Bounds on option prices in point process diffusion models

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Abstract

We obtain lower and upper bounds on option prices in one-dimensional jump-diffusion markets with point process components. Our proofs rely in general on the classical Kolmogorov equation argument and on the propagation of convexity property for Markov semigroups, but the bounds on intensities and jump sizes formulated in our hypotheses are different from the ones already found in the literature [1], [2].

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1 Introduction

Bounds on Black-Scholes prices have first been obtained in [6] in the continuous diffusion case and extended to jump-diffusion processes in several papers [1], [2], [5], assuming the propagation of convexity of diffusion semigroups.

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In this paper we obtain lower and upper bounds for option prices in jump-diffusion models with point process components, which complete the results of [2] by providing new conditions for the ordering of option prices. Namely we show that, in addition to the class of directionally convex functions considered in [2], the class of non-increasing functions can be used as test functions in the generator of the jump part of a jump-diffusion semigroup in order to derive bounds on options prices. As an application we study several particular cases (point processes, point processes with bounded jumps, Poisson random measures) in which our conditions can be formulated explicitly. Our proofs are carried out in the one-dimensional case.

We proceed as follows. In Section 2 we recall the classical Kolmogorov equation in our point process diffusion framework. In Section 3 we present our main result (Theorem 3.2) which states some general conditions for the supermartingale property of option prices to hold. Finally, in Sections 4, 5 and 6 we formulate our results in the case of point processes and Poisson random measures.

2 Backward Kolmogorov equation

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with an increasing filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Consider two assets whose respective prices are modelled via jump-diffusion processes $(S_*(t))_{t\in\mathbb{R}_+}$, $(S^*(t))_{t\in\mathbb{R}_+}$ solutions of the stochastic differential equations

$$\frac{dS^*(t)}{S^*(t^-)} = r^*(t)dt + \sigma^*(t, S^*(t))dW_t + \int_{-\infty}^{\infty} y(\mu^*(dt, dy) - \nu^*(t, S^*(t^-), dy)dt),$$

and

$$\frac{dS_*(t)}{S_*(t^-)} = r_*(t)dt + \sigma_*(t, S_*(t))dW_t + \int_{-\infty}^{\infty} y(\mu_*(dt, dy) - \nu_*(t, S_*(t^-), dy)dt),$$

where $r^*(t)$, $r_*(t)$ are deterministic interest rate functions and $\sigma^*(t, x)$, $\sigma_*(t, x)$ are Lipschitz volatility functions. Here, $(W_t)_{t\in\mathbb{R}_+}$ is a $\mathcal{F}_t$-Brownian motion and $\mu^*(dt, dy)$, $\mu_*(dt, dy)$ are jump measures with respective $\mathcal{F}_t$-compensators $\nu^*(t, S^*(t^-), dy)$ and $\nu_*(t, S_*(t^-), dy)$, see Theorem 13.58, Theorem 14.80 of [7], p. 438 and p. 481, and the results on martingale problems for discontinuous processes of [9].

Let $\mathcal{L}^*$ and $\mathcal{L}_*$ denote the respective generators of $(S^*(t))_{t\in\mathbb{R}_+}$ and of $(S_*(t))_{t\in\mathbb{R}_+}$, i.e.

$$\begin{aligned}\mathcal{L}^* f(t,x) &= r^*(t)x\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}x^2|\sigma^*(t,x)|^2\frac{\partial^2 f}{\partial x^2}(t,x) \\ &\quad + \int_{-\infty}^{\infty}\left(f(t,x(1+y)) - f(t,x) - xy\frac{\partial f}{\partial x}(t,x)\right)\nu^*(t,x,dy) \quad (2.1)\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_* f(t,x) &= r_*(t)x\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}x^2|\sigma_*(t,x)|^2\frac{\partial^2 f}{\partial x^2}(t,x) \\ &\quad + \int_{-\infty}^{\infty}\left(f(t,x(1+y)) - f(t,x) - xy\frac{\partial f}{\partial x}(t,x)\right)\nu_*(t,x,dy). \quad (2.2)\end{aligned}$$

The following lemma is a formulation of the classical Kolmogorov equation. Here the function ϕ plays the role of a payoff function.

Lemma 2.1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function and assume that there exists a function v^* in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ such that*

$$v^*(t,S^*(t)) = \mathbb{E}\left[\phi(S^*(T))\middle| S^*(t)\right], \qquad 0 \le t \le T. \quad (2.3)$$

Then v^* satisfies the partial differential equation (PDE)

$$\begin{cases} \dfrac{\partial v^*}{\partial t}(t,x) + \mathcal{L}^* v^*(t,x) = 0, \\ \\ v^*(T,x) = \phi(x). \end{cases} \quad (2.4)$$

Proof. Itô's formula applied to $v^*(t,S^*(t))$ reads

$$\begin{aligned} v^*(t,S^*(t)) &= v^*(0,S^*(0)) + \int_0^t \frac{\partial v^*}{\partial s}(s,S^*(s))ds \\ &\quad + \int_0^t r^*(s)S^*(s)\frac{\partial v^*}{\partial x}(s,S^*(s))ds + \int_0^t \sigma^*(s,S^*(s))\frac{\partial v^*}{\partial x}(s,S^*(s))dW_s \\ &\quad + \frac{1}{2}\int_0^t |S^*(s)|^2|\sigma^*(s,S^*(s))|^2\frac{\partial^2 v^*}{\partial x^2}(s,S^*(s))ds \\ &\quad + \int_0^t\int_{-\infty}^{\infty}\left(v^*(s,S^*(s)(1+y)) - v^*(s,S^*(s))\right)\mu^*(ds,dy) \\ &\quad - \int_0^t\int_{-\infty}^{\infty} yS^*(s)\frac{\partial v^*}{\partial x}(s,S^*(s))\nu^*(ds,dy) \\ &= v^*(0,S^*(0)) + \int_0^t \frac{\partial v^*}{\partial s}(s,S^*(s))ds \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t r^*(s)S^*(s)\frac{\partial v^*}{\partial x}(s,S^*(s))ds+\int_0^t \sigma^*(s,S^*(s))\frac{\partial v^*}{\partial x}(s,S^*(s))dW_s\\
&+\frac{1}{2}\int_0^t |S^*(s)|^2|\sigma^*(s,S^*(s))|^2\frac{\partial^2 v^*}{\partial x^2}(s,S^*(s))ds\\
&+\int_0^t\int_{-\infty}^{\infty}(v^*(s,S^*(s)(1+y))-v^*(s,S^*(s)))(\mu^*(ds,dy)-\nu^*(s,S^*(s),dy)ds)\\
&+\int_0^t\int_{-\infty}^{\infty}\left(v^*(s,S^*(s)(1+y))-v^*(s,S^*(s))-yS^*(s)\frac{\partial v^*}{\partial x}(s,S^*(s))\right) \qquad (2.5)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\nu^*(s,S^*(s),dy)ds.
\end{aligned}
$$

By construction in (2.3) the process $v^*(t,S^*(t))$ is a martingale, hence from e.g. Cor. 1, p. 64 of [11], the finite variation terms in (2.5) vanishes, i.e.

$$
\begin{aligned}
0&=\frac{\partial v^*}{\partial s}(s,S^*(s))+r^*(s)S^*(s)\frac{\partial v^*}{\partial x}(s,S^*(s))+\frac{1}{2}|S^*(s)|^2|\sigma^*(s,S^*(s))|^2\frac{\partial^2 v^*}{\partial x^2}(s,S^*(s))\\
&+\int_{-\infty}^{\infty}\left(v^*(s,S^*(s)(1+y))-v^*(s,S^*(s))-yS^*(s)\frac{\partial v^*}{\partial x}(s,S^*(s))\right)\nu^*(s,S^*(s),dy),
\end{aligned}
$$

which yields (2.6). $\square$

Similarly, any function $v_*\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ satisfying

$$
v_*(t,S_*(t))=\mathbb{E}\left[\phi(S_*(T))\Big|S_*(t)\right],\qquad 0\le t\le T, \tag{2.6}
$$

will also satisfy the PDE

$$
\begin{cases}
\dfrac{\partial v_*}{\partial t}(t,x)+\mathcal{L}_*v_*(t,x)=0,\\[2ex]
v_*(T,x)=\phi(x).
\end{cases}
$$

The smoothness conditions imposed on v_* and v^* can be satisfied under adequate regularity conditions on the semi-groups of $(S^*(t))_{t\in\mathbb{R}_+}$ and of $(S_*(t))_{t\in\mathbb{R}_+}$.

In the sequel, some of our results will use the following propagation of convexity (PC) assumption on the Markov semigroups of $(S_*(t))_{t\in\mathbb{R}_+}$ and of $(S^*(t))_{t\in\mathbb{R}_+}$.

Assumption (PC). The functions $x\mapsto v_*(t,x)$ and $x\mapsto v^*(t,x)$ defined in (2.3) and (2.6) are convex on $\mathbb{R}$ for all $t\in[0,T]$ when the payoff function ϕ is convex.

Note that in the one-dimensional diffusion case without jumps, propagation of convexity is essentially always satisfied under reasonable smoothness assumptions on the diffusion coefficient, see e.g. [6] and [10]. In one-dimensional models with jumps it suffices that the jump size be a concave and positive or convex and negative function of the state of the process, but no condition on the diffusion coefficient is required, see Theorem 6.1 of [5]. See also [4], Theorem 5.1, on the necessity of this condition.

For the d-dimensional case with or without jumps, see [2], [3] for a condition on the diffusion matrix and more precisely the LCP condition of [5].

3 Supermartingale property

Consider $(X_t)_{t\in\mathbb{R}_+}$ an $\mathcal{F}_t$-martingale with right-continuous paths with left limits. Denote by $(X_t^c)_{t\in\mathbb{R}_+}$ the continuous part of $(X_t)_{t\in\mathbb{R}_+}$, and by

$$\Delta X_t = X_t - X_{t^-}$$

its jumps. The process $(X_t)_{t\in\mathbb{R}_+}$ has jump measure

$$\mu(dt, dy) = \sum_{s>0} 1_{\{\Delta X_s \neq 0\}} \delta_{(s,\Delta X_s)}(dt, dy),$$

where $\delta_{(s,x)}$ denotes the Dirac measure at $(s, x) \in \mathbb{R}_+ \times \mathbb{R}$. Denote by $\nu(dt, dy)$ the $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-dual predictable projection of $\mu(dt, dy)$ and by $([X, X]_t)_{t\in\mathbb{R}_+}$, resp. $(\langle X^c, X^c\rangle_t)_{t\in\mathbb{R}_+}$, the corresponding optional, resp. predictable quadratic variations. The pair

$$(\nu(dt, dy), \langle X^c, X^c\rangle)$$

is called the local characteristics of $(X_t)_{t\in\mathbb{R}_+}$, cf. [8]. We will assume that $\nu(dt, dy)$ has the form

$$\nu(dt, dy) = \nu_t(dy)dt.$$

Consider $(S_t)_{t\in\mathbb{R}_+}$ a jump-diffusion price process of the form

$$\frac{dS_t}{S_{t^-}} = r_t dt + \sigma_t dW_t + \int_{-\infty}^{\infty} y(\mu(dt, dy) - \nu_t(dy)dt), \tag{3.1}$$

with (logarithmic) jump measure $\mu(dt, dy)$ and compensator $\nu_t(dx)dt$, where $(r_t)_{t\in\mathbb{R}_+}$ and $(\sigma_t)_{t\in\mathbb{R}_+}$ are adapted processes. Define the (random) pseudo-generator $\mathcal{L}$ as

$$\begin{aligned}\mathcal{L}f(t,x) &= r_t x\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}x^2\sigma_t^2\frac{\partial^2 f}{\partial x^2}(t,x)\\ &\quad + \int_{-\infty}^{\infty}\left(f(t,x(1+y)) - f(t,x) - yx\frac{\partial f}{\partial x}(t,x)\right)\nu_t(dy).\end{aligned} \tag{3.2}$$

Note that here, S, S_* and S^* all have the same drift coefficient $(r_t)_{t\in\mathbb{R}_+}$. Recall that Itô's formula, applied to $f \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ and to the jump-diffusion S_t, reads

$$\begin{aligned}f(t,S_t) &= f(0,S_0) + \int_0^t r_s S_s\frac{\partial f}{\partial x}(s,S_s)ds + \int_0^t \frac{\partial f}{\partial s}(s,S_s)ds\\ &\quad + \int_0^t \sigma_s S_s\frac{\partial f}{\partial x}(s,S_s)dW_s + \frac{1}{2}\int_0^t S_s^2\sigma_s^2\frac{\partial^2 f}{\partial x^2}(s,S_s)ds\\ &\quad - \int_0^t\int_{-\infty}^{\infty} yS_s\frac{\partial f}{\partial x}(s,S_s)\mu(ds,dy)\\ &\quad + \int_0^t\int_{-\infty}^{\infty}\left(f(s,S_s(1+y)) - f(s,S_s)\right)\mu(ds,dy)\\ &= f(0,S_0) + \int_0^t \sigma_s S_s\frac{\partial f}{\partial x}(s,S_s)dW_s\\ &\quad + \int_0^t \frac{\partial f}{\partial s}(s,S_s)ds + \int_0^t \mathcal{L}f(s,S_s)ds\\ &\quad + \int_0^t\int_{-\infty}^{\infty}\left(f(s,S_s(1+y)) - f(s,S_s)\right)\left(\mu(ds,dy) - \nu_s(dy)\right).\end{aligned}$$

In the following results (Lemma 3.1, Theorem 3.2 and Propositions 4.2, 5.1 and 6.1), assuming only a lower (resp. upper) type bound in the hypothesis conducts to the corresponding lower (resp. upper) bound in (3.4) below.

Lemma 3.1. *The processes $v_*(t,S_t)$ and $v^*(t,S_t)$ are respectively a submartingale and a supermartingale provided*

$$\mathcal{L}_* v_*(t,S_t) \le \mathcal{L}v_*(t,S_t), \qquad \textit{resp.} \qquad \mathcal{L}v^*(t,S_t) \le \mathcal{L}^* v^*(t,S_t), \qquad dtdP\text{-}a.e., \tag{3.3}$$

and in this case we have

$$v_*(t,S_t) \le \mathbb{E}\left[\phi(S_T)\middle|\mathcal{F}_t\right] \le v^*(t,S_t), \qquad 0 \le t \le T, \tag{3.4}$$

and in particular

$$E[\phi(S_*(T)) \mid S_*(0)=x] \le \mathbb{E}\left[\phi(S_T) \mid S_0 = x\right] \le E[\phi(S^*(T)) \mid S^*(0)=x], \qquad x>0.$$

Proof. Using Lemma 2.1 we have

$$\begin{aligned} v^*(t,S_t) &= v^*(0,S_0) + \int_0^t \sigma_s S_s \frac{\partial v^*}{\partial x}(s,S_s)dW_s \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \left(v^*(s,S_s(1+y)) - v^*(s,S_s)\right)\left(\mu(ds,dy) - \nu_s(dy)\right) \\ &\quad + \int_0^t \mathcal{L}v^*(s,S_s)ds - \int_0^t \mathcal{L}^* v^*(s,S_s)ds \end{aligned}$$

and it remains to use the fact that the sum of a martingale and a non-increasing adapted process is a supermartingale. Similarly we have

$$\begin{aligned} v_*(t,S_t) &= v_*(0,S_0) + \int_0^t \sigma_s S_s \frac{\partial v_*}{\partial x}(s,S_s)dW_s \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \left(v_*(s,S_s(1+y)) - v_*(s,S_s)\right)\left(\mu(ds,dy) - \nu_s(dy)\right) \\ &\quad + \int_0^t (\mathcal{L}v_*(s,S_s) - \mathcal{L}_* v_*(s,S_s))ds, \end{aligned}$$

which is a submartingale as the sum of a martingale and a non-decreasing adapted process. Finally the submartingale and supermartingale properties imply

$$v_*(t,S_t) \le \mathbb{E}[v_*(T,S_T) \mid \mathcal{F}_t] = \mathbb{E}[\phi(S_T) \mid \mathcal{F}_t] = \mathbb{E}[v^*(T,S_T) \mid \mathcal{F}_t] \le v^*(t,S_t),$$

$0 \le t \le T$. □

We now present some sufficient conditions for the condition (3.3) to hold and to ensure the inequality (3.4).

Theorem 3.2 below provides an additional sufficient condition for the ordering of option prices as compared to [2], [3]. Precisely, in Theorem 2.3 of [2], resp. in Theorem 2.2 of [3], f is taken in a class of directionally convex functions, while in Theorem 3.2 below we consider f in the class of non-decreasing functions.

Theorem 3.2. *Assume that $r^*(t) = r_t = r_*(t)$, $t \in \mathbb{R}_+$, that the (PC) property holds for S_* and S^*, and either:*

i-a) $|\sigma_*(t,S_t)| \le |\sigma_t| \le |\sigma^*(t,S_t)|$, *and*

i-b) $\nu_*(t,S_t,dy)$, $\nu^*(t,S_t,dy)$, $\nu_t(dy)$ *are supported by* $\mathbb{R}_+$, $dPdt$*-a.e., and*

i-c) for all non-negative and non-decreasing functions f *we have:*

$$\int_0^\infty yf(y)\nu_*(t,S_t,dy) \le \int_0^\infty yf(y)\nu_t(dy) \le \int_0^\infty yf(y)\nu^*(t,S_t,dy), \qquad t \in \mathbb{R}_+,$$

or:

ii-a) $\nu_*(t,S_t,dy)$, $\nu^*(t,S_t,dy)$, $\nu_t(dy)$ *are supported by* $(-1,\infty)$, $dPdt$*-a.e., and*

ii-b) the functions $\frac{\partial v_*}{\partial x}(t,\cdot)$ *and* $\frac{\partial v^*}{\partial x}(t,\cdot)$ *are convex and we have:*

$$\begin{aligned} f(0)|\sigma_*(t,S_t)|^2 + \int_{-\infty}^\infty y^2 f(y)\nu_*(t,S_t,dy) &\le f(0)|\sigma_t|^2 + \int_{-\infty}^\infty y^2 f(y)\nu_t(dy) \\ &\le f(0)|\sigma^*(t,S_t)|^2 + \int_{-\infty}^\infty y^2 f(y)\nu^*(t,S_t,dy), \qquad t \in \mathbb{R}_+, \end{aligned}$$

for all non-negative and non-decreasing functions f.

Then

$$v_*(t,S_t) \le \mathbb{E}\left[\phi(S_T)|\mathcal{F}_t\right] \le v^*(t,S_t), \qquad 0 \le t \le T,$$

holds.

Proof. We only deal with v^*, the case of v_* being treated by similar arguments.
i) We have

$$\begin{aligned} &\mathcal{L}^*v^*(t,S_t) - \mathcal{L}v^*(t,S_t) \\ &= \frac{1}{2}S_t^2\frac{\partial^2 v^*}{\partial x^2}(t,S_t)(|\sigma_t|^2 - |\sigma_t^*|^2) \\ &\quad + \int_{-\infty}^\infty \big(v^*(s,S_s(1+y)) - v^*(s,S_s) - yS_s\frac{\partial v^*}{\partial x}(s,S_s)\big)\nu^*(s,S_s,dy) \\ &\quad - \int_{-\infty}^\infty \big(v^*(s,S_s(1+y)) - v^*(s,S_s) - yS_s\frac{\partial v^*}{\partial x}(s,S_s)\big)\nu_s(dy) \\ &= \frac{1}{2}S_t^2\frac{\partial^2 v^*}{\partial x^2}(t,S_t)(|\sigma_t|^2 - |\sigma_t^*|^2) + S_s\int_{-\infty}^\infty y\varphi_s(S_s,y)\big(\nu^*(s,S_s,dy) - \nu_s(dy)\big), \end{aligned}$$

where

$$\varphi_t(x,y) = \frac{v^*(t,x(1+y)) - v^*(t,x) - xy\frac{\partial v^*}{\partial x}(t,x)}{xy}, \qquad x,y > 0.$$

Since $v^*(t,\cdot)$ is convex, the function $y \mapsto \varphi_t(x,y)$ is non-negative and non-decreasing in $y \in \mathbb{R}$ for all $x \in \mathbb{R}$ and $t \in \mathbb{R}_+$, and $\dfrac{\partial^2 v^*}{\partial x^2}(t,x) \geq 0$ for all $x \in \mathbb{R}$, hence $\mathcal{L}v^*(t,S_t) \leq \mathcal{L}^*v^*(t,S_t)$, $t \in \mathbb{R}_+$. The conclusion follows from Lemma 3.1.

ii) Using the following version of Taylor's formula

$$\phi(y+x) = \phi(y) + x\phi'(y) + |x|^2 \int_0^1 (1-\tau)\phi''(y+\tau x)d\tau, \qquad x,y \in \mathbb{R},$$

we have:

$$\begin{aligned}
&\mathcal{L}v^*(t,S_t) - \mathcal{L}^*v^*(t,S_t) \\
&= \frac{1}{2}S_t^2\sigma_t^2\frac{\partial^2 v^*}{\partial x^2}(t,S_t) + S_t^2\int_{-\infty}^{\infty}|y|^2\int_0^1(1-\tau)\frac{\partial^2 v^*}{\partial x^2}(t,S_t(1+\tau y))\nu_t(dy)d\tau \\
&-\frac{1}{2}S_t^2|\sigma^*(t,S_t)|^2\frac{\partial^2 v^*}{\partial x^2}(t,S_t) - S_t^2\int_{-\infty}^{\infty}|y|^2\int_0^1(1-\tau)\frac{\partial^2 v^*}{\partial x^2}(t,S_t(1+\tau y))\nu^*(t,S_t,dy)d\tau \\
&= \frac{1}{2}S_t^2(\sigma_t^2 - |\sigma^*(t,S_t)|^2)\frac{\partial^2 v^*}{\partial x^2}(t,S_t) \\
&\quad + S_t^2\int_0^1(1-\tau)\int_{-\infty}^{\infty}|y|^2\frac{\partial^2 v^*}{\partial x^2}(t,S_t(1+\tau y))\big(\nu_t(dy) - \nu^*(t,S_t,dy)\big)d\tau \\
&= S_t^2\int_0^1(1-\tau)(\sigma_t^2 - |\sigma^*(t,S_t)|^2)\frac{\partial^2 v^*}{\partial x^2}(t,S_t)d\tau \\
&\quad + S_t^2\int_0^1(1-\tau)\int_{-\infty}^{\infty}|y|^2\frac{\partial^2 v^*}{\partial x^2}(t,S_t(1+\tau y))\big(\nu_t(dy) - \nu^*(t,S_t,dy)\big)d\tau.
\end{aligned}$$

The convexity assumptions on $v^*(t,\cdot)$ and $\dfrac{\partial v^*}{\partial x}(t,\cdot)$ respectively imply that $\dfrac{\partial^2 v^*}{\partial x^2}(t,\cdot)$ is non-negative and non-decreasing, hence by assumption (*ii*) we have

$$\mathcal{L}v^*(t,S_t) - \mathcal{L}^*v^*(t,S_t) \leq 0, \qquad t \in \mathbb{R}_+,$$

and the conclusion follows from Lemma 3.1.

$\square$

Note that Condition (*i*) above requires two distinct bounds on σ_t and ν_t, whereas Condition (*ii*) is formulated using a single condition on both σ_t and on ν_t and requires the convexity of $\dfrac{\partial v_*}{\partial x}(t,\cdot)$, and $\dfrac{\partial v^*}{\partial x}(t,\cdot)$.

In the sequel we assume as in Theorem 3.2 that $r^*(t) = r_t = r_*(t)$, $t \in \mathbb{R}_+$. However the conclusions of Theorem 3.2 still hold when r_t is random and $r^*(t) \leq r_t \leq r_*(t)$, a.s., $t \in \mathbb{R}_+$, provided in addition that $v_*(t,\cdot)$ and $v^*(t,\cdot)$ are non-decreasing functions.

The above two comments also apply to Propositions 4.2, 5.1 and 6.1 below.

4 Bounded jumps

In this section and the following, we study some particular cases of Theorem 3.2. The proofs rely on the following comparison lemma.

Lemma 4.1. *Let m_1, m_2 be two positive measures on $\mathbb{R}$ such that*

$$m_1([x,\infty)) \leq m_2([x,\infty)) < \infty, \tag{4.1}$$

for all $x \in \mathbb{R}$. Then we have

$$\int_{-\infty}^{\infty} f(x) m_1(dx) \leq \int_{-\infty}^{\infty} f(x) m_2(dx) \tag{4.2}$$

for all non-decreasing and non-negative measurable functions f on $\mathbb{R}$.

Proof. Clearly, the implication holds for any linear combination of the form

$$\sum_{i=1}^{n} \alpha_i \mathbf{1}_{[x_i,\infty)}, \qquad x_1,\ldots,x_n \in \mathbb{R}, \quad \alpha_1,\ldots,\alpha_n \in \mathbb{R}_+.$$

The property is extended to the general case by approximating f by a sequence of such step functions. $\square$

From now on we assume that the respective compensators $\nu_*(t,x,dy)$ of S_* and $\nu^*(t,x,dy)$ of S^* have the form

$$\nu_*(t,x,dy) = \lambda_*(t,x)\delta_{k_*}(dy) \qquad \text{and} \qquad \nu^*(t,x,dy) = \lambda^*(t,x)\delta_{k^*}(dy), \tag{4.3}$$

where $-\infty < k_* \leq k^* \leq \infty$ and $(\lambda_*(t,x))_{t\in\mathbb{R}_+}$, $(\lambda^*(t,x))_{t\in\mathbb{R}_+}$ are non-negative functions, with the conventions $\delta_{-\infty} = 0$ and $\delta_{+\infty} = 0$.

Note that in Proposition 4.2 below, part (*ii*) does not apply to European calls with payoff functions of the form $\phi(x) = (x-K)^+$ due to the additional convexity assumption made on $\dfrac{\partial v_*}{\partial x}(t,\cdot)$ and $\dfrac{\partial v^*}{\partial x}(t,\cdot)$.

Proposition 4.2. *Relation* (3.4) *holds for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$, provided the (PC) property holds for S_* and S^*, $r^*(t) = r_t = r_*(t)$, $t \in \mathbb{R}_+$, and one of the following conditions is satisfied for some $-1 < k_* \le k^*$:*

i) we have $0 \le k_* \le \Delta X_t \le k^*$ and

$$|\sigma_*(t,S_t)| \le |\sigma_t| \le |\sigma^*(t,S_t)|, \qquad k_*\lambda_*(t,S_t) \le \int_{k_*}^{k^*} y\nu_t(dy) \le k^*\lambda^*(t,S_t),$$

$dPdt$-a.e.

ii) the functions $\dfrac{\partial v_*}{\partial x}(t,\cdot)$ and $\dfrac{\partial v^*}{\partial x}(t,\cdot)$ are convex and

$$|\sigma_*(t,S_t)|^2 + |k_*|^2\lambda_*(t,S_t) \le |\sigma_t|^2 + \int_{k_*}^{k^*} |y|^2\nu_t(dy) \le |\sigma^*(t,S_t)|^2 + |k^*|^2\lambda^*(t,S_t), \tag{4.4}$$

$dPdt$-a.e. and either:

ii-a) $k_* \le \Delta X_t \le k^* \le 0$ and $|\sigma_*(t,S_t)|^2 \le |\sigma_t|^2 \le |\sigma^*(t,S_t)|^2$, $dPdt$-a.e., or:

ii-b) $k_* \le \Delta X_t \le 0 \le k^*$ and $|\sigma_*(t,S_t)|^2 \le |\sigma_t|^2$, $dPdt$-a.e., or:

ii-c) $k_* \le 0 \le k^*$, $k_* \le \Delta X_t \le k^*$, and

$$|\sigma_*(t,S_t)|^2 \le |\sigma_t|^2 + \int_0^{k^*} |y|^2\nu_t(dy), \quad \int_0^{k^*} |y|^2\nu_t(dy) \le |k^*|^2\lambda^*(t,S_t),$$

$dPdt$-a.e., or:

ii-d) $k_* \le 0 \le \Delta X_t \le k^*$ and $\int_0^{k^*} |y|^2\nu_t(dy) \le |k^*|^2\lambda^*(t,S_t)$, $dPdt$-a.e., or:

ii-e) $0 \le k_* \le \Delta X_t \le k^*$ and $k_*^2\lambda_*(t,S_t) \le \int_{k_*}^{k^*} |y|^2\nu_t(dy) \le |k^*|^2\lambda^*(t,S_t)$, $dPdt$-a.e.

Proof. Note that the condition $0 \le \Delta X_t \le k^*$, resp. $k_* \le \Delta X_t \le k^*$, is equivalent to $\nu_t([0,k^*]^c) = 0$, resp. $\nu_t([k_*,k^*]^c) = 0$. In case (*i*) we apply the comparison Lemma 4.1 to the measures

$$\bar\nu_t(dy) = y\nu_t(dy), \quad \bar\nu_{*,t}(dy) = y\nu_{*,t}(dy), \quad \bar\nu_t^*(dy) = y\nu_t^*(dy),$$

after checking that they satisfy Condition (4.1), and we conclude the proof from Theorem 3.2. In case (ii) we proceed similarly with the measures

$$\begin{aligned}
\tilde{\nu}_t(dy) &= |y|^2\nu_t(dy) + |\sigma_t|^2\delta_0(dy), \\
\tilde{\nu}_{*,t}(dy) &= |y|^2\nu_{*,t}(dy) + |\sigma_*(t,S_t)|^2\delta_0(dy), \\
\tilde{\nu}^*_t(dy) &= |y|^2\nu^*_t(dy) + |\sigma^*(t,S_t)|^2\delta_0(dy),
\end{aligned}$$

noting that here the expression of the condition depends on the position of 0 with respect to k_* and to k^*. $\square$

Again, case (ii) is formulated as a unique assumption on σ_t and ν_t, with additional conditions in $a)$-$e)$. Note further that in ii-$b)$ (resp. ii-$d)$), only (4.4) is required for the upper (resp. lower) bound to hold in (4.3). A similar remark applies also to Propositions 5.1 and 6.1.

On the other hand, in ii-$c)$ above, no hypothesis is made on the sign of ΔX_t. Moreover, in both (i) and ii-$e)$ it is assumed that $0 \le k_* \le \Delta X_t \le k^*$, $dPdt$-a.e., strong conditions on σ_t and ΔX_t in (i), while the convexity of the derivatives is required in ii-$e)$.

5 Point processes

Consider a Brownian motion $(W_t)_{t\in\mathbb{R}_+}$ and a point process $(Z_t)_{t\in\mathbb{R}_+}$ with intensity $(\lambda_t)_{t\in\mathbb{R}_+}$, generating a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. We assume now that $(S_t)_{t\in\mathbb{R}_+}$ in (3.1) has the following form

$$S_t = S_0 + \int_0^t r_u S_u du + \int_0^t \sigma_u S_u dW_u + \int_0^t J_{u^-} S_{u^-}(dZ_u - \lambda_u du), \quad t \in \mathbb{R}_+, \tag{5.1}$$

where $(\sigma_t)_{t\in\mathbb{R}_+}$, $(J_t)_{t\in\mathbb{R}_+}$ are predictable with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. That is, the characteristic measure of $(X_t)_{t\in\mathbb{R}_+}$ in Section 3 is taken to be

$$\nu_t(dx) = \lambda_t \delta_{J_t}(dx). \tag{5.2}$$

In addition we assume that the respective compensators $\nu_*(t,x,dy)$ of S_* and $\nu^*(t,x,dy)$ of S^* have the form

$$\nu_*(t,x,dy) = \lambda_*(t,x)\delta_{J_*(t,x)}(dy) \qquad \text{and} \qquad \nu^*(t,x,dy) = \lambda^*(t,x)\delta_{J^*(t,x)}(dy) \tag{5.3}$$

where $-\infty < J_*(t,x) \le J^*(t,x) \le \infty$, and $\lambda_*(t,x)$, $\lambda^*(t,x)$ are non-negative functions, again with the convention $\delta_\infty = 0$. Applying Theorem 3.2 we derive the following corollary:

Proposition 5.1. *Assume that $(S_t)_{t\in\mathbb{R}_+}$ has the jump characteristics (5.2) and $r^*(t) = r_t = r_*(t)$, $t \in \mathbb{R}_+$. Then the inequality*

$$v_*(t,S_t) \le \mathbb{E}\left[\phi(S_T)|\mathcal{F}_t\right] \le v^*(t,S_t), \qquad 0 \le t \le T, \tag{5.4}$$

holds for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$ provided the (PC) property holds for S_*, S^*, and one of the following conditions is satisfied:

i) we have $0 \le J_*(t,S_t) \le J_t \le J^*(t,S_t)$ and

$$|\sigma_*(t,S_t)| \le |\sigma_t| \le |\sigma^*(t,S_t)|, \qquad \lambda_*(t,S_t)J_*(t,S_t) \le \lambda_t J_t \le \lambda^*(t,S_t)J^*(t,S_t),$$

$dPdt$-a.e.

ii) the functions $\dfrac{\partial v_*}{\partial x}(t,\cdot)$ and $\dfrac{\partial v^*}{\partial x}(t,\cdot)$ are convex and

$$|\sigma_*(t,S_t)|^2 + \lambda_*(t,S_t)|J_*(t,S_t)|^2 \le \sigma_t^2 + \lambda_t|J_t|^2 \le |\sigma^*(t,S_t)|^2 + \lambda^*(t,S_t)|J^*(t,S_t)|^2, \tag{5.5}$$

$dPdt$-a.e. and either:

ii-a) $-1 < J_*(t,S_t) \le J_t \le J^*(t,S_t) \le 0$ and $|\sigma_*(t,S_t)|^2 \le |\sigma_t|^2 \le |\sigma^*(t,S_t)|^2$, $dPdt$-a.e., or:

ii-b) $-1 < J_*(t,S_t) \le J_t \le 0 \le J^*(t,S_t)$ and $|\sigma_*(t,S_t)|^2 \le |\sigma_t|^2$, $dPdt$-a.e., or:

ii-c) $-1 < J_*(t,S_t) \le 0, J_t \le J^*(t,S_t)$ and

$$|\sigma_*(t,S_t)|^2 \le |\sigma_t|^2, \quad \lambda_t|J_t|^2 \le \lambda^*(t,S_t)|J^*(t,S_t)|^2,$$

$dPdt$-a.e., or:

ii-d) $-1 < J_*(t,S_t) \le 0 \le J_t \le J^*(t,S_t)$ and $\lambda_t|J_t|^2 \le \lambda^*(t,S_t)|J_*(t,S_t)|^2$, $dPdt$-a.e., or:

ii-e) $0 \le J_*(t, S_t) \le J_t \le J^*(t, S_t)$ *and*

$$\lambda_*(t, S_t)|J_*(t, S_t)|^2 \le \lambda_t |J_t|^2 \le \lambda_*(t, S_t)|J^*(t, S_t)|^2,$$

dPdt-a.e. .

Proof. We apply the comparison lemma to the same measures as in the proof of Proposition 4.2 and the result follows from Theorem 3.2. Once more, the expression of the condition in (ii) depends on the position of 0 with respect to $J_*(t, S_t)$ and to $J^*(t, S_t)$. □

The remarks formulated after the proof of Proposition 4.2 also apply here.

6 Poisson random measures

We now investigate the consequences of Theorem 3.2 in the setting of Poisson random measures. Let γ be a diffuse Radon measure on $\mathbb{R}^d \setminus \{0\}$ such that

$$\int_{\mathbb{R}^d \setminus \{0\}} (|x|^2 \wedge 1)\gamma(dx) < \infty,$$

and consider a random measure $\omega(dt, dx)$ of the form

$$\omega(dt, dx) = \sum_{i \in \mathbb{N}} \delta_{(t_i, x_i)}(dt, dx),$$

which is assumed to be Poisson distributed with intensity $\gamma(dx)dt$ on $\mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\})$, and consider a standard Brownian motion $(W_t)_{t \in [0,T]}$, independent of $\omega(dt, dx)$, under a probability P on Ω. Here we have

$$\mathcal{F}_t = \sigma\big(W_s,\ \omega([0, s] \times A)\ :\ 0 \le s \le t,\ A \in \mathcal{B}_b(\mathbb{R}^d \setminus \{0\})\big), \qquad t \in \mathbb{R}_+,$$

where $\mathcal{B}_b(\mathbb{R}^d \setminus \{0\}) = \{A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})\ :\ \gamma(A) < \infty\}$. Let S be the solution of

$$dS_t = r_t S_t dt + \sigma_t S_t dW_t + \int_{\mathbb{R}^d \setminus \{0\}} J_{t^-, x} S_{t^-} (\omega(dt, dx) - \gamma(dx)), \tag{6.1}$$

where σ_t is a square-integrable $\mathcal{F}_t$-predictable process and $(J_{t,x})_{(t,x) \in [0,T] \times (\mathbb{R}^d \setminus \{0\})}$ is an $\mathcal{F}_t$-predictable process satisfying the hypotheses of Proposition 6.1 below. In (i) and (ii) below we respectively assume that

$$(J_{t,x})_{(t,x) \in [0,T] \times \mathbb{R}^d \setminus \{0\})} \in L^1(\Omega \times [0, T] \times (\mathbb{R}^d \setminus \{0\}), dP \times dt \times d\gamma),$$

and

$$(J_{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d\setminus\{0\})} \in L^2(\Omega\times[0,T]\times(\mathbb{R}^d\setminus\{0\}), dP\times dt\times d\gamma).$$

Proposition 6.1. *Assume that $(S_t)_{t\in\mathbb{R}_+}$ has the jump characteristics (5.2) and $r^*(t) = r_t = r_*(t)$, $t\in\mathbb{R}_+$. Then*

$$v_*(t,S_t) \le \mathbb{E}\left[\phi(S_T)|\mathcal{F}_t\right] \le v^*(t,S_t), \qquad 0\le t\le T,$$

holds provided the (PC) property holds for S_*, S^*, and one of the following conditions is satisfied:

i) we have $0 \le J_*(t,S_t) \le J_{t,x} \le J^*(t,S_t)$, $|\sigma_*(t,S_t)| \le |\sigma_t| \le |\sigma_t^*(t,S_t)|$, and

$$\lambda_*(t,S_t)J_*(t,S_t) \le \int_{\mathbb{R}^d\setminus\{0\}} J_{t,y}\gamma(dy) \le \lambda^*(t,S_t)J^*(t,S_t), \tag{6.2}$$

$dP\gamma(dx)dt$-a.e.

ii) the functions $\frac{\partial v_*}{\partial x}(t,\cdot)$ and $\frac{\partial v^*}{\partial x}(t,\cdot)$ are convex and

$$|\sigma_*(t,S_t)|^2+\lambda_*(t,S_t)|J_*(t,S_t)|^2 \le \sigma_t^2+\int_{\mathbb{R}^d\setminus\{0\}} |J_{t,y}|^2\gamma(dy) \le |\sigma^*(t,S_t)|^2+\lambda^*(t,S_t)|J^*(t,S_t)|^2, \tag{6.3}$$

$dPdt$-a.e. and either :

ii-a) $-1 < J_*(t,S_t) \le J_{t,x} \le J^*(t,S_t) \le 0$ and $|\sigma_*(t,S_t)|^2 \le |\sigma_t|^2 \le |\sigma_*(t,S_t)|^2$, $dP\gamma(dx)dt$-a.e., or:

ii-b) $-1 < J_*(t,S_t) \le J_{t,x} \le 0 \le J^*(t,S_t)$ and $|\sigma_*(t,S_t)|^2 \le |\sigma_t|^2$, $dP\gamma(dx)dt$-a.e., or:

ii-c) $-1 < J_*(t,S_t) \le 0, J_{t,x} \le J^*(t,S_t)$ and

$$|\sigma_*(t,S_t)|^2 \le |\sigma_t|^2, \quad \int_{\mathbb{R}^d\setminus\{0\}} |J_{t,y}|^2\gamma(dy) \le \lambda^*(t,S_t)|J^*(t,S_t)|^2,$$

$dP\gamma(dx)dt$-a.e., or:

ii-d) $-1 < J_*(t,S_t) \le 0 \le J_{t,x} \le J^*(t,S_t)$ and $\int_{\mathbb{R}^d\setminus\{0\}} |J_{t,y}|^2\gamma(dy) \le \lambda^*(t,S_t)|J^*(t,S_t)|^2$, $dP\gamma(dx)dt$-a.e., or:

ii-e) $0 \leq J_*(t, S_t) \leq J_{t,x} \leq J^*(t, S_t)$ *and*

$$\lambda_*(t, S_t)|J_*(t, S_t)|^2 \leq \int_{\mathbb{R}^d \setminus \{0\}} |J_{t,y}|^2 \gamma(dy) \leq \lambda^*(t, S_t)|J^*(t, S_t)|^2,$$

$dP\gamma(dx)dt$*-a.e.*

Proof. We directly apply Theorem 3.2 instead of Proposition 5.1. Here, $\nu_t(dx)$ denotes the image measure of $\gamma(dx)$ by the mapping $x \mapsto J_{t,x}$, $t \geq 0$, and $\mu(dt, dx)$ denotes the image measure of $\omega(dt, dx)$ by $(s, y) \mapsto (s, J_{s,y})$, i.e.

$$\mu(dt, dx) = \sum_{\omega(\{(s,y)\})=1} \delta_{(s,J_{s,y})}(dt, dx).$$

Let also

$$\nu_*(t, x, dy) = \lambda_*(t, x)\delta_{J_*(t,x)}(dy), \quad \nu^*(t, x, dy) = \lambda^*(t, x)\delta_{J^*(t,x)}(dy).$$

From $J_*(t, S_t) \leq J_{t,x} \leq J^*(t, S_t)$ $dP\gamma(dx)dt$-a.e. and from (6.2) (resp. (6.3), with extra conditions in a)–e) according to the place of 0 with respect to J_*, to J and to J^*), we derive for $p = 1$ (resp. $p = 2$):

$$\begin{aligned}
&\int_x^\infty \left(\lambda_*(t, S_t)y^p\delta_{J_*(t,S(t))}(dy) + |\sigma_*(t, S_t)|^2\delta_0(dy)\delta_{p,2}\right) \\
&\qquad \leq \int_{\mathbb{R}^d \setminus \{0\}} \mathbf{1}_{\{x \leq J_{t,y}\}} \left(J_{t,y}^p \gamma(dy) + |\sigma_t|^2\delta_0(dy)\delta_{p,2}\right) \\
&\qquad \leq \int_x^\infty \left(\lambda^*(t, S_t)y^p\delta_{J^*(t,S(t))}(dy) + |\sigma^*(t, S_t)|^2\delta_0(dy)\delta_{p,2}\right)
\end{aligned}$$

$x \in \mathbb{R}$ and $t \in [0, T]$. Using the comparison lemma, the hypotheses of Theorem 3.2 are derived in i) for $p = 1$ and in ii) for $p = 2$. □

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