a finite support,

$$\begin{aligned} D_k \Delta_k^* F &= D_k(-\tau_k - \tau_k D_k + I)F \\ &= (-\tau_k D_k - \tau_k D_k^2 + 2D_k + I)F \\ &= (\Delta_k^* D_k + I + D_k)F. \end{aligned}$$

The second equality is proved using Prop. 15 and the above relation:

$$\begin{aligned} D_k L F &= D_k \sum_{l=0}^{\infty} \Delta_l^* D_l F \\ &= \sum_{l=0}^{\infty} \Delta_l^* D_k D_l F + (I + D_k) D_k F \\ &= L D_k F + (I + D_k) D_k F. \quad \square \end{aligned}$$

We deduce the following isometry property.

Proposition 17 *If $u : \mathbb{N} \longrightarrow \mathcal{P}$ has a finite support in $\mathbb{N}$,*

$$E[\tilde{\delta} oi(u)^2] = E[\| i(u) \|^2_{L^2(\mathbb{R}_+)}] + E[\| j \circ \tilde{D} u \|^2_{H \otimes H}].$$

Proof.

$$\begin{aligned} E[\tilde{\delta} \circ i(u)^2] &= \sum_{k,l \in \mathbb{N}} E[\Delta_k^* u_k \Delta_l^* u_l] \\ &= \sum_{k,l \in \mathbb{N}} E[\tau_k u_k D_k \Delta_l^* u_l] \\ &= \sum_{k \neq l} E[\tau_k u_k \Delta_l^* D_k u_l] + \sum_{k \in \mathbb{N}} E[\tau_k u_k (\Delta_k^* D_k + I + D_k) u_k] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k,l\in\mathbb{N}} E[\tau_k u_k \Delta_l^* D_k u_l] + \sum_{k\in\mathbb{N}} E[\tau_k u_k(u_k + D_k u_k)] \\
&= \sum_{k,l\in\mathbb{N}} E[\tau_l D_l(\tau_k u_k) D_k u_l] + \sum_{k\in\mathbb{N}} E[\tau_k u_k^2 + \tau_k u_k D_k u_k] \\
&= \sum_{k,l\in\mathbb{N}} E[\tau_l \tau_k D_k u_l D_l u_k] + \sum_{k\in\mathbb{N}} E[\tau_k u_k^2] \\
&= \sum_{k,l\in\mathbb{N}} E[(j\circ \tilde{D}u_l)_k (j\circ \tilde{D}u_k)_l] + \sum_{k\in\mathbb{N}} E[\tau_k u_k^2]. \quad \square
\end{aligned}
$$

The next proposition shows that the "opérateur carré du champ" Γ for L has a classical expression.

Proposition 18 *If $F \in \mathcal{P}$ depends on $(F_0, ..., F_n)$, $F = f(F_0, ..., F_n)$, $F_0, ..., F_n \in \mathcal{P}$, f polynomial in $n+1$ variables, we have*

1. *Chain rule of derivation:*

$$
\tilde{D}F = \sum_{i=0}^{i=n} \partial_i f(F_0, ..., F_n) \tilde{D}F_i.
$$

2. *The "opérateur carré du champ" Γ defined on $\mathcal{P}\times\mathcal{P}$ as:*

$$
\Gamma(F,G) = -\frac{1}{2}(L(FG) - FLG - GLF)
$$

satisfies to $\Gamma(F,G) =< \tilde{D}F, \tilde{D}G >_{L^2(\mathbb{R}_+)}$.

3. *L has the following expression*

$$
LF = \sum_{i=0}^{i=n} \partial_i f(F_0, ..., F_n) LF_i - \sum_{l,k=0}^{n} \partial_l \partial_k f(F_0, ..., F_n) \Gamma(F_l, F_k).
$$

Proof. The first part is obvious from the definitions. From Prop. 11,

$$
\tilde{\delta}(Fu) = F\tilde{\delta}(u) - \int_0^\infty u_t \tilde{D}_t F dt.
$$

This gives

$$
\begin{aligned}
LF &= \tilde{\delta}(\sum_{i=0}^{i=n} \partial_i f(F_0, ..., F_n)\tilde{D}F_i) \\
&= \sum_{i=0}^{i=n} \partial_i f(F_0, ..., F_n)\tilde{\delta}\tilde{D}F_i - < \tilde{D}F_i, \tilde{D}\partial_i f(F_0, ..., F_n) >_{L^2(\mathbb{R}_+)} \\
&= \sum_{i=0}^{i=n} \partial_i f(F_0, ..., F_n) LF_i - \sum_{j=0}^{j=n}\sum_{i=0}^{i=n} \partial_j\partial_i f(F_0, ..., F_n) < \tilde{D}F_i, \tilde{D}F_j >_{L^2(\mathbb{R}_+)} .
\end{aligned}
$$

It remains to prove that $\Gamma(F,G) =< \tilde{D}F, \tilde{D}G >_{L^2(\mathbb{R}_+)}$.
Let us consider $F = f(\tau_0, ..., \tau_n)$ and $G = g(\tau_0, ..., \tau_n)$. From the above chain rule, we have:

$$
\begin{aligned}
L(FG) &= FLG + GLF - \sum_{k,l=0}^{n} (\partial_l g \partial_k f + \partial_k g \partial_l f) < \tilde{D}\tau_l, \tilde{D}\tau_k >_{L^2(\mathbb{R}_+)} \\
&= FLG + GLF - 2 < \tilde{D}F, \tilde{D}G >_{L^2(\mathbb{R}_+)} . \quad \square
\end{aligned}
$$

To end this section, we study the existing relation between the discrete m.s.i. and the Wiener-Poisson m.s.i. We need first to recall the definition of the Wiener-Poisson m.s.i. as given in [8]. Let ω be the random measure defined as

$$\omega(\cdot) = \sum_{k=1}^{\infty} \delta_{T_k}(\cdot)$$

where δ_t is the Dirac distribution at $t \in \mathbb{R}_+$, and $(T_k)_{k\geq 1}$ are the jump times of the Poisson process. Let λ denote the Lebesgue measure on $\mathbb{R}$. The Wiener-Poisson m.s.i. of $h_n \in L^2(\mathbb{R})^{\circ n}$, space of symmetric square-integrable functions on $\mathbb{R}^n$, can be written as

$$\tilde{I}_n(h_n) = n! \int_0^\infty \int_0^{t_n^-} \cdots \int_0^{t_1^-} h_n(t_1, ..., t_n)(\omega - \lambda)(dt_1) \cdots (\omega - \lambda)(dt_n). \qquad (7)$$

Note that the integration is not performed on the diagonals of $\mathbb{R}_+^n$, so that $\tilde{I}_n$ is actually an iterated stochastic integral of predictable processes. If f_n is not symmetric, let

$$\tilde{I}_n(f_n) = \tilde{I}_n(\hat{f}_n)$$

where $\hat{f}_n$ is the symmetrization of f_n in n variables, defined as

$$\hat{f}_n(t_1, \cdots, t_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} f(t_{\sigma(1)}, \ldots, t_{\sigma(n)})$$

and Σ_n is the set of all permutations of $\{1, 2, \ldots, n\}$. The Wiener-Poisson and discrete chaotic decompositions allow to define two kinds of exponential vectors.

Proposition 19 *Let $f \in l^2(\mathbb{N})$ such that $\| f \|_2 < 1$, and $h \in L^2(\mathbb{R}_+)$. We define*

$$\epsilon(f) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(f^{\circ n})$$

$$\tilde{\epsilon}(h) = \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{I}_n(h^{\circ n}).$$

We have the relation

$$\epsilon(f) = \tilde{\epsilon}\left(i(\frac{f}{1-f})\right)$$

where $\frac{f}{1-f}$ is defined to be the sequence $(\frac{f_k}{1-f_k})_{k\in\mathbb{N}}$.

Proof. First, note that the exponential $\epsilon(f)$ is in $L^2(B)$ if $\| f \|_2 < 1$:

$$\begin{aligned} \| \epsilon(f) \|_{L^2(B)}^2 &= \sum_{n\geq 0} \frac{1}{(n!)^2} \| I_n((f)^{\circ n}) \|_2^2 \\ &\leq \sum_{n\geq 0} \| f^{\circ n} \|_2^2 \leq \frac{1}{1 - \| f \|_2^2} \end{aligned}$$

from Prop. 3. If $h = i(\frac{f}{1-f})$ where $f \in l^2(\mathbb{N})$ has a finite support in $\mathbb{N}$ and $\| f \|_2 < 1$, then

$$\tilde{\epsilon}(h) = \sum \frac{1}{n!} \tilde{I}_n(h^{\circ n}) = \prod_{k=1}^{\infty} (1 + h(T_k)) exp\left(-\int_0^\infty h(t)dt\right)$$

Hence

$$\begin{aligned}
\tilde{\epsilon}\left(i(\frac{f}{1-f})\right) &= \prod_{k=0}^{\infty}\left[\left(\frac{1}{1-f_k}\right) exp\left(\frac{-\tau_k f_k}{1-f_k}\right)\right] \\
&= \prod_{k=0}^{\infty}\frac{e^{\frac{-\tau_k fk}{1-f_k}}}{1-f_k} \\
&= \prod_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}(f_k)^n L_n(\tau_k)\right)
\end{aligned}$$

from the definition of the generating function of Laguerre polynomials, and

$$\begin{aligned}
\tilde{\epsilon}\left(i(\frac{f}{1-f})\right) &= \sum_{n=0}^{\infty}\sum_{\substack{k_1<\cdots<k_l \\ n_1+\cdots+n_l=n}} (f_{k_1})^{on_1}\cdots(f_{k_l})^{n_l}L_{n_1}(\tau_{k_1})\cdots L_{n_l}(\tau_{k_l}) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}I_n(f^{on}) = \epsilon(f).
\end{aligned}$$

The relation is extended by density to any $f \in l^2(\mathbb{N})$ such that $\| f \|_2 < 1$. □

6 Wiener-Poisson decompositions of jump times

We apply here the preceding results to find the decomposition of elements of $L^2(B)$ in the Wiener-Poisson chaotic decomposition. The first point that we need to clarify is the relationship between ∇ and $\tilde{D}$. Denote by ${}^p(\quad) : L^2(B) \otimes L^2(\mathbb{R}_+) \longrightarrow L^2(B) \otimes L^2(\mathbb{R}_+)$ the predictable projection operator. More precisely, if $u \in L^2(B) \otimes L^2(\mathbb{R}_+)$ with $u = \sum_{n=0}^{\infty} \tilde{I}_n(u_{n+1})$ where $u_{n+1} \in L^2(\mathbb{R}_+)^{on} \otimes L^2(\mathbb{R}_+)$, we have ${}^p(u) = \sum_{n=0}^{\infty} \tilde{I}_n(u_{n+1}^{\Delta})$ where

$$u_{n+1}^{\Delta}(t_1,\ldots,t_n,t) = u_{n+1}(t_1,\ldots,t_n,t)1_{\{t_1,\ldots,t_n<t\}}.$$

We know already that $\tilde{\delta}$ and ∇^* coincide on predictable processes but not on anticipative processes. The following proposition gives the analog relation between $\tilde{D}$ and ∇.

Proposition 20 *Let $F \in \mathbb{D}_{2,1} \cap Dom(\nabla)$. Then $(\tilde{D}_t F)_{t\in\mathbb{R}_+}$ and $(\nabla_t F)_{t\in\mathbb{R}_+}$ have the same predictable projection:*

$$ {}^p(\tilde{D}F) = {}^p(\nabla F). \tag{8}$$

Proof. We need to show that for any predictable process $u \in L^2(B) \otimes L^2(\mathbb{R}_+)$, $E[({}^p(\tilde{D}F), u)_{L^2(\mathbb{R}_+)}] = E[({}^p(\nabla F), u)_{L^2(\mathbb{R}_+)}]$, or

$$E[(\tilde{D}F, u)_{L^2(\mathbb{R}_+)}] = E[(\nabla F, u)].$$

This follows by duality, since $\tilde{\delta}$ and ∇^* coincide on the square-integrable predictable processes, i.e.

$$E[F\tilde{\delta}(u)] = E[F\nabla^*(u)]. \quad \square$$

Proposition 21 *The predictable projections of ∇F and $\tilde{D}F$, ${}^p(\nabla)$ and ${}^p(\tilde{D})$, extend to continuous operators from $L^2(B)$ into the space of predictable processes in $L^2(B)\otimes L^2(\mathbb{R}_+)$.*

Proof. We have if $F = \sum_{n=0}^\infty \tilde{I}_n(f_n) \in Dom(\nabla)$ and $u = \sum_{n=0}^\infty \tilde{I}_n(u_{n+1})$ with $u_{n+1} \in L^2(\mathbb{R}_+)^{\circ n} \otimes L^2(\mathbb{R}_+)$, $n \in \mathbb{N}$:

$$\begin{aligned}
\mid ({}^p(\nabla F), u)_{L^2(B)\otimes L^2(\mathbb{R}_+)} \mid &\leq \sum_{n=0}^\infty (n+1)! \mid \int_0^\infty (f_{n+1}(*,t)1_{\{*<t\}}, u_{n+1}(*,t))_{L^2(\mathbb{R}^n)} dt \mid \\
&\leq \sum_{n=0}^\infty n! \parallel f_{n+1} \parallel_{L^2(\mathbb{R}_+^{n+1})} \parallel u_{n+1} \parallel_{L^2(\mathbb{R}_+^{n+1})} \sqrt{n+1} \\
&\leq (\sum_{n=0}^\infty n! \parallel f_n \parallel^2_{L^2(\mathbb{R}_+^n)})^{1/2} (\sum_{n=0}^\infty n! \parallel u_{n+1} \parallel^2_{L^2(\mathbb{R}_+^{n+1})})^{1/2} \\
&\leq \parallel F \parallel_{L^2(B)} \parallel u \parallel_{L^2(B)\otimes L^2(\mathbb{R}_+)} .
\end{aligned}$$

The analog statement for $\tilde{D}$ is evident from relation 8, since $Dom(\nabla)\bigcap Dom(\tilde{D})$ is dense in $L^2(B)$. □

The Clark-Hausmann-Üstünel formula for the representation of a random variable as a stochastic integral, cf. [5], [12], has now two different expressions on Poisson space. Let

$$\tilde{\Delta}_n = \left\{(t_1, ..., t_n) \in \mathbb{R}_+^n \ : \ t_1 < \cdots < t_n\right\}.$$

Theorem 1 *If $F \in L^2(B)$, then*

$$F = E[F] + \int_0^\infty {}^p(\tilde{D}F)(t) d(N_t - t) = E[F] + \int_0^\infty {}^p(\nabla F)(t) d(N_t - t).$$

Proof. We start by assuming that $F \in Dom(\nabla) \cap \mathbb{D}_{2,1}$. The relation using the operator ∇ is proved with the Wiener-Poisson chaotic decomposition, as in the Wiener case. We write

$$\begin{aligned}
F &= \sum_{n=0}^\infty \frac{1}{n!} \tilde{I}_n(f_n) \\
&= E[F] + \sum_{n\geq 1} \tilde{I}_n(f_n 1_{\tilde{\Delta}_n}) \\
&= E[F] + \sum_{n\geq 1} \int_0^\infty \tilde{I}_{n-1}(f_n(\cdot, t) 1_{\tilde{\Delta}_n}(\cdot, t)) d\tilde{N}_t \\
&= E[F] + \int_0^\infty \sum_{n\geq 0} {}^p(\tilde{I}_n(f_{n+1} 1_{\tilde{\Delta}_n}))(t) d\tilde{N}_t \\
&= E[F] + \int_0^\infty {}^p(\nabla F)(t) d\tilde{N}_t.
\end{aligned}$$

The other part is proved using relation 8. The result is then extended to $F \in L^2(B)$ using the above proposition. □

We now recall a formula that allows to compute the Wiener chaotic decomposition of a random variable on Wiener space, cf. Stroock[11], and is valid also for the Wiener-Poisson chaotic decomposition, but has no analogue on the discrete chaotic decomposition.

Proposition 22 *If* $F \in \bigcap_{n=0}^{\infty} Dom(\nabla^n)$ *then* $F = \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{I}_n(E[\nabla^n F])$.

However, this result does not allow to compute Wiener-Poisson chaotic developments, because the expression of $\nabla^n F$ is too much complicated for $n \geq 2$. The next proposition gives a similar result that makes explicit calculations possible, using ∇ and $\tilde{D}$.

Proposition 23 *If* $F \in \bigcap_{n=0}^{\infty} Dom(\nabla^n \tilde{D})$,

$$F = E[F] + \sum_{n\geq 0}^{\infty} \tilde{I}_{n+1}(1_{\tilde{\Delta}_{n+1}} E[\nabla^n \tilde{D} F])$$

where $\tilde{\Delta}_n = \left\{(t_1, ..., t_n) \in \mathbb{R}_+^n \; : \; t_1 < \cdots < t_n\right\}$.

Proof. First, note that for any $t \in \mathbb{R}_+$, $\tilde{D}_t F$ is well-defined in $L^2(B)$ since the trajectories of $\tilde{D}F$ are constant between the jump times of the Poisson process. We start by applying Stroock's formula to $\tilde{D}_t F$, $t \in \mathbb{R}_+$:

$$\tilde{D}_t F = E[\tilde{D}_t F] + \sum_{n=1}^{\infty} \tilde{I}_n(1_{\tilde{\Delta}_n} E[\nabla^n \tilde{D}_t F]).$$

Taking the predictable projection of this process, we obtain

$${}^p(\tilde{D}F)(t) = E[\tilde{D}_t F] + \sum_{n=1}^{\infty} \tilde{I}_n(1_{\tilde{\Delta}_{n+1}(*,t)} E[\nabla^n \tilde{D}_t F]).$$

By stochastic integration with respect to $d(N_t - t)$ and use of the Clark formula, we get

$$F - E[F] = \sum_{n=0}^{\infty} \tilde{I}_{n+1}(1_{\tilde{\Delta}_{n+1}} E[\nabla^n \tilde{D} F]). \quad \square$$

As an application, the next result gives the decomposition of T_k, $k \geq 1$, in the Wiener-Poisson chaotic decomposition. The n-th kernel is given by the (n-1)-th derivative of the distribution function of T_k.

Proposition 24 $T_k = k + \sum_{n=1}^{\infty} \tilde{I}_n(f_n^k)$ *with*

$$f_n^k(t_1, ..., t_n) = -\frac{\partial^{n-1}}{\partial t_1^{n-1}} P(T_k \geq t_1) 1_{\tilde{\Delta}_n}(t_1, \ldots, t_n) \quad n \geq 1.$$

This result gives also the decomposition of τ_k, $k \in \mathbb{N}$, which is a (trivial) occupation time for the Poisson process.
Proof. We start with a lemma.

Lemma 1 *From the definition of* ∇ *as a finite difference operator,*

$$\nabla_t \left(1_{[0,T_k]}(u)\right) = 1_{\{t<u\}} \left(1_{[0,T_{k-1}]}(u) - 1_{[0,T_k]}(u)\right) \quad P.a.s.$$

Proof. We have

$$\begin{aligned}
\nabla_t 1_{[0,T_k]}(u) &= 1_{[0,T_k]}(u)1_{\{t>T_k\}} + 1_{[0,t]}(u)1_{\{T_{k-1}<t<T_k\}} \\
&\quad +1_{[0,T_{k-1}]}(u)1_{\{t<T_{k-1}\}} - 1_{[0,T_k]}(u) \\
&= 1_{\{u<t\}}1_{\{T_{k-1}<t<T_k\}} + 1_{[0,T_{k-1}]}(u)1_{\{t<T_{k-1}\}} \\
&\quad -1_{[0,T_{k-1}]}(u)1_{\{t<T_{k-1}\}} - 1_{[0,T_{k-1}]}(u)1_{\{T_{k-1}<t<T_k\}} \\
&\quad -1_{\{T_{k-1}<u<T_k\}}1_{\{t<T_{k-1}\}} - 1_{\{T_{k-1}<u<T_k\}}1_{\{T_{k-1}<t<T_k\}} \\
&= 1_{\{u<t\}}1_{\{T_{k-1}<t<T_k\}} - 1_{[0,T_k]}(u)1_{\{T_{k-1}<t<T_k\}} - 1_{\{T_{k-1}<u<T_k\}}1_{\{t<T_{k-1}\}} \\
&= 1_{\{t<u\}}\left(1_{[0,T_{k-1}]}(u) - 1_{[0,T_k]}(u)\right) \quad P.a.s. \ \square
\end{aligned}$$

Proof of proposition. If $n = 1$, $\tilde{D}_t T_k = -1_{[0,T_k]}(t)$ and $f_1^k(t_1) = -E[1_{[0,T_k]}(t_1)] = -P(T_k \geq t_1)$. If $n > 1$, the above lemma gives $\nabla_{t_{n-1}}\tilde{D}_{t_n}T_k = 1_{\{t_{n-1}<t_n\}}(\tilde{D}_t T_{k-1} - \tilde{D}_t T_k)$ and

$$\nabla_{t_1}\circ\cdots\circ\nabla_{t_{n-1}}\circ\tilde{D}_{t_n}T_k = \left[\nabla_{t_2}\circ\cdots\circ\nabla_{t_{n-1}}\circ\tilde{D}_{t_n}T_{k-1} - \nabla_{t_2}\circ\cdots\circ\nabla_{t_{n-1}}\circ\tilde{D}_{t_n}T_k\right]1_{\tilde{\Delta}_n}(t_1,...,t_n).$$

Taking the expectation on both sides, we have:

$$f_n^k(t_1,...,t_n) = (f_{n-1}^{k-1}(t_2,...,t_n) - f_{n-1}^k(t_2,...,t_n))1_{\tilde{\Delta}_n}(t_1,...,t_n).$$

On the other hand,

$$-\frac{\partial}{\partial t_1}P(T_k \geq t_1) = -(P(T_{k-1} \geq t_1) - P(T_k \geq t_1)),$$

hence

$$-\frac{\partial^{n-1}}{\partial t_1^{n-1}}P(T_k \geq t_1) = -\frac{\partial^{n-2}}{\partial t_1^{n-2}}P(T_{k-1} \geq t_1) + \frac{\partial^{n-2}}{\partial t_1^{n-2}}P(T_k \geq t_1)) \quad n \geq 2.$$

We have two sequences, $(-\frac{\partial^{n-1}}{\partial t_1^{n-1}}P(T_k \geq t_1))_{n\geq 1}$ and $(f_n^k(t_1,...,t_n))_{n\geq 1}$, defined by the same recurrence relation, and having the same first term $\forall k \geq 1$. Those sequences are identical, and this gives the conclusion. $\square$

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