A calculus on Fock space and its probabilistic interpretations

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Abstract.
We introduce on the Fock space $\Gamma(L^2(\mathbf{R}_+))$ two operators $\nabla^{\ominus}$ and $\nabla^{\oplus}$ expressing the infinitesimal perturbations of random variables by time changes in the Wiener and Poisson probabilistic interpretations of $\Gamma(L^2(\mathbf{R}_+))$. These operators have close connections with stochastic integration, regularity of laws, chaotic expansions and complement the annihilation and creation operators ∇^- and ∇^+ that are related to perturbations by shifts of trajectories.

Résumé.
Nous introduisons sur l'espace de Fock $\Gamma(L^2(\mathbf{R}_+))$ deux opérateurs $\nabla^{\ominus}$ qui $\nabla^{\oplus}$ qui permettent d'exprimer les perturbations infinitésimales de variables aléatoires par changements de temps dans les interprétations probabilistes de Wiener et de Poisson de $\Gamma(L^2(\mathbf{R}_+))$. Ces opérateurs complètent les opérateurs d'annihilation et de création ∇^- and ∇^+ relatifs aux perturbations par translations de trajectoires, et sont liés à l'intégration stochastique, à l'absolue continuité des lois de variables aléatoires et aux décompositions chaotiques.

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1 Introduction

Let $\Gamma(H) = \bigoplus_{n\geq 0} H^{\circ n}$ denote the symmetric Fock space over the Hilbert space H, where $H^{\circ n}$ consists of the space of symmetric tensors in the tensor product $H^{\otimes n}$, endowed with the norm $\| \cdot \|^2_{H^{\circ n}} = n! \parallel \cdot \|^2_{H^{\otimes n}}$, $n \in \mathbb{N}$. The annihilation operator $\nabla^- : \Gamma(H) \to \Gamma(H) \otimes H$ is defined by $\nabla^- f^{\circ n} = n f^{\circ(n-1)} \otimes f$, $n \in \mathbb{N}$, while the creation operator $\nabla^+ : \Gamma(H) \otimes H \to \Gamma(H)$ satisfies $\nabla^+ f^{\circ n} \otimes g = f^{\circ n} \circ g$, $n \in \mathbb{N}$. Those operators are extended by polarization, linearity and closability to their respective domains in $\Gamma(H)$ and $\Gamma(H) \otimes H$. In case $H = L^2(\mathbb{R}_+)$, the two main probabilistic interpretations of $\Gamma(L^2(\mathbb{R}_+))$ are the Wiener and Poisson interpretations, which are constructed by identifying $f_n \in L^2(\mathbb{R}_+)^{\circ n}$ with its multiple stochastic integral with respect to the Wiener or Poisson processes. It is well-known, cf. e.g. [4], [5], that in these probabilistic interpretations the annihilation operator acts on random variables by shifts of the Brownian, resp. Poisson trajectories. On the other hand, on the Poisson space, trajectories can be perturbed by time changes, and this yields another construction of the stochastic calculus of variations, cf. [2], [6]. The purpose of this paper is first to determine the action on Fock space of perturbations by time changes of the Poisson process. It turns out that the corresponding operator can be written as $\nabla^{\ominus} + \nabla^{-}$, where $\nabla^{\ominus}$ has a relatively simple description on Fock space. Since $\nabla^{\ominus}$ is expressed on the Fock space, it is a natural question to ask about its action on Wiener space. The results obtained are summarized in the table below.

Type of perturbation and properties		Wiener case	Poisson case
Shifts of trajectories	Operator on Fock space	∇^{-}	∇^{-}
	Absolute continuity	a.c.	not a.c.
Time changes	Operator on Fock space	$\nabla^{\ominus} + \frac{1}{2}\nabla^{-}\nabla^{-}$	$\nabla^{\ominus} + \nabla^{-}$
	Absolute continuity	not a.c.	a.c.

In this table some information has been added concerning the absolute continuity of the considered transformations with respect to the Wiener and Poisson probability measures. Transformations by deterministic shifts of trajectories are absolutely continuous with respect to the Wiener measure (from the Cameron-Martin theorem), but not with respect to the Poisson measure, since the standard Poisson process has fixed height jumps. On the other hand, the action of deterministic time changes on trajectories is absolutely continuous with respect to the Poisson measure (from the Girsanov theorem on Poisson space), but not with respect to the Wiener measure since a time changed Brownian motion does not have the standard quadratic variation. Consequently some smoothness has to be imposed on Wiener and Poisson functionals in order to perturb them by time changes, in particular they need to be defined everywhere, and on Fock space this corresponds to the assumption that the considered functionals have finite developments with smooth kernels. The operators obtained in this way can be extended by closability. The calculus introduced in this paper differs from the chaotic calculus of [6] which uses the polynomials of Laguerre instead of Hermite and can not be based on the Fock space.

This paper is organized as follows. In Sect. 2, definitions and preliminary results are stated. In Sect. 3 we define the operators $\nabla^{\ominus}$ and $\nabla^{\oplus}$ and show that they give a non-commutative decomposition of the number operator on Fock space. Sect. 4 is devoted to the Wiener space interpretation of $\nabla^{\ominus}$. In Sect. 5, we obtain the Poisson probabilistic interpretation of our calculus from the explicit chaotic expansions of functionals of the Poisson process jump times. Part of the results of this paper have been announced in [8]. In the Poisson space case a different approach to this calculus can be found in [3].

2 Preliminaries and notation

For $A \in \mathcal{B}(\mathbb{R}_+)$, let $\pi_A : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ denote the projection operator defined by $\pi_A f = 1_A f$, $f \in L^2(\mathbb{R}_+)$. Let also $\pi_{t]} = \pi_{[0,t]}$, $\pi_{[t} = \pi_{[t,\infty[}$, and $f_{[t} = \pi_{[t} f$, $f_{t]} = \pi_{t]} f$, $t \in \mathbb{R}_+$, $f \in L^2(\mathbb{R}_+)$. The exponential vector $\xi(f)$, $f \in L^2(\mathbb{R}_+)$, or Wick exponential, is defined as

$$\xi(f) = \sum_{n \in \mathbb{N}} \frac{1}{n!} f^{\circ n}.$$

The vector space generated by exponential vectors $\xi(f)$ with $f \in \mathcal{C}_c^1(\mathbb{R}_+)$ is denoted by Ξ. We denote by Φ the Fock space $\Gamma(L^2(\mathbb{R}_+))$ on $L^2(\mathbb{R}_+)$, and call $\mathcal{S}$ the set of elements of Φ which are in a finite number of chaos and whose developments involve only functions which are $\mathcal{C}^1$ with compact supports. We let $\Gamma(U) : \Phi \to \Phi$, densely defined on $\mathcal{S}$ as

$$\Gamma(U) = \bigoplus_{n \geq 0} U^{\circ n},$$

denote the second quantization of any operator $U : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ defined on $\mathcal{C}_c^1(\mathbb{R}_+)$. We say that $F \in \Phi$ is $\mathcal{F}_A$-measurable if $\Gamma(\pi_A)F = F$, $A \in \mathcal{B}(\mathbb{R}_+)$, and let $\mathcal{S}([a,b])$, $0 \leq a < b$, denote the elements of $\mathcal{S}$ that are $\mathcal{F}_{[a,b]}$-measurable. All operators considered in this work are densely defined on $\mathcal{S} \bigcup \Xi$ and closable. We denote by $< \cdot, \cdot >$ the scalar product on Φ.

Definition 1 *The adjoint* $U^* : \Phi \otimes L^2(\mathbb{R}_+) \to \Phi$ *of an operator* $U : \Phi \to \Phi \otimes L^2(\mathbb{R}_+)$ *is said to be an extension of the stochastic integral if* $\Gamma(\pi_{s]}) U_t = \Gamma(\pi_{s]}) \nabla_t^-$, $\quad 0 \leq s \leq t$, $t \in \mathbb{R}_+$.

The space of square-integrable adapted processes is defined to be the completion in $\Phi \otimes L^2(\mathbb{R}_+)$ of the set of simple adapted processes of the form

$$\sum_{k=1}^{k=n-1} F_i 1_{[t_i, t_{i+1}[}, \quad F_i \in \mathcal{S}([0,t_i]), \ i = 1, \ldots, n, \ t_1 < \cdots < t_n, \ n \in \mathbb{N}.$$

For $G \in \mathcal{S}([0,a])$, we have

$$\begin{aligned} < U^*(1_{[a,b]} G), F > &= \int_0^\infty 1_{[a,b]}(s) < G, U_s F > ds \\ &= \int_0^\infty 1_{[a,b]}(s) < G, \nabla_s^- F > ds =< \nabla^+(1_{[a,b]} G), F >, \quad F \in \mathcal{S}. \end{aligned}$$

Hence U^* is an extension of the stochastic integral if and only if U^* and ∇^+ coincide on the square-integrable adapted processes. If the Fock space is identified to the L^2-space of a stochastic process $(Y_t)_{t\in\mathbb{R}_+}$ with stationary independent increments, such as the Wiener and Poisson processes considered below, then the above property means that U^* coincides with the stochastic integral with respect to $(Y_t)_{t\in\mathbb{R}_+}$ on the square-integrable adapted processes. The following result shows that the Clark formula can be stated in general using the adjoint of an extension of the stochastic integral. It can be proved using the same argument as in [6].

Proposition 1 *If U^* is an extension of the stochastic integral on Fock space, then the application $F \mapsto (\Gamma(\pi_{s]})U_s F)_{s\in\mathbb{R}_+}$ is continuous from Φ to $\Phi \otimes L^2(\mathbb{R}_+)$, and any $F \in \Phi$ can be represented as*

$$F =< F, 1 > +U^*(\Gamma(\pi_{\cdot]})U_{\cdot}F).$$

As a consequence, the formula of [9] can be extended as follows.

Proposition 2 *If $U : \Phi \to \Phi \otimes L^2(\mathbb{R}_+)$ is the adjoint of an extension of the stochastic integral and*

$$F \in \bigcap_{n\in\mathbb{N}} Dom((\nabla^-)^n U),$$

then the chaotic development of F can be written as

$$F =< F, 1 > + \sum_{n\in\mathbb{N}} s\left(1_{\Delta_{n+1}} < (\nabla^-)^n UF, 1 >\right), \tag{1}$$

where $\Delta_{n+1} = \{(t_1, \ldots, t_{n+1}) \in \mathbb{R}_+^{n+1} \ : \ t_1 < \cdots < t_{n+1}\}$, and $s(f_n)$, $f_n \in L^2(\mathbb{R}_+)^{\otimes n}$, denotes the symmetrization of f_n in n variables.

3 A non-commutative decomposition of the number operator

In this section we define the operators $\nabla^\ominus$ and $\nabla^\oplus$ and remark that their sum gives the number operator. Some connections between $\nabla^\ominus$, $\nabla^\oplus$, the number operator and the non-commutative stochastic calculus have been studied in [8]. Let $\overset{\circ}{h}$, $h \in L^2(\mathbb{R}_+)$, denote the function defined as $\overset{\circ}{h}(t) = \int_0^t h(s)ds$, and let $f'(t) = \frac{d}{dt}f(t)$, $t \in \mathbb{R}_+$, $f \in \mathcal{C}_c^1(\mathbb{R}_+)$. Let $\mathcal{U}$ denote the set of elements of $\Phi \otimes L^2(\mathbb{R}_+)$ of the form $\sum_{i=1}^{i=n} F_i \otimes h_i$, with $h_1, \ldots, h_n \in \mathcal{C}_c^1(\mathbb{R}_+)$, and $F_1, \ldots, F_n \in \mathcal{S}$, $n \in \mathbb{N}$.

Definition 2 *We define the linear operators* $\nabla^\ominus : \Phi \to \Phi \otimes L^2(\mathbb{R}_+)$ *on* $\mathcal{S}$ *and* $\nabla^\oplus : \Phi \otimes L^2(\mathbb{R}_+) \to \Phi$ *on* $\mathcal{U}$ *by*

$$\nabla^\ominus_t f^{\circ n} = -n f'_{[t} \circ f^{\circ(n-1)} \quad \textit{and } \nabla^\oplus(f^{\circ n} \otimes h) = n \left(f \overset{\circ}{h} \right)' \circ f^{\circ(n-1)},$$

$f, h \in \mathcal{C}^1_c(\mathbb{R}_+)$, $n \in \mathbb{N}$, $t \in \mathbb{R}_+$, *and by linearity and polarization of these expressions.*

We note that for $f_n \in \mathcal{C}^1_c(\mathbb{R}^n_+)$ symmetric, $\nabla^\ominus_t f_n$ can be defined as the symmetrization in n variables of

$$-\sum_{i=1}^{i=n} 1_{[t,\infty[}(t_i) \partial_i f_n(t_1, \ldots, t_n).$$

Proposition 3 *Both* $\nabla^\ominus$, $\nabla^\oplus$ *are closable, and* $\nabla^\oplus : \Phi \otimes L^2(\mathbb{R}_+) \to \Phi$ *is adjoint of* $\nabla^\ominus : \Phi \to \Phi \otimes L^2(\mathbb{R}_+)$.

Proof. By polarization, we need to prove the following.

$$\begin{aligned}
(\nabla^\ominus f^{\circ n}, g^{\circ n} \otimes h)_{\Phi \otimes L^2(\mathbb{R}_+)} &= -n \int_0^\infty (f'_{[t} \circ f^{\circ(n-1)}, g^{\circ n})_{L^2(\mathbb{R}_+)^{\circ n}} h(t) dt \\
&= -n^2 (f^{\circ(n-1)}, g^{\circ(n-1)})_{L^2(\mathbb{R}_+)^{\circ(n-1)}} \int_0^\infty h(t) \int_t^\infty f'(s) g(s) ds dt \\
&= -n^2 (f^{\circ(n-1)}, g^{\circ(n-1)})_{L^2(\mathbb{R}_+)^{\circ(n-1)}} \int_0^\infty f'(t) g(t) \overset{\circ}{h}(t) dt \\
&= n^2 (f^{\circ(n-1)}, g^{\circ(n-1)})_{L^2(\mathbb{R}_+)^{\circ(n-1)}} \int_0^\infty f(t) (\overset{\circ}{h} g)'(t) dt \\
&= < f^{\circ n}, \nabla^\oplus (g^{\circ n} \otimes h) >, \quad f, g, h \in \mathcal{C}^1_c(\mathbb{R}_+),
\end{aligned}$$

hence the relation $(\nabla^\ominus F, u)_{\Phi \otimes L^2(\mathbb{R}_+)} =< F, \nabla^\oplus(u) >$, $F \in \mathcal{S}$, $u \in \mathcal{U}$. The closability of $\nabla^\ominus$ and $\nabla^\oplus$ follows from this relation and from the density of $\mathcal{S}$ and $\mathcal{U}$.

□

For $h \in L^2(\mathbb{R}_+)$, let a°_h denote the number operator defined by linearity and polarization as

$$a^\circ_h f^{\circ n} = n(fh) \circ f^{\circ n-1}, \quad f \in \mathcal{C}^1_c(\mathbb{R}_+), \; n \in \mathbb{N},$$

and let $a^\ominus_h$, $a^\oplus_h$ be defined as

$$a^\ominus_h f^{\circ n} = (\nabla^\ominus f^{\circ n}, h)_{L^2(\mathbb{R}_+)}, \quad a^\oplus_h = \nabla^\oplus (f^{\circ n} \otimes h), \quad n \in \mathbb{N}, \; f, h \in L^2(\mathbb{R}_+).$$

Proposition 4 *The above definitions give a non-commutative decomposition of* a°_h *into the sum of a gradient operator and its adjoint:*

$$a^\circ_h = a^\ominus_h + a^\oplus_h, \quad h \in L^2(\mathbb{R}_+).$$

Proof. We have $(\nabla^\ominus f^{\circ n}, h)_{L^2(\mathbb{R}_+)} + \nabla^\oplus (f^{\circ n} \otimes h) = n f^{\circ(n-1)} \circ (fh) = a^\circ_h f^{\circ n}$, $n \in \mathbb{N}$, $f \in \mathcal{C}^1_c(\mathbb{R}_+)$.

$\square$

Proposition 5 *The exponential vector $\xi(f)$, $f \in L^2(\mathbb{R}_+) \cap L^4(\mathbb{R}_+)$, is in the domain of $\nabla^\ominus$ if $\int_0^\infty t f'(t)^2 dt < \infty$. In this case, $\nabla_t^\ominus \xi(f) = -f'_{t]} \circ \xi(f)$, and*

$$\| \nabla^\ominus \xi(f) \|^2_{\Phi \otimes L^2(\mathbb{R}_+)} = \left(\frac{1}{8} \| f \|^2_{L^4(\mathbb{R}_+)} + \int_0^\infty t f'(t)^2 dt \right) \exp\left((f,f)_{L^2(\mathbb{R}_+)} \right).$$

Proof. We have

$$\begin{aligned} \| \nabla^\ominus f^{\circ n} \|^2_{L^2(\mathbb{R}_+)^{\circ n} \otimes L^2(\mathbb{R}_+)} &= n^2 \int_0^\infty \| f'_{[t} \circ f^{\circ(n-1)} \|^2_{L^2(\mathbb{R}_+)^{\circ n}} dt \\ &= n^2 (n-1)! (f,f)^{n-1}_{L^2(\mathbb{R}_+)} \int_0^\infty \int_t^\infty f'(s)^2 ds dt \\ &\quad + n^2 (n-1)(n-1)! (f,f)^{n-2}_{L^2(\mathbb{R}_+)} \int_0^\infty \left(\int_t^\infty f'(s) f(s) ds \right)^2 dt \\ &= n n! (f,f)^{n-1} \int_0^\infty t f'(t)^2 dt + \frac{n}{8}(n-1) n! (f,f)^{n-2} \int_0^\infty f^4(t) dt, \end{aligned}$$

and

$$\begin{aligned} \| \nabla^\ominus \xi(f) \|^2_\Phi &= \sum_{n \geq 1} \frac{n n!}{n!^2} (f,f)^{n-1}_{L^2(\mathbb{R}_+)} \left(\int_0^\infty t f'(t)^2 dt + \frac{n-1}{2} \right) \\ &= \left(\frac{1}{2} (f,f)_{L^2(\mathbb{R}_+)} + \int_0^\infty t f'(t)^2 dt \right) \exp\left((f,f)_{L^2(\mathbb{R}_+)} \right). \quad \square \end{aligned}$$

4 Wiener space interpretation

Let $(W, L^2(\mathbb{R}_+), \mu)$ denote the classical Wiener space, with Brownian motion $(B_t)_{t \in \mathbb{R}_+}$. Multiple stochastic integrals are defined as

$$\hat{I}_n(f_n) = n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dB_{t_1} \cdots dB_{t_n},$$

$f_n \in L^2(\mathbb{R}_+)^{\circ n}$. These integrals provide an isometric isomorphism between $L^2(W, \mu)$ and Φ, since

$$E\left[\hat{I}_n(f_n) \hat{I}_m(g_m) \right] = 1_{\{n=m\}} (f_n, g_m)_{L^2(\mathbb{R}_+)^{\circ n}}, \quad f_n \in L^2(\mathbb{R}_+)^{\circ n}, \ g_m \in L^2(\mathbb{R}_+)^{\circ m}.$$

This identification will be assumed throughout this section. We are interested here in the properties of $\nabla^\ominus$ in the Wiener interpretation of Φ. We recall that ∇^- is identified to a derivation operator which satisfies

$$(\nabla^- F, h)_{L^2(\mathbb{R}_+)} = \lim_{\varepsilon \to 0} \frac{F\left(B_\cdot + \varepsilon \int_0^\cdot h(s) ds \right) - F}{\varepsilon}, \quad F \in \mathcal{S}, \ h \in L^2(\mathbb{R}_+),$$

cf. e.g. [4], [10] and the references therein.

Lemma 1 *On the Wiener space, $\nabla^\ominus$ satisfies the relation*

$$\nabla_t^\ominus(FG) = F\nabla_t^\ominus G + G\nabla_t^\ominus F - \nabla_t^- F \nabla_t^- G, \quad t \in \mathbb{R}_+, \ F, G \in \mathcal{S}. \tag{2}$$

Proof. We are using the multiplication formula for the Wiener multiple integrals:

$$\hat{I}_n(f^{\circ n})\hat{I}_1(g) = \hat{I}_{n+1}(f^{\circ n} \circ g) + n(f,g)_{L^2(\mathbb{R}_+)}\hat{I}_{n-1}(f^{\circ n-1}), \quad f, g \in L^2(\mathbb{R}_+), \ n \in \mathbb{N}.$$

We first show that

$$\begin{aligned}
\nabla^\ominus\left(\hat{I}_n(f^{\circ n})\hat{I}_1(g)\right) &= \nabla^\ominus\left(\hat{I}_{n+1}(f^{\circ n} \circ g) + n(f,g)\hat{I}_{n-1}(f^{\circ(n-1)})\right) \\
&= -\hat{I}_{n+1}(g'_{[t} \circ f^{\circ n}) - n\hat{I}_{n+1}(f'_{[t} \circ g \circ f^{\circ(n-1)}) - n(n-1)(f,g)_{L^2(\mathbb{R}_+)}\hat{I}_{n-1}(f'_{[t} \circ f^{\circ(n-2)}) \\
&= -n\hat{I}_{n+1}(f'_{[t} \circ f^{\circ(n-1)} \circ g) - n(g, f'_{[t)})_{L^2(\mathbb{R}_+)}\hat{I}_{n-1}(f^{\circ(n-1)}) - \hat{I}_{n+1}(f^{\circ n} \circ g'_{[t)}) \\
&\quad -n(n-1)(f,g)_{L^2(\mathbb{R}_+)}\hat{I}_{n-1}(f^{\circ(n-2)} \circ f'_{[t)}) - n(f, g'_{[t)})_{L^2(\mathbb{R}_+)}\hat{I}_{n-1}(f^{\circ(n-1)}) \\
&\quad +n(f'_{[t}, g)_{L^2(\mathbb{R}_+)}\hat{I}_{n-1}(f^{\circ(n-1)}) + n(g'_{[t}, f)_{L^2(\mathbb{R}_+)}\hat{I}_{n-1}(f^{\circ(n-1)}) \\
&= -n\hat{I}_n(f'_{[t} \circ f^{\circ(n-1)})\hat{I}_1(g) - \hat{I}_n(f^{\circ n})\hat{I}_1(g'_{[t)}) - nf(t)g(t)\hat{I}_{n-1}(f^{\circ(n-1)}) \\
&= \hat{I}_1(g)\nabla_t^\ominus\hat{I}_n(f^{\circ n}) + \hat{I}_n(f^{\circ n})\nabla_t^\ominus\hat{I}_1(g) - \nabla_t^-\hat{I}_1(g)\nabla_t^-\hat{I}_n(f^{\circ n}), \quad t \in \mathbb{R}_+.
\end{aligned}$$

Using the fact that on Wiener space ∇^- is a derivation, we can now work by induction to show that the formula holds for functionals that are polynomials in Wiener multiple stochastic integrals. Assume that for some $k \geq 1$, and $t \in \mathbb{R}_+$,

$$\nabla_t^\ominus(\hat{I}_n(f^{\circ n})\hat{I}_1(g)^k) = \hat{I}_n(f^{\circ n})\nabla_t^\ominus\left(\hat{I}_1(g)^k\right) + \hat{I}_1(g)^k\nabla_t^\ominus\hat{I}_n(f^{\circ n}) - \nabla_t^-\hat{I}_n(f^{\circ n})\nabla_t^-\left(\hat{I}_1(g)^k\right).$$

Then

$$\begin{aligned}
&\nabla_t^\ominus(\hat{I}_n(f^{\circ n})\hat{I}_1(g)^{k+1}) \\
&= \hat{I}_1(g)\nabla_t^\ominus(\hat{I}_n(f^{\circ n})\hat{I}_1(g)^k) + \hat{I}_n(f^{\circ n})\hat{I}_1(g)^k\nabla_t^\ominus\hat{I}_1(g) - \nabla_t^-\hat{I}_1(g)\nabla_t^-(\hat{I}_1(g)^k\hat{I}_n(f^{\circ n})) \\
&= \hat{I}_1(g)\left(\hat{I}_1(g)^k\nabla_t^\ominus\hat{I}_n(f^{\circ n}) + \hat{I}_n(f^{\circ n})\nabla_t^\ominus\left(\hat{I}_1(g)^k\right) - \nabla_t^-\left(\hat{I}_1(g)^k\right)\nabla_t^-\hat{I}_n(f^{\circ n})\right) \\
&\quad +\hat{I}_n(f^{\circ n})\hat{I}_1(g)^k\nabla_t^\ominus\hat{I}_1(g) - \nabla_t^-\hat{I}_1(g)\left(\hat{I}_1(g)^k\nabla_t^-\hat{I}_n(f^{\circ n}) + \hat{I}_n(f^{\circ n})\nabla_t^-\left(\hat{I}_1(g)^k\right)\right) \\
&= \hat{I}_1(g)^{k+1}\nabla_t^\ominus\hat{I}_n(f^{\circ n}) + \hat{I}_n(f^{\circ n})\nabla_t^\ominus\left(\hat{I}_1(g)^{k+1}\right) - \nabla_t^-\left(\hat{I}_1(g)^{k+1}\right)\nabla_t^-\hat{I}_n(f^{\circ n}),
\end{aligned}$$

$t \in \mathbb{R}_+$. □.

For $h \in L^2(\mathbb{R}_+)$, with $\sup_{x\in\mathbb{R}_+} \mid h(x) \mid< 1$, let $\nu_h(t) = t + \int_0^t h(s)ds$, $t \in \mathbb{R}_+$.

Definition 3 *We define a mapping $\mathcal{T}_h : W \to W$, $t, \varepsilon \in \mathbb{R}_+$, as*

$$\mathcal{T}_h(\omega) = \omega \circ \nu_h^{-1}, \quad h \in L^2(\mathbb{R}_+), \ \sup_{x\in\mathbb{R}_+} \mid h(x) \mid< 1. \tag{3}$$

The transformation $\mathcal{T}_h$ acts on the trajectory of $(B_s)_{s\in\mathbb{R}_+}$ by change of time. Although $\mathcal{T}_h$ is not absolutely continuous with respect to the Wiener measure, the functional $F\circ\mathcal{T}_h$ is well-defined for $F\in\mathcal{S}$, since elements of $\mathcal{S}$ are defined trajectory by trajectory.

Proposition 6 *We have for $F\in\mathcal{S}$, under the Wiener identification of Φ:*

$$\int_0^\infty h(t)\left(\nabla_t^\ominus+\frac{1}{2}\nabla_t^-\nabla_t^-\right)Fdt=-\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(F\circ\mathcal{T}_{\varepsilon h}-F).$$

Proof. We first notice that as a consequence of Lemma 1, the operator $\nabla_t^\ominus+\frac{1}{2}\nabla_t^-\nabla_t^-$ is a derivation operator on $\mathcal{S}$, $t\in\mathbb{R}_+$. Moreover, $\mathcal{T}_{\varepsilon h}$ is multiplicative, hence we only need to treat the particular case of $F=\hat{I}_1(f)$. We have

$$\begin{aligned}\hat{I}_1(f)\circ\mathcal{T}_{\varepsilon h}-\hat{I}_1(f) &= \int_0^\infty f(s)dB(\nu_{\varepsilon h}^{-1}(s))-\hat{I}_1(f)\\ &= \int_0^\infty f(\nu_{\varepsilon h}(s))dB_s-\int_0^\infty f(s)dB_s\\ &= \int_0^\infty\left(f\left(t+\varepsilon\int_0^t h(s)ds\right)-f(t)\right)dB_t.\end{aligned}$$

After division by $\varepsilon>0$, this converges in $L^2(W,\mu)$ as $\varepsilon\to 0$ to

$$\begin{aligned}\int_0^\infty f'(t)\int_0^t h(s)dsdB_t &= \int_0^\infty h(t)\int_t^\infty f'(s)dB_sdt=-\int_0^\infty h(t)\nabla_t^\ominus\hat{I}_1(f)dt\\ &= -\int_0^\infty h(t)\left(\nabla_t^\ominus+\frac{1}{2}\nabla_t^-\nabla_t^-\right)\hat{I}_1(f)dt.\ \square\end{aligned}$$

5 Poisson space interpretation

Before dealing with the Poisson interpretation of $\nabla^\ominus$, we will need to compute the explicit chaotic decomposition of functionals of the Poisson process jump times. Let $T_k=\sum_{i=0}^{i=k-1}\tau_i$, $k\geq 1$, denote the sequence of jump times of a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ on a probability space $(\Omega,\mathcal{F},P)$. The Poisson multiple stochastic integral of $h_n\in L^2(\mathbb{R})^{\circ n}$, space of symmetric square-integrable functions on $\mathbb{R}^n$, can be written as

$$\tilde{I}_n(h_n)=n!\int_0^\infty\int_0^{t_n^-}\cdots\int_0^{t_2^-}h_n(t_1,...,t_n)d(N_{t_1}-t_1)\cdots d(N_{t_n}-t_n). \qquad (4)$$

As on the Wiener space, we have the isometry

$$(\tilde{I}_n(f_n),\tilde{I}_m(g_m))_{L^2(B)}=1_{\{n=m\}}(f_n,g_m)_{L^2(\mathbb{R}_+)^{\circ n}},\quad f_n\in L^2(\mathbb{R}_+)^{\circ n},$$

which provides an isometric isomorphism between $L^2(B)$ and Φ. This identification will be used in the remaining of this paper. From [5], the operator ∇^- satisfies

$$\nabla_t F = F(N_\cdot + 1_{[t,\infty[}) - F(N_\cdot), \quad t \in \mathbb{R}_+, \ F \in \mathcal{S}, \tag{5}$$

hence

$$\nabla^-(FG) = F\nabla^- G + G\nabla^- F + \nabla^- F \nabla^- G, \quad F, G \in \mathcal{S}. \tag{6}$$

There exists a different approach to the calculus of variations on Poisson space, cf. [2], [6], which consists in defining a closable operator $\tilde{D} : L^2(B) \to L^2(B) \otimes L^2(\mathbb{R}_+)$ by time changes:

$$(\tilde{D}F, h)_{L^2(\mathbb{R}_+)} = -\lim_{\varepsilon \to 0} \frac{F \circ \mathcal{T}_{\varepsilon h} - F}{\varepsilon}, \quad h \in L^2(\mathbb{R}_+),$$

where the transformation $\mathcal{T}_{\varepsilon h}$ is defined as in (3), by application of a time change to the Poisson process trajectories. This is equivalent to

$$\tilde{D}F = -\sum_{k=0}^{k=n} \partial_k f(T_1, ..., T_n) 1_{[0,T_k]}, \quad F \in \mathcal{S}, \tag{7}$$

for $F = f(T_1, \ldots, T_n)$. The following proposition extends to functionals of jump times the result of [6] which was only proved for jump times.

Proposition 7 *For $k \geq 1$, the chaotic development of $f(T_k)$ is given as*

$$f(T_k) = E[f(T_k)] + \sum_{n \geq 1} \frac{1}{n!} \tilde{I}_n(f_n^k),$$

where $f_n^k(t_1, \ldots, t_n) = \alpha_n^k(f)(t_1 \vee \cdots \vee t_n)$, $t_1, \ldots, t_n \in \mathbb{R}_+$, and

$$\alpha_n^k(f)(t) = f(t)\partial^{n-1} p_k(t) + (f, 1_{[t,\infty[} \partial^n p_k)_{L^2(\mathbb{R}_+)}, \quad t \in \mathbb{R}_+, \quad n \geq 1. \tag{8}$$

For $f \in \mathcal{C}_c^1(\mathbb{R}_+)$ we also have

$$\alpha_n^k(f)(t) = -\int_t^\infty f'(s)\partial^{n-1} p_k(s) ds.$$

Lemma 2 *We have for $f \in \mathcal{C}_c^1(\mathbb{R})$ and $n \geq 1$*

$$\nabla_t^- \tilde{D}_s f(T_n) = \tilde{D}_{s \vee t} f(T_{n-1}) - \tilde{D}_{s \vee t} f(T_n) - 1_{\{s<t\}} 1_{[T_{n-1}, T_n]}(s \vee t) f'(s \vee t), \ a.s., \quad s, t \in \mathbb{R}_+.$$

Proof. We have

$$\begin{aligned} \nabla_t^- \tilde{D}_s f(T_n) &= -1_{[0,T_{n-1}]}(t) \left(1_{[0,T_{n-1}]}(s) f'(T_{n-1}) - 1_{[0,T_n]}(s) f'(T_n)\right) \\ &\quad -1_{[T_{n-1},T_n]}(t) \left(1_{[0,t]}(s) f'(t) - 1_{[0,T_n]}(s) f'(T_n)\right) \\ &= 1_{\{t<s\}} \left(1_{[0,T_n]}(s) f'(T_n) - 1_{[0,T_{n-1}]}(s) f'(T_{n-1}),\right) \\ &\quad +1_{\{s<t\}} \left(1_{[0,T_n]}(t) f'(T_n) - 1_{[0,T_{n-1}]}(t) f'(T_{n-1}) - 1_{[T_{n-1},T_n]}(t) f'(t)\right), \end{aligned}$$

P-a.s.

□

Proof of Prop. 7. Since the adjoint of $\tilde{D}$ extends the stochastic integral, cf. [2], [6], we can apply Prop. 2 with $U = \tilde{D}$. Let us first assume that $f \in \mathcal{C}_c^1(\mathbb{R}_+)$. We have

$$f_1^k(t) = E[\tilde{D}_t f(T_k)] = -E[1_{[0,T_k]}(t) f'(T_k)] = -\int_t^\infty p_k(s) f'(s) ds.$$

Now, from Lemma 2, for $n \geq 2$ and $0 \leq t_1 < \cdots < t_n$,

$$\nabla_{t_1}^- \cdots \nabla_{t_{n-1}}^- \tilde{D}_{t_n} f(T_k) = \nabla_{t_1}^- \cdots \nabla_{t_{n-2}}^- (\tilde{D}_{t_n} f(T_{k-1}) - \tilde{D}_{t_n} f(T_k)),$$

hence

$$f_n^k(t_1, \ldots, t_n) = f_{n-1}^{k-1}(t_1, \ldots, t_{n-2}, t_n) - f_{n-1}^k(t_1, \ldots, t_{n-2}, t_n),$$

and we can show (8) by induction, for $n \geq 2$:

$$\begin{aligned} f_n^k(t_1, \ldots, t_n) &= f_{n-1}^{k-1}(t_1, \ldots, t_{n-2}, t_n) - f_{n-1}^k(t_1, \ldots, t_{n-2}, t_n), \\ &= -\int_{t_n}^\infty f'(s) \partial^{n-2} p_{k-1}(s) ds + \int_{t_n}^\infty f'(s) \partial^{n-2} p_k(s) ds \\ &= -\int_{t_n}^\infty f'(s) \partial^{n-1} p_k(s) ds. \end{aligned}$$

The conclusion is obtained by density of the $\mathcal{C}_c^1$ functions in $L^2(\mathbb{R}_+, p_k(t)dt)$, $k \geq 1$.

□

We note the relation

$$\frac{d}{dt} \alpha_n^k(f)(t) = \alpha_n^k(f')(t) + \alpha_{n+1}^k(f)(t), \quad t \in \mathbb{R}_+, \quad f \in L^2(\mathbb{R}_+, p_k(t)dt). \tag{9}$$

We now prove that $\nabla^\ominus + \nabla^-$ is identified to the operator $\tilde{D}$ under the Poisson identification of Φ and $L^2(B)$.

Lemma 3 *On the Poisson space, $\nabla^\ominus$ satisfies the relation*

$$\nabla_t^\ominus(FG) = F \nabla_t^\ominus G + G \nabla_t^\ominus F - \nabla_t^- F \nabla_t^- G, \quad t \in \mathbb{R}_+, \ F, G \in \mathcal{S}. \tag{10}$$

Note that $\nabla^\ominus$ and ∇^- satisfy the same relation on Wiener space, cf. (2).
Proof. We need the following multiplication formula for Poisson multiple stochastic integrals, known as the Kabanov formula:

$$\tilde{I}_n(f^{\circ n}) \tilde{I}_1(g) = \tilde{I}_{n+1}(f^{\circ n} \circ g) + n(f, g) \tilde{I}_{n-1}(f^{\circ n-1}) + n \tilde{I}_n((fg) \circ f^{\circ n-1}), \ \ f, g \in L^4(\mathbb{R}_+).$$

We first show that

$$\nabla_t^\ominus \left(\tilde{I}_n(f^{\circ n}) \tilde{I}_1(g) \right) = \tilde{I}_n(f^{\circ n}) \nabla_t^\ominus \tilde{I}_1(g) + \tilde{I}_1(g) \nabla_t^\ominus \tilde{I}_n(f^{\circ n}) - \nabla_t^- \tilde{I}_1(g) \nabla_t^- \tilde{I}_n(f^{\circ n}), \quad t \in \mathbb{R}_+,$$

with $f,g\in\mathcal{C}_c^1(\mathbb{R}_+)$ and $(f,f)_{L^2(\mathbb{R}_+)}=1$. We have

$$\begin{aligned}
&\tilde I_n(f^{\circ n})\nabla_t^{\ominus}\tilde I_1(g)+\tilde I_1(g)\nabla_t^{\ominus}\tilde I_n(f^{\circ n})\\
&\quad= -n\tilde I_1(g)\tilde I_n(f'_{[t}\circ f^{\circ(n-1)})-\tilde I_n(f^{\circ n})\tilde I_1(g'_{[t})\\
&\quad= -n\Big(\tilde I_{n+1}(f'_{[t}\circ f^{\circ(n-1)}\circ g)+(n-1)\tilde I_n((fg)\circ f'_{[t}\circ f^{\circ(n-2)})+\tilde I_n((gf'_{[t})\circ f^{\circ(n-1)})\\
&\qquad +(f'_{[t},g)_{L^2(\mathbb{R}_+)}\tilde I_{n-1}(f^{\circ(n-1)})+(n-1)(f,g)_{L^2(\mathbb{R}_+)}\tilde I_{n-1}(f'_{[t}\circ f^{\circ(n-2)})\Big)\\
&\qquad -\tilde I_{n+1}(g'_{[t}\circ f^{\circ n})-n\tilde I_n((g'_{[t}f)\circ f^{\circ(n-1)})-n(g'_{[t},f)_{L^2(\mathbb{R}_+)}\tilde I_{n-1}(f^{\circ(n-1)})\\
&\quad= -n\tilde I_{n+1}(f'_{[t}\circ f^{\circ(n-1)}\circ g)-\tilde I_{n+1}(g'_{[t}\circ f^{\circ n})-n(n-1)\tilde I_n(f'_{[t}\circ(fg)\circ f^{\circ(n-2)})\\
&\qquad -n\tilde I_n((gf'_{[t})\circ f^{\circ(n-1)})-n\tilde I_n((fg'_{[t})\circ f^{\circ(n-1)})\\
&\qquad +nf(t)g(t)\tilde I_{n-1}(f^{\circ(n-1)})-n(n-1)(f,g)_{L^2(\mathbb{R}_+)}\tilde I_{n-1}(f'_{[t}\circ f^{\circ(n-2)})\\
&\quad= \nabla_t^{\ominus}\Big(\tilde I_{n+1}(f^{\circ n}\circ g)+n\tilde I_n(f^{\circ(n-1)}\circ(fg))+n(f,g)_{L^2(\mathbb{R}_+)}\tilde I_{n-1}(f^{\circ(n-1)})\Big)\\
&\qquad +nf(t)g(t)\tilde I_{n-1}(f^{\circ(n-1)})\\
&\quad= \nabla_t^{\ominus}(\tilde I_n(f^{\circ n})\tilde I_1(g))+\nabla_t^-\tilde I_1(g)\nabla_t^-\tilde I_n(f^{\circ n}),\quad f,g\in\mathcal{C}_c^1(\mathbb{R}_+).
\end{aligned}$$

We now make use of (6) to prove the result on $\mathcal{S}$ by induction. Assume that (10) holds for $F=\tilde I_n(f^{\circ n})$ and $G=\tilde I_1(g)^k$ for some $k\geq 1$. Then

$$\begin{aligned}
&\nabla_t^{\ominus}(\tilde I_n(f^{\circ n})\tilde I_1(g)^{k+1})\\
&\quad= \tilde I_1(g)\nabla_t^{\ominus}(\tilde I_n(f^{\circ n})\tilde I_1(g)^k)+\tilde I_n(f^{\circ n})\tilde I_1(g)^k\nabla_t^{\ominus}\tilde I_1(g)-\nabla_t^-\tilde I_1(g)\nabla_t^-(\tilde I_1(g)^k\tilde I_n(f^{\circ n}))\\
&\quad= \tilde I_1(g)\Big(\tilde I_1(g)^k\nabla_t^{\ominus}\tilde I_n(f^{\circ n})+\tilde I_n(f^{\circ n})\nabla_t^{\ominus}\Big(\tilde I_1(g)^k\Big)-\nabla_t^-\Big(\tilde I_1(g)^k\Big)\nabla_t^-\tilde I_n(f^{\circ n})\Big)\\
&\qquad +\tilde I_n(f^{\circ n})\tilde I_1(g)^k\nabla_t^{\ominus}\tilde I_1(g)-\nabla_t^-\tilde I_1(g)\Big(\tilde I_1(g)^k\nabla_t^-\tilde I_n(f^{\circ n})+\tilde I_n(f^{\circ n})\nabla_t^-\Big(\tilde I_1(g)^k\Big)\Big)\\
&\qquad -\nabla_t^-\tilde I_1(g)\nabla_t^-\tilde I_1(g)^k\nabla_t^-\tilde I_n(f^{\circ n})\\
&\quad= \tilde I_1(g)^{k+1}\nabla_t^{\ominus}\tilde I_n(f^{\circ n})+\tilde I_n(f^{\circ n})\nabla_t^{\ominus}\Big(\tilde I_1(g)^{k+1}\Big)-\nabla_t^-\Big(\tilde I_1(g)^{k+1}\Big)\nabla_t^-\tilde I_n(f^{\circ n}),
\end{aligned}$$

$t\in\mathbb{R}_+$. □

Proposition 8 *Under the Poisson probabilistic interpretation of* Φ, $\tilde D=\nabla^{\ominus}+\nabla^-$.

Proof. From Lemma 3, we know that $(\nabla^{\ominus}+\nabla^-)$ is a derivation operator. Thus it is sufficient to show that $(\nabla^-+\nabla^{\ominus})f(T_k)=\tilde Df(T_k)$, $k\geq 1$. We have from Prop. 7

$$\begin{aligned}
(\nabla_t^-+\nabla_t^{\ominus})f(T_k) &= (\nabla_t^-+\nabla_t^{\ominus})\sum_{n\in\mathbb{N}}\frac{1}{n!}\tilde I_n(f_n^k),\\
&= \sum_{n\geq 1}\frac{1}{(n-1)!}\tilde I_{n-1}(f_n^k(\cdot,t))-\sum_{n\geq 1}\frac{1}{(n-1)!}\tilde I_n(\pi_{[t}\otimes I_d^{\otimes(n-1)}\partial_1 f_n^k)\\
&= \sum_{n\in\mathbb{N}}\frac{1}{n!}\tilde I_n\left(f_{n+1}^k(\cdot,t)-n\pi_{[t}\otimes I_d^{\otimes(n-1)}\partial_1 f_n^k\right),
\end{aligned}$$

where $I_d : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ denotes the identity operator. Now from (9),

$$\begin{aligned} & f^k_{n+1}(t,t_1,\ldots,t_n) - n\pi_{[t} \otimes I_d^{\otimes(n-1)} \partial_1 f^k_n(t_1,\ldots,t_n) \\ &= \alpha^k_{n+1}(f)(t_1 \vee \cdots \vee t_n \vee t) - 1_{\{t<t_1\vee\cdots\vee t_n\}} \left(\alpha^k_n(f') + \alpha^k_{n+1}(f)\right)(t_1 \vee \cdots \vee t_n) \\ &= \alpha^k_{n+1}(f) 1_{\{t_1\vee\cdots\vee t_n<t\}} - \alpha^k_n(f')(t_1 \vee \cdots \vee t_n) 1_{\{t_1\vee\cdots\vee t_n>t\}} \\ &= \alpha^k_n(-f'_{[t)})(t_1 \vee \cdots \vee t_n), \quad n \geq 1. \end{aligned}$$

Now from Prop. 7, $\alpha^k_n(-f'_{[t)})(t_1 \vee \cdots \vee t_n)$ is precisely the n-th chaos term of the expansion of $-1_{[0,T_k]} f'(T_k)$, $n \in \mathbb{N}$. Hence $\nabla^- + \nabla^\ominus = \tilde{D}$.

□

Since both ∇^+ and $\tilde{\delta}$ coincide with the Itô integral on adapted processes, it follows from Prop. 8 that $\nabla^\oplus$ vanishes on adapted processes. By duality this implies that the adapted projection of $\nabla^\ominus$ is zero.

Proposition 9 *On the Poisson space, we have for $f \in \mathcal{C}^1_c(\mathbb{R}_+)$:*

$$\tilde{D}_t \xi(f) = -\left(\int_t^\infty \frac{f'(s)}{1+f(s)} 1_{\{f(s)\neq -1\}} dN_s\right) \xi(f), \quad t \in \mathbb{R}_+.$$

Proof. We have

$$\xi(f) = \exp\left(-\int_0^\infty f(s)ds\right) \prod_{k\geq 1} (1 + f(T_k)),$$

hence

$$\begin{aligned} \tilde{D}_t \xi(f) &= -\exp\left(-\int_0^\infty f(s)ds\right) \sum_{i\geq 1} 1_{[0,T_i]}(t) f'(T_i) \prod_{k\neq i} (1 + f(T_k)) \\ &= -\exp\left(-\int_0^\infty f(s)ds\right) \int_t^\infty 1_{\{f(s)\neq -1\}} \frac{f'(s)}{1+f(s)} dN_s \prod_{k\geq 1} (1 + f(T_k)). \end{aligned}$$

□

As an application of this calculus, we obtain the following absolute continuity criterion for Poisson stochastic integrals.

Proposition 10 *Let $f \in L^2(\mathbb{R}_+)$ such that $\int_0^\infty t f'(t)^2 dt < \infty$ and*

$$\lim_{n\to\infty} \int_{\{f'=0\}} p_n(t) dt = 0. \tag{11}$$

Then the law of $\int_0^\infty f(t) d(N_t - t)$ is absolutely continuous with respect to the Lebesgue measure.

This condition is satisfied in particular if $\{f' = 0\}$ has finite Lebesgue measure.

Proof. From the proof of Prop. 5, $\tilde{I}_1(f) \in Dom(\tilde{D})$. We have

$$\begin{aligned}\left(\tilde{D}_t \tilde{I}_1(f)\right)^2 &= \left(\sum_{k=1}^{\infty} 1_{[0,T_k]}(t) f'(T_k)\right)^2 = \left(\sum_{k=0}^{\infty} 1_{]T_k,T_{k+1}]}(t) \sum_{i=k+1}^{\infty} f'(T_i)\right)^2 \\ &= \sum_{k=0}^{\infty} 1_{]T_k,T_{k+1}]}(t) \left(\sum_{i=k+1}^{\infty} f'(T_i)\right)^2, \quad t \in \mathbb{R}_+,\end{aligned}$$

hence

$$\| \tilde{D}\tilde{I}_1(f) \|_{L^2(\mathbb{R}_+)}^2 = \sum_{k=0}^{\infty} \tau_k \left(\sum_{i=k+1}^{\infty} f'(T_i)\right)^2 .$$

If the law of $\tilde{I}_1(f)$ were not absolutely continuous, then according to the criterion of [1], (cf. [7] for its Poisson space version), there would exist $A \in \mathcal{F}$ such that $P(A) > 0$ and $\| \tilde{D}\tilde{I}_1(f) \|_{L^2(\mathbb{R}_+)} = 0$, everywhere on A. The above calculation implies then that $f'(T_k) = 0$ on A, $k \geq 1$. Hence $T_n(A) \subset \{f' = 0\}$, $n \geq 1$, and from (11),

$$\lim_{n\to\infty} \int_0^{\infty} 1_{T_n(A)}(t) p_n(t) dt = 0.$$

This contradicts the fact that

$$\int_0^{\infty} 1_{T_n(A)}(t) p_n(t) dt = P(\{\omega \in \Omega \ : \ T_n(\omega) \in T_n(A)\}) \geq P(A) > 0, \quad n \geq 1. \quad \square$$

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