Hedging in bond markets by the Clark-Ocone formula

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Abstract

Hedging strategies in bond markets are computed by martingale representation and the choice of a suitable of numeraire, based on the Clark-Ocone formula in a model driven by the dynamics of bond prices. Applications are given to the hedging of swaptions and other interest rate derivatives and we compare our approach to delta hedging when the underlying swap rate is modeled by a diffusion process.

Key words: Bond markets, hedging, forward measure, Clark-Ocone formula under change of measure, bond options, swaptions.
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1 Introduction

The pricing of interest rate derivatives is usually performed by the change of numeraire technique under a suitable forward measure $\hat{\mathbb{P}}$. On the other hand, the computation of hedging strategies for interest rate derivatives presents several difficulties, in particular, hedging strategies appear not to be unique and one is faced with the problem of choosing an appropriate tenor structure of bond maturities in order to correctly hedge maturity-related risks, see e.g. [2] in the jump case.

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In this paper we consider the application of the change of numeraire technique to the computation of hedging strategies for interest rate derivatives. The payoff of an interest derivative is usually based on an underlying asset priced $\hat{X}_t$ at time t (e.g. a swap rate) which is defined from a family $(P_t(T_i))_i$ of bond prices with maturities $(T_i)_i$.

We will distinguish between two different modeling situations.

(1) Modeling $\hat{X}_t$ as a Markov diffusion process

$$d\hat{X}_t = \hat{\sigma}_t(\hat{X}_t)d\hat{W}_t \tag{1.1}$$

where $(\hat{W}_t)_{t \in \mathbb{R}_+}$ is a Brownian motion under the forward measure $\hat{\mathbb{P}}$. In this case delta hedging can be applied and this approach has been adopted in [7] to compute self-financing hedging strategies for swaptions based on geometric Brownian motion. In Section 4 of this paper we review and extend this approach.

(2) Modeling each bond price $P_t(T)$ by a stochastic differential equation of the form

$$dP_t(T) = r_t P_t(T)dt + P_t(T)\zeta_t(T)dW_t, \tag{1.2}$$

where W_t is a standard Brownian motion under the risk-neutral measure $\mathbb{P}$. In this case the process $\hat{X}_t$ may no longer have a simple Markovian dynamics under $\hat{\mathbb{P}}$ (cf. Lemma 3.2 or (3.17) below) and we rely on the Clark-Ocone formula which is commonly used for the hedging of path-dependent options. Precisely, due to the use of forward measures we will apply the Clark-Ocone formula under change of measure of [9]. This approach is carried out in Section 3.

We consider a bond price curve $(P_t)_{t \in \mathbb{R}_+}$, valued in a real separable Hilbert space G, usually a weighted Sobolev space of real-valued functions on $\mathbb{R}_+$, cf. [4] and § 6.5.2 of [1], and we denote by G^* the dual space of continuous linear mappings on G.

Given $\mu \in G^*$ a signed finite measure on $\mathbb{R}_+$ with support in $[T, \infty)$, we consider

$$P_t(\mu) := \langle \mu, P_t \rangle_{G^*, G} = \int_T^\infty P_t(y)\mu(dy),$$

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which represents a basket of bonds whose maturities are beyond the exercise date $T > 0$ and distributed according to the measure μ. The value of a portfolio strategy $(\phi_t)_{t \in [0,T]}$ is given by

$$V_t := \langle \phi_t, P_t \rangle_{G^*, G} = \int_T^\infty P_t(y) \phi_t(dy) \tag{1.3}$$

where the measure $\phi_t(dy)$ represents the amount of bonds with maturity in $[y, y + dy]$ in the portfolio at time $t \in [0, T]$.

Given $\nu \in G^*$ another positive finite measure on $\mathbb{R}_+$ with support in $[T, \infty)$, we consider the generalized annuity numeraire

$$P_t(\nu) := \langle \nu, P_t \rangle_{G^*, G} = \int_T^\infty P_t(y) \nu(dy),$$

and the forward bond price curve

$$\hat{P}_t = \frac{P_t}{P_t(\nu)}, \qquad 0 \le t \le T,$$

which is a martingale under the forward measure $\hat{\mathbb{P}}$ defined by

$$\mathbb{E}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} \Big| \mathcal{F}_S \right] = e^{-\int_0^S r_s ds} \frac{P_S(\nu)}{P_0(\nu)}, \tag{1.4}$$

where the maturity S is such that $S \ge T$.

In practice, $\mu(dy)$ and $\nu(dy)$ will be finite point measures, i.e. sums

$$\sum_{k=i}^j \alpha_k \delta_{T_k}(dy)$$

of Dirac measures based on the maturities $T_i, \ldots, T_j \ge T$ of a given a tenor structure, in which α_k represents the amount allocated to a bond with maturity T_k, $k = i, \ldots, j$. In this case we are interested in finding a hedging strategy $\phi_t(dy)$ of the form

$$\phi_t(dy) = \sum_{k=i}^j \alpha_k(t) \delta_{T_k}(dy)$$

in which case (1.3) reads

$$V_t = \sum_{k=i}^{j} \alpha_k(t) P_t(T_k), \qquad 0 \le t \le T,$$

and similarly for $P_t(\mu)$ and $P_t(\nu)$ using $\mu(dx)$ and $\nu(dx)$ respectively.

Lemma 2.1 below shows how to compute self-financing hedging strategies from the decomposition

$$\hat{\xi} = \hat{\mathbb{E}}[\hat{\xi}] + \int_0^T \langle \phi_s, d\hat{P}_s \rangle_{G^*, G}, \tag{1.5}$$

of a forward claim payoff $\hat{\xi} = \xi/P_S(\nu)$, where $(\phi_t)_{t \in [0,T]}$ is a square-integrable G^*-valued adapted process of continuous linear mappings on G. The representation (1.5) can be obtained from the predictable representation

$$\hat{\xi} = \hat{\mathbb{E}}[\hat{\xi}] + \int_0^T \langle \hat{\alpha}_t, d\hat{W}_t \rangle_H, \tag{1.6}$$

where $(\hat{W}_t)_{t \in \mathbf{R}_+}$ is a Brownian motion under $\hat{\mathbb{P}}$ with values in a separable Hilbert space H, cf. (2.7) below, and $(\hat{\alpha}_t)_{t \in \mathbf{R}_+}$ is an H-valued square-integrable $\mathcal{F}_t$-adapted process.

In case the forward price process $\hat{P}_t = P_t/P_t(\nu)$, $t \in \mathbf{R}_+$, follows the dynamics

$$d\hat{P}_t = \hat{\sigma}_t d\hat{W}_t, \tag{1.7}$$

where $(\hat{\sigma}_t)_{t \in \mathbf{R}_+}$ is an $\mathcal{L}_{HS}(H, G)$-valued adapted process of Hilbert-Schmidt operators from H to G, cf. [1], and $\hat{\sigma}_t^* : H \to G^*$ is invertible, $0 \le t \le T$, Relation (1.7) shows that the process $(\phi_t)_{t \in \mathbf{R}_+}$ in Lemma 2.1 is given by

$$\phi_t = (\hat{\sigma}_t^*)^{-1} \hat{\alpha}_t, \qquad 0 \le t \le T. \tag{1.8}$$

However this invertibility condition can be too restrictive in practice.

On the other hand the invertibility of $\sigma_t^* : G^* \to H$ as an operator is not required in order to hedge the claim ξ. As an illustrative example, when $H = \mathbf{R}$ we have

$$\hat{\xi} = \mathbb{E}[\hat{\xi}] + \int_0^T \hat{\alpha}_t d\hat{W}_t = \mathbb{E}[\hat{\xi}] + \sum_{i=1}^{n} c_i \int_0^T \frac{\hat{\alpha}_t}{\hat{\sigma}_t(T_i)} d\hat{P}_t(T_i),$$

where $\{T_1, \ldots, T_n\} \subset \mathbf{R}_+$ is a given tenor structure and $c_1, \ldots, c_n \in \mathbf{R}_+$ satisfy $c_1 + \cdots + c_n = 1$, and we can take

$$\phi_t = \sum_{i=1}^n c_i \frac{\hat{\alpha}_t}{\hat{\sigma}_t(T_i)} \delta_{T_i}.$$

Such a hedging strategy $(\phi_t)_{t \in [0,T]}$ depends as much on the bond structure (through the volatility process $\sigma_t(x)$) as on the claim ξ itself (through α_t), in connection with the problem of hedging maturity-related risks.

The predictable representation (1.6) can be computed from the Clark-Ocone formula for the Malliavin gradient $\hat{D}$ with respect to $(\hat{W}_t)_{t \in \mathbf{R}_+}$, cf. e.g. Proposition 6.7 in § 6.5.5 of [1] when the numeraire is the money market account, cf. also [13] for examples of explicit calculations in this case. This approach is more suitable to a non-Markovian or path-dependent dynamics specified for $(\hat{P}_t)_{t \in \mathbf{R}_+}$ as a functional of $(\hat{W}_t)_{t \in \mathbf{R}_+}$. However this is not the approach chosen here since the dynamics assumed for the bond price is either Markovian as in (1.1), cf. Section 4, or written in terms of W_t as in (1.2), cf. Section 3.

In this paper we specify the dynamics of $(P_t)_{t \in \mathbf{R}_+}$ under the risk-neutral measure and we apply the Clark-Ocone formula under a change of measure [9], using the Malliavin gradient D with respect to W_t, cf. (2.10) below. In Proposition 3.1 below we compute self-financing hedging strategies for contingent claims with payoff of the form $\xi = P_S(\nu)\hat{g}(P_T(\mu)/P_T(\nu))$.

This paper is organized as follows. Section 2 contains the preliminaries on the derivation of self-financing hedging strategies by change of numeraire and the Clark-Ocone formula under change of measure. In Section 3 we use the Clark-Ocone formula under a change of measure to compute self-financing hedging strategies for swaptions and other derivatives based on the dynamics of $(P_t)_{t \in \mathbf{R}_+}$. In Section 4 we compare the above results with the delta hedging approach when the dynamics of the swap rate $(\hat{X}_t)_{t \in \mathbf{R}_+}$ is based on a diffusion process.

2 Preliminaries

In this section we review the hedging of options by change of numeraire, cf. e.g. [5], [14], in the framework of [1]. We also quote the Clark-Ocone formula under change of measure.

Hedging by change of numeraire

Consider a numeraire $(M_t)_{t \in \mathbb{R}_+}$ under the risk-neutral probability measure $\mathbb{P}$ on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$, that is, $(M_t)_{t \in \mathbb{R}_+}$ is a continuous, strictly positive, $\mathcal{F}_t$-adapted asset price process such that the discounted price process $e^{-\int_0^t r_s ds} M_t$ is an $\mathcal{F}_t$-martingale under $\mathbb{P}$.

Recall that an option with payoff ξ, exercise date T and maturity S, is priced at time t as

$$\mathbb{E}\left[e^{-\int_t^S r_s ds} \xi \mid \mathcal{F}_t\right] = M_t \hat{\mathbb{E}}[\hat{\xi} | \mathcal{F}_t], \qquad 0 \le t \le T, \tag{2.1}$$

under the forward measure $\hat{\mathbb{P}}$ defined by

$$\mathbb{E}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} \Big| \mathcal{F}_S\right] = e^{-\int_0^S r_s ds} \frac{M_S}{M_0}, \tag{2.2}$$

$S \ge T$, where

$$\hat{\xi} = \frac{\xi}{M_S} \in L^1(\hat{\mathbb{P}}, \mathcal{F}_S)$$

denotes the forward payoff of the claim ξ.

In the framework of [1], consider $(W_t)_{t \in \mathbb{R}_+}$ a cylindrical Brownian motion taking values in a separable Hilbert space H with covariance

$$E[W_s(h) W_t(k)] = (s \wedge t)\langle h, k \rangle_H, \qquad h, k \in H, \quad s, t \in \mathbb{R}_+,$$

and generating the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$. Consider a continuous $\mathcal{F}_t$-adapted asset price process $(X_t)_{t \in \mathbb{R}_+}$ taking values in a real separable Hilbert space G, and assume that both $(X_t)_{t \in \mathbb{R}_+}$ and $(M_t)_{t \in \mathbb{R}_+}$ are Itô processes in the sense of § 4.2.1 of [1]. The forward asset price

$$\hat{X}_t := \frac{X_t}{M_t}, \qquad 0 \le t \le T,$$

is a martingale in G under the forward measure $\hat{\mathbb{P}}$, provided it is integrable under $\hat{\mathbb{P}}$.

The next lemma will be key to compute self-financing portfolio strategies in the assets (X_t, M_t) by numeraire invariance, cf. [14], [6] for the finite dimensional case. We say that a portfolio $(\phi_t, \eta_t)_{t \in [0,T]}$ with value

$$\langle \phi_t, X_t \rangle_{G^*, G} + \eta_t M_t, \qquad 0 \le t \le T,$$

is self-financing if

$$dV_t = \langle \phi_t, dX_t \rangle_{G^*, G} + \eta_t dM_t. \tag{2.3}$$

The portfolio $(\phi_t, \eta_t)_{t \in [0,T]}$ is said to hedge the claim $\xi = M_S \hat{\xi}$ if

$$\langle \phi_t, X_t \rangle_{G^*, G} + \eta_t M_t = \mathbb{E} \left[e^{- \int_t^S r_s ds} M_S \hat{\xi} \, \Big| \mathcal{F}_t \right], \qquad 0 \le t \le T.$$

Lemma 2.1 *Assume that the forward claim price $\hat{V}_t := \hat{\mathbb{E}}[\hat{\xi}|\mathcal{F}_t]$ has the predictable representation*

$$\hat{V}_t = \hat{\mathbb{E}}[\hat{\xi}] + \int_0^t \langle \phi_s, d\hat{X}_s \rangle_{G^*, G}, \qquad 0 \le t \le T, \tag{2.4}$$

where $(\phi_t)_{t \in [0,T]}$ is a square-integrable G^*-valued adapted process of continuous linear mappings on G. Then the portfolio $(\phi_t, \eta_t)_{t \in [0,T]}$ defined with

$$\eta_t = \hat{V}_t - \langle \phi_t, \hat{X}_t \rangle_{G^*, G}, \qquad 0 \le t \le T, \tag{2.5}$$

and priced as

$$V_t = \langle \phi_t, X_t \rangle_{G^*, G} + \eta_t M_t, \qquad 0 \le t \le T,$$

is self-financing and hedges the claim $\xi = M_S \hat{\xi}$.

Proof. For completeness we provide the proof of this lemma, although it is a direct extension of classical results. In order to check that the portfolio $(\phi_t, \eta_t)_{t \in [0,T]}$ hedges the claim $\xi = M_S \hat{\xi}$ it suffices to note that by (2.1) and (2.5) we have

$$\langle \phi_t, X_t \rangle_{G^*, G} + \eta_t M_t = M_t \hat{V}_t = \mathbb{E} \left[e^{- \int_t^S r_s ds} M_S \hat{\xi} \, \Big| \mathcal{F}_t \right], \qquad 0 \le t \le T.$$

The portfolio $(\phi_t, \eta_t)_{t \in [0,T]}$ is clearly self-financing for $(\hat{X}_t, 1)$ by (2.4), and by the semi-martingale version of numeraire invariance, cf. e.g. page 184 of [14], and [6], it is also

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self-financing for (X_t, M_t), cf. also § 3.2 of [8] and references therein.

For completeness we quote the proof of the self-financing property, as follows:

$$
\begin{aligned}
dV_t &= d(M_t \hat{V}_t) \\
&= \hat{V}_t dM_t + M_t d\hat{V}_t + dM_t \cdot d\hat{V}_t \\
&= \hat{V}_t dM_t + M_t \langle \phi_t, d\hat{X}_t \rangle_{G^*,G} + dM_t \cdot \langle \phi_t, d\hat{X}_t \rangle_{G^*,G} \\
&= \langle \phi_t, \hat{X}_t \rangle_{G^*,G} dM_t + M_t \langle \phi_t, d\hat{X}_t \rangle_{G^*,G} + dM_t \cdot \langle \phi_t, d\hat{X}_t \rangle_{G^*,G} \\
&\quad + (\hat{V}_t - \langle \phi_t, \hat{X}_t \rangle_{G^*,G}) dM_t \\
&= \langle \phi_t, d(M_t \hat{X}_t) \rangle_{G^*,G} + (\hat{V}_t - \langle \phi_t, \hat{X}_t \rangle_{G^*,G}) dM_t \\
&= \langle \phi_t, dX_t \rangle_{G^*,G} + \eta_t dM_t.
\end{aligned}
$$

$\square$

Lemma 2.1 yields a self-financing portfolio $(\phi_t, \eta_t)_{t \in [0,T]}$ with value

$$
V_t = V_0 + \int_0^t \eta_s dM_s + \int_0^t \langle \phi_s, dX_s \rangle_{G^*,G}, \qquad 0 \le t \le T, \tag{2.6}
$$

given by (2.3), which hedges the claim with exercise date T and random payoff ξ.

Clark formula under change of measure

Recall that by the Girsanov theorem, cf. Theorem 10.14 of [3] or Theorem 4.2 of [1], the process $(\hat{W}_t)_{t \in \mathbb{R}_+}$ defined by

$$
d\hat{W}_t = dW_t - \frac{1}{M_t} dM_t \cdot dW_t, \qquad t \in \mathbb{R}_+, \tag{2.7}
$$

is a H-valued Brownian motion under $\hat{\mathbb{P}}$. Let D denote the Malliavin gradient with respect to $(W_t)_{t \in \mathbb{R}_+}$, defined on smooth functionals

$$
\hat{\xi} = f(W_{t_1}, \ldots, W_{t_n})
$$

of Brownian motion, $f \in \mathcal{C}_b(\mathbb{R}^n)$, as

$$
D_t \hat{\xi} = \sum_{k=1}^n \mathbf{1}_{[0,t_k]}(t) \frac{\partial f}{\partial x_k}(W_{t_1}, \ldots, W_{t_n}), \qquad t \in \mathbb{R}_+,
$$

and extended by closability to its domain $\mathrm{Dom}\,(D)$. The proof of Proposition 3.1 relies on the following Clark-Ocone formula under a change of measure, cf. [9], which can be extended to H-valued Brownian motion by standard arguments.

Lemma 2.2 *Let $(\gamma_t)_{t \in \mathbb{R}_+}$ denote a H-valued square-integrable $\mathcal{F}_t$-adapted process such that $\gamma_t \in \mathrm{Dom}\,(D)$, $t \in \mathbb{R}_+$, and*

$$dW_t = \gamma_t dt + d\hat{W}_t.$$

Let $\hat{\xi} \in \mathrm{Dom}\,(D)$ such that

$$\hat{E}\left[\int_0^T \|D_t\hat{\xi}\|_H^2 dt\right] < \infty \tag{2.8}$$

and

$$\hat{E}\left[|\hat{\xi}| \int_0^T \left\|\int_0^T D_t\gamma_s d\hat{W}_s\right\|_H^2 dt\right] < \infty. \tag{2.9}$$

Then the predictable representation

$$\hat{\xi} = \hat{\mathbb{E}}[\hat{\xi}] + \int_0^T \langle \hat{\alpha}_t, d\hat{W}_t \rangle_H$$

is given by

$$\hat{\alpha}_t = \hat{\mathbb{E}}\left[D_t\hat{\xi} + \hat{\xi}\int_t^T D_t\gamma_s d\hat{W}_s \Big| \mathcal{F}_t\right], \qquad 0 \le t \le T. \tag{2.10}$$

3   Hedging by the Clark-Ocone formula

In this section we present a computation of hedging strategies using the Clark-Ocone formula under change of measure and we assume that the dynamics of $(P_t)_{t \in \mathbb{R}_+}$ is given by the stochastic differential equation

$$dP_t = r_t P_t dt + P_t \zeta_t dW_t, \tag{3.1}$$

in the Sobolev space G which is assumed to be an algebra of real-valued functions on $\mathbb{R}_+$. The process $(r_t)_{t \in \mathbb{R}_+}$ represents a short term interest rate process adapted to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ generated by $(W_t)_{t \in \mathbb{R}_+}$, and $(\zeta_t)_{t \in \mathbb{R}_+}$ is an $\mathcal{L}_{HS}(H, G)$-valued deterministic function.

The aim of this section is to prove Proposition 3.1 below under the non-restrictive integrability conditions

$$\int_0^T \int_T^\infty \|\zeta_t(y)\|_H^2 \hat{\mathbb{E}}[|\hat{P}_T|^2(y)] \mu(dy) dt < \infty \tag{3.2}$$

and

$$\int_0^T \int_T^\infty \|\zeta_t(y)\|_H^2 \hat{\mathbb{E}}[|\hat{P}_T(\mu)|^2(|\hat{P}_T|^2(y) + |\hat{P}_t|^2(y))] \nu(dy) dt < \infty. \tag{3.3}$$

which are respectively derived from (2.8) and (2.9). The next proposition provides an alternative to Proposition 3.3 in [13] by applying to a different family of payoff functions. It coincides with Proposition 3.3 of [13] in case $S = T$ and $\nu = \delta_T$.

Proposition 3.1 *Consider the claim with payoff*

$$\xi = P_S(\nu)\hat{g}\left(\frac{P_T(\mu)}{P_T(\nu)}\right),$$

where $\hat{g} : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function. Then the portfolio

$$\begin{aligned}
\phi_t(dy) \quad &:= \quad \hat{\mathbb{E}}\left[\frac{\hat{P}_T(y)}{\hat{P}_t(y)}\hat{g}'(\hat{P}_T(\mu))\Big|\mathcal{F}_t\right] \mu(dy) \\
&\quad + \hat{\mathbb{E}}\left[(\hat{g}(\hat{P}_T(\mu)) - \hat{P}_T(\mu)\hat{g}'(\hat{P}_T(\mu)))\frac{\hat{P}_T(y)}{\hat{P}_t(y)}\Big|\mathcal{F}_t\right] \nu(dy) \tag{3.4}
\end{aligned}$$

$0 \leq t \leq T$, *is self-financing and hedges the claim ξ.*

Before proving Proposition 3.1 we check that the portfolio ϕ_t hedges the claim $\xi = P_S(\nu)\hat{g}(\hat{P}_T(\mu))$ by construction, since we have

$$\begin{aligned}
\langle\phi_t, P_t\rangle_{G^*,G} \quad &= \quad \int_T^\infty P_t(y)\phi_t(dy) \\
&= \quad \int_T^\infty \hat{\mathbb{E}}\left[\frac{\hat{P}_T(y)}{\hat{P}_t(y)}\hat{g}'(\hat{P}_T(\mu))\Big|\mathcal{F}_t\right] P_t(y)\mu(dy) \\
&\quad + \int_T^\infty \hat{\mathbb{E}}\left[(\hat{g}(\hat{P}_T(\mu)) - \hat{P}_T(\mu)\hat{g}'(\hat{P}_T(\mu)))\frac{\hat{P}_T(y)}{\hat{P}_t(y)}\Big|\mathcal{F}_t\right] P_t(y)\nu(dy) \\
&= \quad P_t(\nu)\hat{\mathbb{E}}\left[\hat{g}(\hat{P}_T(\mu))\Big|\mathcal{F}_t\right] \\
&\quad - P_t(\nu)\int_T^\infty \hat{\mathbb{E}}\left[\hat{P}_T(y)\hat{g}'(\hat{P}_T(\mu))\Big|\mathcal{F}_t\right] \mu(dy)
\end{aligned}$$

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$$+P_t(\nu) \int_T^\infty \hat{\mathbb{E}}\left[\hat{P}_T(\mu)\hat{g}'(\hat{P}_T(\mu))\hat{P}_T(y)\Big|\mathcal{F}_t\right] \nu(dy)$$

$$= P_t(\nu)\hat{\mathbb{E}}\left[\hat{g}(\hat{P}_T(\mu))\Big|\mathcal{F}_t\right] = \mathbb{E}\left[e^{-\int_t^S r_s ds}\xi \Big|\mathcal{F}_t\right].$$

by (2.1). Hence

$$\langle \phi_t, \hat{P}_t \rangle_{G^*,G} = \hat{\mathbb{E}}\left[\hat{g}(\hat{P}_T(\mu))\Big|\mathcal{F}_t\right] = \hat{V}_t \tag{3.5}$$

The identity (3.5) will also be used in the proof of Lemma 3.5 below.

Before moving to the proof of Proposition 3.1 we consider some examples of applications of the results of Proposition 3.1, in which the dynamics of $(P_t)_{t\in\mathbb{R}_+}$ is given by (1.2), cf. e.g. Chapters 7 and 10 of [11] for an introduction to the derivatives considered in the following examples.

Exchange options

In the case of an exchange option with $S = T$ and payoff $(P_T(\mu) - \kappa P_T(\nu))^+$, Proposition 3.1 yields the self-financing hedging strategy

$$\phi_t(dy) = \hat{\mathbb{E}}\left[\mathbf{1}_{\{\hat{P}_T(\mu)>\kappa\}}\frac{\hat{P}_T(y)}{\hat{P}_t(y)}\Big|\mathcal{F}_t\right]\mu(dy) - \kappa\hat{\mathbb{E}}\left[\mathbf{1}_{\{\hat{P}_T(\mu)>\kappa\}}\frac{\hat{P}_T(y)}{\hat{P}_t(y)}\Big|\mathcal{F}_t\right]\nu(dy)$$

$$= \hat{\mathbb{E}}\left[\mathbf{1}_{\{\hat{P}_T(\mu)>\kappa\}}\frac{\hat{P}_T(y)}{\hat{P}_t(y)}\Big|\mathcal{F}_t\right](\mu(dy) - \kappa\nu(dy)).$$

Bond options

In the case of a bond call option with $S = T$ and payoff $(P_T(U) - \kappa)^+$ and $\mu = \delta_U$, $\nu = \delta_T$, this yields

$$\phi_t(dy) = \frac{P_t(T)}{P_t(U)}\hat{\mathbb{E}}\left[\mathbf{1}_{\{\hat{P}_T(U)>\kappa\}}\hat{P}_T(U)\Big|\mathcal{F}_t\right]\delta_U(dy) - \kappa\hat{\mathbb{E}}\left[\mathbf{1}_{\{\hat{P}_T(U)>\kappa\}}\Big|\mathcal{F}_t\right]\delta_T(dy). \tag{3.6}$$

This particular setting of bond options can be modeled using the diffusions of Section 4 since in that case $\hat{P}_t(\mu) = P_t(U)/P_t(T)$ is a geometric Brownian motion under $\hat{\mathbb{P}}$ with volatility

$$\hat{\sigma}(t) = \zeta_t(U) - \zeta_t(T) \tag{3.7}$$

given by (3.12) below, in which case the above result coincides with the delta hedging formula (4.10) below.

Caplets on the LIBOR rate

In the case of a caplet with payoff

$$(S - T)(L(T, T, S) - \kappa)^+ = (P_T(S)^{-1} - (1 + \kappa(S - T)))^+, \tag{3.8}$$

on the LIBOR rate

$$L(t, T, S) = \frac{P_t(T) - P_t(S)}{(S - T)P_t(S)}, \qquad 0 \le t \le T < S, \tag{3.9}$$

and $\mu = \delta_T$, $\nu = \delta_S$, Proposition 3.1 yields

$$
\begin{aligned}
\phi_t(dy) &= \frac{P_t(S)}{P_t(T)} \hat{\mathbb{E}} \left[\frac{1}{P_T(S)} \mathbf{1}_{\{P_T(S) < 1/(1+\kappa(S-T))\}} \Big| \mathcal{F}_t \right] \delta_T(dy) \\
&\quad - (1 + \kappa(S - T)) \hat{\mathbb{E}} \left[\mathbf{1}_{\{P_T(S) < 1/(1+\kappa(S-T))\}} \Big| \mathcal{F}_t \right] \delta_S(dy)
\end{aligned}
\tag{3.10}
$$

In this case, $\hat{P}_t(\mu) = P_t(T)/P_t(S)$ is modeled by a geometric Brownian motion with volatility $\hat{\sigma}(t) = \zeta_t(T) - \zeta_t(S)$ as in Section 4 and the above result coincides with the formula (4.11) below.

Swaptions

In this case the modeling of the swap rate differs from the diffusion model of Section 4. For a swaption with $S = T$ and payoff

$$(P_T(T_i) - P_T(T_j) - \kappa P_T(\nu))^+$$

on the LIBOR, where

$$\mu(dy) = \delta_{T_i}(dy) - \delta_{T_j}(dy) \quad \text{and} \quad \nu(dy) = \sum_{k=i}^{j-1} \tau_k \delta_{T_{k+1}}(dy),$$

with $\tau_k = T_{k+1} - T_k$, $k = i, \ldots, j - 1$, we obtain

$$
\begin{aligned}
\phi_t(dy) &= \hat{\mathbb{E}} \left[\mathbf{1}_{\{\hat{P}_T(\mu) > \kappa\}} \frac{\hat{P}_{T_i}(T_i)}{\hat{P}_t(T_i)} \Big| \mathcal{F}_t \right] \delta_{T_i}(dy) \\
&\quad - (1 + \kappa \tau_{j-1}) \hat{\mathbb{E}} \left[\mathbf{1}_{\{\hat{P}_T(\mu) > \kappa\}} \frac{\hat{P}_{T_i}(T_j)}{\hat{P}_t(T_j)} \Big| \mathcal{F}_t \right] \delta_{T_j}(dy)
\end{aligned}
$$

$$-\kappa \sum_{k=i+1}^{j-1} \tau_{k-1} \hat{\mathbb{E}} \left[\mathbf{1}_{\{\hat{P}_T(\mu) > \kappa\}} \frac{\hat{P}_{T_i}(T_k)}{\hat{P}_t(T_k)} \Big| \mathcal{F}_t \right] \delta_{T_k}(dy). \qquad (3.11)$$

The above consequence of Proposition 3.1 differs from (4.13) in Section 4 because of different modeling assumptions. Moreover, in this case the volatility of $(\hat{P}_t(\mu))_{t \in [0,T]}$ may not be deterministic, cf. (3.14), (3.17) below.

Proof of Proposition 3.1. By Lemma 3.5 below the forward claim price $\hat{V}_t$ has the predictable representation

$$\hat{V}_t = \hat{\mathbb{E}}[\hat{\xi}] + \int_0^t \langle \phi_s, d\hat{P}_s \rangle_{G^*, G}, \qquad 0 \le t \le T.$$

Hence by Lemma 2.1 the portfolio priced as

$$V_t = \langle \phi_t, P_t \rangle_{G^*, G}, \qquad 0 \le t \le T,$$

is self-financing and it hedges the claim $\xi = P_S(\nu)\hat{g}(P_T(\mu)/P_T(\nu))$, since $\eta_t = 0$ by (2.5) and (3.5). $\qquad \square$

The next lemma, which will be used in the proof of Lemma 3.4 below, shows in particular that for fixed $U > 0$, $(\hat{P}_t(U))_{t \in \mathbb{R}_+}$ is usually not a geometric Brownian motion, except in the case of bond options with $\mu(dy) = \delta_U(dy)$ and $\nu(dy) = \delta_T(dy)$, where we get

$$d\frac{P_t(U)}{P_t(T)} = \frac{P_t(U)}{P_t(T)}(\zeta_t(U) - \zeta_t(T))d\hat{W}_t,$$

and

$$\hat{\sigma}(t) = \zeta_t(U) - \zeta_t(T), \qquad 0 \le t \le T. \qquad (3.12)$$

Lemma 3.2 *For all $y \in \mathbb{R}_+$ we have*

$$d\hat{P}_t(y) = \hat{\sigma}_t(\hat{P}_t, y)d\hat{W}_t, \qquad t, y \in \mathbb{R}_+, \qquad (3.13)$$

where

$$\hat{\sigma}_t(\hat{P}_t, y) := \hat{P}_t(y) \int_T^\infty \hat{P}_t(z)(\zeta_t(y) - \zeta_t(z))\nu(dz), \qquad t, y \in \mathbb{R}_+. \qquad (3.14)$$

Proof. Defining the discounted bond price $\tilde{P}_t$ by

$$\tilde{P}_t = \exp\left(-\int_0^t r_s ds\right) P_t, \qquad t \in \mathbb{R}_+, \tag{3.15}$$

we have

$$
\begin{aligned}
d\hat{P}_t(y) &= d\left(\frac{\tilde{P}_t(y)}{\tilde{P}_t(\nu)}\right)\\
&= \frac{d\tilde{P}_t(y)}{\tilde{P}_t(\nu)} + \tilde{P}_t(y)d\left(\frac{1}{\tilde{P}_t(\nu)}\right) + d\tilde{P}_t(y) \cdot d\left(\frac{1}{\tilde{P}_t(\nu)}\right)\\
&= \frac{d\tilde{P}_t(y)}{\tilde{P}_t(\nu)} + \frac{\tilde{P}_t(y)}{\tilde{P}_t(\nu)}\left(-\frac{d\tilde{P}_t(\nu)}{\tilde{P}_t(\nu)} + \left(\frac{d\tilde{P}_t(\nu)}{\tilde{P}_t(\nu)}\right)^2\right) - \frac{d\tilde{P}_t(y)}{\tilde{P}_t(\nu)} \cdot \frac{d\tilde{P}_t(\nu)}{\tilde{P}_t(\nu)}\\
&= \frac{d\tilde{P}_t(y)}{\tilde{P}_t(\nu)} - \hat{P}_t(y)\frac{d\tilde{P}_t(\nu)}{\tilde{P}_t(\nu)}\\
&\quad + \hat{P}_t(y)\int_T^\infty \hat{P}_t(s)\int_T^\infty \hat{P}_t(z)\zeta_t(z)\zeta_t(s)\nu(dz)\nu(ds)dt\\
&\quad - \zeta_t(y)\hat{P}_t(y)\int_T^\infty \hat{P}_t(z)\zeta_t(z)\nu(dz)dt\\
&= \hat{P}_t(y)\zeta_t(y)dW_t - \hat{P}_t(y)\int_T^\infty \hat{P}_t(z)\zeta_t(z)\nu(dz)dW_t\\
&\quad - \hat{P}_t(y)\int_T^\infty \hat{P}_t(s)\int_T^\infty \hat{P}_t(z)(\zeta_t(y) - \zeta_t(z))\zeta_t(s)\nu(dz)\nu(ds)dt\\
&= \hat{P}_t(y)\int_T^\infty \hat{P}_t(z)(\zeta_t(y) - \zeta_t(z))\nu(dz)dW_t\\
&\quad - \hat{P}_t(y)\int_T^\infty \hat{P}_t(z)(\zeta_t(y) - \zeta_t(z))\int_T^\infty \hat{P}_t(s)\zeta_t(s)\nu(ds)\nu(dz)dt\\
&= \hat{P}_t(y)\int_T^\infty \hat{P}_t(z)(\zeta_t(y) - \zeta_t(z))\nu(dz)d\hat{W}_t,
\end{aligned}
$$

by the relation

$$d\hat{W}_t = dW_t - \int_T^\infty \hat{P}_t(s)\zeta_t(s)\nu(ds)dt, \qquad t \in \mathbb{R}_+, \tag{3.16}$$

which follows from (2.7). $\qquad\square$

In the case of a swaption with

$$\mu(dy) = \delta_{T_i}(dy) - \delta_{T_j}(dy) \quad \text{and} \quad \nu(dy) = \sum_{k=i}^{j-1} \tau_k \delta_{T_{k+1}}(dy),$$

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$\hat{P}_t(\mu)$ becomes the corresponding swap rate and Lemma 3.2 yields

$$d\frac{P_t(\mu)}{P_t(\nu)}$$

$$= \frac{P_t(\mu)}{P_t(\nu)}\left(\frac{P_t(T_j)}{P_t(\mu)}(\zeta_t(T_i) - \zeta_t(T_j)) + \sum_{k=i}^{j-1}\tau_k\frac{P_t(T_{k+1})}{P_t(\nu)}(\zeta_t(T_i) - \zeta_t(T_{k+1}))\right)d\hat{W}_t,$$

which shows that

$$\hat{\sigma}(t) = \frac{P_t(T_j)}{P_t(\mu)}(\zeta_t(T_i) - \zeta_t(T_j)) + \sum_{k=i}^{j-1}\tau_k\frac{P_t(T_{k+1})}{P_t(\nu)}(\zeta_t(T_i) - \zeta_t(T_{k+1})), \qquad (3.17)$$

$0 \le t \le T$, and coincides with the dynamics of the LIBOR swap rate in Relation (1.28), page 17 of [15].

Lemma 3.3 has been used in the proof of Proposition 3.1.

Lemma 3.3 *We have*

$$D_t\hat{P}_u(y) = \hat{\sigma}_t(\hat{P}_u, y), \qquad 0 \le t \le u, \quad y \in \mathbb{R}_+, \qquad (3.18)$$

where

$$\hat{\sigma}_t(\hat{P}_u, y) = \hat{P}_u(y)\int_T^\infty \hat{P}_u(z)(\zeta_t(y) - \zeta_t(z))\nu(dz), \qquad (3.19)$$

$0 \le t \le u, \ y \in \mathbb{R}_+$.

Proof. The discounted bond price $\tilde{P}_t$ defined in (3.15) satisfies the relation

$$\tilde{P}_u(y) = \tilde{P}_0(y)\exp\left(\int_0^u \zeta_t(y)dW_t - \frac{1}{2}\int_0^u |\zeta_t(y)|^2\,dt\right), \qquad y \in \mathbb{R}_+,$$

with

$$D_u\tilde{P}_T(y) = \tilde{P}_T(y)\zeta_u(y), \qquad 0 \le u \le T, \quad y \in \mathbb{R}_+.$$

Hence we get

$$D_t\hat{P}_u(y) = D_t\frac{\tilde{P}_u(y)}{\tilde{P}_u(\nu)}$$

$$= \frac{D_t\tilde{P}_u(y)}{\tilde{P}_u(\nu)} - \frac{\tilde{P}_u(y)}{\tilde{P}_u(\nu)}\frac{D_t\tilde{P}_u(\nu)}{\tilde{P}_u(\nu)}$$

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$$= \frac{\tilde{P}_u(y)}{\tilde{P}_u(\nu)} \left(\zeta_t(y) - \int_T^\infty \zeta_t(z) \frac{\tilde{P}_u(z)}{\tilde{P}_u(\nu)} \nu(dz) \right)$$

$$= \hat{P}_u(y) \int_T^\infty \hat{P}_u(z)(\zeta_t(y) - \zeta_t(z))\nu(dz)$$

$$= \hat{\sigma}_t(\hat{P}_u, y),$$

$0 \le t \le u, \ y \in \mathbf{R}_+.$ \hfill $\square$

The following lemma has been used in the proof of Lemma 3.5.

Lemma 3.4 *Taking $\hat{\xi} = \hat{g}(\hat{P}_T(\mu))$, the process in Lemma 2.2 is given by*

$$\begin{aligned}
\hat{\alpha}_t &= \int_T^\infty \hat{\mathbb{E}}\left[\hat{g}'(\hat{P}_T(\mu))\hat{P}_T(y)\Big|\mathcal{F}_t\right] \zeta_t(y)\mu(dy) \\
&\quad - \int_T^\infty \hat{\mathbb{E}}\left[\hat{P}_T(\mu)\hat{g}'(\hat{P}_T(\mu))\hat{P}_T(y)\Big|\mathcal{F}_t\right] \zeta_t(y)\nu(dy) \\
&\quad + \int_T^\infty \hat{\mathbb{E}}\left[\hat{g}(\hat{P}_T(\mu))(\hat{P}_T(y) - \hat{P}_t(y))\Big|\mathcal{F}_t\right] \zeta_t(y)\nu(dy)
\end{aligned}$$

Proof. By (3.16), the process $(\gamma_t)_{t\in\mathbf{R}_+}$ in (2.10) is given by

$$\gamma_t = \int_T^\infty \hat{P}_t(s)\zeta_t(s)\nu(ds) \in H, \qquad t \in \mathbf{R}_+.$$

Taking $\hat{\xi} = \hat{g}(\hat{P}_T(\mu))$, Lemma 2.2 yields

$$\hat{V}_t = \hat{\mathbb{E}}[\hat{g}(\hat{P}_T(\mu))] + \int_0^t \langle \hat{\alpha}_s, d\hat{W}_s\rangle_H, \qquad 0 \le t \le T,$$

where

$$\hat{\alpha}_s = \hat{\mathbb{E}}\left[D_s\hat{g}(\hat{P}_T(\mu)) + \hat{g}(\hat{P}_T(\mu)) \int_s^T D_s \int_T^\infty \hat{P}_u(y)\zeta_u(y)\nu(dy)d\hat{W}_u\Big|\mathcal{F}_s\right], \qquad (3.20)$$

$0 \le s \le T$. By integration with respect to $\mu(dy)$ in (3.18) we get

$$D_t\hat{P}_T(\mu) = \int_T^\infty \zeta_t(y)\hat{P}_T(y)\mu(dy) - \hat{P}_T(\mu) \int_T^\infty \zeta_t(y)\hat{P}_T(y)\nu(dy),$$

which allows us to compute $D_t\hat{g}(\hat{P}_T(\mu)) = \hat{g}'(\hat{P}_T(\mu))D_t\hat{P}_T(\mu)$ in (3.20), $0 \le t \le T$. On the other hand, by Relations (3.14) and (3.19) in Lemmas 3.2 and Lemma 3.3 we have

$$\int_T^\infty \hat{\sigma}_t(\hat{P}_u, y)\zeta_u(y)\nu(dy) = \int_T^\infty \hat{P}_u(y) \int_T^\infty \hat{P}_u(z)(\zeta_t(y) - \zeta_t(z))\nu(dz)\zeta_u(y)\nu(dy)$$

$$= \int_T^\infty \hat{P}_u(y) \int_T^\infty \hat{P}_u(z)\zeta_t(y)(\zeta_u(y) - \zeta_u(z))\nu(dz)\nu(dy)$$

$$= \int_T^\infty \zeta_t(y)\hat{\sigma}_u(\hat{P}_u, y)\nu(dy),$$

hence from Relations (3.13) and (3.18) the second term in (3.20) can be computed as

$$\int_t^T D_t \int_T^\infty \hat{P}_u(y)\zeta_u(y)\nu(dy)d\hat{W}_u = \int_t^T \int_T^\infty D_t\hat{P}_u(y)\zeta_u(y)\nu(dy)d\hat{W}_u$$

$$= \int_t^T \int_T^\infty \hat{\sigma}_t(\hat{P}_u, y)\zeta_u(y)\nu(dy)d\hat{W}_u$$

$$= \int_t^T \int_T^\infty \hat{\sigma}_u(\hat{P}_u, y)\zeta_t(y)\nu(dy)d\hat{W}_u$$

$$= \int_T^\infty \int_t^T \zeta_t(y)\hat{\sigma}_u(\hat{P}_u, y)d\hat{W}_u\nu(dy)$$

$$= \int_T^\infty \zeta_t(y) \int_t^T d\hat{P}_u(y)\nu(dy)$$

$$= \int_T^\infty (\hat{P}_T(y) - \hat{P}_t(y))\zeta_t(y)\nu(dy),$$

where $\hat{\sigma}_t(\hat{P}_u, y)$ is given by (3.19) above. Hence we have

$$D_t\hat{g}(\hat{P}_T(\mu)) + \hat{g}(\hat{P}_T(\mu)) \int_t^T D_t \int_T^\infty \hat{P}_u(y)\zeta_t(y)\nu(dy)d\hat{W}_u$$

$$= \hat{g}'(\hat{P}_T(\mu))D_t\hat{P}_T(\mu) + \hat{g}(\hat{P}_T(\mu)) \int_t^T D_t \int_T^\infty \hat{P}_u(y)\zeta_t(y)\nu(dy)d\hat{W}_u$$

$$= \hat{g}'(\hat{P}_T(\mu)) \int_T^\infty \zeta_t(y)\hat{P}_T(y)\mu(dy) - \hat{P}_T(\mu)\hat{g}'(\hat{P}_T(\mu)) \int_T^\infty \zeta_t(y)\hat{P}_T(y)\nu(dy)$$

$$+ \int_T^\infty \hat{g}(\hat{P}_T(\mu))(\hat{P}_T(y) - \hat{P}_t(y))\zeta_t(y)\nu(dy),$$

which is square-integrable by Conditions (3.2) and (3.3). By (3.20), this yields

$$\hat{\alpha}_t = \int_T^\infty \hat{\mathbb{E}}\left[\hat{g}'(\hat{P}_T(\mu))\hat{P}_T(y)\Big|\mathcal{F}_t\right]\zeta_t(y)\mu(dy)$$

$$- \int_T^\infty \hat{\mathbb{E}}\left[\hat{P}_T(\mu)\hat{g}'(\hat{P}_T(\mu))\hat{P}_T(y)\Big|\mathcal{F}_t\right]\zeta_t(y)\nu(dy)$$

$$+ \int_T^\infty \hat{\mathbb{E}}\left[\hat{g}(\hat{P}_T(\mu))(\hat{P}_T(y) - \hat{P}_t(y))\Big|\mathcal{F}_t\right]\zeta_t(y)\nu(dy)$$

$$\square$$

The next lemma has been used in the proof of Proposition 3.1.

Lemma 3.5 *The process ϕ_t in the predictable representation*

$$\hat{V}_t = \hat{\mathbb{E}}[\hat{\xi}] + \int_0^t \langle \phi_s, d\hat{P}_s \rangle_{G^*,G}, \qquad 0 \le t \le T,$$

of the forward claim price $\hat{V}_t := \hat{\mathbb{E}}[\hat{\xi}|\mathcal{F}_t]$, cf. (2.4), is given by

$$
\begin{aligned}
\phi_t(dy) \;=\; & \hat{\mathbb{E}}\left[\frac{\hat{P}_T(y)}{\hat{P}_t(y)} \hat{g}'(\hat{P}_T(\mu)) \Big| \mathcal{F}_t \right] \mu(dy) \\
& + \hat{\mathbb{E}}\left[(\hat{g}(\hat{P}_T(\mu)) - \hat{P}_T(\mu)\hat{g}'(\hat{P}_T(\mu))) \frac{\hat{P}_T(y)}{\hat{P}_t(y)} \Big| \mathcal{F}_t \right] \nu(dy),
\end{aligned}
$$

$0 \le t \le T,$

Proof. By Lemma 3.4 above we have, since $\hat{P}_t(\nu) = \int_T^\infty \frac{P_t(y)}{P_t(\nu)} \nu(dy) = 1$,

$$
\begin{aligned}
\langle \hat{\alpha}_t, dW_t \rangle_H \;=\; & \int_T^\infty \hat{\mathbb{E}}\left[\hat{g}'(\hat{P}_T(\mu))\hat{P}_T(y) \Big| \mathcal{F}_t \right] \zeta_t(y)\mu(dy)dW_t \\
& - \int_T^\infty \hat{\mathbb{E}}\left[\hat{P}_T(\mu)\hat{g}'(\hat{P}_T(\mu))\hat{P}_T(y) \Big| \mathcal{F}_t \right] \zeta_t(y)\nu(dy)dW_t \\
& + \int_T^\infty \hat{\mathbb{E}}\left[\hat{g}(\hat{P}_T(\mu))(\hat{P}_T(y) - \hat{P}_t(y)) \Big| \mathcal{F}_t \right] \zeta_t(y)\nu(dy)dW_t \\
\;=\; & \int_T^\infty \hat{\mathbb{E}}\left[\hat{g}'(\hat{P}_T(\mu))\hat{P}_T(y) \Big| \mathcal{F}_t \right] \mu(dy)\left(\frac{dP_t(y)}{P_t(y)} - r_t dt \right) \\
& - \int_T^\infty \hat{\mathbb{E}}\left[\hat{P}_T(\mu)\hat{g}'(\hat{P}_T(\mu))\hat{P}_T(y) \Big| \mathcal{F}_t \right] \nu(dy)\left(\frac{dP_t(y)}{P_t(y)} - r_t dt \right) \\
& + \int_T^\infty \hat{\mathbb{E}}\left[\hat{g}(\hat{P}_T(\mu))(\hat{P}_T(y) - \hat{P}_t(y)) \Big| \mathcal{F}_t \right] \nu(dy)\left(\frac{dP_t(y)}{P_t(y)} - r_t dt \right) \\
\;=\; & \int_T^\infty \hat{\mathbb{E}}\left[\hat{g}'(\hat{P}_T(\mu))\hat{P}_T(y) \Big| \mathcal{F}_t \right] \mu(dy)\frac{dP_t(y)}{P_t(y)} \\
& - \int_T^\infty \hat{\mathbb{E}}\left[\hat{P}_T(\mu)\hat{g}'(\hat{P}_T(\mu))\hat{P}_T(y) \Big| \mathcal{F}_t \right] \nu(dy)\frac{dP_t(y)}{P_t(y)} \\
& + \int_T^\infty \hat{\mathbb{E}}\left[\hat{g}(\hat{P}_T(\mu))(\hat{P}_T(y) - \hat{P}_t(y)) \Big| \mathcal{F}_t \right] \nu(dy)\frac{dP_t(y)}{P_t(y)} \\
\;=\; & \int_T^\infty \hat{\mathbb{E}}\left[\hat{P}_T(y)\hat{g}'(\hat{P}_T(\mu)) \Big| \mathcal{F}_t \right] \mu(dy)\frac{dP_t(y)}{P_t(y)} \\
& + \int_T^\infty \hat{\mathbb{E}}\left[\hat{P}_T(y)(\hat{g}(\hat{P}_T(\mu)) - \hat{P}_T(\mu)\hat{g}'(\hat{P}_T(\mu))) \Big| \mathcal{F}_t \right] \nu(dy)\frac{dP_t(y)}{P_t(y)}
\end{aligned}
$$

$$-\hat{\mathbb{E}}\left[\hat{g}(\hat{P}_T(\mu))\Big|\mathcal{F}_t\right]\frac{dP_t(\nu)}{P_t(\nu)}$$

$$= \frac{1}{M_t}\langle\phi_t, dP_t\rangle_{G^*,G} - \hat{V}_t\frac{dP_t(\nu)}{P_t(\nu)},$$

and by (2.7) and (3.5) we have

$$\langle\hat{\alpha}_t, d\hat{W}_t\rangle_H = \langle\hat{\alpha}_t, dW_t\rangle_H - \frac{1}{M_t}dM_t \cdot \langle\hat{\alpha}_t, dW_t\rangle_H$$

$$= \langle\hat{\alpha}_t, dW_t\rangle_H - \frac{1}{M_t}dM_t \cdot \left(\frac{1}{M_t}\langle\phi_t, dP_t\rangle_{G^*,G} - \frac{1}{M_t}\hat{V}_t dM_t\right)$$

$$= \langle\hat{\alpha}_t, dW_t\rangle_H - \frac{1}{M_t}dM_t \cdot \left(\langle\phi_t, d\hat{P}_t\rangle_{G^*,G} + \frac{1}{M_t}\langle\phi_t, \hat{P}_t\rangle_{G^*,G}dM_t\right.$$

$$\left. + \frac{1}{M_t}dM_t \cdot \langle\phi_t, d\hat{P}_t\rangle_{G^*,G} - \frac{1}{M_t}\hat{V}_t dM_t\right)$$

$$= \langle\hat{\alpha}_t, dW_t\rangle_H - \frac{1}{M_t}dM_t \cdot \left(\langle\phi_t, d\hat{P}_t\rangle_{G^*,G} + \frac{1}{M_t}dM_t \cdot \langle\phi_t, d\hat{P}_t\rangle_{G^*,G}\right)$$

$$= \frac{1}{M_t}\langle\phi_t, dP_t\rangle_{G^*,G} - \frac{1}{M_t}\hat{V}_t dM_t - \frac{1}{M_t}dM_t \cdot \langle\phi_t, d\hat{P}_t\rangle_{G^*,G}$$

$$= \langle\phi_t, d\hat{P}_t\rangle_{G^*,G}, \tag{3.21}$$

since

$$dP_t = M_t d\hat{P}_t + \hat{P}_t dM_t + dM_t \cdot d\hat{P}_t.$$

$\square$

When the forward price process $(\hat{P}_t)_{t\in\mathbf{R}_+}$ follows the dynamics (1.7), Relation (3.21) above shows that we have the relation

$$\langle\hat{\alpha}_t, d\hat{W}_t\rangle_H = \langle\phi_t, d\hat{P}_t\rangle_{G^*,G} = \langle\phi_t, \hat{\sigma}_t d\hat{W}_t\rangle_{G^*,G},$$

which shows that

$$\hat{\alpha}_t = \hat{\sigma}_t^* \phi_t,$$

and recovers (1.8).

4   Delta hedging

In this section we consider a G-valued asset price process $(X_t)_{t\in\mathbf{R}_+}$ and a numeraire $(M_t)_{t\in\mathbf{R}_+}$, and we assume that the forward asset price $\hat{X}_t := \hat{X}_t/M_t$, $t \in \mathbf{R}_+$, is

modeled by the diffusion equation

$$d\hat{X}_t = \hat{\sigma}_t(\hat{X}_t)d\hat{W}_t, \tag{4.1}$$

under the forward measure $\hat{\mathbb{P}}$ defined by (2.2), where $x \mapsto \hat{\sigma}_t(x) \in \mathcal{L}_{HS}(H,G)$ is a Lipschitz function from G into the space of Hilbert-Schmidt operators from H to G, uniformly in $t \in \mathbb{R}_+$,

Vanilla options

In this Markovian setting a Vanilla option with payoff

$$\xi = M_S \hat{g}(\hat{X}_T)$$

is priced at time t as

$$\mathbb{E}\left[e^{-\int_t^S r_s ds} M_S \hat{g}(\hat{X}_T)\Big|\mathcal{F}_t\right] = M_t \hat{\mathbb{E}}\left[\hat{g}(\hat{X}_T)\Big|\mathcal{F}_t\right] = M_t \hat{C}(t, \hat{X}_t), \tag{4.2}$$

for some measurable function $\hat{C}(t,x)$ on $\mathbb{R}_+ \times G$, and Lemma 2.1 has the following corollary.

Corollary 4.1 *Assume that the function $\hat{C}(t,x)$ is $\mathcal{C}^2$ on $\mathbb{R}_+ \times G$, and let*

$$\eta_t = \hat{C}(t, \hat{X}_t) - \langle \nabla \hat{C}(t, \hat{X}_t), \hat{X}_t \rangle_{G^*, G}, \qquad 0 \le t \le T.$$

Then the portfolio $(\nabla \hat{C}(t, \hat{X}_t), \eta_t)_{t \in [0,T]}$ with value

$$V_t = \eta_t M_t + \langle \nabla \hat{C}(t, \hat{X}_t), X_t \rangle_{G^*, G}, \qquad 0 \le t \le T,$$

is self-financing and hedges the claim $\xi = M_S \hat{g}(\hat{X}_T)$.

Proof. By Itô's formula, cf. Theorem 4.17 of [3], and the martingale property of $\hat{V}_t$ under $\hat{\mathbb{P}}$, the process $(\phi_t)_{t \in [0,T]}$ in the predictable representation (2.4) is given by

$$\phi_t = \nabla \hat{C}(t, \hat{X}_t), \qquad 0 \le t \le T.$$

$\square$

When

$$X_t = P_t(\mu) := \langle \mu, P_t \rangle_{G^*,G} = \int_T^\infty P_t(y)\mu(dy),$$

and

$$M_t = P_t(\nu) = \langle \nu, P_t \rangle_{G^*,G} = \int_T^\infty P_t(y)\nu(dy),$$

Corollary 4.1 shows that the portfolio

$$\phi_t(dy) = \frac{\partial \hat{C}}{\partial x}(t, \hat{X}_t)\mu(dy) + \left(\hat{C}(t, \hat{X}_t) - \hat{X}_t \frac{\partial \hat{C}}{\partial x}(t, \hat{X}_t)\right)\nu(dy), \qquad (4.3)$$

$0 \leq t \leq T$, where $\hat{C}(t,x)$ is defined in (4.2), is a self-financing hedging strategy for the claim

$$\xi = P_S(\nu)\hat{g}\left(\frac{P_T(\mu)}{P_T(\nu)}\right),$$

with $M_t = P_t(\nu)$, $t \in \mathbb{R}_+$.

When $G = \mathbb{R}$ and $(\hat{X}_t)_{t \in \mathbb{R}_+}$ is a geometric Brownian motion with deterministic volatility H-valued function $(\hat{\sigma}(t))_{t \in \mathbb{R}_+}$ under the forward measure $\hat{\mathbb{P}}$, i.e.

$$d\hat{X}_t = \hat{X}_t \hat{\sigma}_t(t)d\hat{W}_t, \qquad (4.4)$$

the exchange call option with payoff

$$M_S(\hat{X}_T - \kappa)^+,$$

is priced by the Black-Scholes-Margrabe [10] formula

$$\mathbb{E}\left[e^{-\int_t^S r_s ds}M_S(\hat{X}_T - \kappa)^+ \Big| \mathcal{F}_t\right] = X_t\Phi_+^0(t, \kappa, \hat{X}_t) - \kappa M_t\Phi_-^0(t, \kappa, \hat{X}_t), \qquad t \in \mathbb{R}_+,$$
$$(4.5)$$

where

$$\Phi_+^0(t, \kappa, x) = \Phi\left(\frac{\log(x/\kappa)}{v(t,T)} + \frac{v(t,T)}{2}\right), \quad \Phi_-^0(t, \kappa, x) = \Phi\left(\frac{\log(x/\kappa)}{v(t,T)} - \frac{v(t,T)}{2}\right),$$
$$(4.6)$$

and

$$v^2(t,T) = \int_t^T \hat{\sigma}^2(s)ds,$$

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cf. e.g. § 10.4 of [12]. By Corollary 4.1 and the relation

$$\frac{\partial \hat{C}}{\partial x}(t,x) = \Phi\left(\frac{\log(x/\kappa)}{v(t,T)} + \frac{v(t,T)}{2}\right) = \Phi_+^0(t,\kappa,x),$$

this yields a self-financing portfolio

$$(\Phi_+^0(t,\kappa,\hat{X}_t), -\kappa\Phi_-^0(t,\kappa,\hat{X}_t))_{t\in[0,T]}$$

in (X_t, M_t) that hedges the claim $\xi = M_S(\hat{X}_T - \kappa)^+$. In particular, when the short rate process $(r_t)_{t\in\mathbb{R}_+}$ is a deterministic function and $M_t = e^{-\int_t^T r_s ds}$, $0 \le t \le T$, (4.5) is Merton's "zero interest rate" version of the Black-Scholes formula, a property which has been used in [7] for the hedging of swaptions.

In particular, from (4.5) we have

$$\mathbb{E}\left[e^{-\int_t^S r_s ds} P_S(\nu)(\hat{X}_T - \kappa)^+ \Big| \mathcal{F}_t\right] = P_t(\nu)\hat{C}(t,\hat{X}_t) \tag{4.7}$$
$$= P_t(\mu)\Phi_+^0(t,\kappa,\hat{X}_t) - \kappa P_t(\nu)\Phi_-^0(t,\kappa,\hat{X}_t),$$

and the portfolio

$$\phi_t(dy) = \Phi_+^0(t,\kappa,\hat{X}_t)\mu(dy) - \kappa\Phi_-^0(t,\kappa,\hat{X}_t)\nu(dy), \qquad 0 \le t \le T, \tag{4.8}$$

is self-financing, hedges the claim $P_S(\nu)(\hat{X}_T - \kappa)^+$, and is evenly distributed with respect to $\mu(dy)$ and to $\nu(dy)$.

As applications of (4.3) and (4.7), we consider some examples of delta hedging, in which the asset allocation is uniform on $\mu(dy)$ and $\nu(dy)$ with respect to the bond maturities $y \in [T,\infty)$.

Bond options

Taking $S = T$, the bond option with payoff

$$\xi = M_T \hat{g}(P_T(U)), \qquad 0 \le T \le U,$$

belongs to the above framework with

$$\mu(dy) = \delta_U(dy) \quad \text{and} \quad \nu(dy) = \delta_T(dy),$$

hence $M_t = P_t(\nu) = P_t(T)$ and when $\hat{X}_t = P_t(U)/P_t(T)$ is Markov as in (4.1), the self-financing hedging strategy is given from (4.3) by

$$\phi_t(dy) = \frac{\partial \hat{C}}{\partial x}(t, \hat{X}_t)\delta_U(dy) + \left(\hat{C}(t, \hat{X}_t) - \hat{X}_t \frac{\partial \hat{C}}{\partial x}(t, \hat{X}_t)\right)\delta_T(dy). \qquad (4.9)$$

Furthermore, when $(\hat{X}_t)_{t \in \mathbb{R}_+}$ is a geometric Brownian motion given by (4.4) under $\hat{\mathbb{P}}$, the bond call option with payoff

$$(P_T(\mu) - \kappa P_T(\nu))^+ = (P_T(U) - \kappa)^+$$

is priced as

$$\mathbb{E}\left[e^{-\int_t^T r_s ds}(P_T(U) - \kappa)^+ \Big| \mathcal{F}_t\right] = P_t(U)\Phi_+^0(t, \kappa, \hat{X}_t) - \kappa P_t(T)\Phi_-^0(t, \kappa, \hat{X}_t),$$

and the corresponding hedging strategy is therefore given by

$$\phi_t(dy) = \Phi_+^0(t, \kappa, \hat{X}_t)\delta_U(dy) - \kappa\Phi_-^0(t, \kappa, \hat{X}_t)\delta_T(dy), \qquad (4.10)$$

from (4.8). When the dynamics of $(P_t)_{t \in \mathbb{R}_+}$ is given by (3.1) where $\zeta_t(y)$ is deterministic, $\hat{\sigma}(t)$ is given from (3.12) and Lemma 3.2 as

$$\hat{\sigma}(t) = \zeta_t(U) - \zeta_t(T), \qquad 0 \le t \le T \le U,$$

and we check that (4.10) coincides with the result (3.6) obtained in Section 3, cf. also page 207 of [13].

Caplets

Here we take $T < S$, $X_t = P_t(\mu) = P_t(T)$, $M_t = P_t(\nu) = P_t(S)$, with

$$\mu(dy) = \delta_T(dy) \quad \text{and} \quad \nu(dy) = \delta_S(dy),$$

and we consider the caplet with payoff (3.8) on the LIBOR rate (3.9), i.e.

$$\xi = (S - T)(L(T, T, S) - \kappa)^+ = (\hat{X}_T - (1 + \kappa(S - T)))^+.$$

Assuming that $\hat{X}_t = P_t(T)/P_t(S)$ is a (driftless) geometric Brownian motion under $\hat{\mathbb{P}}$ with $\hat{\sigma}(t)$ a deterministic function, this caplet is priced as in (4.7) as

$$(S - T)\,\mathbb{E}\left[e^{-\int_t^S r_s ds}(L(T, T, S) - \kappa)^+ \Big| \mathcal{F}_t\right]$$

$$
\begin{aligned}
&= M_t \hat{\mathbb{E}} \left[(\hat{X}_T - (1 + \kappa(S - T)))^+ \Big| \mathcal{F}_t \right] \\
&= P_t(T) \Phi_+^0(t, 1 + \kappa(S - T), \hat{X}_t) \\
&\quad - (1 + \kappa(S - T)) \Phi_-^0(t, 1 + \kappa(S - T), \hat{X}_t) P_t(S),
\end{aligned}
$$

since $P_S(\nu) = 1$, and the corresponding hedging strategy is given as in (4.8) by

$$
\phi_t(dy) = \Phi_+^0(t, 1 + \kappa(S - T), \hat{X}_t) \delta_T(dy) - (1 + \kappa(S - T)) \Phi_-^0(t, 1 + \kappa(S - T), \hat{X}_t) \delta_S(dy).
$$
(4.11)

When the dynamics of $(P_t)_{t \in \mathbf{R}_+}$ is given by (3.1), where $\zeta_t(y)$ in (3.1) is deterministic, Lemma 3.2 shows that $\hat{\sigma}(t)$ in (4.4) can be taken as

$$
\hat{\sigma}(t) = \zeta_t(T) - \zeta_t(S), \qquad 0 \le t \le T \le S,
$$

and in this case (4.11) coincides with Relation (3.10) above.

Hedging strategies for caps are easily computed by summation of hedging strategies for caplets.

Swaptions on LIBOR rates

Consider a tenor structure $\{T \le T_i, \ldots, T_j\}$ and the swaption on the LIBOR rate with payoff

$$
\xi = P_T(\nu) \hat{g} \left(\frac{P_T(T_i) - P_T(T_j)}{P_T(\nu)} \right),
$$
(4.12)

where

$$
\hat{X}_t = \frac{P_t(\mu)}{P_t(\nu)} = \frac{P_t(T_i) - P_t(T_j)}{P_t(\nu)}, \qquad 0 \le t \le T,
$$

is the swap rate, which is a martingale under $\hat{\mathbb{P}}$, in which case we have

$$
\mu(dy) = \delta_{T_i}(dy) - \delta_{T_j}(dy) \quad \text{and} \quad \nu(dy) = \sum_{k=i}^{j-1} \tau_k \delta_{T_{k+1}}(dy)
$$

and

$$
M_t = P_t(\nu) = \sum_{k=i}^{j-1} \tau_k P_t(T_{k+1})
$$

is the annuity numeraire.

When $(\hat{X}_t)_{t \in \mathbb{R}_+}$ is Markov as in (4.1), the self-financing hedging strategy of the swaption with payoff (4.12) is given by (4.3) as

$$\phi_t(dy) \;=\; \frac{\partial \hat{C}}{\partial x}(t, \hat{X}_t)\delta_{T_i}(dy) + \left(\hat{C}(t, \hat{X}_t) - \hat{X}_t\frac{\partial \hat{C}}{\partial x}(t, \hat{X}_t) \right) \sum_{k=i+1}^{j-1} \tau_{k-1}\delta_{T_k}(dy)$$
$$+ \left(\tau_{j-1}\hat{C}(t, \hat{X}_t) - (1 + \tau_{j-1}\hat{X}_t)\frac{\partial \hat{C}}{\partial x}(t, \hat{X}_t) \right) \delta_{T_j}(dy),$$

$0 \le t \le T$.

Finally we assume that the swap rate

$$\hat{X}_t := \frac{P_t(T_i) - P_t(T_j)}{\sum_{k=i}^{j-1} \tau_k P_t(T_{k+1})}, \qquad 0 \le t \le T,$$

is modeled according to a driftless geometric Brownian motion under the forward swap measure $\hat{\mathbb{P}}$ determined by $M_t := \sum_{k=i}^{j-1} \tau_k P_t(T_{k+1})$, $t \in \mathbb{R}_+$, with $(\hat{\sigma}(t))_{t \in [0,T]}$ a deterministic function. In this case the swaption with payoff

$$(P_T(\mu) - \kappa P_T(\nu))^+ = (P_T(T_i) - P_T(T_j) - \kappa P_T(\nu))^+,$$

priced from (4.7) as

$$\mathbb{E}\left[e^{-\int_t^T r_s ds}(P_T(T_i) - P_T(T_j) - \kappa P_T(\nu))^+ \Big| \mathcal{F}_t \right]$$
$$= (P_t(T_i) - P_t(T_j))\Phi_+^0(t, \kappa, \hat{X}_t) - \kappa P_t(\nu)\Phi_-^0(t, \kappa, \hat{X}_t)$$

has the self-financing hedging strategy

$$\phi_t(dy) \;=\; \Phi_+^0(t, \kappa, \hat{X}_t)\delta_{T_i}(dy) - (\Phi_+^0(t, \kappa, \hat{X}_t) + \kappa\tau_{j-1}\Phi_-^0(t, \kappa, \hat{X}_t))\delta_{T_j}(dy)$$
$$- \kappa\Phi_-^0(t, \kappa, \hat{X}_t) \sum_{k=i+1}^{j-1} \tau_{k-1}\delta_{T_k}(dy), \tag{4.13}$$

by (4.8). This recovers the self-financing hedging strategy of [7]. The above hedging strategy (4.13) also shares the same maturity dates as (3.11) above, although it is stated in a different model.

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