Markovian bridges and reversible diffusion processes with jumps

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Abstract

Markovian bridges driven by Lévy processes are constructed from the data of an initial and a final distribution, as particular cases of a family of time reversible diffusions with jumps. The processes obtained in this way are essentially the only (not necessarily continuous) Markovian Bernstein processes. These processes are also characterized using the theory of stochastic control for jump processes. Our construction is motivated by Euclidean quantum mechanics in momentum representation, but the resulting class of processes is much bigger than the one needed for this purpose. A large collection of examples is included.

Key words: Lévy processes, bridges, time reversal, Euclidean quantum mechanics.
Classification: 60J25, 81S20, 47D07.

1 Introduction

Euclidean quantum mechanics yields a probabilistic approach to Schrödinger equations, which relies on the construction of time reversible stochastic processes. A probabilistic counterpart of a quantum system with symmetric (more precisely, self-adjoint) Hamiltonian H is provided by considering positive solutions of two heat

*This work has been initiated during a visit of the first author to the GFM (Group of Mathematical Physics) under a Grant conceded by the FCT in the "Programatico" framework.

equations which are adjoint with respect to the time parameter:

$$-\hbar \frac{\partial \eta_t^*}{\partial t}(q) = H\eta_t^*(q) \quad \text{and} \quad \hbar \frac{\partial \eta_t}{\partial t}(q) = H\eta_t(q), \qquad t \in [r, v], \quad q \in \mathbb{R}^d, \tag{1.1}$$

where $[r, v]$ is a fixed interval, and by postulating that the density of the law at time t of the system is given by the product $\eta_t(q)\eta_t^*(q)$, instead of the product of the solution of the Schrödinger equation with its complex conjugate. This approach allows moreover to construct time reversible diffusion processes which precisely have the law $\eta_t(q)\eta_t^*(q)dq$ at time t, see [29], [7], [2] when the Hamiltonian is a self-adjoint Schrödinger operator of the form $H = -\frac{\hbar^2}{2}\Delta + V(q)$ and V is a scalar potential in Kato's class. We refer the reader to [6] for a detailed survey of the relations between this method, and in particular the Feynman path integral approach to quantum mechanics, when the processes have continuous trajectories.

In this paper we generalize this construction in the case where the above Schrödinger operator is replaced by a pseudo-differential operator. Our motivations are of two types: the first one is the study of the probabilistic counterpart of quantum mechanics in the momentum representation and its relation with the one of the position representation, the link between these representations being given by the Fourier transform which maps position operators to momentum operators, and scalar potentials to a pseudo-differential operators. This illustrates the more general aim of this program of construction of quantum-like reversible measures. They provide (through their Hilbert space analytical models) fresh structural relations between stochastic processes generally regarded as unrelated. Our second motivation is to treat relativistic Hamiltonians along the line of [15], but in a fully time reversible framework.

Lévy bridges have been studied and constructed by several authors, see e.g. [12] and Section VIII.3 in [4]. However, an absolute continuity condition with respect to Lebesgue measure is generally imposed on the law of the process, thus excluding simple Poisson bridges and many other more complex processes. Our construction of reversible diffusions with jumps provides, in particular, a general construction of Markovian bridges with given initial and final distributions π_r and π_v. For this we use a result of Beurling [5] which, under the assumption of existence of densities with respect to a fixed reference measure, asserts the existence of initial and final conditions η_r and η_v^* for (1.1) such that $\pi_r = \eta_r \eta_r^*$ and $\pi_v = \eta_v \eta_v^*$. This allows us to construct forward and backward Lévy processes with Dirac measures as initial, resp. final, laws. In this case we extend existing results on the martingale representation of time-reversed processes, cf. e.g. [19]. We also show how time reversible processes can be constructed from non-symmetric Lévy processes and generators.

We use the term "bridge" in the wide sense, i.e. a process which is determined from its initial and final laws (which are not necessarily Dirac measures) will be called a bridge. Bridges and more generally diffusions with jumps, reversible on $[r, v]$, are constructed via the forward and backward Markov semi-groups

$$p(t,k,u,dl) = \frac{\eta_u(l)}{\eta_t(k)} h(t,k,u,dl),$$

and

$$p^*(s,dj,t,k) = \frac{\eta_s^*(j)}{\eta_t^*(k)} h^\dagger(s,dj,t,k),$$

for $s \le t \le u$ in $[r, v]$, $j, k, l \in \mathbb{R}^d$, where $p(t, k, u, dl)$ and $p^*(s, dj, t, k)$ are the kernels associated to $\exp(-(u-t)H)$ and $\exp(-(t-s)H^\dagger)$. In the time homogeneous case (i.e. when η, η^* depend trivially on time) this construction of Markov semi-groups in relation to time reversal seems to go back to [14] (see also [10] where it is applied to conditioned processes), but does not seem to have been the object of systematic studies when η_t and η_t^* are given as the solutions of "heat equations" for general H. This also provides a construction of Bernstein process [3] in the jump case, i.e. we construct stochastic processes $(z_t)_{t\in[r,v]}$ that satisfy the relation

$$P(z_t \in dk \mid \mathcal{P}_s \vee \mathcal{F}_u) = P(z_t \in dk \mid z_s, z_u), \qquad r \le s < t < u \le v,$$

where $(\mathcal{P}_t)_{t\in[r,v]}$, resp. $(\mathcal{F}_t)_{t\in[r,v]}$, denotes the increasing, resp. decreasing, filtration generated by $(z_t)_{t\in[r,v]}$. When their paths are continuous, as we said such processes have been constructed in the framework of Euclidean quantum mechanics. We also show that the general processes constructed in this paper are essentially the only Markovian Bernstein processes.

We proceed as follows. After recalling some notation on Lévy processes and their generators in Section 2, the main results of the paper are presented in Section 3. The construction of Bernstein processes with jumps is given in Section 5. In Section 6, we compute the generators of Markovian bridges and derive the associated stochastic differential equations driven by Lévy processes. The uniqueness of Markovian Bernstein processes with jumps is discussed in Section 7. A variational characterization is obtained in Section 8, and the associated almost-sure equations of motion encoding all the dynamical properties of these processes are stated in Section 9. In particular, the construction provides time reversible jump diffusions whose law is given in terms of positive solutions of "heat equations" associated to the Schrödinger operator in the momentum representation.

2 Notation - Lévy processes and generators

We refer to the survey [17] and to the references therein for the notions recalled in this section. Let $V : \mathbb{R}^d \longrightarrow \mathbb{C}$ such that $V(0) \geq 0$ and $\exp(-tV(q))$ is continuous in q and positive definite. The function V admits the Lévy-Khintchine representation

$$V(q) = \frac{1}{\hbar}\left(a + i\langle c, \hbar q\rangle + \frac{1}{2}\langle \hbar q, \hbar q\rangle_B - \int_{\mathbb{R}^d} (e^{-i\langle \hbar q, y\rangle} - 1 + i\langle \hbar q, y\rangle 1_{\{|\hbar y| \leq 1\}})\nu(dy)\right),$$

where $a, r \in \mathbb{R}^d$, B is a positive definite $d \times d$ matrix, $\langle q, q\rangle_B = \langle Bq, q\rangle$, ν is a Lévy measure on $\mathbb{R}^d \setminus \{0\}$ satisfying $\int_{\mathbb{R}^d}(|y|^2 \wedge 1)\nu(dy) < \infty$, and $\hbar$ is a fixed strictly positive parameter which will be, later on, identified with Planck's constant. In the following we assume that $a = 0$, i.e. $V(0) = 0$. Then the Lévy process is conservative (i.e. it has an infinite life time). We will write

$$V(q) = i\langle c, q\rangle + \frac{1}{2}\langle q, q\rangle_{\hbar B} - \int_{\mathbb{R}^d} (e^{-i\langle q, y\rangle} - 1 + i\langle q, y\rangle 1_{\{|y| \leq 1\}})\nu_\hbar(dy),$$

where $\nu_\hbar$ is $\hbar^{-1}$ times the image measure of ν by $y \mapsto \hbar y$. Let ξ_t denote the Lévy process with characteristic exponent $V(q)$, i.e. such that

$$E\left[e^{-i\langle \xi_t, q\rangle}\right] = e^{-tV(q)}, \quad q \in \mathbb{R}^d, \quad t \in \mathbb{R},$$

or

$$E\left[\exp\left(-\frac{i}{\hbar}\langle \xi_t, q\rangle\right)\right] = \exp\left(-\frac{t}{\hbar}\left(i\langle c, q\rangle + \frac{1}{2}\langle q, q\rangle_B\right.\right.$$

$$-\int_{\mathbb{R}^d}(e^{-i\langle q,y\rangle}-1+i\langle q,y\rangle 1_{\{|\hbar y|\leq 1\}})\nu(dy)\Big)\Big),$$

$q\in\mathbb{R}^d$, $t\in\mathbb{R}$, i.e. the physically meaningful potential is $\hbar V(q/\hbar)$. The process $(\xi_t)_{t\in[r,v]}$ admits the (forward) Lévy-Itô decomposition with respect to the filtration $(\mathcal{P}_t)_{t\in[r,v]}$:

$$\xi_t = W_t + \int_0^t\int_{\{|y|\leq 1\}} y(\mu(dy,ds)-\nu_\hbar(dy)ds) + \int_0^t\int_{\{|y|\geq 1\}} y\mu(dy,ds) + ct,$$

where W_t is a Brownian motion with covariance matrix $\hbar B$, and $\mu(dy,ds)$ is the Poisson random measure

$$\mu(dy,ds) = \sum_{\Delta\xi_s\neq 0}\delta_{(\Delta\xi_s,s)}(dy,ds)$$

with compensator $E[\mu(dy,ds)] = \nu_\hbar(dy)ds$. Let μ_t denote the law of ξ_t, and let $\mu_t(dk)=\mu_{-t}(-dk)$ when $t<0$. The (forward) generator of $(\xi_t)_{t\in[r,v]}$ is the following pseudo-differential operator

$$\begin{aligned}&-V(\nabla)f(k) \qquad (2.1)\\ &= \langle c,\nabla f(k)\rangle + \frac{1}{2}\Delta_{\hbar B}f(k) + \int_{\mathbb{R}^d}(f(k+y)-f(k)-\langle y,\nabla f(k)\rangle 1_{\{|y|\leq 1\}})\nu_\hbar(dy),\end{aligned}$$

where $\Delta_{\hbar B} = \hbar\mathrm{div}B\nabla$. We shall also need the reversed Lévy process $(\xi_t^*)_{t\in[0,v]} = (-\xi_{v-t})_{t\in[0,v]}$ whose backward generator is

$$\begin{aligned}&\bar{V}(\nabla)f(k) = V(-\nabla)f(k) \qquad (2.2)\\ &= \langle c,\nabla f(k)\rangle - \frac{1}{2}\Delta_{\hbar B}f(k) - \int_{\mathbb{R}^d}(f(k-y)-f(k)+\langle y,\nabla f(k)\rangle 1_{\{|y|\leq 1\}})\nu_\hbar(dy).\end{aligned}$$

Note that when $V(\nabla)$ is not symmetric (i.e. V is not real-valued), the reversal of $(\xi_t)_{t\in[0,v]}$ is both in space and time, whereas in the symmetric (e.g. Brownian) case a time reversal suffices.

In view of our applications to mathematical physics we consider a perturbation of the generator of $(\xi_t)_{t\in[r,v]}$ by a potential $U:\mathbb{R}^d\to\mathbb{R}$, continuous and bounded below:

Definition 2.1 *Let $H = U + V(\nabla)$, i.e. for $u\in\mathcal{S}(\mathbb{R}^d)$:*

$$\begin{aligned}Hf(k) &= U(k)f(k) - \langle c,\nabla f(k)\rangle - \frac{1}{2}\Delta_{\hbar B}f(k)\\ &\quad - \int_{\mathbb{R}^d}(f(k+y)-f(k)-\langle y,\nabla f(k)\rangle 1_{\{|y|\leq 1\}})\nu_\hbar(dy), \qquad k\in\mathbb{R}^d.\end{aligned}$$

The operator $V(\nabla)$ is obtained from the potential V by considering (Euclidean) momentum ∇ as a variable. The potential U is symmetrically deduced from a differential operator, e.g. the quadratic potential $\hbar U(k/\hbar) = k^2/2$ in momentum representation correspond to the Laplacian Δ in position representation. The adjoint $H^\dagger$ of H with respect to dk is given by $H^\dagger = U + \bar{V}(\nabla)$ with $\bar{V}(q) = V(-q)$, i.e.

$$\begin{aligned} H^\dagger f(k) &= U(k)f(k) + \langle c, \nabla f(k)\rangle - \frac{1}{2}\Delta_{\hbar B} f(k) \\ &\quad - \int_{\mathbb{R}^d} (f(k-y) - f(k) + \langle y, \nabla f(k)\rangle 1_{\{|y|\le 1\}})\nu_\hbar(dy), \qquad k \in \mathbb{R}^d, \end{aligned}$$

where $\nu_\hbar(-dy)$ denotes the image measure of $\nu_\hbar$ under $y \mapsto -y$. If $c = 0$, the operator H is symmetric when V is real-valued, that is when ν is symmetric with respect to $y \mapsto -y$.

Let $T_{t,u}$, $t < u$, resp. $T^\dagger_{s,t}$, $s < t$, denote the positive operator defined through the Feynman-Kac formula

$$T_{t,u} f(k) = E\left[f(\xi_u) e^{-\int_t^u U(\xi_\tau)d\tau} \mid \xi_t = k \right] = E\left[f(k + \xi_{u-t}) e^{-\int_0^{u-t} U(k+\xi_\tau)d\tau} \right], \quad t < u,$$

respectively

$$T^\dagger_{s,t} f(k) = E\left[f(\xi^*_s) e^{-\int_s^t U(\xi^*_\tau)d\tau} \mid \xi^*_t = k \right] = E\left[f(k + \xi^*_{s-t}) e^{-\int_0^{t-s} U(k+\xi^*_\tau)d\tau} \right], \quad s < t.$$

Since $-V(\nabla)$ is the generator of $(\xi_t)_{t\in[0,u]}$ and $\bar{V}(\nabla) = V(-\nabla)$ is the generator of the reversed Lévy process $(\xi^*_s)_{s\in[0,t]} = (-\xi_{t-s})_{s\in[0,t]}$, we have

$$\frac{\partial}{\partial u} T_{t,u} = -T_{t,u} H, \quad \text{and} \quad \frac{\partial}{\partial s} T^\dagger_{s,t} = T^\dagger_{s,t} H^\dagger,$$

for $T_{t,u} = \exp(-(u-t)H)$ and $T^\dagger_{s,t} = \exp(-(t-s)H^\dagger)$, i.e. these semi-groups are time homogeneous since V and U are independent of time.

We denote by $h^\dagger(s, dj, t, k)$ and $h(t, k, u, dl)$, $0 < s < t < u$, $j, k, l \in \mathbb{R}^d$, the "integral kernels" of $\exp(-(t-s)H^\dagger)$ and $\exp(-(u-t)H)$, defined by

$$\exp(-(t-s)H^\dagger) f(k) = \int_{\mathbb{R}^d} f(j) h^\dagger(s, dj, t, k),$$

and

$$\exp(-(u-t)H) f(k) = \int_{\mathbb{R}^d} f(l) h(t, k, u, dl).$$

Examples

1) Deterministic process.

Here, U does not necessarily vanish. Then $B = 0$ and $\nu_\hbar = 0$ but $c \neq 0$. So $V(q) = icq$ and

$$Hf(k) = -c\nabla f(k), \quad H^\dagger f(k) = c\nabla f(k),$$

with integral kernels

$$h^\dagger(s, dj, t, k) = \delta_{k-c(t-s)}(dj)e^{-\int_s^t U(k+c(\tau-t))d\tau},$$

$$h(t, k, u, dl) = \delta_{k+c(u-t)}(dl)e^{-\int_t^u U(k+c(\tau-t))d\tau}.$$

2) Poisson process ($U = 0$).

The kinetic term U vanishes, as well as B, and $c = 1$. Moreover $\nu_\hbar = \delta_1$, so $V(q) = -(e^{-iq} - 1)$ and we have

$$-Hf(k) = f(k+1) - f(k), \quad -H^\dagger f(k) = f(k-1) - f(k).$$

The associated integral kernels reduce to

$$h(t, k, u, dl) = \sum_{p=0}^{\infty} e^{-(u-t)} \frac{(u-t)^p}{p!} \delta_{k+p}(dl),$$

and

$$h^\dagger(s, dj, t, k) = \sum_{p=0}^{\infty} e^{-(t-s)} \frac{(t-s)^p}{p!} \delta_{k-p}(dj).$$

3) Lévy case ($U = 0$).

In this case, $e^{-(u-t)H}$ and $e^{-(t-s)H^\dagger}$ are respectively given by the convolutions with the law μ_t of the Lévy process

$$\int_{\mathrm{R}^d} \eta_s^*(dj)h(s, j, t, dk) = \eta_s^* * \mu_{t-s}(dk), \qquad s < t,$$

and

$$\int_{\mathrm{R}^d} \eta_u(dl)h^\dagger(t, dk, u, l) = \eta_u * \mu_{t-u}(dk), \qquad t < u.$$

If moreover $\eta_s^*(dj) = \eta_s^*(j)\lambda(dj)$ and $\eta_u(dl) = \eta_u(l)\lambda(dl)$, are absolutely continuous with respect to λ, then

$$e^{-(t-s)H^\dagger}\eta_s^*(k)\lambda(dk) \;\; = \;\; \int_{\mathrm{R}^d} \eta_s^*(j)h^\dagger(s, dj, t, k)\lambda(dk)$$

$$= \int_{\mathbb{R}^d} \eta_s^*(dj) h(s,j,t,dk) = \eta_s^* * \mu_{t-s}(k)\lambda(dk), \qquad s < t,$$

and

$$\begin{aligned} e^{-(u-t)H}\eta_u(k)\lambda(dk) &= \int_{\mathbb{R}^d} \eta_u(l) h(t,k,u,dl)\lambda(dk) \\ &= \int_{\mathbb{R}^d} \eta_u(dl) h^\dagger(t,dk,u,l) = \eta_u * \mu_{t-u}(k)\lambda(dk), \qquad t < u, \end{aligned}$$

hence

$$e^{-(t-s)H^\dagger}\eta_s^*(k) = \eta_s^* * \mu_{t-s}(k), \quad s < t,$$

and

$$e^{-(u-t)H}\eta_u(k) = \eta_u * \mu_{t-u}(k), \quad t < u,$$

where the convolution of functions is with respect to $\lambda(dk)$. The formulas for $h(t,k,u,dl)$ and $h^\dagger(s,dj,t,k)$ are given in Relations (2.3) and (2.4) below.

4) General case ($U \neq 0$).

We have, by definition

$$\begin{aligned} &\int_{\mathbb{R}^d} h(t,k,u,dl) f(l) = e^{-(u-t)H} f(k) \\ &= E\left[f(\xi_u) e^{-\int_t^u U(\xi_\tau)d\tau} \mid \xi_t = k \right] = E\left[f(\xi_{u-t}+k) e^{-\int_0^{u-t} U(k+\xi_\tau)d\tau} \right] \\ &= \int_{\mathbb{R}^d} E\left[f(\xi_{u-t}+k) e^{-\int_0^{u-t} U(k+\xi_\tau)d\tau} \mid \xi_{u-t} = l \right] \mu_{u-t}(dl) \\ &= \int_{\mathbb{R}^d} f(k+l) E\left[e^{-\int_0^{u-t} U(k+\xi_\tau)d\tau} \mid \xi_{u-t} = l \right] \mu_{u-t}(dl) \\ &= \int_{\mathbb{R}^d} f(l) E\left[e^{-\int_0^{u-t} U(k+\xi_\tau)d\tau} \mid \xi_{u-t} = l-k \right] \mu_{u-t}(k+dl), \end{aligned}$$

where $\mu_{u-t}(k+dl)$ denotes the image measure of μ_{u-t} under $l \mapsto k+l$. Consequently we obtain

$$h(t,k,u,dl) = \alpha(u-t,l-k)\mu_{u-t}(k+dl), \tag{2.3}$$

with

$$\alpha(u-t,l-k) = E\left[e^{-\int_0^{u-t} U(k+\xi_\tau)d\tau} \mid \xi_{u-t} = l-k \right].$$

Similarly we have

$$\int_{\mathbb{R}^d} h^\dagger(s,dj,t,k)f(j) = e^{-(t-s)H^\dagger} f(k)$$
$$\begin{aligned}
&= E\left[f(\xi^*_s)e^{-\int_s^t U(\xi^*_\tau)d\tau} \mid \xi^*_t = k\right] = E\left[f(k-\xi_{t-s})e^{-\int_s^t U(k-\xi_{t-\tau})d\tau}\right] \\
&= \int_{\mathbb{R}^d} E\left[f(k-\xi_{t-s})e^{-\int_0^{t-s} U(k-\xi_{t-s-\tau})d\tau} \mid \xi_{t-s} = j\right]\mu_{t-s}(dj) \\
&= \int_{\mathbb{R}^d} f(k-j)E\left[e^{-\int_0^{t-s} U(k-\xi_\tau)d\tau} \mid \xi_{t-s} = j\right]\mu_{t-s}(dj) \\
&= \int_{\mathbb{R}^d} f(j)E\left[e^{-\int_0^{t-s} U(k-\xi_\tau)d\tau} \mid \xi_{t-s} = j-k\right]\mu_{t-s}(k+dj),
\end{aligned}$$

Hence

$$h^\dagger(s,dj,t,k) = \alpha^\dagger(t-s,k-j)\mu_{t-s}(k+dj), \tag{2.4}$$

where

$$\alpha^\dagger(t-s,k-j) = E\left[e^{-\int_0^{t-s} U(k-\xi_\tau)d\tau} \mid \xi_{u-t} = j-k\right].$$

We end this section with a lemma that will be useful in Section 3.2.

Lemma 2.2 *We have, for all $k \in \mathbb{R}^d$;*

$$\int_{\mathbb{R}^d} y\frac{\mu_t(k-y)}{\mu_t(k)}\nu_\hbar(dy) = \frac{k}{t} - c + \hbar B\nabla \log \mu_t(k) + \int_{\mathbb{R}^d} y1_{\{|y|\le 1\}}\nu_\hbar(dy),$$

and

$$\int_{\mathbb{R}^d} y\frac{\mu_{t-v}(k+y)}{\mu_{t-v}(k)}\nu_\hbar(dy) = -\frac{k}{v-t} - c - \hbar B\nabla \log \mu_{t-v}(k) + \int_{\mathbb{R}^d} y1_{\{|y|\le 1\}}\nu_\hbar(dy).$$

Proof. We have for $k, q \in \mathbb{R}^d$:

$$-i\int_{\mathbb{R}^d} ke^{-ikq}\mu_t(k)dk = \nabla_q e^{-tV(q)} = -t\nabla V(q)e^{-tV(q)}$$
$$\begin{aligned}
&= -t\int_{\mathbb{R}^d} e^{-ikq}\mu_t(k)dk\left(ic + \hbar Bq + i\int_{\mathbb{R}^d} y(e^{-iqy} - 1_{\{|y|\le 1\}})\nu_\hbar(dy)\right) \\
&= -it\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-ikq}\mu_t(k-y)dk\nu_\hbar(dy) \\
&\quad -t(ic+\hbar Bq)\int_{\mathbb{R}^d} e^{-ikq}\mu_t(k)dk + it\mu_t(k)\int_{\mathbb{R}^d} y1_{\{|y|\le 1\}}\nu_\hbar(dy) \\
&= -it\int_{\mathbb{R}^d} e^{-ikq}\int_{\mathbb{R}^d}\mu_t(k-y)dk\nu_\hbar(dy) - ict\int_{\mathbb{R}^d} e^{-ikq}\mu_t(k)dk
\end{aligned}$$

$$+it\int_{\mathrm{R}^d} e^{-ikq}\hbar B\nabla\mu_t(k)dk + it\mu_t(k)\int_{\mathrm{R}^d} y1_{\{|y|\leq 1\}}\nu_\hbar(dy).$$

For the second relation we have $\mu_{t-v}(k)=\mu_{v-t}(-k)$ and

$$[\nabla\log\mu_{v-t}](-k) = -\nabla\log\mu_{v-t}(-k) = -\hbar B\nabla\log\mu_{t-v}(k),$$

hence

$$\begin{aligned}\int_{\mathrm{R}^d} y\frac{\mu_{t-v}(k+y)}{\mu_{t-v}(k)}\nu_\hbar(dy) &= \int_{\mathrm{R}^d} y\frac{\mu_{v-t}(-k-y)}{\mu_{v-t}(-k)}\nu_\hbar(dy)\\ &= -\frac{k}{v-t} - c + \hbar B[\nabla\log\mu_{v-t}](-k) + \int_{\mathrm{R}^d} y1_{\{|y|\leq 1\}}\nu_\hbar(dy)\\ &= -\frac{k}{v-t} - c - \hbar B\nabla\log\mu_{t-v}(k) + \int_{\mathrm{R}^d} y1_{\{|y|\leq 1\}}\nu_\hbar(dy).\end{aligned}$$

□

3 Construction of Markovian bridges - main results

Among the objectives of this paper is the proof of Prop. 3.1 below. Assume that

$h^\dagger(s,dj,t,k) = h^\dagger(s,j,t,k)\lambda(dj)$ and $h(t,k,u,dl) = h(t,k,u,l)\lambda(dl)$ are absolutely continuous with respect to λ,

H and $H^\dagger$ are mutually adjoint under λ, i.e. $h^\dagger(s,j,t,k) = h(s,j,t,k)$,

$h^\dagger(s,j,t,k) = h(s,j,t,k)$ is continuous in (j,k) and strictly positive for all $0<s<t$.

Let z_{t^-} denotes the left limit of z at $t\in[r,v]$. The following proposition holds under the assumptions A and B of Section 3.2, before Prop. 3.5.

Proposition 3.1 *Let $\pi_r(dk)$ and $\pi_v(dk)$ be two given probability measures on $\mathbb{R}^d$, which are assumed to be absolutely continuous with a.e. strictly positive densities with respect to a fixed reference measure λ. There exists a $\mathbb{R}^d$-valued process $(z_t)_{t\in[r,v]}$ with initial distribution $\pi_r(dk)$ and final distribution $\pi_v(dk)$, driven by $(\xi_t)_{t\in[r,v]}$, i.e. such that $(z_t)_{t\in[r,v]}$ solves in the weak sense the stochastic integro-differential equation*

$$dz_t = cdt + dW_t + \int_{\mathrm{R}^d} y\left(\mu(dy,dt) - \frac{\eta_t(z_{t^-}+y)}{\eta_t(z_{t^-})}1_{\{|y|\leq 1\}}\nu_\hbar(dy)dt\right) \tag{3.1}$$

$$+\int_{\mathrm{R}^d} y\left(\frac{\eta_t(z_{t^-}+y)-\eta_t(z_{t^-})}{\eta_t(z_{t^-})}\right)1_{\{|y|\leq 1\}}\nu_\hbar(dy)dt+\hbar B\nabla\log\eta_t(z_{t^-})dt,$$

and the law of z_t at time t is $\eta_t(k)\eta_t^*(k)\lambda(dk)$, where

W is a Brownian motion with covariance $\hbar B$,

the canonical point process $\mu(dy,dt)$ has compensator $\frac{\eta_t(z_{t^-}+y)}{\eta_t(z_{t^-})}\nu_\hbar(dy)dt$,

$\eta_t = e^{-(v-t)H}\eta_v$, $r\leq t\leq v$,

$\eta_t^* = e^{-(t-r)H^\dagger}\eta_r^*$, $r\leq t\leq v$,

and η_r^*, η_v are two positive initial and final conditions which are determined from π_r and π_v.

Moreover the process $(z_t)_{t\in[r,v]}$ in question is a Bernstein process, i.e.

$$P(z_t\in dk\mid \mathcal{P}_s\vee\mathcal{F}_u)=P(z_t\in dk\mid z_s,z_u),$$

and the joint law $P(z_r\in A,\ z_v\in B)$, for A,B two borelians of $\mathbb{R}^d$, is of the form

$$P(z_r\in A,\ z_v\in B)=\int_{A\times B}\eta_r^*(i)h(r,i,v,m)\eta_v(m)\lambda(di)\lambda(dm).$$

Conversely we will also prove a uniqueness result, i.e. if $(z_t)_{t\in[r,v]}$ is a Markovian Bernstein process with Bernstein kernel $h(s,j,t,dk,u,l)=P(z_t\in dk\mid z_s=j,z_u=l)$ such that

$$h(s,j,t,dk,u,l)h(s,j,u,dl)=h(s,j,t,dk)h(t,k,u,dl),$$

or

$$h(s,j,t,dk,u,l)h^\dagger(s,dj,u,l)=h^\dagger(s,dj,t,k)h^\dagger(t,dk,u,l),$$

$s\leq t\leq u$, $j,k,l\in\mathbb{R}^d$, then there exists positive density functions $\eta_r^*(i)$ and $\eta_v(m)$ such that

$$P(z_r\in A,\ z_v\in B)=\int_{A\times B}\eta_r^*(i)h(r,i,v,m)\eta_v(m)\lambda(di)\lambda(dm),$$

cf. Th. 7.1. These results will be precisely stated in different forms and under weaker assumptions in the following section. Proofs will be provided afterwards in several steps, which consist of more refined statements.

3.1 Existence of Markovian bridges

In the following result, $h(t,k,u,dl)$ and $h^\dagger(s,dj,t,k)$ need not be absolutely continuous with respect to a fixed reference measure λ.

Theorem 3.2 *Let λ be a fixed reference measure such that H and $H^\dagger$ are adjoint with respect to λ, and let $\eta_r^*, \eta_v : \mathbb{R}^d \to \mathbb{R}_+$ be two λ-a.e. strictly positive initial and final conditions such that for some $t \in [r,v]$ (and therefore for any such t),*

$$\int_{\mathbb{R}^d} \eta_t^*(k)\eta_t(k)\lambda(dk) = 1,$$

where

$$\eta_t^*(k) = e^{-(t-r)H^\dagger}\eta_r^*(k) = \int_{\mathbb{R}^d} \eta_r^*(i)h^\dagger(r,di,t,k),$$

and

$$\eta_t(k) = e^{-(v-t)H}\eta_v(k) = \int_{\mathbb{R}^d} \eta_v(m)h(t,k,v,dm), \qquad r \le t \le v.$$

Then there exists a $\mathbb{R}^d$-valued process $(z_t)_{t\in[r,v]}$ whose density at time t with respect to λ is $\rho_t(k) = \eta_t^*(k)\eta_t(k)$, which is forward and backward Markovian, with forward transition kernel

$$p(t,k,u,dl) = \frac{\eta_u(l)}{\eta_t(k)}h(t,k,u,dl), \tag{3.2}$$

and backward transition kernel

$$p^*(s,dj,t,k) = \frac{\eta_s^*(j)}{\eta_t^*(k)}h^\dagger(s,dj,t,k). \tag{3.3}$$

In particular, the initial and final laws of $(z_t)_{t\in[r,v]}$ are $\pi_r(di) = \eta_r(i)\eta_r^*(i)\lambda(di)$ and $\pi_v(dm) = \eta_v(m)\eta_v^*(m)\lambda(dm)$.

The above functions $\eta_t^*(k)$ and $\eta_t(k)$ satisfy the partial integro-differential equations

$$-\frac{\partial \eta_t^*}{\partial t}(k) = H^\dagger\eta_t^*(k) \quad \text{and} \quad \frac{\partial \eta_t}{\partial t}(k) = H\eta_t(k), \qquad t \in [r,v]. \tag{3.4}$$

The proof of Th. 3.2 follows from Prop. 5.1 and Prop. 5.2 below. Once Th. 3.2 is proved, Prop. 3.1 follows from Prop. 3.5 and Th. 3.3 below which states that given two probability measures $\pi_r(di) = \pi_r(i)\lambda(di)$ and $\pi_v(dm) = \pi_v(m)\lambda(dm)$, absolutely continuous with respect to λ, it is possible to determine two positive initial and final functions $\eta_r^*, \eta_v : \mathbb{R}^d \to \mathbb{R}_+$ from the data of the initial and final laws π_r, π_v of the process, such that

$$\pi_r(i) = \eta_r^*(i)\eta_r(i), \qquad \pi_v(m) = \eta_v(m)\eta_v^*(m),$$

where

$$\eta_r(i) = \int_{\mathrm{R}^d} \eta_v(m)h(r,i,v,dm),$$

and

$$\eta_v^*(m) = \int_{\mathrm{R}^d} \eta_r^*(i)h^\dagger(r,di,v,m),$$

provided $h(s,k,t,dj)$ and $h^\dagger(s,dj,t,k)$ are absolutely continuous with respect to λ:

$$h(s,k,t,dj) = h(s,k,t,j)\lambda(dj), \tag{3.5}$$

$$h^\dagger(s,dj,t,k) = h^\dagger(s,j,t,k)\lambda(dj), \tag{3.6}$$

with $h(s,k,t,j) = h^\dagger(s,k,t,j)$ since H is adjoint of $H^\dagger$ with respect to λ. More precisely we have the following result, cf. Th. 1 of [5], Th. 3.2 of [20], and Th. 3.4 of [29]:

Theorem 3.3 *Let π_r and π_v be two probability measures. Assume that $h(s,j,k,t)$ is a continuous in (j,k) and strictly positive function. Then there exist two measures $\eta_r^*(di)$ and $\eta_v(dm)$ such that*

$$\pi_r(di) = \eta_r^*(di)\int_{\mathrm{R}^d} h(r,i,v,m)\eta_v(dm),$$

and

$$\pi_v(dm) = \eta_v(dm)\int_{\mathrm{R}^d} h(r,i,v,m)\eta_r^*(di).$$

We present several families of processes satisfying the above hypothesis, starting with the simplest examples. Note that in the first example, the mutual adjointness of H and $H^\dagger$ with respect to the (Lebesgue) measure λ is satisfied without requiring the absolute continuity of $h(t,k,u,dl)$ and $h^\dagger(s,dj,t,k)$ with respect to λ. Also we will present some examples where the initial and final laws can not be arbitrarily chosen, when the hypothesis of Th. 3.3 are not fulfilled. This list of examples includes the classical Brownian bridges. However the aim of this paper is not to focus on the Brownian case which has already been the object of several studies, cf. [20], [2], [29], [7].

Examples

1. Deterministic process.

   The adjoint relation between H and $H^\dagger$ is satisfied in the deterministic case for λ the Lebesgue measure, i.e.

$$h^\dagger(s, dj, t, k) = e^{-\int_s^t U(k+c(\tau-t))d\tau}\delta_{k-c(t-s)}(dj), \quad r \le s < t \le v,$$

$$h(t, k, u, dl) = e^{-\int_t^u U(k+c(\tau-t))d\tau}\delta_{k+c(u-t)}(dl), \quad r \le t < u \le v.$$

   Therefore, for any $r < s < t < u < v$,

$$\eta_t^*(k) = \eta_s^*(k - c(t-s))e^{-\int_s^t U(k+c(\tau-t))d\tau}, \tag{3.7}$$

$$\eta_t(k) = \eta_u(k + c(u-t))e^{-\int_t^u U(k+c(\tau-t))d\tau}. \tag{3.8}$$

   Applying (3.7) and (3.8) successively in $t = s$ and $t = u$ we obtain several expressions for the density of z_t at time t with respect to the Lebesgue measure λ:

$$\begin{aligned}\eta_t^*(k)\eta_t(k) &= \eta_s^*(k - c(t-s))\eta_u(k - c(t-u))e^{-\int_s^u U(k-c(t-\tau))d\tau} \\ &= \eta_s^*(k - c(t-s))\eta_s(k - c(t-s)) \\ &= \eta_u^*(k - c(t-u))\eta_u(k - c(t-u)),\end{aligned}$$

   Note that here $h(s, k, t, dl)$ is not absolutely continuous with respect to the Lebesgue measure $\lambda(dl)$ and that it is clearly not possible to choose independently the initial and final laws.

2. Poisson bridge starting from $a \in \mathbb{N}$ at time r and ending at $b \in \mathbb{N}$ at time v.

   The standard Poisson bridge provides another example where the initial and final laws cannot be chosen arbitrarily, this time because $h(t, k, u, l)$ is not everywhere strictly positive. Take $U = 0$, a reference measure

$$\lambda = \sum_{n=-\infty}^{+\infty} \delta_n,$$

$c = 1$, $\nu_\hbar = \delta_1$, and

$$h(t,k,u,dl) = e^{-(u-t)} \frac{(u-t)^{l-k}}{(l-k)!} 1_{[0,l]}(k) \lambda(dl),$$

$$h^\dagger(s,dj,t,k) = e^{-(t-s)} \frac{(t-s)^{k-j}}{(k-j)!} 1_{[0,k]}(j) \lambda(dj).$$

The simple Poisson bridge with $z_r = a$ and $z_v = b$ is constructed from the boundary conditions

$$\eta^*_r = C(r,v,a,b) 1_{\{a\}}, \qquad \eta_v = 1_{\{b\}},$$

where $C(r,v,a,b)$ is a normalization constant. Then

$$\begin{aligned}
\eta^*_t(k) &= \int_{\mathrm{R}} \eta^*_r(i) h^\dagger(r,di,t,k) \\
&= C(r,v,a,b) e^{-(t-r)} \int_{\mathrm{R}} 1_{\{a\}}(i) \frac{(t-r)^{k-i}}{(k-i)!} 1_{[0,k]}(i) \lambda(di) \\
&= C(r,v,a,b) e^{-(t-r)} \frac{(t-r)^{k-a}}{(k-a)!} 1_{\{k \geq a\}} \\
&= C(r,v,a,b) h(r,a,t,k) = C(r,v,a,b) \mu_{t-r}(k-a),
\end{aligned}$$

$$\begin{aligned}
\eta_t(k) &= \int_{\mathrm{R}} \eta_v(m) h(t,k,v,dm) \\
&= e^{-(v-t)} \int_{\mathrm{R}} 1_{\{b\}}(m) \frac{(v-t)^{m-k}}{(m-k)!} 1_{\{m-k \geq 0\}} \lambda(dm) \\
&= e^{-(v-t)} \frac{(v-t)^{b-k}}{(b-k)!} 1_{[0,b]}(k) \\
&= h(t,k,v,b) = \mu_{v-t}(b-k),
\end{aligned}$$

with the convention $0^0 = 1$. The resulting density at time t with respect to λ is, therefore,

$$\eta^*_t(k)\eta_t(k) = 1_{[a,b]}(k) C(r,v,a,b) e^{r-v} \frac{(t-r)^k}{(k-a)!} \frac{(v-t)^{b-k}}{(b-k)!}.$$

Taking the normalization constant $C(r,v,a,b)$ equal to $e^{v-r} \frac{(b-a)!}{(v-r)^{(b-a)}}$ we obtain

$$\eta^*_t(k)\eta_t(k) = 1_{[a,b]}(k) \binom{b-a}{k-a} \left(\frac{t-r}{v-r}\right)^{k-a} \left(\frac{v-t}{v-r}\right)^{b-k}, \quad r \leq t \leq v,$$

which is the expected binomial law on $\{a, \ldots, b\}$, with parameter $(t-r)/(v-r)$. Note that here, $h(t,k,u,l) = e^{-(u-t)}\frac{(u-t)^{l-k}}{(l-k)!}1_{[0,l]}(k)$ is not $(\lambda \otimes \lambda)(dl, dk)$-strictly positive, and the initial and final laws cannot be chosen arbitrarily, e.g. one cannot have $b < a$. Also this setting is not directly relevant to physics in the momentum representation since $U = 0$.

3. Brownian bridge.

   The Brownian bridge starting at $a \in \mathbb{R}$ and ending at $b \in \mathbb{R}$ is constructed by taking $U = 0$, $\mu_t(k) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}k^2/(\hbar t)}$, and

$$\eta_t^*(k) = \frac{1}{\sqrt{2\pi(t-r)}} e^{-\frac{1}{2}(k-a)^2/(\hbar(t-r))}, \quad \eta_t(k) = \frac{1}{\sqrt{2\pi(v-t)}} e^{-\frac{1}{2}(b-k)^2/(\hbar(v-t))}.$$

4. Lévy bridges from $a \in \mathbb{R}^d$ to $b \in \mathbb{R}^d$.

   Take $U = 0$, and assume that $\mu_t(dk)$ has a density with respect to a fixed reference measure λ, i.e. $\mu_t(dk) = \mu_t(k)\lambda(dk)$. Then

$$\eta_t^*(k) = \mu_{t-r}(k-a), \quad \eta_t(k) = \mu_{v-t}(b-k) = \mu_{t-v}(k-b).$$

   The resulting density at time t with respect to λ is

$$\eta_t^*(k)\eta_t(k) = \mu_{t-r}(k-a)\mu_{v-t}(b-k).$$

   This example includes the Poisson and Brownian bridges seen above. Note that the absolute continuity of $\eta_t(dk)$ with respect to λ at the origin $t = r = 0$ is satisfied for the Poisson bridge but not for the Brownian bridge.

5. Forward and backward Brownian motions ($U = 0$).

   Let $U = 0$, $\nu_\hbar = 0$, $c = 0$, $B = 1$ with $d = 1$ and λ the Lebesgue measure. Taking $\eta_0^*(di) = \delta_0(di)$, $\eta_v(m) = 1$, we have

$$\eta_t^*(dk) = \eta_r^* * \mu_{t-r}(dk) = \mu_t(dk) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}k^2/(\hbar t)} dk, \quad \eta_t(k) = 1, \quad r < t < v,$$

   hence $(z_t)_{t\in[r,v]} = (W_t)_{t\in[r,v]}$ is a (forward) Brownian motion.

   If $\eta_v(dm) = \delta_0(dm)$ and $\eta_r^*(i) = 1$, we have

$$\eta_t(dk) = \eta_v * \mu_{t-v}(dk) = \mu_{t-v}(dk) = \frac{1}{\sqrt{2\pi(v-t)}} e^{-\frac{1}{2}k^2/(\hbar(v-t))} dk, \quad \eta_t^*(k) = 1,$$

$0 < r \le t \le v$, hence $(z_t)_{t\in[r,v]}$ is the backward Brownian motion $(W_t^*)_{t\in[r,v]}$ with final condition $W_v^* = 0$.

6. Forward and backward Poisson processes ($U = 0$ and the absolute continuity of $\mu_t(dk)$ with respect to a fixed reference measure is not required.)
   Let $U = 0$, $B = 0$, $c = 0$ with $d = 1$, and let $\lambda = \sum_{n=-\infty}^{+\infty} \delta_n$. In the forward Poisson case we have $\nu_\hbar(dy) = \delta_1$. If $\eta_r^*(di) = 1_{\{0\}}(i)\lambda(di)$ and $\eta_v(m) = 1$, then $\eta_t^*(dk) = \eta_r^* * \mu_{t-r}(dk) = \mu_t(dk)$ and
   $$\eta_t^*(k) = e^{-(t-r)}\frac{(t-r)^k}{k!}1_{\{k\ge 0\}}, \quad \eta_t(k) = 1, \quad r < t < v,$$
   in this case $(z_t)_{t\in[r,v]} = (N_t)_{t\in[0,v]}$ is the (forward) standard Poisson process.
   The backward Poisson process $(N_t^*)_{t\in[r,v]}$ with final condition $N_v^* = 0$ is constructed with $\nu_\hbar(dy) = \delta_{-1}$, $\eta_v(dm) = 1_{\{0\}}(m)\lambda(dm)$ and $\eta_r^*(i) = 1$, i.e. $\eta_t(dk) = \eta_v * \mu_{t-v}(dk) = \mu_{t-v}(dk)$ and
   $$\eta_t(k) = e^{-(v-t)}\frac{(v-t)^{-k}}{(-k)!}1_{\{k\le 0\}}, \quad \eta_t^*(k) = 1, \quad r < t < v.$$

7. Forward and backward Lévy processes ($U = 0$).
   This example includes the forward and backward Wiener and Poisson processes as particular cases. Taking $\eta_r^*(di) = \mu_r(di)$ and $\eta_v(m) = 1$, we have
   $$\eta_t^*(dk) = \eta_r^* * \mu_{t-r}(dk) = \mu_t(dk), \quad \eta_t(k) = 1, \quad r < t < v,$$
   hence $(z_t)_{t\in[r,v]}$ is the (forward) Lévy process $(\xi_t)_{t\in[r,v]}$: $z_t = \xi_t$, $r < t < v$.
   If $\eta_v(dm) = \mu_0(dm)$ and $\eta_r^*(i) = 1$, we have
   $$\eta_t(dk) = \eta_v * \mu_{t-v}(dk) = \mu_{t-v}(dk), \quad \eta_t^*(k) = 1, \quad r < t < v,$$
   hence $(z_t)_{t\in[r,v]}$ is a backward Lévy process. This is an example of process with initial law $\mu_{r-v}(dk)$ and final condition $z_v = 0$, resp. initial law $\mu_r(dk)$ and final law $\mu_v(dk)$.

8. Processes with densities with respect to the Lebesgue measure.
   Here, U does not necessarily vanish. From (2.3) and (2.4), the absolute continuity conditions (3.5) and (3.6) are satisfied if the law of ξ_t, $t > 0$, has a

density with respect to the Lebesgue measure, e.g. in the case of stable processes (namely such that $V(q) = c|q|^\alpha$ for some $\alpha \in (0,2]$ and $c > 0$), and for Lévy processes with Brownian component ($B \neq 0$). Moreover H is adjoint of $H^\dagger$ with respect to λ when λ is the Lebesgue measure.

9. General case ($U \neq 0$).

   The condition $U \neq 0$ is necessary in the context of Euclidean quantum mechanics. If λ is a given measure (not necessarily the Lebesgue measure), we may work under the absolute continuity hypothesis

$$h(t,k,u,dl) = h(t,k,u,l)\lambda(dl), \qquad \lambda(dk) - a.e., \tag{3.9}$$

$$h^\dagger(s,dj,t,k) = h^\dagger(s,j,t,k)\lambda(dj), \qquad \lambda(dk) - a.e. \tag{3.10}$$

   which imply that H and $H^\dagger$ are also adjoint with respect to λ if $h(s,j,t,k) = h^\dagger(t,k,s,j)$:

$$\begin{aligned} h(s,j,t,dk)\lambda(dj) &= h(s,j,t,k)\lambda(dk)\lambda(dj) \\ &= h^\dagger(s,j,t,k)\lambda(dk)\lambda(dj) = h^\dagger(s,dj,t,k)\lambda(dk). \end{aligned}$$

   In view of (2.3) and (2.4), the conditions (3.9) and (3.10) are satisfied in particular if $\mu_{t-s}(j+dk)$ has a density with respect to $\lambda(dk)$, $\lambda(dj)$-a.e. This relation will hold e.g. if λ is absolutely continuous under the translation $j \mapsto j+k$, $\lambda(dj)$-a.e., and μ_{t-s} is absolutely continuous with respect to λ:

$$\mu_{t-s}(dk) = \mu_{t-s}(k)\lambda(dk).$$

   This hypothesis is satisfied, in particular, for the Poisson bridge, cf. Example 2 above, with $\mu_{t-s}(k) = e^{-(t-s)}\frac{(t-s)^k}{k!}1_{\{k\geq 0\}}$ and $\lambda = \sum_{k=-\infty}^{k=\infty} \delta_k$.

3.2 Stochastic differential equations and generators

In this section we present the description of Markovian bridges of Th. 3.2 in terms of forward and backward stochastic integro-differential equations driven by $(\xi_t)_{t\in[r,v]}$. Let for $f \in \mathcal{S}(\mathbb{R}^d)$ and $g : \mathbb{R}^d \mapsto]0,\infty[$:

$$\mathcal{L}_g f(k) \quad = \quad \langle c, \nabla f(k)\rangle + \frac{1}{2}\Delta_{\hbar B} f(k)$$

$$+\int_{\mathrm{R}^d}(f(k+y)-f(k)-\langle y,\nabla f(k)\rangle 1_{\{|y|\le 1\}})\frac{g(k+y)}{g(k)}\nu_\hbar(dy)$$
$$+\int_{\mathrm{R}^d}\frac{g(k+y)-g(k)}{g(k)}\langle y,\nabla f(k)\rangle 1_{\{|y|\le 1\}}\nu_\hbar(dy)+\langle\nabla\log g(k),\nabla f(k)\rangle_{\hbar B},$$

and

$$\mathcal{L}^*_g f(k)=\langle c,\nabla f(k)\rangle-\frac{1}{2}\Delta_{\hbar B}f(k)$$
$$-\int_{\mathrm{R}^d}(f(k-y)-f(k)+\langle y,\nabla f(k)\rangle 1_{\{|y|\le 1\}})\frac{g(k-y)}{g(k)}\nu_\hbar(dy)$$
$$+\int_{\mathrm{R}^d}\frac{g(k-y)-g(k)}{g(k)}\langle y,\nabla f(k)\rangle 1_{\{|y|\le 1\}}\nu_\hbar(dy)-\langle\nabla\log g(k),\nabla f(k)\rangle_{\hbar B}.$$

The following result is a consequence of Prop. 6.2, which will be proved in Section 6.

Proposition 3.4 *The process $(z_t)_{t\in[r,v]}$ constructed in Th. 3.2 has the forward infinitesimal generator, for $f\in\mathcal{S}(\mathbb{R}^d)$:*

$$\begin{aligned}\mathcal{L}_{\eta_t}f(k) &= \langle c,\nabla f(k)\rangle+\frac{1}{2}\Delta_{\hbar B}f(k)\\ &+\int_{\mathrm{R}^d}(f(k+y)-f(k)-\langle y,\nabla f(k)\rangle 1_{\{|y|\le 1\}})\frac{\eta_t(k+y)}{\eta_t(k)}\nu_\hbar(dy)\\ &+\int_{\mathrm{R}^d}\frac{\eta_t(k+y)-\eta_t(k)}{\eta_t(k)}\langle y,\nabla f(k)\rangle 1_{\{|y|\le 1\}}\nu_\hbar(dy)+\langle\nabla\log\eta_t(k),\nabla f(k)\rangle_{\hbar B},\end{aligned}$$

and the backward infinitesimal generator

$$\mathcal{L}^*_{\eta^*_t}f(k)=\langle c,\nabla f(k)\rangle-\frac{1}{2}\Delta_{\hbar B}f(k)$$
$$-\int_{\mathrm{R}^d}(f(k-y)-f(k)+\langle y,\nabla f(k)\rangle 1_{\{|y|\le 1\}})\frac{\eta^*_t(k-y)}{\eta^*_t(k)}\nu_\hbar(dy)$$
$$+\int_{\mathrm{R}^d}\frac{\eta^*_t(k-y)-\eta^*_t(k)}{\eta^*_t(k)}\langle y,\nabla f(k)\rangle 1_{\{|y|\le 1\}}\nu_\hbar(dy)-\langle\nabla\log\eta^*_t(k),\nabla f(k)\rangle_{\hbar B},$$

The knowledge of the generators of $(z_t)_{t\in[r,v]}$ provides the forward and backward representations of $(z_t)_{t\in[r,v]}$ as weak solutions of stochastic integro-differential equations. We assume that (cf. p. 434 of [18]):

A) the functions

$(t,k)\mapsto\int_{\mathrm{R}^d}(1\wedge|y|^2)\frac{\eta_t(k+y)}{\eta_t(k)}\nu_\hbar(dy)$,

$(t,k) \mapsto \int_{\{|y|\le 1\}} y \frac{\eta_t(k+y)-\eta_t(k)}{\eta_t(k)} \nu_\hbar(dy),$

$(t,k) \mapsto \nabla \log \eta_t(k),$

resp.

$(t,k) \mapsto \int_{\mathrm{R}^d} (1 \wedge |y|^2) \frac{\eta_t^*(k-y)}{\eta_t^*(k)} \nu_\hbar(dy),$

$(t,k) \mapsto \int_{\{|y|\le 1\}} y \frac{\eta_t^*(k-y)-\eta_t^*(k)}{\eta_t^*(k)} \nu_\hbar(dy),$

$(t,k) \mapsto \nabla \log \eta_t^*(k),$

are bounded on compacts of $\mathbb{R}_+ \times \mathbb{R}^d$,

The next proposition is a representation result that follows from Prop. 3.4 and Th. 13.58, Th. 14.80 of [18], p. 438 and p. 481, using the results on martingale problems for discontinuous processes of [21], [22], [26].

Proposition 3.5 *The process $(z_t)_{t\in[r,v]}$ is solution, in the weak sense and with respect to the forward filtration $(\mathcal{P}_t)_{t\in[r,v]}$, of*

$$\begin{aligned} dz_t &= cdt + dW_t + \int_{\mathrm{R}^d} y \left(\mu(dy,dt) - \frac{\eta_t(z_{t^-}+y)}{\eta_t(z_{t^-})} 1_{\{|y|\le 1\}} \nu_\hbar(dy)dt \right) \\ &\quad + \int_{\mathrm{R}^d} y \frac{\eta_t(z_{t^-}+y)-\eta_t(z_{t^-})}{\eta_t(z_{t^-})} 1_{\{|y|\le 1\}} \nu_\hbar(dy)dt + \hbar B \nabla \log \eta_t(z_{t^-})dt, \end{aligned}$$

under a probability P for which W_t is a (forward) Brownian motion with covariance $\hbar B$, and $\mu(dy,ds)$ is the canonical point process with compensator $\frac{\eta_t(z_{t^-}+y)}{\eta_t(z_{t^-})}\nu_\hbar(dy)dt$. In terms of backward differentials we have as well, with respect to the decreasing filtration $(\mathcal{F}_t)_{t\in[r,v]}$,

$$\begin{aligned} d_* z_t &= cdt + d_* W_t^* + \int_{\mathrm{R}^d} y \left(\mu_*(dy,dt) - \frac{\eta_t^*(z_{t^+}-y)}{\eta_t^*(z_{t^+})} 1_{\{|y|\le 1\}} \nu_\hbar(dy)dt \right) \\ &\quad + \int_{\mathrm{R}^d} y \frac{\eta_t^*(z_{t^+}-y)-\eta_t^*(z_{t^+})}{\eta_t^*(z_{t^+})} 1_{\{|y|\le 1\}} \nu_\hbar(dy)dt - \hbar B \nabla \log \eta_t^*(z_{t^+})dt, \end{aligned}$$

where W_t^* denotes a backward Brownian motion with covariance $\hbar B$, and $\mu_*(dy,dt)$ is the backward Poisson random measure with compensator $\frac{\eta_t^*(z_{t^-}-y)}{\eta_t^*(z_{t^-})}\nu_\hbar(dy)dt$.

This also provides the $(\mathcal{P}_t)_{t\in[r,v]}$-decomposition

$$\begin{aligned} z_t &= z_r + c(t-r) + M_t \\ &+ \int_r^t \int_{\mathrm{R}^d} y \left(\frac{\eta_s(z_{s^-}+y)}{\eta_s(z_{s^-})} - 1_{\{|y|\le 1\}} \right) \nu_\hbar(dy)ds + \hbar \int_r^t B\nabla \log \eta_s(z_{s^-})ds, \end{aligned}$$

where $(M_t)_{t\in[0,v]}$ is a $(\mathcal{P}_t)_{t\in[r,v]}$-martingale, and the $(\mathcal{F}_t)_{t\in[r,v]}$-decomposition

$$\begin{aligned} z_t &= z_v - c(v-t) + M_t^* \\ &+ \int_t^v \int_{\mathrm{R}^d} y \left(\frac{\eta_s^*(z_{s^+}-y)}{\eta_s^*(z_{s^+})} - 1_{\{|y|\le 1\}} \right) \nu_\hbar(dy)ds - \hbar \int_t^v B\nabla \log \eta_s^*(z_{s^+})ds, \end{aligned}$$

where $(M_t^*)_{t\in[0,v]}$ is a (backward) $(\mathcal{F}_t)_{t\in[r,v]}$-martingale.

<u>Examples</u> and particular cases of Prop. 3.5.

1. Deterministic process.
   In this case $(z_t)_{t\in[r,v]}$ satisfy the ordinary differential equation

   $$dz_t = d\xi_t = c dt,$$

   both in forward and backward cases, hence

   $$z_t = z_r + c(t-r) = z_v + c(v-t), \quad r < t < v,$$

   with random initial condition z_r and final condition z_v. The influence of U lies in the initial and final laws, not in the dynamics.

2. Poisson bridge starting at $a \in \mathbb{N}$ at time r and ending at $b \in \mathbb{N}$ at time v.
   If $U = 0$, the forward (i.e. $(\mathcal{P}_t)_{t\in[r,v]}$-) stochastic equation (3.1) satisfied by the Poisson bridge is written

   $$dz_t = dN_t^\eta, \quad z_r = a,$$

   where $(N_t^\eta)_{t\in[r,v]}$ is a point process starting from 0 at time r, with compensator

   $$d\langle N_t^\eta \rangle = \frac{\eta_t(z_{t^-}+1)}{\eta_t(z_{t^-})}dt = \frac{b-a-N_{t^-}^\eta}{v-t}dt.$$

This means (see e.g. Th. 7.4. p. 93 of [16] and references therein) that $(z_t)_{t\in[r,v]}$ can be constructed by a time change on a standard Poisson process $(N(t))_{t\in\mathbb{R}_+}$, i.e. the sequence of jump times $(T_k^\eta)_{1\leq k\leq m-i}$ of $(z_t)_{t\in[r,v]} = (a+N_t^\eta)_{t\in[r,v]}$ can be obtained by induction from the jump times $(T_k)_{k\geq 1}$ of $(N(t))_{t\in\mathbb{R}_+}$, as

$$T_k = \sum_{i=1}^{k} \int_{T_{i-1}^\eta}^{T_i^\eta} \frac{b-a-(i-1)}{v-s} ds, \quad 1 \leq k \leq b-a.$$

The backward equation satisfied by $(z_t)_{t\in[r,v]}$ is

$$d_* z_t = d_* N_t^{\eta^*}, \quad z_v = b,$$

where $(N_t^{\eta^*})_{t\in[r,v]} = (-N_{v-t}^\eta)_{t\in[r,v]}$ is a point process starting from 0 at time v, with backward compensator

$$d_*\langle N_t^{\eta^*}\rangle = \frac{\eta_t^*(z_{t^+}-1)}{\eta_t^*(z_{t^+})}dt = -\frac{N_{t^+}^{\eta^*}+(b-a)}{r-t}dt.$$

3. Brownian bridge.

   We have $\mu_t(k) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}k^2/(\hbar t)}$, hence the forward and backward stochastic differential equations satisfied by $(z_t)_{t\in[r,v]}$ are

$$dz_t = dW_t - \frac{z_t-b}{v-t}dt, \quad z_r = a,$$

   and

$$d_* z_t = d_* W_t^* + \frac{z_t-a}{t-r}dt, \quad z_v = b.$$

4. Lévy bridges.

   Take $U=0$, and assume that $\mu_t(dk) = \mu_t(k)\lambda(dk)$ has a density with respect to a fixed reference measure λ. The forward stochastic integro-differential equation satisfied by $(z_t)_{t\in[r,v]}$ is

$$\begin{aligned} dz_t &= cdt + dW_t + \int_{\mathbb{R}^d} y\left(\mu(dy,dt) - \frac{\mu_{t-v}(z_{t^-}+y-b)}{\mu_{t-v}(z_{t^-}-b)} 1_{\{|y|\leq 1\}}\nu_\hbar(dy)dt\right) \\ &\quad + \int_{\mathbb{R}^d} y \frac{\mu_{t-v}(z_{t^-}+y-b)-\mu_{t-v}(z_{t^-}-b)}{\mu_{t-v}(z_{t^-}-b)} 1_{\{|y|\leq 1\}}\nu_\hbar(dy)dt \\ &\quad -\hbar B\nabla \log \mu_{t-v}(z_{t^-}-b)dt, \end{aligned}$$

i.e. using the Lemma 2.2:

$$dz_t = dW_t + \int_{\mathrm{R}^d} y\left(\mu(dy,dt) - \frac{\mu_{t-v}(z_{t^-}+y-b)}{\mu_{t-v}(z_{t^-}-b)}\nu_\hbar(dy)dt\right) - \frac{z_{t^-}-b}{v-t}dt.$$

The backward stochastic differential equation satisfied by the same process $(z_t)_{t\in[r,v]}$ is:

$$\begin{aligned} d_*z_t &= cdt + d_*W_t^* + \int_{\mathrm{R}^d} y\left(\mu_*(dy,dt) - \frac{\mu_{t-r}(z_{t^+}+y-a)}{\mu_{t-r}(z_{t^+}-a)}1_{\{|y|\le 1\}}\nu_\hbar(dy)dt\right) \\ &+ \int_{\mathrm{R}^d} y\frac{\mu_{t-r}(z_{t^+}+y-a)-\mu_{t-r}(z_{t^+}-a)}{\mu_{t-r}(z_{t^+}-a)}1_{\{|y|\le 1\}}\nu_\hbar(dy)dt \\ &-\hbar B\nabla\log\mu_{t-r}(z_{t^+}-a)dt, \end{aligned}$$

i.e. by Lemma 2.2:

$$d_*z_t = d_*W_t^* + \int_{\mathrm{R}^d} y\left(\mu_*(dy,dt) - \frac{\mu_{t-r}(z_{t^+}+y-a)}{\mu_{t-r}(z_{t^+}-a)}\nu_\hbar(dy)dt\right) + \frac{z_{t^+}-a}{t-r}dt.$$

5. Forward and backward Brownian motions ($U=0$).

   The forward Brownian motion $(z_t)_{t\in[0,v]} = (W_t)_{t\in[0,v]}$ satisfies the forward "equation"

$$dz_t = dW_t, \quad z_0 = 0,$$

   and the backward $(\mathcal{F}_t)_{t\in[r,v]}$-equation

$$d_*z_t = d_*W_t^* + \frac{z_t}{t}dt,$$

   where $(W_t^*)_{t\in[r,v]}$ is a backward Brownian motion (starting from 0 at time v). The backward Brownian motion $(z_t)_{t\in[r,v]} = (W_t^*)_{t\in[r,v]}$ satisfies the forward equation

$$dz_t = dW_t - \frac{z_t}{v-t}dt,$$

   and the backward "equation"

$$d_*z_t = d_*W_t^*, \quad z_v = 0.$$

6. Forward and backward Poisson processes ($U=0$).

   In the standard Poisson case, $\mu_t(k) = e^{-(t-r)}(t-r)^k/k!$, and we can compute directly the backward compensator of the standard forward Poisson process $(z_t)_{t\in[0,v]} = (N_t)_{t\in[0,v]}$ as

$$d_*\langle z_t\rangle = \frac{\eta_t^*(z_{t^+}-1)}{\eta_t^*(z_{t^+})} = \frac{\mu_t(z_{t^+}-1)}{\mu_t(z_{t^+})} = \frac{z_{t^+}}{t}dt.$$

The forward compensator of the backward Poisson process $(z_t)_{t\in[r,v]} = (N_t^*)_{t\in[r,v]} = (-N_{v-t})_{t\in[r,v]}$ is similarly given as

$$d\langle z_t\rangle = \frac{\eta_t(z_{t^-}+1)}{\eta_t(z_{t^-})} = \frac{\mu_{v-t}(-z_{t^-}-1)}{\mu_{v-t}(-z_{t^-})} = -\frac{z_{t^-}}{v-t}dt.$$

This structure remains in the forward and backward Lévy cases described next.

7. Forward Lévy processes ($U = 0$).
Assuming that $\mu_t(dk) = \mu_t(k)\lambda(dk)$ is absolutely continuous with respect to $\lambda(dk)$ we have

$$dz_t = cdt + dW_t + \int_{\mathrm{R}^d} y\left(\mu(dy,dt) - 1_{\{|y|\le 1\}}\nu_\hbar(dy)dt\right), \quad z_r = \xi_r,$$

i.e. $z_t = \xi_t$, $r < t < v$. Besides the forward generator $-V(\nabla)$ of $(\xi_t)_{t\in[r,v]}$ (see (2.1)) we obtain the backward generator

$$\begin{aligned}\mathcal{L}^*_{\eta_t^*} f(k) &= \langle c, \nabla f(k)\rangle - \frac{1}{2}\Delta_{\hbar B} f(k)\\ &- \int_{\mathrm{R}^d} (f(k-y) - f(k) + \langle y, \nabla f(k)\rangle 1_{\{|y|\le 1\}})\frac{\mu_t(k-y)}{\mu_t(k)}\nu_\hbar(dy)\\ &+ \int_{\mathrm{R}^d} \frac{\mu_t(k-y)-\mu_t(k)}{\mu_t(k)}\langle y, \nabla f(k)\rangle 1_{\{|y|\le 1\}}\nu_\hbar(dy) - \langle \nabla \log \mu_t(k), \nabla f(k)\rangle_{\hbar B},\end{aligned}$$

or by Lemma 2.2 above:

$$\begin{aligned}\mathcal{L}^*_{\eta_t^*} f(k) &= -\frac{1}{2}\Delta_{\hbar B} f(k) - \int_{\mathrm{R}^d} (f(k-y) - f(k) + \langle y, \nabla f(k)\rangle)\frac{\mu_t(k-y)}{\mu_t(k)}\nu_\hbar(dy)\\ &\quad + \frac{1}{t}\langle k, \nabla f(k)\rangle.\end{aligned}$$

The backward stochastic differential equation satisfied by $(\xi_t)_{t\in[r,v]}$ is:

$$\begin{aligned}d_* z_t &= cdt + d_* W_t^* + \int_{\mathrm{R}^d} y\left(\mu_*(dy,dt) - \frac{\mu_t(z_{t^+}-y)}{\mu_t(z_{t^+})}1_{\{|y|\le 1\}}\nu_\hbar(dy)dt\right)\\ &\quad + \int_{\mathrm{R}^d} y\frac{\mu_t(z_{t^+}-y)-\mu_t(z_{t^+})}{\mu_t(z_{t^+})}1_{\{|y|\le 1\}}\nu_\hbar(dy)dt - \hbar B\nabla \log \mu_t(z_{t^+})dt,\end{aligned}$$

i.e. from Lemma 2.2:

$$d_* z_t = d_* W_t^* + \int_{\mathrm{R}^d} y \left(\mu_*(dy, dt) - \frac{\mu_t(z_{t^+} - y)}{\mu_t(z_{t^+})} \nu_\hbar(dy) dt \right) + \frac{z_{t^+}}{t} dt.$$

In other terms we have the backward martingale decomposition

$$z_t = M_t^* + \int_t^v \frac{z_s}{s} ds, \quad r < t < v,$$

which allows to recover and extend some results in [19].

8. Backward Lévy processes.

Taking $\eta_v(dm) = \mu_0(m)\lambda(dm) = \mu_0(dm)$ and $\eta_r^*(i) = 1$, we have $\eta_t(k) = \eta_v * \mu_{t-v}(k) = \mu_{t-v}(k) = \mu_{v-t}(-k)$, and $\eta_t^*(k) = 1$, $r < t < v$, hence $(z_t)_{t \in [0,v]}$ is the backward Lévy process given by

$$d_* z_t = c dt + d_* W_t^* + \int_{\mathrm{R}^d} y \left(\mu_*(dy, dt) - 1_{\{|y| \le 1\}} \nu_\hbar(dy) dt \right),$$

which has same law as the reversed Lévy process $(\xi_s^*)_{s \in [0,v]} = (-\xi_{v-s})_{s \in [0,v]}$. The forward generator of $(z_t)_{t \in [0,v]}$ is

$$\begin{aligned} \mathcal{L}_{\eta_t} f(k) &= \langle c, \nabla f(k) \rangle + \frac{1}{2} \Delta_{\hbar B} f(k) \\ &+ \int_{\mathrm{R}^d} (f(k+y) - f(k) - \langle y, \nabla f(k) \rangle 1_{\{|y| \le 1\}}) \frac{\mu_{t-v}(k+y)}{\mu_{t-v}(k)} \nu_\hbar(dy) \\ &+ \int_{\mathrm{R}^d} \frac{\mu_{t-v}(k+y) - \mu_{t-v}(k)}{\mu_{t-v}(k)} \langle y, \nabla f(k) \rangle 1_{\{|y| \le 1\}} \nu_\hbar(dy) + \langle \nabla \log \mu_{t-v}(k), \nabla f(k) \rangle_{\hbar B}, \end{aligned}$$

or from Lemma 2.2:

$$\begin{aligned} \mathcal{L}_{\eta_t} f(k) &= \frac{1}{2} \Delta_{\hbar B} f(k) + \int_{\mathrm{R}^d} (f(k+y) - f(k) \\ &\quad - \langle y, \nabla f(k) \rangle) \frac{\mu_{t-v}(k+y)}{\mu_{t-v}(k)} \nu_\hbar(dy) - \frac{1}{v-t} \langle k, \nabla f(k) \rangle. \end{aligned}$$

The forward stochastic differential equation satisfied by $(z_t)_{t \in [0,v]}$ is, therefore,

$$\begin{aligned} dz_t &= c dt + dW_t + \int_{\mathrm{R}^d} y \left(\mu(dy, dt) - \frac{\mu_{t-v}(z_{t^-} + y)}{\mu_{t-v}(z_{t^-})} 1_{\{|y| \le 1\}} \nu_\hbar(dy) dt \right) \\ &- \int_{\mathrm{R}^d} y \frac{\mu_{t-v}(z_{t^-}) - \mu_{t-v}(z_{t^-} + y)}{\mu_{t-v}(z_{t^-})} 1_{\{|y| \le 1\}} \nu_\hbar(dy) dt + \hbar B \nabla \log \mu_{t-v}(z_{t^-}) dt, \end{aligned}$$

where W_t is a forward Brownian motion with covariance $\hbar B$, and $\mu(dy,dt)$ is the forward Poisson random measure with compensator $\frac{\mu_{t-v}(z_{t^-}+y)}{\mu_{t-v}(z_{t^-})}$, i.e. by Lemma 2.2:

$$dz_t = dW_t + \int_{\mathrm{R}^d} y \left(\mu(dy,dt) - \frac{\mu_{t-v}(z_{t^-}+y)}{\mu_{t-v}(z_{t^-})} \nu_\hbar(dy)dt \right) - \frac{z_{t^-}}{v-t}. \qquad (3.11)$$

and we have the forward martingale decomposition

$$z_t = M_t - \int_r^t \frac{z_s}{v-s} ds, \quad r < t < v,$$

to compare to [19] (note that here we have $z_v = 0$). The forward compensator of $(z_t)_{t\in[0,v]}$ is again

$$d\langle z_t \rangle = -\frac{z_{t^-}}{v-t} dt.$$

4 Girsanov theorem

The next proposition shows that the law of the process $(z_t)_{t\in[r,v]}$ of Prop. 3.1 is absolutely continuous with respect to the law of the Lévy process $(\xi_t)_{t\in[r,v]}$.

Proposition 4.1 *Assume that $c = 0$, $\nu_\hbar(\{|y| \geq 1\}) = 0$ and either $B = 0$ or $\nu_\hbar = 0$, i.e. we are in the Brownian case or in the jump case. Under the hypothesis of Th. 3.2, the law Q of $(z_t)_{t\in[r,v]}$ is absolutely continuous with respect to P, with density given by*

$$\frac{dQ}{dP}_{|\mathcal{P}_t} = \frac{\eta_t(z_t)}{\eta_r(z_r)} e^{-\int_r^t U(z_\tau)d\tau}, \quad r \leq t \leq v,$$

i.e. under Q, $(z_r + \xi_t)_{t\in[r,v]}$ has the law of $(z_t)_{t\in[r,v]}$ under P. Similarly we have

$$\frac{dQ}{dP}_{|\mathcal{F}_t} = \frac{\eta_t^*(z_t)}{\eta_v^*(z_v)} e^{-\int_t^v U(z_\tau)d\tau}, \quad r \leq t \leq v,$$

i.e. $(z_v - \xi_{v-t})_{t\in[r,v]}$ has the law of $(z_t)_{t\in[r,v]}$ under P.

Proof. Let

$$L_t = \frac{\eta_t(z_t)}{\eta_r(z_r)} e^{-\int_r^t U(z_\tau)d\tau}, \quad t \in [r,v].$$

Assume that under P, $\mu(dy,dt)$ is the random measure with compensator $\nu_\hbar(dy)dt$ in the Poisson case, with

$$dz_t = \int_{\mathrm{R}^d} y \left(\mu(dy,dt) - 1_{\{|y|\leq 1\}} \nu_\hbar(dy)dt \right),$$

or that $(z_t)_{t\in[r,v]}$ is a standard Brownian motion in the Brownian case, with $dW_t = dz_t - \hbar B\nabla \log \eta_t(z_{t^-})dt$, i.e.

$$dz_t = dW_t + \hbar B\nabla \log \eta_t(z_{t^-})dt.$$

Let us compute

$$\begin{aligned} d\eta_t(z_t) &= \eta_t(z_{t^-})\int_{|y|\le 1} \frac{\eta_t(z_{t^-}+y)-\eta_t(z_{t^-})}{\eta_t(z_{t^-})}\left(\mu(dy,dt) - \frac{\eta_t(z_{t^-}+y)}{\eta_t(z_{t^-})}\nu_\hbar(dy)dt\right) \\ &\quad +\eta_t(z_{t^-})\langle \nabla \log \eta_t(z_{t^-}), dW_t\rangle_{\hbar B} + \mathcal{L}_{\eta_t}\eta_t(z_t)dt + \frac{\partial \eta_t}{\partial t}(z_t)dt \\ &= U(z_{t^-})\eta_t(z_{t^-})dt + \eta_t(z_{t^-})\int_{|y|\le 1}\frac{\eta_t(z_{t^-}+y)-\eta_t(z_{t^-})}{\eta_t(z_{t^-})}(\mu(dy,dt)-\nu_\hbar(dy)dt) \\ &\quad +\eta_t(z_{t^-})\langle \nabla \log \eta_t(z_{t^-}), dW_t\rangle_{\hbar B} + \langle \nabla \log \eta_t(z_{t^-}), \nabla \eta_t(z_{t^-})\rangle_{\hbar B}, \end{aligned}$$

where we used equation 6.1 and the forward infinitesimal generator

$$\begin{aligned} \mathcal{L}_{\eta_t}\eta_t(k) &= -H\eta_t(k) + U(k)\eta_t(k) + \langle \nabla \log \eta_t(k), \nabla \eta_t(k)\rangle_{\hbar B} \\ &\quad + \int_{\mathrm{R}^d}\left(\frac{\eta_t^2(k+y)}{\eta_t(k)} - \eta_t(k)\right)\nu_\hbar(dy)dt + \frac{\partial \eta_t}{\partial t}(z_t). \end{aligned}$$

Hence $(L_t)_{t\in[r,v]}$ satisfies the (forward) stochastic integro-differential equation

$$\begin{aligned} dL_t &= L_{t^-}\cdot \int_{|y|\le 1}\frac{\eta_t(z_{t^-}+y)-\eta_t(z_{t^-})}{\eta_t(z_{t^-})}(\mu(dy,dt)-\nu_\hbar(dy)dt) \\ &\quad +L_{t^-}\cdot\langle \nabla \log \eta_t(z_{t^-}), dW_t + \nabla \log \eta_t(z_{t^-})dt\rangle_{\hbar B}, \quad t\in[r,v]. \end{aligned}$$

Under P we have

$$\frac{1}{L_{s^-}}d\langle L_s, z_s\rangle = \int_{|y|\le 1}\frac{\eta_t(z_{t^-}+y)-\eta_t(z_{t^-})}{\eta_t(z_{t^-})}\nu_\hbar(dy)dt + \hbar B\nabla \log \eta_t(z_{t^-}),$$

and from the Girsanov theorem,

$$\begin{aligned} z_t - \int_r^t \frac{1}{L_{s^-}}d\langle L_s, z_s\rangle &= W_t - W_r + \hbar\int_r^t B\nabla \log \eta_s(z_{s^-})ds \\ &\quad + \int_r^t\int_{\{|y|\le 1\}} y(\mu(dy,ds)-\nu_\hbar(dy)ds) - \int_r^t \frac{1}{L_{s^-}}d\langle L_s, z_s\rangle \\ &= W_t + \int_r^t\int_{\{|y|\le 1\}} y\left(\mu(dy,ds) - \frac{\eta_s(z_{s^-}+y)}{\eta_s(z_{s^-})}\nu_\hbar(dy)ds\right), \end{aligned}$$

for $t \in [r,v]$, is a $(\mathcal{P}_t)_{t\in[r,v]}$-martingale under the probability Q defined by

$$\frac{dQ}{dP}|_{\mathcal{P}_t} = L_t, \quad r \le t \le v,$$

hence under Q, W_t is a Brownian motion (in the Brownian case) and $\mu(dy, ds)$ has the compensator $\frac{\eta_s(z_{s-}+y)}{\eta_s(z_{s-})}\nu_\hbar(dy)$ (in the pure jump case), i.e. $(z_r + \xi_{t-r})_{t\in[r,v]}$ has the law of $(z_t)_{t\in[r,v]}$ under P.

The proof in the backward case is similar and relies on the following calculations. Using the definition of the backward infinitesimal generator,

$$\begin{aligned}
\mathcal{L}^*_{\eta^*_t}\eta^*_t(k) &= H^\dagger\eta^*_t(k) + U(k)\eta^*_t(k) + \langle\nabla\log\eta^*_t(k), \nabla\eta^*_t(k)\rangle_{\hbar B} \\
&\quad + \int_{\mathrm{R}^d}\left(\frac{(\eta^*_t)^2(k+y)}{\eta_t(k)} - \eta^*_t(k)\right)\nu_\hbar(dy)dt + \frac{\partial\eta^*_t}{\partial t}(z_t) \\
&= -\frac{\partial\eta^*_t}{\partial t}(k) + U(k)\eta^*_t(k) + \nabla\log\eta^*_t(k), \nabla\eta^*_t(k)\rangle_{\hbar B} \\
&\quad + \int_{\mathrm{R}^d}\left(\frac{(\eta^*_t)^2(k+y)}{\eta^*_t(k)} - \eta^*_t(k)\right)\nu_\hbar(dy)dt + \frac{\partial\eta^*_t}{\partial t}(k),
\end{aligned}$$

we have

$$\begin{aligned}
d\eta^*_t(z_t) &= \eta^*_t(z_{t-})\int_{|y|\le 1}\frac{\eta^*_t(z_{t^+}-y) - \eta^*_t(z_{t^+})}{\eta^*_t(z_{t^+})}\left(\mu_*(dy,dt) - \frac{\eta^*_t(z_{t^+}-y)}{\eta^*_t(z_{t^+})}\nu_\hbar(dy)dt\right) \\
&\quad +\eta^*_t(z_{t^+})\langle\nabla\log\eta^*_t(z_{t^+}), dW^*_t\rangle_{\hbar B} + \mathcal{L}^*_{\eta^*_t}\eta^*_t(z_t)dt - \frac{\partial\eta^*_t}{\partial t}(z_t)dt \\
&= U(z_{t^+})\eta_t(z_{t^+})dt + \eta^*_t(z_{t^+})\int_{|y|\le 1}\frac{\eta^*_t(z_{t^+}-y) - \eta^*_t(z^*_{t^+})}{\eta^*_t(z_{t^+})}(\mu_*(dy,dt) - \nu_\hbar(dy)dt) \\
&\quad +\eta^*_t(z_{t^+})\langle\nabla\log\eta^*_t(z_{t^+}), dW^*_t\rangle_{\hbar B} + \langle\nabla\log\eta^*_t(z_{t^+}), \nabla\log\eta^*_t(z_{t^+})\rangle_{\hbar B}.
\end{aligned}$$

Hence

$$\begin{aligned}
d_*L^*_t &= L^*_{t^+}\cdot\int_{|y|\le 1}\frac{\eta^*_t(z_{t^+}-y) - \eta^*_t(z_{t^+})}{\eta^*_t(z_{t^+})}(\mu_*(dy,dt) - \nu_\hbar(dy)dt) \\
&\quad +L^*_{t^+}\cdot\langle\nabla\log\eta^*_t(z_{t^+}), dW^*_t + \nabla\log\eta^*_t(z_{t^+})dt\rangle_{\hbar B}, \quad t\in[r,v].
\end{aligned}$$

Under P we have

$$\frac{1}{L^*_{s^+}}d_*\langle L^*_s, z_s\rangle = \int_{|y|\le 1}\frac{\eta^*_t(z_{t^+}-y) - \eta^*_t(z_{t^+})}{\eta^*_t(z_{t^+})}\nu_\hbar(dy)dt + \hbar B\nabla\log\eta^*_t(z_{t^+}),$$

hence

$$W^*_t + \hbar\int_t^v B\nabla\log\eta^*_s(z_{s^+})ds + \int_t^v\int_{\{|y|\le 1\}} y(\mu_*(dy,ds) - \nu_\hbar(dy)ds) - \int_t^v\frac{1}{L^*_{s^+}}d_*\langle L^*_s, z_s\rangle$$

$$= \quad W_t^* + \int_t^v \int_{\{|y|\le 1\}} y \left(\mu_*(dy, ds) - \frac{\eta_s^*(z_{s^+} - y)}{\eta_s^*(z_{s^+})} \nu_\hbar(dy) ds \right),$$

for $t \in [r, v]$, is a backward martingale under the probability Q defined by

$$\frac{dQ}{dP}|_{\mathcal{F}_t} = L_t^*, \quad r \le t \le v.$$

We also use the (backward, i.e. $(\mathcal{F}_t)_{t\in[r,v]}$-) representations

$$d_* z_t = \int_{\mathrm{R}^d} y \left(\mu_*(dy, dt) - 1_{\{|y|\le 1\}} \nu_\hbar(dy) dt \right)$$

in the Poisson case, and

$$d_* z_t = d_* W_t^* - \hbar B \nabla \log \eta_t^*(z_{t^+}) dt$$

in the Brownian case. □

Examples for Prop. 4.1.

1. Deterministic process. In this case we have $Q = P$. More precisely, $\eta_t(z_t) = \eta_r(z_r)$, $\eta_t^*(z_t) = \eta_v^*(z_v)$, $r \le t \le v$, and in fact $(z_t)_{t\in[r,v]} = (z_r)_{t\in[r,v]}$.

2. Poisson bridge from $a \in \mathbb{N}$ to $b \in \mathbb{N}$, $a \le b$.

   In this case the law of $(z_t)_{t\in[r,v]}$ is absolutely continuous with respect to P, with

$$\frac{dQ}{dP}|_{\mathcal{P}_t} = \frac{\eta_t(z_t)}{\eta_r(z_r)} = e^{t-r} \frac{(v-t)^{b-z_t}}{(v-r)^{b-a}} \frac{(b-a)!}{(b-z_t)!} 1_{[a,b]}(z_t), \quad r \le t \le v,$$

   hence

$$\frac{dQ}{dP} = e^{v-r} \frac{(b-a)!}{(v-r)^{b-a}} 1_{\{z_v = b\}},$$

   with $z_r = a$, i.e. under Q, the standard Poisson process $(a + \xi_{t-r})_{t\in[r,v]}$ has the law of the Poisson bridge $(z_t)_{t\in[r,v]}$. We also have

$$\frac{dQ}{dP}|_{\mathcal{F}_t} = \frac{\eta_t^*(z_t)}{\eta_v^*(z_v)} = e^{-(t-v)} \frac{(t-r)^{z_t}}{z_t!} \frac{(b-a)!}{(v-r)^{(b-a)}} 1_{[a,b]}(z_t), \quad r \le t \le v,$$

   hence

$$\frac{dQ}{dP} = e^{-(r-v)} \frac{(b-a)!}{(v-r)^{(b-a)}} 1_{\{z_r = a\}},$$

   with $z_v = b$, i.e. under Q, $(b - \xi_{v-t})_{t\in[r,v]}$ has the law of the Poisson bridge.

3. Brownian bridge.

   The law of the Brownian bridge starting from $a \in \mathbb{R}$ at time r and ending at $b \in \mathbb{R}$ is not absolutely continuous with respect to the Wiener measure, since $\mu_r(di) = \delta_a(di)$ and $\mu_v(dm) = \delta_b(dm)$.

4. Lévy bridges starting at $a \in \mathbb{R}^d$ and ending at $b \in \mathbb{R}^d$.

   Take $U = 0$, and assume that $\mu_t(dk) = \mu_t(k)\lambda(dk)$ has a density with respect to λ. We have

$$\frac{dQ}{dP}|_{\mathcal{P}_t} = \frac{\mu_{v-t}(b - z_t)}{\mu_v(b - z_r)}, \quad r \le t \le v,$$

   and

$$\frac{dQ}{dP}|_{\mathcal{F}_t} = \frac{\mu_{t-r}(z_t - a)}{\mu_{v-r}(z_v - a)}, \quad r \le t \le v.$$

5. Forward and backward Brownian motion ($U = 0$).

   We have either $Q = P$ or Q is not absolutely continuous with respect to P.

6. Forward and backward Poisson processes ($U = 0$).

   In the standard Poisson case, backward Lévy processes give examples of jump processes with $z_v = 0$ and initial Poisson distribution $\rho_r(k)\lambda(dk)$, $k \le 0$, on $-\mathbb{N}$. We have $\mu_{t-v}(k) = e^{-(v-t)}(v-t)^{-k}/(-k)!$, $t \le v$, and

$$\frac{dQ}{dP}|_{\mathcal{P}_t} = \frac{\mu_{t-v}(z_t)}{\mu_{r-v}(z_r)} = e^{-(r-t)} \frac{(v-t)^{-z_t}}{(-z_t)!} \frac{(-z_r)!}{(v-r)^{-z_r}},$$

   hence

$$\frac{dQ}{dP} = e^{v-r} \frac{(-z_r)!}{(v-r)^{-z_r}} 1_{\{z_v=0\}}.$$

   It follows from Prop. 4.1, under Q the process $(z_r + \xi_{t-r})_{t\in[r,v]}$ has the law of $(z_t)_{t\in[r,v]}$, where $(\xi_t)_{t\in[0,+\infty[}$ is the canonical Lévy process. Similarly we have, if $(z_t)_{t\in[r,v]}$ is a standard Poisson process under P:

$$\frac{dQ}{dP}|_{\mathcal{F}_t} = \frac{\mu_t(z_t)}{\mu_v(z_v)} = e^{-(t-v)} \frac{(t-r)^{z_t}}{z_t!} \frac{z_v!}{(v-r)^{z_v}},$$

   hence

$$\frac{dQ}{dP} = e^{v-r} \frac{z_v!}{(v-r)^{z_v}} 1_{\{z_r=0\}}, \quad r \ge 0,$$

   i.e. under Q, $(z_v + \xi^*_t)_{t\in[r,v]} = (z_v - \xi_{v-t})_{t\in[r,v]}$ has same the law as the standard (forward) Poisson process $(\xi_t)_{t\in[r,v]}$.

7. Forward Lévy processes ($U = 0$ and μ_r, μ_v are absolutely continuous with respect to λ).

   Here the probability Q is naturally equal to P, and the process $(z_t)_{t\in[r,v]}$ has same law as the forward Lévy process $(\xi_t)_{t\in[r,v]}$. If $(\xi^*_t)_{t\in[r,v]} = (-\xi_{v-t})_{t\in[r,v]}$ is a backward Lévy process under P, the density $\frac{dQ}{dP}|_{\mathcal{F}_t}$ is given by

$$\frac{dQ}{dP}|_{\mathcal{F}_t} = \frac{\mu_t(z_t)}{\mu_v(z_v)},$$

   and

$$\frac{dQ}{dP} = \frac{\mu_r(z_r)}{\mu_v(z_v)},$$

   i.e. $(z_v + \xi^*_t)_{t\in[r,v]}(z_v - \xi_{v-t})_{t\in[r,v]}$ is a forward Lévy process under Q.

8. Backward Lévy processes ($U = 0$ and μ_r, μ_v are absolutely continuous with respect to λ).

   The process $(z_t)_{t\in[r,v]}$ has the same law as the backward Lévy process $(\xi^*_t)_{t\in[r,v]} = (-\xi_{t-v})_{t\in[r,v]}$. The density $\frac{dQ}{dP}|_{\mathcal{P}_t}$ is given by

$$\frac{dQ}{dP}|_{\mathcal{P}_t} = \frac{\mu_{t-v}(z_t)}{\mu_{r-v}(z_r)},$$

   and

$$\frac{dQ}{dP} = \frac{\mu_0(z_v)}{\mu_{r-v}(z_r)}.$$

   From Prop. 4.1, under Q the process $(z_r + \xi_{t-r})_{t\in[r,v]}$ has the law of the canonical Lévy process $(\xi_t)_{t\in[r,v]}$.

5 Reversible diffusion processes with jumps

In this section we prove Th. 3.2 and some extensions. This provides a construction of Markovian "bridges" with given initial and final laws since from Th. 3.3, η^*_r and η_v can be chosen so that the products $\eta^*_r\eta_r$ and $\eta^*_v\eta_v$ equal any positive initial and final distribution densities fixed in advance. Define the forward and backward Markov semi-groups for $s \le r \le t \le u$ and $j, k, l \in \mathbb{R}^d$:

$$p(t, k, u, dl) = \frac{\eta_u(l)}{\eta_t(k)} h(t, k, u, dl), \tag{5.1}$$

and

$$p^*(s, dj, t, k) = \frac{\eta_s^*(j)}{\eta_t^*(k)} h^\dagger(s, dj, t, k). \tag{5.2}$$

The adjointness relation between H and $H^\dagger$:

$$h(s, j, t, dk)\lambda(dj) = h^\dagger(s, dj, t, k)\lambda(dk)$$

shows that the following reversibility condition holds

$$\begin{aligned} \eta_s^*(dj)\eta_s(j)p(s, j, t, dk) &= \eta_s^*(dj)h(s, j, t, dk)\eta_t(k) \\ &= \eta_s^*(j)h^\dagger(s, dj, t, k)\eta_t(dk) \\ &= p^*(s, dj, t, k)\eta_t^*(dk)\eta_t(k). \end{aligned} \tag{5.3}$$

Let us stress that this property generalizes the one understood, since Kolmogorov, as defining the reversibility of a probability measure (cf., for example, [9]). More generally we have

$$\begin{aligned} &\eta_{t_1}^*(dk_1)\eta_{t_1}(k_1)p(t_1, k_1, t_2, dk_2)\cdots p(t_{n-1}, k_{n-1}, t_n, dk_n) \\ &= \eta_{t_1}^*(dk_1)h(t_1, k_1, t_2, dk_2)\cdots h(t_{n-1}, k_{n-1}, t_n, dk_n)\eta_{t_n}(k_n). \\ &= \eta_{t_1}^*(k_1)h^\dagger(t_1, dk_1, t_2, k_2)\cdots h^\dagger(t_{n-1}, dk_{n-1}, t_n, k_n)\eta_{t_n}(dk_n). \\ &= p^*(t_1, k_1, t_2, dk_2)\cdots p^*(t_{n-1}, dk_{n-1}, t_n, k_n)\eta_{t_n}^*(k_n)\eta_{t_n}(dk_n), \end{aligned}$$

hence the forward Markov process with transition $p(s, j, t, dk)$ and initial law $\eta_s^*(dj)\eta_s(j)$ has the same law $\eta_t^*(dk)\eta_t(k)$ as the backward Markov process with transition $p^*(t, dk, u, l)$ and final law $\eta_u^*(dl)\eta_u(l)$, $s \le t \le u$.

This argument is made precise in the next two propositions, without assuming that $\eta_r^*(dk)$, resp. $\eta_v(dk)$, has a density with respect to $\lambda(dk)$.

Proposition 5.1 *Let $\eta_r^*(di)$ and $\eta_v : \mathbb{R}^d \to \mathbb{R}_+$ be initial and final conditions such that for some $t \in [r, v]$,*

$$\int_{\mathbb{R}^d} \eta_t^*(dk)\eta_t(k) = 1,$$

where

$$\eta_t^*(dk) = \int_{\mathbb{R}^d} \eta_r^*(di)h(r, i, t, dk), \quad \eta_t(k) = \int_{\mathbb{R}^d} \eta_v(m)h(t, k, v, dm) = e^{-(v-t)H}\eta_v(k), \tag{5.4}$$

$r < t < v$, and let us define

$$p(t,k,u,dl) = \frac{\eta_u(l)}{\eta_t(k)} h(t,k,u,dl), \tag{5.5}$$

Then

i) $p(t,k,u,dl)$ is a forward Markov transition kernel,

ii) the inhomogeneous Markov process $(z_t)_{t\in[r,v]}$ with forward transition kernel $p(t,k,u,dl)$ and initial distribution $\eta_s(j)\eta_s^*(dj)$ satisfies

$$P(z_t \in dk \mid \mathcal{P}_s \vee \mathcal{F}_u) = P(z_t \in dk \mid z_s, z_u), \quad r \le s < t < u \le v, \tag{5.6}$$

i.e. it is a Bernstein (or reciprocal, or "local Markov") process,

iii) the law at time t of z_t is $\rho_t(dk) = \eta_t(k)\eta_t^*(dk)$, $t \in [r,v]$.

If moreover H and $H^\dagger$ are adjoint with respect to some fixed reference measure λ:

$$h(s,j,t,dk)\lambda(dj) = h^\dagger(s,dj,t,k)\lambda(dk), \tag{5.7}$$

and $\eta_s^*(dj) = \eta_s^*(j)\lambda(dj)$ is absolutely continuous with respect to λ, then

iv) for all $t < u$, $\eta_t^*(dk)$ is absolutely continuous with respect to λ, with density

$$\eta_t^*(k) = \int_{\mathrm{R}^d} \eta_s^*(j) h^\dagger(s,dj,t,k) = e^{-(t-s)H^\dagger}\eta_s^*(k), \quad r \le s < t,$$

v) $(z_t)_{t\in[r,v]}$ is also a backward Markov process with transition kernel

$$p^*(s,dj,t,k) = \frac{\eta_s^*(j)}{\eta_t^*(k)} h^\dagger(s,dj,t,k), \quad r \le s < t, \tag{5.8}$$

vi) the law of z_t at time t is $\eta_t(k)\eta_t^*(k)\lambda(dk)$.

Proof. The fact (i) that $p(s,j,t,dk)$ is a Markov transition kernel follows from the definition of $\eta_t(k)$ itself:

$$\begin{aligned}\int_{\mathrm{R}^d} p(t,k,u,dl)p(u,l,v,dm) &= \frac{\eta_v(m)}{\eta_t(k)} \int_{\mathrm{R}^d} h(t,k,u,dl)h(u,l,v,dm) \\ &= \frac{\eta_v(m)}{\eta_t(k)} h(t,k,v,dm) = p(t,k,v,dm).\end{aligned}$$

The existence of the inhomogenous Markov process $(z_t)_{t\in[r,v]}$ follows from e.g. Th. 4.1.1 of [11] applied on the (complete separable) space $\mathbb{R}^d$. More precisely, [11] yields the existence of the space-time homogeneous Markov process $(t, z_t)_{t\in[r,v]}$ with transition semigroup

$$\tilde{p}((t,k), s, (du, dl)) = p(t, k, u, dl)\delta_{t+s}(du).$$

Let us show that (5.6) holds for this forward Markov process. We have, for $r \leq t_1 < t_2 < \cdots < t_n \leq v$,

$$\begin{aligned} & P(z_{t_1} \in dk_1, \dots, z_{t_n} \in dk_n) \\ &= \eta^*_{t_1}(dk_1)\eta_{t_1}(k_1)p(t_1, k_1, t_2, dk_2)\cdots p(t_{n-1}, k_{n-1}, t_n, dk_n) \\ &= \eta^*_{t_1}(dk_1)h(t_1, k_1, t_2, dk_2)\cdots h(t_{n-1}, k_{n-1}, t_n, dk_n)\eta_{t_n}(k_n). \end{aligned}$$

In particular, using (5.5),

$$P(z_s \in dj,\ z_u \in dl) = \eta^*_s(dj)\eta_s(j)p(s, j, u, dl) = \eta^*_s(dj)h(s, j, u, dl)\eta_u(l),$$

and

$$P(z_s \in dj,\ z_t \in dk,\ z_u \in dl) = \eta^*_s(dj)h(s, j, t, dk)h(t, k, u, dl)\eta_u(l).$$

Hence $P(z_t \in dk \mid z_s = j, z_u = l)$ satisfies $\eta^*_s(dj)$-a.e.:

$$P(z_t \in dk \mid z_s = j, z_u = l)h(s, j, u, dl) = h(s, j, t, dk)h(t, k, u, dl).$$

This gives, with $s_1 < s_2 < \cdots < s_n < t < u_1 < \cdots < u_m$, and introducing the Bernstein kernel $h(s_n, j_n, t, A, u_1, l_1) = P(z_t \in A \mid z_{s_n} = j_n, z_{u_1} = l_1)$ of Sect. 3,

$$\begin{aligned} & P(z_{s_1} \in dj_1, \dots, z_{s_n} \in dj_n,\ z_t \in A,\ z_{u_1} \in dl_1, \dots, z_{u_m} \in dl_m) \\ &= \int_A \eta^*_{s_1}(dj_1)h(s_1, j_1, s_2, dj_2)\cdots h(s_n, j_n, t, dk) \\ &\quad h(t, k, u_1, dl_1)\cdots h(u_{m-1}, l_{m-1}, u_m, dl_m)\eta_{u_m}(l_m) \\ &= h(s_n, j_n, t, A, u_1, l_1)\eta^*_{s_1}(dj_1)h(s_1, j_1, s_2, dj_2)\cdots h(s_{n-1}, j_{n-1}, s_n, dj_n)h(s_n, j_n, u_1, dl_1) \\ &\quad h(u_1, l_1, u_2, dl_2)\cdots h(u_{m-1}, l_{m-1}, u_m, dl_m)\eta_{u_m}(l_m) \\ &= h(s_n, j_n, t, A, u_1, l_1)P(z_{s_1} \in dj_1, \dots, z_{s_n} \in dj_n,\ z_{u_1} \in dl_1, \dots, z_{u_m} \in dl_m), \end{aligned}$$

hence

$$P(z_t \in dk \mid \mathcal{P}_{s_n} \vee \mathcal{F}_{u_1}) = h(s_n, z_{s_n}, t, dk, u_1, z_{u_1}) = P(z_t \in dk \mid z_{s_n}, z_{u_1}).$$

Finally, under the condition (5.7) we have

$$\begin{aligned}
\eta_t^*(dk) &= \int_{\mathrm{R}^d} \eta_s^*(dj) h(s,j,t,dk) \\
&= \int_{\mathrm{R}^d} \eta_s^*(j) h(s,j,t,dk)\lambda(dj) = \int_{\mathrm{R}^d} \eta_s^*(j) h^\dagger(s,dj,t,k)\lambda(dk).
\end{aligned}$$

The process $(z_t)_{t\in[r,v]}$ being constructed from the forward kernel (3.2), we show that its backward kernel is given by (3.3) when (5.7) holds; we have

$$\begin{aligned}
& P(z_s \in A,\ z_{t_1} \in dk_1, \ldots, z_{t_n} \in dk_n) \\
&= \int_A \eta_s^*(k) h^\dagger(s,dj,t_1,k_1) \cdots h^\dagger(t_{n-1},dk_{n-1},t_n,dk_n)\eta_{t_n}(dk_n) \\
&= \int_A p^*(s,dj,t_1,k_1) \cdots p^*(t_{n-1},dk_{n-1},t_n,k_n)\eta_{t_n}^*(dk_n)\eta_{t_n}(k_n) \\
&= p^*(s,A,t_1,k_1)p^*(t_1,dk_1,t_2,k_2) \cdots p^*(t_{n-1},dk_{n-1},t_n,k_n,)\eta_{t_n}^*(dk_n)\eta_{t_n}(k_n) \\
&= p^*(s,A,t_1,k_1)P(z_{t_1} \in dk_1, \ldots, z_{t_n} \in dk_n,),
\end{aligned}$$

hence $(z_t)_{t\in[r,v]}$ is also backward Markovian with transition kernel $p^*(s,dj,t_1,k_1)$. $\square$

Relation (5.4) can be written as

$$-\frac{\partial \eta_t^*(dk)}{\partial t} = H^\dagger \eta_t^*(dk) \quad \text{and} \quad \frac{\partial \eta_t}{\partial t}(k) = H\eta_t(k), \qquad t \in [r,v].$$

The following similar proposition shows that Markovian bridges can also be constructed from backward Markov processes. Prop. 5.1 and Prop. 5.2 complete the proof of Th. 3.2.

Proposition 5.2 *Let $\eta_r^* : \mathbb{R}^d \to \mathbb{R}_+$ and $\eta_v(dm)$ be initial and final conditions such that for some $t \in [r,v]$,*

$$\int_{\mathrm{R}^d} \eta_t^*(k)\eta_t(dk) = 1,$$

where

$$\eta_t(dk) = \int_{\mathrm{R}^d} \eta_v(dm) h^\dagger(t,dk,v,m), \qquad \eta_t^*(k) = \int_{\mathrm{R}^d} \eta_r^*(i) h^\dagger(r,di,t,k) = e^{-(t-r)H^\dagger}\eta_r^*(k), \tag{5.9}$$

$r < t < v$, and

$$p^*(s,dj,t,k) = \frac{\eta_s^*(j)}{\eta_t^*(k)} h^\dagger(s,dj,t,k). \tag{5.10}$$

Then

i) $p^*(s,dj,t,k)$ is a backward Markov transition kernel,

ii) the inhomogeneous backward Markov process $(z_t)_{t\in[s,u]}$ with transition kernel $p^*(s,dj,t,k)$ and final distribution $\eta_u(dl)\eta^*_u(l)$ satisfies

$$P(z_t \in dk \mid \mathcal{P}_s \vee \mathcal{F}_u) = P(z_t \in dk \mid z_s, z_u), \tag{5.11}$$

i.e. it is a Bernstein process.

iii) the law at time t of z_t is $\rho_t(k) = \eta_t(k)\eta^*_t(dk)$.

If moreover H and $H^\dagger$ are adjoint with respect to a fixed reference measure, i.e.

$$h(t,k,u,dl)\lambda(dk) = h^\dagger(t,dk,u,l)\lambda(dl), \tag{5.12}$$

and $\eta_u(dl) = \eta_u(l)\lambda(dl)$ is absolutely continuous with respect to λ, then

iv) $\eta_t(dk)$ is absolutely continuous with respect to λ, with density

$$\eta_t(k) = \int_{\mathrm{R}^d} \eta_u(l)h(t,k,u,dl) = e^{-(u-t)H}\eta_u(k), \qquad t < u \leq v,$$

v) $(z_t)_{t\in[r,v]}$ is a forward Markov process with transition kernel

$$p(t,k,u,dl) = \frac{\eta_u(l)}{\eta_t(k)} h(t,k,u,dl),$$

vi) the law at time t of z_t is $\eta_t(k)\eta^*_t(k)\lambda(dk)$.

Proof. (similar to the proof of Prop. 5.1, and stated for completeness.) We have

$$\begin{aligned} &P(z_{t_1} \in dk_1, \ldots, z_{t_n} \in dk_n) \\ &= p^*(t_1,dk_1,t_2,k_2)\cdots p^*(t_{n-1},dk_{n-1},t_n,k_n)\eta^*_{t_n}(k_n)\eta_{t_n}(dk_n) \\ &= \eta^*_{t_1}(k_1)h^\dagger(t_1,dk_1,t_2,k_2)\cdots h^\dagger(t_{n-1},dk_{n-1},t_n,k_n)\eta_{t_n}(dk_n). \end{aligned}$$

In particular,

$$P(z_s \in dj,\ z_u \in dl) = p^*(s,dj,u,l)\eta^*_u(l)\eta_u(dl) = \eta^*_s(j)h^\dagger(s,dj,u,l)\eta_u(dl),$$

and

$$P(z_s \in dj,\ z_t \in dk,\ z_u \in dk) = \eta^*_s(j)h^\dagger(s,dj,t,k)h^\dagger(t,dk,u,l)\eta_u(dl).$$

Hence $\eta_u(dl)$-a.e.:

$$P(z_t \in dk \mid z_s = j, z_u = l) h^\dagger(s, j, u, dl) = h^\dagger(s, dj, t, k) h^\dagger(t, dk, u, l).$$

This gives:

$$\begin{aligned}
& P(z_{s_1} \in dj_1, \ldots, z_{s_n} \in dj_n,\ z_t \in A,\ z_{u_1} \in dl_1, \ldots, z_{u_m} \in dl_m) \\
&= \int_A \eta^*_{s_1}(j_1) h^\dagger(s_1, dj_1, s_2, j_2) \cdots h(s_n, dj_n, t, k) \\
&\quad h(t, dk, u_1, dl_1) \cdots h(u_{m-1}, dl_{m-1}, u_m, l_m) \eta_{u_m}(dl_m) \\
&= h(s_n, j_n, t, A, u_1, l_1) \eta^*_{s_1}(j_1) h^\dagger(s_1, dj_1, s_2, j_2) \cdots h^\dagger(s_{n-1}, dj_{n-1}, s_n, j_n) \\
&\quad h^\dagger(s_n, dj_n, u_1, l_1) h^\dagger(u_1, dl_1, u_2, l_2) \cdots h^\dagger(u_{m-1}, dl_{m-1}, u_m, l_m) \eta_{u_m}(dl_m) \\
&= h^\dagger(s_n, j_n, t, A, u_1, l_1) P(z_{s_1} \in dj_1, \ldots, z_{s_n} \in dj_n,\ z_{u_1} \in dl_1, \ldots, z_{u_m} \in dl_m),
\end{aligned}$$

hence (5.11) holds. Finally under (5.12) we have

$$\begin{aligned}
\eta_t(dk) &= e^{-(u-t)H^\dagger} \eta^*_u(dk) = \int_{\mathrm{R}^d} \eta_u(dl) h^\dagger(t, dk, u, l) \\
&= \int_{\mathrm{R}^d} \eta^*_u(l) h^\dagger(t, dk, u, l) \lambda(dl) \\
&= \int_{\mathrm{R}^d} \eta^*_u(k) h(t, k, u, dl) \lambda(dk),
\end{aligned}$$

It remains to shows that if $(z_t)_{t\in[r,v]}$ is constructed from the backward kernel (3.3), then its forward kernel is given by (3.2): using again (5.12) we have

$$\begin{aligned}
& P(z_{t_1} \in dk_1, \ldots, z_{t_n} \in dk_n, z_u \in C) \\
&= \int_C \eta^*_{t_1}(k_1) h^\dagger(t_1, dk_1, t_2, k_2) \cdots h^\dagger(t_{n-1}, dk_{n-1}, t_n, k_n) \eta_u(dl) \\
&= \int_C \eta^*_{t_1}(k_1) \eta_{t_1}(dk_1) p(t_1, k_1, t_2, dk_2) \cdots p(t_{n-1}, k_{n-1}, t_n, dk_n) p(t_n, k_n, u, C) \\
&= \eta^*_{t_1}(k_1) \eta_{t_1}(dk_1) p(t_1, k_1, t_2, dk_2) \cdots p(t_{n-1}, k_{n-1}, t_n, dk_n,) p(t_n, k_n, u, C) \\
&= P(z_{t_1} \in dk_1, \ldots, z_{t_n} \in dk_n) p(t_n, k_n, u, C),
\end{aligned}$$

hence $(z_t)_{t\in[r,v]}$ is Markovian with forward transition kernel $p(t_n, k_n, t, C)$. □

Relation (5.9) can be written as

$$-\frac{\partial \eta^*_t}{\partial t}(k) = H^\dagger \eta^*_t(k) \quad \text{and} \quad \frac{\partial \eta_t(dk)}{\partial t} = H \eta_t(dk) \qquad t \in [r, v].$$

6 Generators

In this section we study the generators of Bernstein diffusions with jumps, solutions of forward and backward stochastic integro-differential equations, under the assumptions of Th. 3.2.

Definition 6.1 *For $f \in \mathcal{S}(\mathbb{R}^d)$ we define the forward generator (cf. Prop. 3.2) by*

$$\begin{aligned}
\mathcal{L}_{\eta_t} f(k) &= \langle c, \nabla f(k)\rangle + \frac{1}{2}\Delta_{\hbar B} f(k) \\
&+ \int_{\mathrm{R}^d} (f(k+y) - f(k) - \langle y, \nabla f(k)\rangle 1_{\{|y|\leq 1\}})\frac{\eta_t(k+y)}{\eta_t(k)}\nu_\hbar(dy) \\
&+ \int_{\mathrm{R}^d} \frac{\eta_t(k+y)-\eta_t(k)}{\eta_t(k)}\langle y, \nabla f(k)\rangle 1_{\{|y|\leq 1\}}\nu_\hbar(dy) + \langle \nabla \log \eta_t(k), \nabla f(k)\rangle_{\hbar B},
\end{aligned}$$

and its backward counterpart by

$$\begin{aligned}
\mathcal{L}^*_{\eta^*_t} f(k) &= \langle c, \nabla f(k)\rangle - \frac{1}{2}\Delta_{\hbar B} f(k) \\
&- \int_{\mathrm{R}^d} (f(k-y) - f(k) + \langle y, \nabla f(k)\rangle 1_{\{|y|\leq 1\}})\frac{\eta^*_t(k-y)}{\eta^*_t(k)}\nu_\hbar(dy) \\
&+ \int_{\mathrm{R}^d} \frac{\eta^*_t(k-y)-\eta^*_t(k)}{\eta^*_t(k)}\langle y, \nabla f(k)\rangle 1_{\{|y|\leq 1\}}\nu_\hbar(dy) - \langle \nabla \log \eta^*_t(k), \nabla f(k)\rangle_{\hbar B}.
\end{aligned}$$

Note that $\mathcal{L}^*_{\eta^*_t}$ is not the adjoint of $\mathcal{L}_{\eta_t}$, which will be denoted, when needed, by $(\mathcal{L}_{\eta_t})^\dagger$. The proof of Prop. 3.4 follows from the next proposition.

Proposition 6.2 *The kernels $p(t,k,u,dl)$ and $p^*(s,dj,t,k)$ of Props. 5.1, 5.2, satisfy the partial integro-differential equations*

$$\frac{\partial p}{\partial u}(t,k,u,dl) = (\mathcal{L}_{\eta_u})^\dagger_l p(t,k,u,dl) \tag{6.1}$$

(Kolmogorov forward or Fokker-Planck equation), and

$$\frac{\partial p^*}{\partial s}(s,dj,t,k) = (\mathcal{L}^*_{\eta^*_s})^\dagger_j p(s,dj,t,k).$$

The notation $(\mathcal{L}_{\eta_u})^\dagger_l p(t,k,u,dl)$, resp. $(\mathcal{L}^*_{\eta^*_s})^\dagger_j p(s,dj,t,k)$, means that $\mathcal{L}_{\eta_u}$, resp. $\mathcal{L}^*_{\eta^*_s}$, acts on the variable l, resp. k, i.e. Prop. 6.2 states that

$$\frac{\partial}{\partial u}\int_{\mathrm{R}^d} f(l)p(t,k,u,dl) = \int_{\mathrm{R}^d} \mathcal{L}_{\eta_u} f(l)p(t,k,u,dl),$$

resp.

$$\frac{\partial}{\partial s}\int_{\mathrm{R}^d} f(j)p^*(s,dj,t,k) = \int_{\mathrm{R}^d} \mathcal{L}^*_{\eta^*_s} f(j)p(s,dj,t,k).$$

In order to prove Prop. 6.2 we will need the following.

Lemma 6.3 *For $f, g \in \mathcal{S}(\mathbb{R}^d)$, the carré du champ operators [24] associated to $-H$ and $-H^\dagger$ are given respectively by*

$$\Gamma(f,g)(k) = U(k)f(k)g(k)+\langle\nabla f(k),\nabla g(k)\rangle_{\hbar B}+\int_{\mathrm{R}^d}(f(k+y)-f(k))(g(k+y)-g(k))\nu_\hbar(dy),$$

and

$$\Gamma^\dagger(f,g)(k) = U(k)f(kg(k)+\langle\nabla f(k),\nabla g(k)\rangle_{\hbar B}+\int_{\mathrm{R}^d}(f(k-y)-f(k))(g(k-y)-g(k))\nu_\hbar(dy).$$

Proof. An elementary computation shows that

$$-H(fg) = -fHg - gHf + \Gamma(f,g),$$

and

$$-H^\dagger(fg) = -fHg - gHf + \Gamma^\dagger(f,g),$$

which is the definition of $\Gamma(f,g)$ and $\Gamma^\dagger(f,g)$. □

Let the operators D_t and D_t^* be defined informally by

$$D_t f = \frac{1}{\eta_t}\left(\frac{\partial}{\partial t} - H\right)(\eta_t f)$$

and

$$D_t^* f = \frac{1}{\eta_t^*}\left(\frac{\partial}{\partial t} + H^\dagger\right)(\eta_t^* f).$$

By an adaptation of the method of [1] one shows that D_t and D_t^* are densely defined operators in $L^2(\mathbb{R}^d, \eta_t^*(k)\eta_t(k)\lambda(dk))$. They will be called afterwards, the forward and backward derivatives, respectively.

The following Lemma provides a decomposition of D_t and D_t^* which will be useful in the proof of Prop. 6.2.

Lemma 6.4 *We have*

$$D_t = \frac{\partial}{\partial t} + \mathcal{L}_{\eta_t} \quad \text{and} \quad D_t^* = \frac{\partial}{\partial t} + \mathcal{L}^*_{\eta_t^*}.$$

Proof. We have

$$
\begin{aligned}
D_t f_t(k) &= \left(\frac{\partial}{\partial t} - H\right) f_t(k) + \frac{f_t(k)}{\eta_t(k)} \left(\frac{\partial}{\partial t} - H\right) \eta_t(k) + \frac{1}{\eta_t(k)} \Gamma(\eta_t, f_t)(k) \\
&= \frac{\partial f_t}{\partial t}(k) - V(\nabla) f_t(k) + \langle \nabla \log \eta_t(k), \nabla f_t(k)\rangle_{\hbar B} \\
&\quad + \int_{\mathrm{R}^d} \frac{\eta_t(k+y) - \eta_t(k)}{\eta_t(k)} (f_t(k+y) - f_t(k)) \nu_\hbar(dy) \\
&= \frac{\partial f_t}{\partial t}(k) + \langle c, \nabla f_t(k)\rangle + \frac{1}{2} \Delta_{\hbar B} f_t(k) \\
&\quad + \int_{\mathrm{R}^d} (f_t(k+y) - f_t(k) - \langle y, \nabla f_t(k)\rangle 1_{\{|y|\le 1\}}) \nu_\hbar(dy) \\
&\quad + \langle \nabla \log \eta_t, \nabla f_t(k)\rangle_{\hbar B} + \int_{\mathrm{R}^d} \frac{\eta_t(k+y) - \eta_t(k)}{\eta_t(k)} (f_t(k+y) - f_t(k)) \nu_\hbar(dy) \\
&= \frac{\partial f_t}{\partial t}(k) + \langle c, \nabla f_t(k)\rangle + \frac{1}{2} \Delta_{\hbar B} f_t(k) \\
&\quad + \int_{\mathrm{R}^d} \left(\frac{\eta_t(k+y)}{\eta_t(k)} (f_t(k+y) - f_t(k)) - \langle y, \nabla f_t(k)\rangle 1_{\{|y|\le 1\}} \right) \nu_\hbar(dy) \\
&\quad + \langle \nabla \log \eta_t, \nabla f_t(k)\rangle_{\hbar B} \\
&= \frac{\partial f_t}{\partial t}(k) + \langle c, \nabla f_t(k)\rangle + \langle \nabla \log \eta_t, \nabla f_t(k)\rangle_{\hbar B} + \frac{1}{2} \Delta_{\hbar B} f_t(k) \\
&\quad + \int_{\mathrm{R}^d} (f_t(k+y) - f_t(k) - \langle y, \nabla f_t(k)\rangle 1_{\{|y|\le 1\}}) \frac{\eta_t(k+y)}{\eta_t(k)} \nu_\hbar(dy) \\
&\quad + \int_{\mathrm{R}^d} \frac{\eta_t(k+y) - \eta_t(k)}{\eta_t(k)} \langle y, \nabla f_t(k)\rangle 1_{\{|y|\le 1\}} \nu_\hbar(dy) \\
&= \left(\frac{\partial}{\partial t} + \mathcal{L}_{\eta_t}\right) f_t(k).
\end{aligned}
$$

Concerning D_t^* we have

$$
\begin{aligned}
D_t^* f_t(k) &= \left(\frac{\partial}{\partial t} + H^\dagger\right) f_t(k) + \frac{f_t(k)}{\eta_t^*(k)} \left(\frac{\partial}{\partial t} + H^\dagger\right) \eta_t^*(k) - \frac{1}{\eta_t^*(k)} \Gamma^\dagger(\eta_t^*, f_t)(k) \\
&= \frac{\partial f_t}{\partial t}(k) + \bar{V}(\nabla) f_t(k) - \langle \nabla \log \eta_t^*(k), \nabla f_t(k)\rangle_{\hbar B} \\
&\quad - \int_{\mathrm{R}^d} \frac{\eta_t^*(k-y) - \eta_t^*(k)}{\eta_t^*(k)} (f_t(k-y) - f_t(k)) \nu_\hbar(dy) \\
&= \frac{\partial f_t}{\partial t}(k) + \langle c, \nabla f_t(k)\rangle - \frac{1}{2} \Delta_{\hbar B} f_t(k) \\
&\quad - \int_{\mathrm{R}^d} (f_t(k-y) - f_t(k) + \langle y, \nabla f_t(k)\rangle 1_{\{|y|\le 1\}}) \nu_\hbar(dy) \\
&\quad - \langle \nabla \log \eta_t^*, \nabla f_t(k)\rangle_{\hbar B} - \int_{\mathrm{R}^d} \frac{\eta_t^*(k-y) - \eta_t^*(k)}{\eta_t^*(k)} (f_t(k-y) - f_t(k)) \nu_\hbar(dy)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\partial f_t}{\partial t}(k) + \langle c, \nabla f_t(k)\rangle - \frac{1}{2}\Delta_{\hbar B} f_t(k) \\
&\quad - \int_{\mathrm{R}^d} \left(\frac{\eta_t^*(k-y)}{\eta_t^*(k)} (f_t(k-y) - f_t(k)) + \langle y, \nabla f_t(k)\rangle 1_{\{|y|\le 1\}} \right) \nu_\hbar(dy) \\
&\quad - \langle \nabla \log \eta_t^*, \nabla f_t(k)\rangle_{\hbar B} \\
&= \frac{\partial f_t}{\partial t}(k) + \langle c, \nabla f_t(k)\rangle - \frac{1}{2}\Delta_{\hbar B} f_t(k) \\
&\quad - \int_{\mathrm{R}^d} \frac{\eta_t^*(k-y)}{\eta_t^*(k)} (f_t(k-y) - f_t(k) + \langle y, \nabla f_t(k)\rangle 1_{\{|y|\le 1\}}) \nu_\hbar(dy) \\
&\quad + \int_{\mathrm{R}^d} \frac{\eta_t^*(k-y) - \eta_t^*(k)}{\eta_t^*(k)} \langle y, \nabla f_t(k)\rangle 1_{\{|y|\le 1\}} \nu_\hbar(dy) - \langle \nabla \log \eta_t^*, \nabla f_t(k)\rangle_{\hbar B} \\
&= \left(\frac{\partial}{\partial t} + \mathcal{L}^*_{\eta_t^*} \right) f_t(k).
\end{aligned}
$$

□

Now we can easily prove Prop. 6.2.

Proof We have for any $f \in \mathcal{S}(\mathbb{R}^d)$, using the decompositions of Lemma 6.4:

$$
\begin{aligned}
&\eta_t(k) \frac{\partial}{\partial u} \int_{\mathrm{R}^d} f(l) p(t,k,u,dl) \\
&= \frac{\partial}{\partial u} \int_{\mathrm{R}^d} f(l)\eta_u(l) h(t,k,u,dl) = \frac{\partial}{\partial u} \left[e^{-(u-t)H} (f\eta_u)(k) \right] \\
&= \int_{\mathrm{R}^d} f(l) \frac{\partial \eta_u}{\partial u}(l) h(t,k,u,dl) - \int_{\mathrm{R}^d} H f(l) \eta_u(l) h(t,k,u,dl) \\
&= \int_{\mathrm{R}^d} \eta_t(l) D_u f(l) h(t,k,u,dl) \\
&= \int_{\mathrm{R}^d} \eta_u(l) \mathcal{L}_{\eta_u} f(l) h(t,k,u,dl) \\
&= \eta_t(k) \int_{\mathrm{R}^d} \mathcal{L}_{\eta_u} f(l) p(t,k,u,dl) \\
&= \eta_t(k) \int_{\mathrm{R}^d} f(l) (\mathcal{L}_{\eta_u})_l^\dagger p(t,k,u,dl).
\end{aligned}
$$

Concerning the dual statement we have

$$
\begin{aligned}
&\eta_t^*(k) \frac{\partial}{\partial s} \int_{\mathrm{R}^d} f(j) p^*(s,dj,t,k) \\
&= \frac{\partial}{\partial s} \int_{\mathrm{R}^d} f(j) \eta_s^*(j) h^\dagger(s,dj,t,k) = \frac{\partial}{\partial s} \left[e^{-(t-s)H^\dagger} (f\eta_s^*)(k) \right] \\
&= \int_{\mathrm{R}^d} f(j) \frac{\partial \eta_s^*}{\partial s}(j) h^\dagger(s,dj,t,dk) + \int_{\mathrm{R}^d} H^\dagger (f\eta_s^*)(j) h(s,dj,t,k)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathrm{R}^d} \eta_t^*(k) D_s^* f(j) h^\dagger(s, dj, t, k) \\
&= \int_{\mathrm{R}^d} \eta_t^*(k) \mathcal{L}_{\eta_s^*}^* f(j) h(s, dj, t, k) \\
&= \eta_t^*(k) \int_{\mathrm{R}^d} \mathcal{L}_{\eta_s^*}^* f(j) p(s, dj, t, k) \\
&= \eta_t^*(k) \int_{\mathrm{R}^d} f(j) (\mathcal{L}_{\eta_s^*}^*)_j^\dagger p(s, dj, t, k),
\end{aligned}
$$

and so Prop. 6.2 holds. □

Proof of Prop. 3.4. Prop. 6.2 shows that $f(z_v) - \int_0^v \mathcal{L}_{\eta_u} f(z_u) du$, $v \geq 0$, is a martingale for $f \in \mathcal{S}(\mathbb{R}^d)$:

$$
E[f(z_v) - f(z_v) \mid \mathcal{F}_t] = \int_t^v \frac{\partial}{\partial u} E[f(z_u) \mid \mathcal{F}_t] du = \int_t^v E[\mathcal{L}_{\eta_u} f(z_u) \mid \mathcal{F}_t] du,
$$

and $f(z_v) - \int_0^v \mathcal{L}_{\eta_u} f(z_u) du$, $v \geq 0$, is a local martingale for $f \in \mathcal{C}^2(\mathbb{R}^d)$. A similar argument holds in the backward case. □

7 Uniqueness of reversible diffusions

In this section we show that the processes constructed in this paper are essentially the only Markovian reversible diffusions with jumps. As defined in Prop. 5.1 let us recall that, more generally, a Bernstein process is a process $(z_t)_{t \in [r,v]}$ such that

$$
P(z_t \in dk \mid \mathcal{P}_s \vee \mathcal{F}_u) = P(z_t \in dk \mid z_s,\ z_u), \qquad r \leq s < t < u \leq v, \tag{7.1}
$$

where $(\mathcal{P}_t)_{t \in [r,v]}$, respectively $(\mathcal{F}_t)_{t \in [r,v]}$, denotes the increasing, resp. decreasing, filtration generated by $(z_t)_{t \in [r,v]}$. Jamison's construction of Bernstein processes [20] is still valid in the jump case. It requires the data of a probability measure ν on $\mathbb{R}^d \times \mathbb{R}^d$ and a Bernstein transition kernel, i.e. a kernel $h(s, j, t, dk, u, l)$ satisfying the counterpart of the Chapman-Kolmogorov equation:

$$
\int_A h(s, j, t, B, u, l) h(s, j, u, dl, v, m) = \int_B h(s, j, t, dk, v, m) h(t, k, u, A, v, m), \tag{7.2}
$$

for $A, B \in \mathcal{B}(\mathbb{R}^d)$. From [20] we know that there exists a unique (generally not Markovian) Bernstein process $(z_t)_{t \in [r,v]}$ such that

a) $P(z_r \in B,\ z_v \in C) = \nu(B \times C)$,

b) $P(z_t \in B \mid z_s, z_u) = h(s, z_s, t, B, u, z_u), \quad r \leq s < t < u \leq v.$

The finite dimensional distribution of $(z_t)_{t\in[r,v]}$ is given by

$$\begin{aligned} & P(z_r \in A,\ z_{t_1} \in B_1, \dots, z_{t_n} \in B_n,\ z_v \in C) \\ & \quad = \int_{A\times C} \nu(dj, dl) \int_{B_1} h(r, i, t_1, dk_1, v, m) \cdots \int_{B_n} h(t_{n-1}, k_{n-1}, t_n, dk_n, v, m), \end{aligned} \tag{7.3}$$

cf. [20].

Our construction of Markovian Bernstein processes did not follow, however, the above procedure. Instead, we started from the data of U and V, defining $H = U + V(\nabla)$ (Def. 2.1), i.e. from the Lévy process $(\xi_t)_{t\in[r,v]}$, and from boundary conditions η_r^* and η_v, allowing to construct a Markov transition kernels with the solutions of the adjoint heat equations (3.4). Then we showed that the corresponding Markov process is a Bernstein process.

Conversely, under the additional hypothesis (7.4), (7.5) on the kernel $h(s, j, t, dk, u, l)$ of a Bernstein process, it is possible to show that if a Bernstein process is Markovian then it is the process described in Th. 3.2. This extends Th. 3.1 of [20] and Th. 3.3 of [29] to the case where $h(t, dk, u, dl)$ and $h^\dagger(s, dj, t, k)$ are not absolutely continuous with respect to a reference measure.

Theorem 7.1 *Assume that H and $H^\dagger$ are adjoint with respect to a measure λ. Then the conditions*

$$h(s, j, t, dk, u, l)h(s, j, u, dl) = h(s, j, t, dk)h(t, k, u, dl), \quad \lambda(dj) - a.e., \tag{7.4}$$

and

$$h(s, j, t, dk, u, l)h^\dagger(s, dj, u, l) = h^\dagger(s, dj, t, k)h^\dagger(t, dk, u, l), \quad \lambda(dl) - a.e., \tag{7.5}$$

are equivalent. Moreover,

a) Let $(z_t)_{t\in[r,v]}$ denote the Bernstein process with kernel $h(s, j, t, dk, u, l)$ satisfying (7.4). Then the following are equivalent:

(i) the process $(z_t)_{t\in[r,v]}$ *is forward Markovian and* $p(t,k,u,dl)$ *is absolutely continuous with respect to* $h(t,k,u,dl)$*,*

(ii) there exists a measure $\eta_r^*(di)$ *and a positive density function* $\eta_v(m)$ *such that*

$$P(z_r \in A,\ z_v \in B) = \int_{A\times B} \eta_r^*(di) h(r,i,v,dm)\eta_v(m).$$

b) Assume that $h(s,j,t,dk,u,l)$ *satisfies* (7.5)*. Then the following are equivalent:*

(iii) there exists a positive density function $\eta_r^*(i)$ *and a probability measure* $\eta_v(dm)$ *such that*

$$P(z_r \in A,\ z_v \in B) = \int_{A\times B} \eta_r^*(i) h^\dagger(r,di,v,m)\eta_v(dm).$$

(iv) the process $(z_t)_{t\in[r,v]}$ *is backward Markovian and* $p^*(s,dj,t,k)$ *is absolutely continuous with respect to* $h^\dagger(s,dj,t,k)$*.*

If $\eta_r^*(di) = \eta_r^*(i)\lambda(di)$ *and* $\eta_v(dl) = \eta_v(l)\lambda(dl)$ *are absolutely continuous with respect to a fixed measure* λ*, then* (i)*,* (ii)*,* (iii) *and* (iv) *are equivalent.*

Proof. Under the adjointness hypothesis of H and $H^\dagger$ with respect to λ:

$$h^\dagger(s,dj,t,k)\lambda(dk) = h(u,j,t,dk)\lambda(dj),$$

conditions (7.4) and (7.5) are equivalent since, then,

$$h(s,j,t,dk,u,l)h(s,j,u,dl)\lambda(dj) = h(s,j,t,dk,u,l)h^\dagger(s,dj,u,l)\lambda(dl),$$

and

$$h(s,j,t,dk)h(t,k,u,dl)\lambda(dj) = h^\dagger(s,dj,t,k)\lambda(dl)h^\dagger(t,dk,u,l).$$

The implications $(ii) \Rightarrow (i)$, $(iv) \Rightarrow (iii)$ follows from Propositions 7.4, 7.5, and $(i) \Rightarrow (ii)$, $(iii) \Rightarrow (iv)$ will follow from Propositions 7.2, 7.3. Under the self-adjointness assumption (5.7), the equivalence $(i) \Leftrightarrow (iii)$ follows from Propositions 5.1 and 5.2. which show that the Bernstein process $(z_t)_{t\in[r,v]}$ is forward Markovian if and only if it is backward Markovian. □

Proposition 7.2 *Assume that the Bernstein kernel $h(s,j,t,dk,u,l)$ satisfies*

$$h(s,j,t,dk,u,l)h(s,j,u,dl) = h(s,j,t,dk)h(t,k,u,dl), \quad \rho_s(dj) - a.e., \tag{7.6}$$

where ρ_s is the law of z_s, $r \le s \le v$. If the Bernstein process $(z_t)_{t\in[r,v]}$ is forward Markovian and $p(t,k,v,dm)$ is absolutely continuous with respect to $h(t,k,v,dm)$, then there exists a measure $\eta_r^*(di)$ and a positive density function $\eta_v(m)$ such that

$$P(z_r \in di, z_v \in dm) = \eta_r^*(di)h(r,i,v,dm)\eta_v(m), \quad r < v. \tag{7.7}$$

Moreover we have

$$p(r,i,t,dk) = \frac{\eta_t(k)}{\eta_r(i)}h(r,i,t,dk), \tag{7.8}$$

with

$$\eta_t(k) = \int_{\mathrm{R}^d} \eta_v(m)h(t,k,v,dm), \qquad \eta_t^*(dk) = \int_{\mathrm{R}^d} \eta_r^*(di)h(r,i,t,dk), \quad r \le t \le v. \tag{7.9}$$

Proof. Let us assume that $(z_t)_{t\in[r,v]}$ is Markovian, with transition kernel $p(t,k,u,dl)$. Let $\rho_r(di)$ denote an initial law of $(z_t)_{t\in[r,v]}$. We have

$$P(z_r \in A,\ z_t \in B,\ z_v \in C) = \int_A \rho_r(di) \int_B p(r,i,t,dk) \int_C p(t,k,v,dm). \tag{7.10}$$

On the other hand,

$$P(z_r \in A,\ z_t \in B,\ z_v \in C) = \int_A \rho_r(di) \int_C p(r,i,v,dm) \int_B h(r,i,t,dk,v,m). \tag{7.11}$$

Equating (7.10) and (7.11), we obtain

$$p(r,i,t,dk)p(t,k,v,dm) = p(r,i,v,dm)h(r,i,t,dk,v,m),$$

which, using (7.6), gives

$$p(r,i,v,dm) = h(r,i,v,dm)\frac{p(r,i,t,dk)}{h(r,i,t,dk)}\frac{p(t,k,v,dm)}{h(t,k,v,dm)}, \tag{7.12}$$

and

$$\nu(A \times C) \ = \ \int_A \rho_r(di) \int_C p(r,i,v,dm)$$

$$= \int_{A\times C} \rho_r(di)\frac{p(r,i,t,dk)}{h(r,i,t,dk)}h(r,i,v,dm)\frac{p(t,k,v,dm)}{h(t,k,v,dm)}.$$

Let us fix $(t_0, k_0) \in \mathbb{R}_+ \times \mathbb{R}^d$, and define

$$\eta_v(m) = c(t_0,k_0)\frac{p(t_0,k_0,v,dm)}{h(t_0,k_0,v,dm)}, \tag{7.13}$$

and

$$\eta_r^*(di) = \frac{1}{c(t_0,k_0)}\frac{p(r,i,t_0,dk_0)}{h(r,i,t_0,dk_0)}\rho_r(di), \tag{7.14}$$

where $c(t_0, k_0)$ is a normalization constant equal to $\eta_{t_0}(k_0)$ after integrating in dm the relation

$$\eta_v(m)h(t_0,k_0,v,dm) = c(t_0,k_0)p(t_0,k_0,v,dm).$$

From (7.10), (7.12), (7.13) and (7.14) we have

$$P(z_r \in di, z_v \in dm) = \rho_r(di)p(r,i,v,dm) = \eta_r^*(di)h(r,i,v,dm)\eta_v(m),$$

i.e. (7.7) holds. Finally, from (7.3) and (7.6) we have

$$\begin{aligned} P(z_r \in di, z_t \in dk) &= \int_{\mathbb{R}^d} \eta_r^*(di)h(r,i,v,dm)\eta_v(m)h(r,i,t,dk,v,m) \\ &= \int_{\mathbb{R}^d} \eta_r^*(di)h(r,i,t,dk)h(t,k,v,dm)\eta_v(m) \\ &= \eta_r^*(di)h(r,i,t,dk)\eta_t(k), \end{aligned}$$

and $P(z_r \in di) = \eta_r^*(di)\eta_r(i)$, which proves (7.8). □

In the backward Markovian case we have the following result.

Proposition 7.3 *Assume that the Bernstein kernel $h(s,j,t,dk,u,l)$ satisfies*

$$h(s,j,t,dk,u,l)h^\dagger(s,dj,u,l) = h^\dagger(s,dj,t,k)h^\dagger(t,dk,u,l), \quad \rho_u(dl) - a.e., \tag{7.15}$$

where ρ_u is the law of z_u, $r \le u \le v$. If the Bernstein process $(z_t)_{t\in[r,v]}$ is backward Markovian and $p^*(r,di,t,k)$ is absolutely continuous with respect to $h^\dagger(r,di,t,k)$, then there exists a positive density function $\eta_r^*(i)$ and a measure $\eta_v(dm)$ such that

$$P(z_r \in di, z_v \in dm) = \eta_r(i)h^\dagger(r,di,v,m)\eta_v^*(dm).$$

Moreover we have

$$p^*(t, dk, v, m) = \frac{\eta_t^*(k)}{\eta_v^*(m)} h^\dagger(t, dk, v, m), \tag{7.16}$$

with

$$\eta_t(dk) = \int_{\mathrm{R}^d} \eta_v(dm) h^\dagger(t, dk, v, m), \quad \eta_t^*(k) = \int_{\mathrm{R}^d} \eta_r^*(i) h^\dagger(r, di, t, k), \quad r \le t \le v.$$

Proof. (similar to the proof of Prop. 7.2, and stated for completeness.) Let us assume that $(z_t)_{t\in[r,v]}$ has the backward Markov transition kernel $p^*(t, dk, u, l)$. Let ρ_v denote the final law of $(z_t)_{t\in[r,v]}$ at time v. We have

$$P(z_r \in A,\ z_t \in B,\ z_v \in C) = \int_A p^*(r, di, t, k) \int_B p^*(t, dk, v, m) \int_C \rho_v(dm). \tag{7.17}$$

On the other hand,

$$P(z_r \in A,\ z_t \in B,\ z_v \in C) = \int_A h(r, i, t, dk, v, m) \int_C p^*(r, di, v, m) \int_B \rho_v(dm) \tag{7.18}$$

Equating (7.17) and (7.18), we obtain

$$p^*(r, di, t, k) p^*(t, dk, v, m) = p^*(r, di, v, m) h(r, i, t, dk, v, m),$$

which from (7.15) gives

$$p^*(r, di, v, m) = h^\dagger(r, di, v, m) \frac{p^*(r, di, t, k)}{h^\dagger(r, di, t, k)} \frac{p^*(t, dk, v, m)}{h^\dagger(t, dk, v, m)},$$

and

$$\begin{aligned} \nu(A \times C) &= \int_A p^*(r, di, v, m) \int_C \rho_v(dm) \\ &= \int_{A\times C} \frac{p^*(r, di, t, k)}{h^\dagger(r, di, t, k)} h^\dagger(r, di, v, m) \frac{p^*(t, dk, v, m)}{h^\dagger(t, dk, v, m)} \rho_v(dm), \end{aligned}$$

which leads to

$$\eta_r^*(i) = c(t_0, k_0) \frac{p^*(r, di, t_0, k_0)}{h^\dagger(r, di, t_0, k_0)}, \tag{7.19}$$

and

$$\eta_v(dm) = \frac{1}{c(t_0, k_0)} \frac{p^*(t_0, dk_0, v, m)}{h^\dagger(t_0, dk_0, v, m)} \rho_v(dm),$$

where $c(t_0, k_0)$ is equal to $\eta_{t_0}^*(k_0)$. This shows (7.16). Moreover we have

$$P(z_r \in di, z_v \in dm) = \int_{\mathrm{R}^d} \rho_v(dm) p^*(r, di, v, m) = \eta_r^*(i) h^\dagger(r, di, v, m) \eta_v^*(dm),$$

Finally we have

$$\begin{aligned} P(z_t \in dk, z_v \in dm) &= \int_{\mathrm{R}^d} \eta_r^*(i) h^\dagger(r, di, v, m) \eta_v(dm) h(r, i, t, dk, v, m) \\ &= \int_{\mathrm{R}^d} \eta_r^*(i) h^\dagger(r, di, t, k) h^\dagger(t, dk, v, m) \eta_v(dm) \\ &= \eta_t^*(k) h^\dagger(t, dk, v, m) \eta_v(dm), \end{aligned}$$

and $P(z_v \in dm) = \eta_v^*(m)\eta_v(dm)$. □

The following is a converse to Prop. 7.2.

Proposition 7.4 *Assume that there exists a measure $\eta_r^*(di)$ and a positive density function $\eta_v(m)$ such that*

$$\nu(A \times B) = P(z_r \in A,\ z_v \in B) = \int_{A \times B} \eta_r^*(di) h(r, i, v, dm) \eta_v(m). \tag{7.20}$$

Then the Bernstein process $(z_t)_{t \in [r,v]}$ with kernel $h(s, j, t, dk, u, l)$ satisfying the conditions (7.4) *or* (7.5) *is forward Markovian and $p(t, k, u, dl)$ is absolutely continuous with respect to $h(t, k, u, dl)$, and given by* (7.8).

Proof. From (7.3), (7.4) and (7.20) we have

$$\begin{aligned} &P(z_{t_1} \in dk_1, \ldots, z_{t_n} \in dk_n, z_u \in dl) \\ &= \int_{\mathrm{R}^d} \eta_r^*(di) h(r, i, t_1, dk_1) \cdots h(t_n, k_n, u, dl) \int_{\mathrm{R}^d} \eta_v(m) h(u, l, v, dm) \\ &= P(z_{t_1} \in dk_1, \ldots, z_{t_n} \in dk_n) h(t_n, k_n, u, dl) \frac{\int_{\mathrm{R}^d} \eta_v(m) h(u, l, v, dm)}{\int_{\mathrm{R}^d} \eta_v(m) h(t_n, k_n, v, dm)}, \end{aligned}$$

hence

$$p(t_n, k_n, u, dl) = \frac{\int_{\mathrm{R}^d} \eta_v(m) h(u, l, v, dm)}{\int_{\mathrm{R}^d} \eta_v(m) h(t_n, k_n, v, dm)} h(t_n, k_n, u, dl).$$

□

Of course, it is also true that

Proposition 7.5 *Assume that there exists a positive density function $\eta_r^*(i)$ and a measure $\eta_v(dm)$ such that*

$$P(z_r \in A,\ z_v \in B) = \int_{A \times B} \eta_r^*(i) h^\dagger(r, di, v, m) \eta_v(dm). \tag{7.21}$$

Then the Bernstein process $(z_t)_{t\in[r,v]}$ *with kernel* $h(s,j,t,dk,u,l)$ *satisfying* (7.4) *or* (7.5) *is backward Markovian and* $p^*(s,dj,t,k)$ *is absolutely continuous with respect to* $h^\dagger(s,dj,t,k)$*, and given by* (7.16).

Proof. From (7.3), (7.5) and (7.21) we have

$$\begin{aligned}
&P(z_s \in dj, z_{t_1} \in dk_1, \ldots, z_{t_n} \in dk_n) \\
&= \int_{\mathrm{R}^d} \eta_r^*(i) h^\dagger(r, di, s, j) h^\dagger(s, dj, t_1, k_1) \cdots h^\dagger(t_{n-1}, dk_{n-1}, t_n, k_n) \int_{\mathrm{R}^d} \eta_v(dm) h^\dagger(t_n, dk_n, v, m) \\
&= h^\dagger(s, dj, t_1, k_1) \frac{\int_{\mathrm{R}^d} \eta_r(i) h^\dagger(r, di, s, j)}{\int_{\mathrm{R}^d} \eta_r(i) h^\dagger(r, di, t_1, k_1)} P(z_{t_1} \in dk_1, \ldots, z_{t_n} \in dk_n),
\end{aligned}$$

hence

$$p^*(s, dj, t_1, k_1) = \frac{\int_{\mathrm{R}^d} \eta_r(i) h^\dagger(r, di, s, j)}{\int_{\mathrm{R}^d} \eta_r(i) h^\dagger(r, di, t_1, k_1)} h^\dagger(s, dj, t_1, k_1).$$

□

8 Variational characterization

In this section we use the approach to stochastic control for jump processes of [13], [25], to obtain a variational characterization of the Markovian Bernstein processes (or reversible diffusions) with jumps considered before. We consider the stochastic control problem $\inf_f J(t,k;f)$ with action functional

$$J(t,k;f) = E_{(t,k)} \left[\int_t^v L(z(s), f) ds - \log \eta_v(z(v)) \right], \tag{8.1}$$

where $E_{(t,k)}$ denotes the conditional expectation given $\{z_t = k\}$, and the Lagrangian $L(k,f)$ is defined informally as

$$L(k,f) = \mathcal{L}_f \log f(k) + \frac{1}{f(k)} H f(k), \qquad f \in \mathcal{S}(\mathbb{R}^d), \ f > 0,$$

where $\mathcal{L}_f$ is defined at the beginning of Sect. 3.2. We have explicitly

$$\begin{aligned}
L(k,f) &= \int_{\mathrm{R}^d} \left(\frac{f(k) - f(k+y)}{f(k)} + \frac{f(k+y)}{f(k)} \log \frac{f(k+y)}{f(k)} \right) \nu_\hbar(dy) \\
&\quad + \frac{1}{2} \langle \nabla \log f(k), \nabla \log f(k) \rangle_{\hbar B} + U(k)
\end{aligned}$$

$$= \int_{\mathrm{R}^d} g\left(\frac{\delta f(k,y)}{f(k)}\right)\nu_\hbar(dy) + \frac{1}{2}\langle \nabla \log f(k), \nabla \log f(k)\rangle_{\hbar B} + U(k),$$

with $g(x) = (1+x)\log(1+x) - x$ and $\delta f(k,y) = f(k+y) - f(k)$. In particular, when $f = \eta_t$,

$$\begin{aligned} L(k,\eta_t) &= \mathcal{L}_{\eta_t}\log\eta_t(k) + \frac{1}{\eta_t(k)}H\eta_t(k) \\ &= \mathcal{L}_{\eta_t}\log\eta_t(k) + \frac{1}{\eta_t(k)}\frac{\partial}{\partial t}\eta_t(k) \\ &= \mathcal{L}_{\eta_t}\log\eta_t(k) + \frac{\partial}{\partial t}\log\eta_t(k) \\ &= D_t\log\eta_t(k). \end{aligned}$$

Proposition 8.1 *The dynamic programming equation with final boundary condition*

$$\frac{\partial A_t}{\partial t}(k) + \min_f\left[\mathcal{L}_f A_t(k) + L(k,f)\right] = 0, \quad A_v = -\log\eta_v, \tag{8.2}$$

associated to the action functional (8.1) *has the solution $A_t = -\log\eta_t$, the minimum in f being attained on $f_t(k) = \eta_t(k)$, i.e. when A_t is solution of the Hamilton-Jacobi-Bellman equation*

$$\begin{aligned} \frac{\partial A_t}{\partial t}(k) &= U(k) - \frac{1}{2}(\Delta_{\hbar B}A_t(k) - \langle\nabla A_t(k),\nabla A_t(k)\rangle_{\hbar B}) \\ &\quad - \int_{\mathrm{R}^d}(e^{-A_t(k+y)+A_t(k)} - 1 + 1_{\{|y|\le 1\}}\langle y,\nabla A_t(k)\rangle)d\nu_\hbar(y) + \langle c,\nabla A_t(k)\rangle. \end{aligned} \tag{8.3}$$

Proof. We first show that for $g_t(k) > 0$:

$$\min_f\left[-\mathcal{L}_{f_t}\log g_t(k) + L(k,f_t)\right] = \frac{1}{g_t(k)}Hg_t(k), \tag{8.4}$$

and that the minimum is attained for $f_t = g_t$. Let us define

$$F(k,y) = -\frac{f_t(k+y)}{f_t(k)}\log\frac{g_t(k+y)}{g_t(k)} + \frac{f_t(k+y)}{f_t(k)}\log\frac{f_t(k+y)}{f_t(k)} - \frac{f_t(k+y)}{f_t(k)} + \frac{g_t(k+y)}{g_t(k)}.$$

We have

$$\begin{aligned} &L(k,f_t) - \mathcal{L}_{f_t}\log\eta_t(k) - \frac{1}{\eta_t(k)}H\eta_t(k) \\ &= -\mathcal{L}_{f_t}\log\frac{\eta_t(k)}{f_t(k)} + \frac{1}{f_t(k)}Hf_t(k) - \frac{1}{\eta_t(k)}H\eta_t(k) \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathrm{R}^d} F(k,y)\eta_t(y)\nu_\hbar(dy) + \frac{1}{2}\left\langle \frac{\nabla g_t}{g_t}(k) - \frac{\nabla f_t}{f_t}(k), \frac{\nabla g_t}{g_t}(k) - \frac{\nabla f_t}{f_t}(k)\right\rangle_{\hbar B} \\
&\geq \int_{\mathrm{R}^d} F(k,y) g_t(y)\nu_\hbar(dy),
\end{aligned}
$$

Now, for all $a > 0$,

$$
\min_{z\in\mathrm{R}} (za + a\log a - a + e^{-z}) = 0,
$$

hence taking $z = -\log(g_t(k+y)/g_t(k))$ and $a = f_t(k+y)/f_t(k)$, we have $F(k,y) \geq 0$, and

$$
L(k,f) - \mathcal{L}_{f_t}\log g_t - \frac{1}{g_t}Hg_t = -\mathcal{L}_{f_t}\log\frac{g_t}{f_t} + \frac{1}{f_t}Hf_t - \frac{1}{g_t}Hg_t \geq 0.
$$

the minimum (zero) being attained with $f = g_t$, i.e.:

$$
\min_f [L(k,f) - \mathcal{L}_{f_t}\log g_t] = \frac{1}{g_t}Hg_t.
$$

Letting $A_t = -\log g_t$, the dynamic programming equation (8.2) becomes

$$
\frac{\partial A_t}{\partial t} + e^{A_t}He^{-A_t} = 0,
$$

with solution $A_t = -\log\eta_t$. Finally, from the relation

$$
\Delta_B A_t = \frac{1}{g_t}\Delta_B g_t - \left\langle \frac{\nabla g_t}{g_t}, \frac{\nabla g_t}{g_t}\right\rangle_B,
$$

we have

$$
\begin{aligned}
\frac{1}{g_t(k)}Hg_t(k) &= \frac{1}{g_t(k)}\Bigg(U(k)g_t - \langle c, \nabla g_t(k)\rangle - \frac{1}{2}\Delta_{\hbar B}g_t(k) \\
&\quad - \int_{\mathrm{R}^d}(g_t(k+y) - g_t(k) - \langle y, \nabla g_t(k)\rangle 1_{\{|y|\leq 1\}})\nu_\hbar(dy)\Bigg) \\
&= U(k) + \langle c, \nabla A_t(k)\rangle - \frac{1}{2}(\Delta_{\hbar B}A_t - \langle \nabla A_t, \nabla A_t\rangle_{\hbar B}) \\
&\quad - \int_{\mathrm{R}^d}(e^{-A_t(k+y)+A_t(k)} - 1 + 1_{\{|y|\leq 1\}}\langle y, \nabla A_t\rangle)d\nu_\hbar,
\end{aligned}
$$

which yields (8.3). □

In the backward case we consider the action functional which is, informally, the time reversed of (8.1):

$$
J^*(t,k;f^*) = E_{(t,k)}\left[\int_r^t L^*(z(s), f^*)ds - \log\eta_r^*(z(r))\right]. \tag{8.5}
$$

with Lagrangian $L^*(k, f^*)$ defined now as

$$L^*(k, f^*) = -\mathcal{L}^*_{f^*} \log f^*(k) + \frac{1}{f^*(k)} H^\dagger f^*(k), \qquad f^* \in \mathcal{S}(\mathbb{R}^d).$$

We have

$$L^*(k, f^*) = \int_{\mathrm{R}^d} g\left(\frac{\delta^* f^*(k, y)}{f^*(k)}\right) \nu_\hbar(dy) + \frac{1}{2} \langle \nabla \log f^*(k), \nabla \log f^*(k) \rangle_{\hbar B} + U(k),$$

with $\delta^* f^*(k, y) = f^*(k - y) - f^*(k)$. In particular,

$$L^*(k, \eta^*_t) = -D^*_t \log \eta^*_t(k).$$

Proposition 8.2 *The backward dynamic programming equation with initial boundary condition*

$$\frac{\partial A^*_t}{\partial t}(k) + \min_{f^*} \left[-\mathcal{L}^*_{f^*} A^*_t(k) + L(k, f^*)\right] = 0, \quad A^*_r = -\log \eta^*_r, \tag{8.6}$$

associated to (8.5) *has solution* $A^*_t = -\log \eta^*_t$*, the minimum in* f^* *being attained at* $f^*_t(k) = \eta^*_t(k)$*, and* A^*_t *is solution of the backward Hamilton-Jacobi-Bellman equation*

$$\begin{aligned} \frac{\partial A^*_t}{\partial t}(k) &= U(k) + \frac{1}{2}(\Delta_{\hbar B} A^*_t(k) - \langle \nabla A^*_t(k), \nabla A^*_t(k) \rangle_{\hbar B}) \qquad (8.7) \\ &\quad - \int_{\mathrm{R}^d} (e^{-A^*_t(k-y) + A^*_t(k)} - 1 - 1_{\{|y| \le 1\}} \langle y, \nabla A^*_t(k) \rangle) d\nu_\hbar(y) - \langle c, \nabla A^*_t(k) \rangle. \end{aligned}$$

Proof. The proof, symmetric to the preceding one, is given for completeness. We first show that for $g_t(k) > 0$:

$$\min_{f^*} \left[-\mathcal{L}^*_{f^*_t} \log g_t(k) + L^*(k, f^*_t)\right] = -\frac{1}{g_t(k)} H^\dagger g_t(k), \tag{8.8}$$

and the minimum is attained for $f^*_t = g_t$. Let

$$F^*(k, y) = -\frac{f^*_t(k - y)}{f^*_t(k)} \log \frac{g_t(k + y)}{g_t(k)} + \frac{f^*_t(k - y)}{f^*_t(k)} \log \frac{f^*_t(k + y)}{f^*_t(k)} - \frac{f^*_t(k - y)}{f^*_t(k)} + \frac{g_t(k - y)}{g_t(k)}.$$

We have

$$\begin{aligned} & L^*(k, f^*) + \mathcal{L}^*_{f^*_t} \log g_t(k) - \frac{1}{g_t(k)} H^\dagger g_t(k) \\ &= \int_{\mathrm{R}^d} F^*(k, y) g_t(y) \nu_\hbar(dy) + \frac{1}{2} \left\langle \frac{\nabla g_t}{g_t}(k) - \frac{\nabla f^*_t}{f^*_t}(k), \frac{\nabla g_t}{g_t}(k) - \frac{\nabla f^*_t}{f^*_t}(k) \right\rangle_{\hbar B} \end{aligned}$$

$$\geq \int_{\mathbb{R}^d} F^*(k,y) g_t(y) \nu_\hbar(dy),$$

Proceeding as in the forward case we obtain (8.8) and letting $A_t^* = -\log g_t$, the dynamic programming equation (8.2) becomes

$$\frac{\partial A_t^*}{\partial t} - e^{A_t^*} H^\dagger e^{-A_t^*} = 0,$$

with solution $A_t^* = -\log \eta_t^*$. Finally we have

$$\begin{aligned}
\frac{1}{\eta_t^*(k)} H^\dagger \eta_t^*(k) &= \frac{1}{\eta_t^*(k)} \Bigg(U(k)\eta_t^* + \langle c, \nabla \eta_t^*(k)\rangle - \frac{1}{2}\Delta_{\hbar B}\eta_t^*(k) \\
&\quad - \int_{\mathbb{R}^d} (\eta_t^*(k-y) - \eta_t^*(k) + \langle y, \nabla \eta_t^*(k)\rangle 1_{\{|y|\leq 1\}}) \nu_\hbar(dy) \Bigg) \\
&= U(k) - \langle c, \nabla A_t(k)\rangle + \frac{1}{2}(\Delta_{\hbar B} A_t - \langle \nabla A_t, \nabla A_t\rangle_{\hbar B}) \\
&\quad - \int_{\mathbb{R}^d} (e^{-A_t(k-y)+A_t(k)} - 1 - 1_{\{|y|\leq 1\}}\langle y, \nabla A_t\rangle) d\nu_\hbar,
\end{aligned}$$

which yields (8.3). □

In summary, we have shown here that the diffusion processes with jumps constructed before can also be regarded as minima of some stochastic action functionals associated with the starting H.

9 Equations of motion

In this section we derive the a.s. equations of motion associated to $(z_t)_{t\in[r,v]}$. This is useful in the context of physical applications, especially when $\hbar U(\frac{k}{\hbar}) = \frac{1}{2}\langle k, k\rangle_C$, i.e. when $H = U + V(\nabla)$ is obtained by the action of a Fourier transform with parameter $\hbar$ on the Hamiltonian $\frac{\hbar}{2}\Delta_C + V(\frac{q}{\hbar})$. In this case, $(z_t)_{t\in[r,v]}$ represents the process associated to the system in the momentum representation, and the expectations of the equations of motion is the probabilistic counterpart of the Ehrenfest theorem in quantum mechanics.

The forward and backward derivatives are given in terms of the generators $D_t = \frac{\partial}{\partial t} + \mathcal{L}_{\eta_t}$ and $D_t^* = \frac{\partial}{\partial t} + \mathcal{L}^*_{\eta_t^*}$ as the following limits of conditional expectations:

$$D_t f_t(z_t) = \lim_{\Delta t \downarrow 0} E\left[\frac{f_{t+\Delta t}(z_{t+\Delta t}) - f_t(z_t)}{\Delta t} \mid \mathcal{P}_t\right] = E\left[\frac{d}{dt^+} f_t(z_t) \mid \mathcal{P}_t\right], \quad (9.1)$$

and

$$D_t^* f_t(z_t) = \lim_{\Delta t \downarrow 0} E\left[\frac{f_t(z_t) - f_{t-\Delta t}(z_{t-\Delta t})}{\Delta t} \mid \mathcal{F}_t\right] = E\left[\frac{d}{dt^-} f_t(z_t) \mid \mathcal{F}_t\right], \tag{9.2}$$

for $\Delta t \geq 0$, and $\frac{d}{dt^+} f$, $\frac{d}{dt^-} f$ denote the right hand side and left hand side derivatives corresponding to the formal limits of (9.1) and (9.2) when Planck's constant $\hbar$ is equal to 0 (cf. Sect. 2). In the next proposition we make the assumption:

$$\int_{\mathrm{R}^d} |y| \nu_\hbar(dy) < \infty, \tag{9.3}$$

and let $D_t z_t$ denote $D_t k_{|k=z_t}$.

Proposition 9.1 *The process $(z_t)_{t\in[r,v]}$ which is the minumum of the action functional of Prop. 8.1 solves the almost sure equations of motion*

$$D_t z_t = \frac{1}{\eta_t(z_t)} (-i\nabla_q V)(\nabla_k)\eta_t(z_t), \qquad D_t\left(\frac{\nabla \eta_t}{\eta_t}\right)(z_t) = \nabla U(z_t). \tag{9.4}$$

Proof. We have

$$-i\nabla V(q) = c - i\hbar B q + \int_{\mathrm{R}^d} y(e^{-i\langle q,y\rangle} - 1_{\{|y|\leq 1\}})\nu(dy),$$

hence

$$(-i\nabla V)(\nabla)\eta_t(k) = c\eta_t(k) + \hbar B \nabla \eta_t(k) + \int_{\mathrm{R}^d} y(\eta_t(k+y) - \eta_t(k) 1_{\{|y|\leq 1\}})\nu_\hbar(dy).$$

On the other hand we have

$$D_t k \ = \ \mathcal{L}_{\eta_t} k = c + \int_{\mathrm{R}^d} y\left(\frac{\eta_t(k+y)}{\eta_t(k)} - 1_{\{|y|\leq 1\}}\right)\nu_\hbar(dy) + \hbar B \nabla \log \eta_t(k),$$

which proves the first relation. Concerning the second relation we have:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\nabla \eta_t}{\eta_t}\right)(k) &= \frac{1}{\eta_t}\frac{\partial}{\partial t}\nabla\eta_t(k) - \frac{1}{\eta_t^2}\nabla\eta_t(k)\frac{\partial \eta_t}{\partial t}(k) \\
&= \frac{1}{\eta_t(k)}\nabla\frac{\partial \eta_t}{\partial t}(k) - \frac{1}{\eta_t^2(k)}\nabla\eta_t(k)\frac{\partial \eta_t}{\partial t}(k) \\
&= \frac{1}{\eta_t(k)}\nabla H\eta_t(k) - \frac{1}{\eta_t^2(k)}\nabla\eta_t(k) H\eta_t(k) \\
&= \frac{1}{\eta_t(k)}\nabla\left(U(k)\eta_t + V(\nabla)\eta_t(k)\right) - \frac{1}{\eta_t^2(k)}\left(U(k)\eta_t(k) + V(\nabla)\eta_t(k)\right)\nabla\eta_t(k)
\end{aligned}$$

$$
\begin{aligned}
&= \nabla U(k) + \frac{1}{\eta_t(k)}\nabla(V(\nabla)\eta_t(k)) - \frac{1}{\eta_t^2(k)}\left(V(\nabla)\eta_t(k)\right)\nabla\eta_t(k) \\
&= \nabla U(k) \\
&\quad + \frac{1}{\eta_t(k)}\nabla\left(-\langle c, \nabla\eta_t(k)\rangle - \frac{1}{2}\Delta_{\hbar B}\eta_t(k) - \int_{\mathrm{R}^d}(\eta_t(k+y) - \eta_t(k) - \langle y, \nabla\eta_t(k)\rangle)\nu_\hbar(dy)\right) \\
&\quad - \frac{1}{\eta_t^2(k)}\nabla\eta_t(k)\left(-\langle c, \nabla\eta_t(k)\rangle - \frac{1}{2}\Delta_{\hbar B}\eta_t(k) - \int_{\mathrm{R}^d}(\eta_t(k+y) - \eta_t(k) - \langle y, \nabla\eta_t(k)\rangle)\nu_\hbar(dy)\right) \\
&= \nabla U(k) - \left\langle c, \nabla\left(\frac{\nabla\eta_t(k)}{\eta_t(k)}\right)\right\rangle - \Delta_{\hbar B}\left(\frac{\nabla\eta_t(k)}{\eta_t(k)}\right) \\
&\quad + \int_{\mathrm{R}^d}\left(\frac{\nabla\eta_t(k+y)}{\eta_t(k+y)} - \frac{\nabla\eta_t(k)}{\eta_t(k)} - \left\langle y, \nabla\left(\frac{\nabla\eta_t}{\eta_t}\right)\right\rangle\right)\frac{\eta_t(k+y)}{\eta_t(k)}\nu_\hbar(dy) \\
&\quad + \int_{\mathrm{R}^d}\frac{\eta_t(k+y) - \eta_t(k)}{\eta_t(k)}\left\langle y, \nabla\left(\frac{\nabla\eta_t(k)}{\eta_t(k)}\right)\right\rangle\nu_\hbar(dy) - \left\langle\frac{\nabla\eta_t(k)}{\eta_t(k)}, \nabla\left(\frac{\nabla\eta_t(k)}{\eta_t(k)}\right)\right\rangle_{\hbar B} \\
&= \nabla U(k) - \mathcal{L}_{\eta_t(k)}\left(\frac{\nabla\eta_t(k)}{\eta_t(k)}\right),
\end{aligned}
$$

where we used the relation

$$
\begin{aligned}
\Delta_{\hbar B}\left(\frac{\nabla\eta_t(k)}{\eta_t(k)}\right) &= \frac{1}{\eta_t(k)}\nabla\Delta_{\hbar B}\eta_t(k) - \frac{\nabla\eta_t(k)}{\eta_t^2(k)}\Delta_{\hbar B}\eta_t(k) \\
&\quad - 2\frac{1}{\eta_t^2(k)}\langle\nabla^2\eta_t(k), \nabla\eta_t(k)\rangle_{\hbar B} - 2\frac{1}{\eta_t^3(k)}\langle\nabla\eta_t(k), \nabla\eta_t(k)\otimes\nabla\eta_t(k)\rangle_{\hbar B} \\
&= \frac{1}{\eta_t(k)}\nabla\Delta_{\hbar B}\eta_t(k) - \frac{\nabla\eta_t(k)}{\eta_t^2(k)}\Delta_{\hbar B}\eta_t(k) - 2\left\langle\frac{\nabla\eta_t(k)}{\eta_t(k)}, \nabla\left(\frac{\nabla\eta_t(k)}{\eta_t(k)}\right)\right\rangle_{\hbar B}.
\end{aligned}
$$

□

In the backward case, similar calculations yield

$$
D_t^* z_t = \frac{1}{\eta_t^*(z_t)}(i\nabla_q\bar{V})(\nabla_k)\eta_t^*(z_t), \qquad D_t^*\left(\frac{\nabla\eta_t^*}{\eta_t^*}\right)(z_t) = -\nabla U(z_t).
$$

The (forward) analog of the Newton equation in momentum representation becomes

$$
D_t D_t\left(\frac{\nabla\eta_t}{\eta_t}\right)(z_t) = \frac{1}{\eta_t(z_t)}(-i\nabla V)(\nabla)\eta_t(z_t).
$$

The relation with quantum dynamics is clearer when expressed in terms of expectations. For this purpose it is sufficient to observe that

Corollary 9.2 *Under expectations, the a.s. equations of motion are:*

$$
\frac{d}{dt}E[f_t(z_t)]. = E[D_t f_t(z_t)] = E[D_t^* f_t(z_t)], \quad f \in \mathcal{S}(\mathbb{R}^d).
$$

Proof. This follows from the Itô formula, written as

$$\begin{aligned} df_t(z_t) &= D_t f_t(z_t)dt + \langle \nabla f_t(z_t), dW_t \rangle \\ &\quad + \int_{\mathrm{R}^d} (f_t(z_{t^-} + y) - f_t(z_{t^-})) \left(\mu(dy, dt) - \frac{\eta_t(z_{t^-} + y)}{\eta_t(z_{t^-})} 1_{\{|y| \le 1\}} \nu_\hbar(dy) dt \right), \end{aligned}$$

and

$$\begin{aligned} d_* f(z_t) &= D_t^* f_t(z_t)dt + \langle \nabla f(z_t), d_* W_t^* \rangle \\ &\quad + \int_{\mathrm{R}^d} (f(z_{t^+}) - f(z_{t^+} - y)) \left(\mu_*(dy, dt) - \frac{\eta_t^*(z_{t^+} - y)}{\eta_t^*(z_{t^+})} 1_{\{|y| \le 1\}} \nu_\hbar(dy) dt \right), \end{aligned}$$

□

If $\hbar V(\frac{q}{\hbar^2}) = \frac{1}{2}q^2$, $(B = \mathrm{Id})$ we obtain $DDz_t = \nabla U(z_t)$ and $D_t z_t = \hbar \nabla \log \eta_t$. This is the purely diffusive case, already known.

A number of the properties of these processes remains to be investigated. Many of those known to hold for pure diffusions should survive for the much richer class of diffusions with jumps considered here. In particular, a systematic study of their symmetries, in term of a Noether Theorem, on the model of [27], [28], is possible and should provide further informations on the general structure of the construction. A more geometrical approach to these symmetries [23] can probably be extended as well to this class. Moreover, the almost sure equations of motion could be more elegantly deduced from an appropriate generalization of the stochastic calculus of variations used in [8].

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