Large time behavior of reaction-diffusion equations with Bessel generators

José Alfredo López-Mimbela Nicolas Privault

Abstract

We investigate explosion in finite time of one-dimensional semilinear equations of the form

$$\frac{\partial u_t}{\partial t}(x) = \frac{1}{2}\frac{\partial^2 u_t}{\partial x^2}(x) + \frac{\varphi'(x)}{\varphi(x)}\frac{\partial u_t}{\partial x}(x) - \frac{a}{x^2}u_t(x) + u_t^{1+\beta}(x)$$

with initial value $\phi \geq 0$, where $\varphi \in \mathcal{C}^2(\mathbb{R})$ is positive and $a \geq 0$, $\beta > 0$ are constants. In the free case $a = 0$ we provide conditions on φ under which any positive nontrivial solution is non-global. In the case $a > 0$ and $\varphi(x) = x^{\mu+1/2}$, $\mu \in \mathbb{R}$, which includes in the special case $\mu = -1/2$ the equation

$$\frac{\partial u_t}{\partial t}(x) = \frac{1}{2}\frac{\partial^2 u_t}{\partial x^2}(x) - \frac{a}{x^2}u_t(x) + u_t^{1+\beta}(x),$$

we use the Feynman-Kac formula for Bessel processes to give conditions on the equation parameters ensuring finite-time blowup and existence of nontrivial positive global solutions.

Key words: Semilinear PDEs, Bessel processes, Feynman-Kac representation, critical exponent, finite time blow-up, global solution.

Mathematics Subject Classification: 60H30, 35K55, 35K57, 35B35.

1 Introduction

Consider a semilinear PDE of the form

$$\begin{cases} \dfrac{\partial u_t}{\partial t}(x) = Au_t(x) - V(x)u_t(x) + G(u_t(x)), & t > 0, \\ u_0(x) = \phi(x), & x \in \mathbb{R}^N, \end{cases}$$

where A is the generator of a nice Markov process in $\mathbb{R}^N$ and $V(x)$, $x \in \mathbb{R}^N$, is a nonnegative potential, $G(z) \geq 0$ is a nonlinear term which is locally Lipschitz, say of the form $G(z) = z^{1+\beta}$ for some $\beta > 0$, and the initial value $u_0(x) = \phi(x)$ is bounded and nonnegative. In this setting, a positive solution $u_t(x)$ will either exist up to a finite life span τ_f, and in this case $\|u_t\|_\infty$ explodes as t approaches τ_f from below, or exists globally in the sense that $\|u_t\|_\infty < \infty$ for all $t > 0$. In the former case it is said that $u_t(x)$ blows up in finite time, and in the latter that $u_t(x)$ is a global solution.

It is well-known that, when $V = 0$ and $A = \Delta$ is the Laplacian in the N-dimensional Euclidean space, the ratio $2/N$ rules out the asymptotic growth of $u_t(x)$. More precisely, if $\beta \leq 2/N$ then, apart from 0, there are no nonnegative global solutions, whereas if $\beta > 2/N$, the equation admits nontrivial positive global solutions.

When $V(x) > 0$ is constant one can prove under mild conditions on A that $u_t(x)$ must be global if we choose ϕ appropriately small. This follows from the fact that, since $V > 0$ and e^{tA} is a contraction for every $t > 0$, $\|e^{-tV}e^{tA}\varphi\|_\infty$ decays exponentially fast as $t \to \infty$, and this suffices to conclude that $u_t(x)$ is global if $\phi(x)$ decays quickly enough to 0 as $\|x\|$ goes to infinity.

The case when $V(x) \geq 0$ is non-constant is less clear and has been studied both in the analytic and probabilistic literature. For example, critical exponents for the finite time blow-up of the semilinear equation

$$\begin{cases} \dfrac{\partial u_t}{\partial t}(x) = \dfrac{1}{2}\Delta u_t(x) - V(x)u_t(x) + u_t^{1+\beta}(x), \quad t > 0, \\[2ex] u_0(x) = \phi(x), \quad x \in \mathbb{R}^N, \end{cases}$$

in dimensions $N \geq 3$, where $\phi(x) \geq 0$ and $V(x)$ is bounded above (or below) by $a/(1 + |x|^b)$, $x \in \mathbb{R}^N$, with $a > 0$ and $b \in [2, \infty)$, have been studied in [11], [14], [15]. In [7] we treated this type of problem using heat kernel estimates and the Feynman-Kac representation, again for $N \geq 3$, but including in the critical value $b = 2$. It turns out that, if $N \geq 3$ and

$$0 \leq V(x) \leq \frac{a}{1 + \|x\|^b}, \qquad x \in \mathbb{R}^N,$$

where $a > 0$ and $b > 2$, then $u_t(x)$ blows up in finite time for all $0 < \beta < 2/N$. On the contrary, if

$$V(x) \geq \frac{a}{1 + \|x\|^b}, \qquad x \in \mathbb{R}^N,$$

with $a > 0$ and $0 \leq b < 2$, then $u_t(x)$ can be global for any $\beta > 0$ provided $\phi(x)$ decays sufficiently fast to 0 as x goes to infinity. For the "critical" value $b = 2$, blowup occurs when $0 < \beta < \beta_*(a)$, where the upper bound $\beta_*(a)$ depends rather sensitively on a; see [6], [7], [11]. Thus, substracting a nontrivial potential to the diffusion term in our equation may be a delicate question.

The unbounded potential $V(x) = a/|x|^2$ has been considered when $N \geq 3$ in [5], [1], and [2], where it is shown that (1.6) admits a unique critical exponent $\beta(a) < 2/N$, given by

$$\beta(a) = \frac{4}{2 + N + \sqrt{8a + (N-2)^2}}. \tag{1.1}$$

Namely, if $V(x) = a|x|^{-2}$, then no global nontrivial solution of (1.6) exists if $\beta < \beta(a)$, whereas global solutions exist if $\beta > \beta(a)$.

As noted above, most of the existing literature on the blowup of semilinear PDEs with potential deals with the case $N \geq 3$. In this paper we investigate the one-dimensional case of semilinear equations of the form

$$\begin{cases} \dfrac{\partial u_t}{\partial t}(x) = \dfrac{1}{2}\dfrac{\partial^2 u_t}{\partial x^2}(x) + \dfrac{\varphi'(x)}{\varphi(x)}\dfrac{\partial u_t}{\partial x}(x) - \dfrac{a}{x^2}u_t(x) + u_t^{1+\beta}(x), \quad t > 0, \\ \\ u_0(x) = \phi(x) \geq 0, \quad x > 0, \end{cases} \tag{1.2}$$

where $\beta > 0$ and $a \geq 0$ are constants, and ϕ is a non identically vanishing measurable function on $\mathbb{R}$. The case $N = 1$ has several features that make it different from that of $N \geq 3$, in particular the potential $V(x) = a/x^2$, $a > 0$, is not integrable around 0 and the underlying Brownian motion returns to 0 with probability one.

We proceed as follows. In Section 2 we first consider the "free case" $a = 0$ and $\mathcal{C}^2$ functions $\varphi \in L^2(\mathbb{R})$ such that φ'/φ is bounded. Using Jensen's inequality, together with the fact that $\mu(dx) = \varphi^2(x)\,dx$ is an invariant measure of the semigroup $(T_t)_{t\in\mathbb{R}_+}$ with generator

$$L^\varphi f(x) := \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x) + \frac{\varphi'(x)}{\varphi(x)}\frac{\partial f}{\partial x}(x),$$

we arrive at the conclusion that $\|u_t\|_\infty$ explodes in finite time.

In the remaining sections 3, 4, 5 and 6 we consider the case $a \geq 0$ with

$$\varphi(x) = x^{\mu+1/2}, \quad x \in \mathbb{R},$$

where $\mu \in \mathbb{R}$, which renders the equation

$$\begin{cases} \dfrac{\partial u_t}{\partial t}(x) = L_\mu u_t(x) - \dfrac{a}{x^2}u_t(x) + u_t^{1+\beta}(x), \quad t > 0, \\ \\ u_0(x) = \phi(x), \quad x > 0, \end{cases} \tag{1.3}$$

where

$$L_\mu = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{2\mu+1}{2x}\frac{\partial}{\partial x}$$

is the generator of the Bessel process of index $\mu \in \mathbb{R}$, and $\beta > 0$ is a constant. This case falls apart from the setting outlined above, including in the case of $a = 0$, therefore we need to resort to other techniques. Our main tools here are the Feynman-Kac representation of (1.3), as well as certain bounds established in Sections 3 and 4 for the heat kernel and for the conditional moments of the Bessel generator perturbed by $V(x)$. This approach also allows us to deal with a class of convex increasing nonlinearities $G(z)$, and with certain time-dependent nonlinear terms of the form $t^\zeta G(u_t(x))$.

Next in Section 5 we prove that for $\mu \in \mathbb{R}$, if $a > 0$ and

$$\beta < \frac{2+\mu-\sqrt{\mu^2+2a}}{2+\mu+\sqrt{\mu^2+2a}}, \tag{1.4}$$

then (1.3) possesses no nontrivial positive global solutions, cf. Corollary 5.3. We also deal with a semilinear equation whose nonlinear term is of the form $t^\zeta G(u_t(x))$, where $\zeta \geq 0$ is a constant and $G(z)$ is a positive increasing convex function satisfying certain growth conditions, see (5.3) below.

In Section 6 we show that for $\mu \in \mathbb{R}$ and $a > 0$, (1.3) admits a global positive nontrivial solution when

$$\beta > \frac{2}{2+\mu+\sqrt{\mu^2+2a}}, \tag{1.5}$$

cf. Theorem 6.2. When $\mu > -1$ and $a = 0$ we find $2/d := (1+\mu)^{-1}$ as the critical exponent for explosion of (1.3), where $d = 2 + 2\mu$ denotes the dimension of the underlying Bessel process with parameter $\mu \in \mathbb{R}$. Notice that when $\mu \geq 0$, (1.4) and (1.5) recover the critical exponent $(1+\mu)^{-1}$ as a tends to 0, while in case $\mu < 0$ the equation exhibits a discontinuous behavior and the bounds (1.4) and (1.5) do not apply when $a = 0$. This is due to the fact that the Bessel process almost surely does not return to 0 when $\mu \geq 0$, and returns to 0 in finite time with strictly positive probability when $\mu < 0$. In particular when $\mu = -1/2$ and $a > 0$, i.e. $d = 1$, this shows that the equation

$$\frac{\partial u_t}{\partial t}(x) = \frac{1}{2}\frac{\partial^2 u_t}{\partial x^2}(x) - \frac{a}{x^2}u_t(x) + u_t^{1+\beta}(x), \tag{1.6}$$

(with a nontrivial positive u_0) blows up in finite time for

$$\beta < \frac{3-\sqrt{1+8a}}{3+\sqrt{1+8a}} < \beta(a),$$

and admits a global solution when

$$\beta > \frac{4}{3+\sqrt{1+8a}} = \beta(a),$$

which partly extends (1.1) to the case $N = 1$. As noted above, when a tends to 0 the above thresholds do not recover the critical exponent 2, which is obtained separately in Corollary 5.3 and Theorem 6.2 below.

2 The free case

In this section we consider the case $a = 0$ and prove that (1.2) blows-up in finite time when $\varphi : \mathbb{R} \to (0, \infty)$ is in $L^2(\mathbb{R})$, of class $\mathcal{C}^2$, and such that the function

$$x \longmapsto \frac{\varphi'(x)}{\varphi(x)}, \quad x \in \mathbb{R},$$

is bounded. First, let us explain our method in the particular case

$$\varphi(x) = e^{-x^2/2}, \qquad x \in \mathbb{R},$$

in which (1.2) becomes

$$\frac{\partial u_t}{\partial t}(x) = \frac{1}{2}\frac{\partial^2 u_t}{\partial x^2}(x) - x\frac{\partial u_t}{\partial x}(x) + u_t^{1+\beta}(x), \qquad t > 0,$$

where

$$L^\varphi = \frac{1}{2}\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial x}$$

is the generator of the Ornstein-Uhlenbeck process with semigroup $(T_t)_{t\in\mathbb{R}_+}$. Then the semigroup $(Q_t)_{t\in\mathbb{R}_+}$ defined by

$$Q_t f(x) := e^{-x^2/2} T_t \left(e^{x^2/2} f(x) \right), \qquad t \in \mathbb{R}_+,$$

is the Markov semigroup corresponding to the harmonic oscillator, a real-valued Gauss-Markov process having generator

$$H = \frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{x^2}{2} + \frac{1}{2}. \tag{2.1}$$

Since Q_t can be written as

$$Q_t f(x) = \int_{-\infty}^{\infty} K_t(x,y) f(y)\, dy,$$

where the kernel $K_t(x,y)$ is given by Mehler's formula, it is easy to verify that for any positive $\phi \in L^2(\mu)$ and all sufficiently large t we have

$$Q_t\phi(x) \geq c\kappa(\beta t)^{-1/\beta}, \qquad x \in \mathbb{R}, \tag{2.2}$$

for some constant $c > 0$, and

$$\kappa = \inf_{t\geq 1} \inf_{|x|,|y|<1} K_t(x,y) > 0,$$

provided ϕ does not identically vanish. The inequality (2.2) above ensures finite-time blowup of the solution of

$$\begin{cases} \dfrac{\partial u_t}{\partial t}(x) = H u_t(x) + u_t^{1+\beta}(x), \\ \\ u_0(x) = \phi(x), \qquad x \in \mathbb{R}, \quad t > 0, \end{cases}$$

see [8] and [9]. From here we can infer that the norm $\|u_t\|_\infty$ of the solution $u_t(x)$ to (1.2) with $a = 0$ explodes in finite time.

In the general case, when the function $\varphi(x)$ is not specified, one cannot expect to know explicitly the transition densities of $(Q_t)_{t\in\mathbb{R}_+}$, however we obtain the following result.

Theorem 2.1 *Let $\varphi \in L^2(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R})$ be a positive function such that*

$$x \mapsto \frac{\varphi'(x)}{\varphi(x)}, \quad x \in \mathbb{R},$$

is bounded. Let $\mu(dx) := \varphi^2(x)\, dx$, and let G be a positive convex function such that

$$G(z) \geq c z^{1+\beta} \textit{ for all } z \geq 0,$$

where $\beta > 0$ and $c > 0$ is constant. Then the norm $\|u_t\|_{L^1(\mu)}$ of any positive solution of

$$\begin{cases} \dfrac{\partial u_t}{\partial t}(x) = L^\varphi u_t(x) + G(u_t(x)), \\ \\ u_0(x) = \phi(x) \geq 0, \qquad x \in \mathbb{R}, \quad t > 0, \end{cases} \tag{2.3}$$

blows up in finite time, provided its initial value $\phi \geq 0$ has the form

$$\phi(x) = \frac{h(x)}{\varphi(x)} \geq 0, \quad x \in \mathbb{R}, \tag{2.4}$$

for some positive nontrivial $h \in \mathcal{C}^2(\mathbb{R}) \bigcap L^2(\mu)$.

Proof. By a classical comparison argument, see e.g. Lemma 3.1 in [6], it suffices to consider the case $G(z) = z^{1+\beta}$, $z \geq 0$. Writing again $(T_t)_{t\in\mathbb{R}_+}$ for the semigroup with generator L^φ, which now is given by

$$L^\varphi := \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\varphi'(x)}{\varphi(x)}\frac{\partial}{\partial x},$$

we get

$$u_t(x) = T_t\phi(x) + \int_0^t T_{t-s}u_s^{1+\beta}(x)\,ds, \qquad x \in \mathbb{R}, \quad t \geq 0. \tag{2.5}$$

By multiplying the above equation by $\varphi(x)$, letting $w_t(x) := \varphi(x)u_t(x)$ and using (2.4), we obtain

$$w_t(x) = \varphi(x)T_t\left(\frac{h}{\varphi}\right)(x) + \int_0^t \varphi(x)T_{t-s}\left(w_s^{1+\beta}\varphi^{-\beta-1}\right)(x)\,ds.$$

Notice that

$$f \mapsto Q_t^\varphi f := \varphi T_t\left(\frac{f}{\varphi}\right), \quad t \geq 0, \tag{2.6}$$

defines a semigroup $(Q_t^\varphi)_{t\in\mathbb{R}_+}$ of bounded linear operators on $L^2(\mu)$ with infinitesimal generator given by

$$H^\varphi f = \frac{1}{2}\frac{\partial^2}{\partial x}f - \frac{1}{2}\frac{\varphi''}{\varphi}f, \qquad f \in \mathcal{C}^2(\mathbb{R}), \tag{2.7}$$

and that

$$w_t(x) = (Q_t^\varphi h)(x) + \int_0^t Q_{t-s}^\varphi\left(w_s^{1+\beta}(\cdot)\varphi^{-\beta}\right)(x)\,ds. \tag{2.8}$$

Let now

$$\mathcal{E}(f) = \langle f, \varphi\rangle_{L^2(\mathbb{R})}, \quad f \in L^2(\mu).$$

Since $\tilde{h} := \varphi^2$ satisfies

$$\frac{1}{2}\frac{\partial^2\tilde{h}}{\partial x^2}(x) - \frac{\partial}{\partial x}\left(\tilde{h}(x)\frac{\varphi'(x)}{\varphi(x)}\right) = 0,$$

it turns out that $\mu(dx) = \varphi^2(x)\,dx$ is an invariant probability measure of $(T_t)_{t\in\mathbb{R}_+}$ up to normalization, and it follows from (2.8) that

$$\mathcal{E}(w_t) = \mathcal{E}(h) + \int_0^t \mathcal{E}\left(w_s^{1+\beta}\varphi^{-\beta}\right)\,ds. \tag{2.9}$$

Now,

$$\begin{aligned}
\mathcal{E}\left(w_s^{1+\beta}\varphi^{-\beta}\right) &= \|\varphi\|_{L^2}^2 \int_{-\infty}^{\infty} \left(\frac{w_s^{1+\beta}}{\varphi^{1+\beta}}\right)(x)\frac{\varphi^2(x)}{\|\varphi\|_{L^2}}\,dx \\
&\geq \frac{\|\varphi\|_{L^2}^2}{\|\varphi\|_{L^2}^{2+2\beta}} \left(\int_{\mathbb{R}} \frac{w_s(x)}{\varphi(x)}\varphi^2(x)\,dx\right)^{1+\beta} \\
&= \|\varphi\|_{L^2}^{-2\beta}(\mathcal{E}(w_s))^{1+\beta},
\end{aligned}$$

where we used Jensen's inequality. Hence from (2.9) we have

$$\mathcal{E}(w_t) \geq \mathcal{E}(w_0) + \|\varphi\|_{L^2}^{-2\beta} \int_0^t (\mathcal{E}(w_s))^{1+\beta} ds.$$

Let now $y(t)$ be the solution to the ordinary differential equation

$$y(t) = \mathcal{E}(w_0) + \|\varphi\|_{L^2}^{-2\beta} \int_0^t y^{1+\beta}(s) ds. \tag{2.10}$$

Since $\mathcal{E}(w_t)$ is a supersolution of (2.10) and $y(t)$ explodes at time

$$t_{h,\varphi} = \frac{\|\varphi\|_{L^2}^{2\beta}}{\beta\left(\int_{-\infty}^{\infty} h(x)\varphi(x)\,dx\right)^{\beta}},$$

it follows that

$$\mathcal{E}(w_t) = \|u_t\|_{L^1(\mu)} = +\infty, \qquad t \geq t_{h,\varphi}.$$

□

From the inequality

$$\int \varphi^2(x) u_t(x)\,dx \;\leq\; \|u_t\|_\infty \int_{\mathbb{R}} \varphi^2(x)\,dx, \qquad t > 0,$$

we get the following corollary of Theorem 2.1.

Corollary 2.2 *Let $\beta > 0$, and let $\phi \geq 0$ be of the form (2.4). Then under the assumptions of Theorem 2.1, the solution of* (2.3) *blows up in finite time.*

3 Heat kernel estimates

In order to apply the Feynman-Kac formula to the study of equations of the form (1.3) we need to study the transition function of the Bessel semigroup.

Let $\mathbb{P}_\mu$ denote the law of the Bessel process $(R_t)_{t\in\mathbb{R}_+}$ with parameter $\mu \in \mathbb{R}$, generator

$$L_\mu := \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{2\mu+1}{2x}\frac{\partial}{\partial x}, \qquad x > 0,$$

and transition density $q_t^\mu(x,y)$, given when $\mu > -1$ by

$$q_t^\mu(x,y) = \frac{1}{t}\left(\frac{y}{x}\right)^\mu y e^{-\frac{x^2+y^2}{2t}} I_\mu\left(\frac{xy}{t}\right), \qquad x, y, t > 0,$$

where I_μ denotes the modified Bessel function of the first kind of order $\mu > -1$, cf. e.g. [3], Theorem 9.1.

The measure $\mathbb{P}_\mu$ is a probability measure on the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+})$, where Ω is the space $C(\mathbb{R}_+, \mathbb{R}_+)$ of nonnegative continuous functions on $\mathbb{R}_+$, $\mathcal{F} = \sigma\{R_s,\ s \geq 0\}$ and $\mathcal{F}_t = \sigma\{R_s,\ 0 \leq s \leq t\}$, $t \geq 0$. Here $R_s(\omega) = \omega(s)$ for all $s \in \mathbb{R}_+$ and $\omega \in \Omega$. Recall that when

$$d := 2 + 2\mu$$

is a positive integer we have $R_t = \|W_t\|$ under $\mathbb{P}_\mu$, where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion in $\mathbb{R}^d$, and $(R_t)_{t\in\mathbb{R}_+}$ solves the stochastic differential equation

$$dR_t = \frac{2\mu+1}{2}\frac{dt}{R_t} + dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a one-dimensional standard Brownian motion. Moreover, for $\mu, \nu \in \mathbb{R}$ and F any $\mathcal{F}_t$-measurable positive random variable, we have

$$\begin{aligned} & \mathbb{E}_\mu\left[\left(\frac{x}{R_t}\right)^\mu F \exp\left(-\frac{\nu^2}{2}\int_0^t \frac{ds}{R_s^2}\right)\mathbf{1}_{\{t<\tau_0\}}\Big| R_0 = x\right] \qquad (3.1) \\ = \ & \mathbb{E}_\nu\left[\left(\frac{x}{R_t}\right)^\nu F \exp\left(-\frac{\mu^2}{2}\int_0^t \frac{ds}{R_s^2}\right)\mathbf{1}_{\{t<\tau_0\}}\Big| R_0 = x\right], \qquad x \geq 0, \end{aligned}$$

where $\mathbb{E}_\mu$ denotes the expectation under $\mathbb{P}_\mu$ and τ_0 denotes the first time $(R_t)_{t\in\mathbb{R}_+}$ reaches 0, cf. [12] Lemma (4.5), [10] Chapter XI, Exercise 1.22, and [13] Chapter 6, § 2.2.

For any $a \geq 0$ let $p_t^a(x,y)$, $t > 0$, denote the transition densities of the Markov process $(X_t)_{t\in\mathbb{R}_+}$ with generator $L_\mu - V$, where $V(x) = a/x^2$, $x > 0$. Recall that from the Feynman-Kac formula we have

$$p_t^a(x,y) = q_t^\mu(x,y)\,\mathbb{E}_\mu\left[\exp\left(-a\int_0^t \frac{1}{R_s^2}\,ds\right)\Big| R_t = y,\ R_0 = x\right], \qquad x, y \geq 0, \quad (3.2)$$

for all $\mu \in \mathbb{R}$ and $a \geq 0$. On the other hand, letting

$$\tau_0 = \inf\{t > 0 \ : \ R_t = 0\}$$

denote the first hitting time of 0 by $(R_t)_{t\in\mathbb{R}_+}$, it is known that $\tau_0 = +\infty$ when $\mu \geq 0$, and the integral $\displaystyle\int_0^t \frac{ds}{R_s^2}$ diverges a.s. when $\mu < 0$, cf. [12] and [13] Chapter 6. Hence we also have

$$p_t^a(x,y) = q_t^\mu(x,y)\,\mathbb{E}_\mu\left[\exp\left(-a\int_0^t \frac{1}{R_s^2}\,ds\right)\mathbf{1}_{\{t<\tau_0\}}\Big| R_t = y,\ R_0 = x\right], \qquad x,y \geq 0,$$

for all $\mu \in \mathbb{R}$ and $a > 0$. Note that in general, (3.2) yields

$$\lim_{a\searrow 0} p_t^a(x,y) = p_t^0(x,y)\mathbb{P}_\mu(t < \tau_0),$$

thus, when $\mu < 0$, $p_t^a(x,y)$ does not converge to $p_t^0(x,y)$ as a tends to 0.

For any $\mu \in \mathbb{R}$ and $a \geq 0$, let

$$\nu := \begin{cases} \sqrt{\mu^2+2a}, & \text{if } a > 0, \\ \mu, & \text{if } a = 0, \end{cases}$$

and

$$n := \frac{\nu-\mu}{2} = \begin{cases} \dfrac{\sqrt{\mu^2+2a}-\mu}{2}, & \text{if } a > 0, \\ 0, & \text{if } a = 0. \end{cases} \tag{3.3}$$

Moreover, when $a = 0$ we will assume that $\nu = \mu > -1$.

Lemma 3.1 *For all $\mu \in \mathbb{R}$ and $a \geq 0$ we have*

$$p_t^a(x,y) = x^{2n}y^{-2n}q_t^\nu(x,y), \qquad x,y,t \geq 0. \tag{3.4}$$

Proof. Clearly it suffices to consider the case $a > 0$. By an application of (3.1) to

$$F := \left(\frac{x}{R_t}\right)^{-\mu} U \exp\left(\frac{\mu^2}{2}\int_0^t \frac{ds}{R_s^2}\right),$$

where U is an $\mathcal{F}_t$-measurable non-negative random variable, we get

$$\mathbb{E}_\mu\left[U\exp\left(-\frac{\nu^2-\mu^2}{2}\int_0^t\frac{ds}{R_s^2}\right)\mathbf{1}_{\{t<\tau_0\}}\Big| R_0 = x\right] = \mathbb{E}_\nu\left[\left(\frac{x}{R_t}\right)^{\nu-\mu} U\mathbf{1}_{\{t<\tau_0\}}\Big| R_0 = x\right], \tag{3.5}$$

hence

$$\begin{aligned}
&\mathbb{E}_\mu \left[\exp\left(-\frac{\nu^2-\mu^2}{2} \int_0^t \frac{ds}{R_s^2} \right) \mathbf{1}_{\{R_t \in dy\}} \mathbf{1}_{\{t<\tau_0\}} \Big| R_0 = x \right] \\
&\quad = \mathbb{E}_\nu \left[\left(\frac{x}{R_t} \right)^{\nu-\mu} \mathbf{1}_{\{R_t \in dy\}} \mathbf{1}_{\{t<\tau_0\}} \Big| R_0 = x \right] \\
&\quad = \left(\frac{x}{y} \right)^{\nu-\mu} \mathbb{P}_\nu(R_t \in dy \mid R_0 = x),
\end{aligned}$$

and

$$\mathbb{E}_\mu \left[\exp\left(-\frac{\nu^2-\mu^2}{2} \int_0^t \frac{ds}{R_s^2} \right) \mathbf{1}_{\{t<\tau_0\}} \Big| R_t = y, \ R_0 = x \right] = \left(\frac{x}{y} \right)^{\nu-\mu} \frac{q_t^\nu(x,y)}{q_t^\mu(x,y)},$$

which yields (3.4) due to (3.2). □

Note that in case $\mu < -1$, letting $\tilde{\nu} = -\sqrt{\mu^2+2a} < -1$ and $\tilde{n} = (\tilde{\nu}-\mu)/2$, $a \geq 0$, the above argument would yield the upper bound

$$p_t^a(x,y) \leq x^{2\tilde{n}} y^{-2\tilde{n}} q_t^{\tilde{\nu}}(x,y), \qquad x,y,t \geq 0, \tag{3.6}$$

which is weaker than (3.4) but remains valid in the limit as a tends to 0.

From Lemma 3.1 we deduce the following lower bounds. In the sequel, $c > 0$ denotes a generic positive constant whose value depends on the context.

Lemma 3.2 *Let $\mu \in \mathbb{R}$ and $a > 0$. For all sufficiently large $t > t_0$ we have*

$$p_t^a(x,y) \geq c t^{-\nu-1} x^{2n} y^{2n+d-1} \mathbf{1}_{[0,\sqrt{t}]}(x) \mathbf{1}_{[0,\sqrt{t}]}(y), \qquad x,y \geq 0. \tag{3.7}$$

Proof. From the equivalence

$$I_\nu(z) \simeq c z^\nu$$

as z tends to 0, which is valid for $\nu > -1$, we get

$$q_t^\nu(x,y) \simeq c t^{-\nu-1} y^{2\nu+1} e^{-\frac{x^2+y^2}{2t}}, \qquad \text{as } t \to \infty,$$

which, due to Lemma 3.1, shows that for all $x,y \geq 0$ we have

$$p_t^a(x,y) \simeq c t^{-2n-d/2} x^{2n} y^{2n+d-1} e^{-\frac{x^2+y^2}{2t}}, \qquad \text{as } t \to \infty, \tag{3.8}$$

hence (3.7) holds. □

When $\mu > -1$ and $a = 0$, an argument similar to that of Lemma 3.2 yields

$$p_t^0(x,y) \geq ct^{-\mu-1}y^{d-1}\mathbf{1}_{[0,\sqrt{t}]}(x)\mathbf{1}_{[0,\sqrt{t}]}(y).$$

As a consequence of Lemma 3.2, for all sufficiently large $t > 0$ we have for all $x, y \geq 0$,

$$p_t^a(x,y) \geq ct^{-\nu-1}y^{2n+d-1}\mathbf{1}_{[\alpha,\sqrt{t}]}(x)\mathbf{1}_{[0,\sqrt{t}]}(y), \tag{3.9}$$

and

$$p_t^a(x,y) \geq ct^{-\nu-1}x^{2n}\mathbf{1}_{[0,\sqrt{t}]}(x)\mathbf{1}_{[\alpha,\sqrt{t}]}(y). \tag{3.10}$$

The next lemma provides upper bounds for the transition densities $p_t^a(x,y)$.

Lemma 3.3 *Let $\mu \in \mathbb{R}$ and $a > 0$. There exists $t_0 > 0$ such that for all $t > t_0$ we have*

$$p_t^a(x,y) \leq ct^{-n-d/2}y^{2n+d-1}, \qquad x, y \geq 0. \tag{3.11}$$

Proof. Due to (3.8) and the fact that $x^{2n}e^{-\frac{x^2}{2t}} \leq Ct^n$ for all $x \geq 0$, where $C > 0$ is a constant, we have

$$\begin{aligned} p_t^a(x,y) &\leq ct^{-2n-d/2}x^{2n}y^{2n+d-1}e^{-\frac{x^2}{2t}} \\ &\leq ct^{-n-d/2}y^{2n+d-1}, \qquad x, y \geq 0. \end{aligned}$$

□

When $\mu > -1$ and $a = 0$, by the argument leading to (3.11) we get

$$p_t^0(x,y) = q_t^\mu(x,y) \leq ct^{-d/2}y^{d-1}, \qquad x, y \geq 0. \tag{3.12}$$

On the other hand, when $\mu < 0$, using (3.6) yields another upper bound

$$p_t^a(x,y) \leq ct^{-\tilde{n}-d/2}y^{2\tilde{n}+d-1}, \qquad x, y \geq 0,$$

with $\tilde{\nu} = -\sqrt{\mu^2 + 2a} > -1$, $a \geq 0$, and $\tilde{n} = (\tilde{\nu} - \mu)/2$, which is not directly comparable with (3.11) but, unlike (3.11), recovers (3.12) as a tends to 0, and conducts to (6.7) below.

4 Semigroup bounds

In this section, from the heat kernel bounds of the previous section, we derive the semigroup bounds that will be used to prove the results of Sections 5 and 6.

In the next lemma we give a lower bound for the semigroup $(T_t^a)_{t\geq 0}$ generated by $L_\mu - V$, i.e.

$$T_t^a \phi(x) = \int_0^\infty \phi(y) p_t^a(x,y)\, dy, \tag{4.1}$$

is the solution $f_t(x)$ of the linear equation

$$\begin{cases} \dfrac{\partial f_t}{\partial t}(x) = \dfrac{1}{2}\Delta f_t(x) + \dfrac{2\mu+1}{2x}\nabla f_t(x) - \dfrac{a}{x^2} f_t(x), & t > 0, \\ f_0(x) = \phi(x), \qquad x > 0. \end{cases}$$

Without loss of generality we assume that $\phi > 0$ a.e. on a bounded interval $(s,t) \subset \mathbb{R}_+$, and that $\int_0^\infty \phi(y)\, dy < \infty$. The next lemma uses the number n defined in (3.3).

Lemma 4.1 *Assume that $\mu \in \mathbb{R}$ and $a > 0$. Then for all sufficiently large $t > 1$ there holds*

$$T_t^a \phi(x) \geq c t^{-2n-d/2} x^{2n} \mathbf{1}_{[0,\sqrt{t}]}(x), \qquad x \geq 0. \tag{4.2}$$

Proof. We use the bound (3.10), which yields

$$\begin{aligned} f_t(x) &= \int_0^\infty \phi(y) p_t^a(x,y)\, dy \\ &\geq c t^{-2n-d/2} x^{2n} \mathbf{1}_{[0,\sqrt{t}]}(x) \int_\alpha^{\sqrt{t}} \phi(y)\, dy \\ &\geq c' t^{-2n-d/2} x^{2n} \mathbf{1}_{[0,\sqrt{t}]}(x) \int_0^\infty \phi(y)\, dy, \qquad x \geq 0, \end{aligned}$$

for all sufficiently large t, due to our assumptions on $\phi(y)$. □

Note that the bound (4.2) is also valid if $\mu > -1$ and $a = 0$, in which case it reads

$$T_t^0 \phi(x) \geq c t^{-d/2} \mathbf{1}_{[0,\sqrt{t}]}(x), \qquad x \geq 0. \tag{4.3}$$

Let $(X_t)_{t\in\mathbb{R}_+}$ be the Markov process with generator $L_\mu - V$.

Lemma 4.2 *Assume that $\mu \in \mathbb{R}$ and $a > 0$. Let $\alpha \in (0,1)$ and $x \in [\alpha, 1]$. For all t large enough, all $s \in [1, t/2]$ and all $y \in [0, \sqrt{t-s}]$, there holds*

$$\mathbb{E}\left[f_{t-s}^\beta(X_s) \mathbf{1}_{[0,\sqrt{t-s}]}(X_s) \middle| X_t = y, X_0 = x \right] \geq c t^{-(n+2\beta n+\beta d/2)} s^{\beta n}. \tag{4.4}$$

Proof. From Lemma 4.1 above, for all t large enough we have

$$f_{t-s}(X_s) \geq c(t-s)^{-2n-d/2} X_s^{2n} \mathbf{1}_{[0,\sqrt{t-s}]}(X_s), \qquad 0 \leq s \leq t, \tag{4.5}$$

and therefore,

$$\mathbb{E}\left[f^{\beta}_{t-s}(X_s)\mathbf{1}_{[0,\sqrt{t-s}]}(X_s)\Big| X_t = y, X_0 = x\right] \quad (4.6)$$
$$\geq \;\; c(t-s)^{-\beta(2n+d/2)}\, \mathbb{E}\left[X_s^{2\beta n}\mathbf{1}_{[0,\sqrt{t-s}]}(X_s)\big| X_t = y, X_0 = x\right].$$

We now proceed to obtain a lower bound for the moments of the process $(X_t)_{t\in\mathbb{R}_+}$. Let $\alpha \in (0,1)$ be given, and let $x \in [\alpha, 1]$. Due to (3.9) we have, for all sufficiently large $t > 2$ and all $1 \leq s \leq t/2$,

$$p^a_s(x,z) \geq cs^{-2n-d/2} z^{2n+d-1}\mathbf{1}_{[0,\sqrt{s}]}(z), \qquad x, y \geq 0,$$

and

$$p^a_{t-s}(z,y) \geq c(t-s)^{-2n-d/2} z^{2n} y^{2n+d-1}\mathbf{1}_{[0,\sqrt{t-s}]}(z)\mathbf{1}_{[0,\sqrt{t-s}]}(y), \qquad x, y \geq 0.$$

Together with Lemma 3.3, the above inequalities render

$$\frac{p^a_s(x,z)p^a_{t-s}(z,y)}{p^a_t(x,y)} \geq cz^{4n+d-1}\frac{s^{-2n-d/2}(t-s)^{-2n-d/2}}{t^{-n-d/2}}\mathbf{1}_{[0,\sqrt{s}]}(z), \qquad x, y, z \geq 0.$$

It follows that

$$\begin{aligned} &\mathbb{E}\left[X_s^{2\beta n}\mathbf{1}_{[0,\sqrt{t-s}]}(X_s)\Big|X_t = y,\ X_0 = x\right] \\ &= \int_0^\infty z^{2\beta n}\mathbf{1}_{[0,\sqrt{t-s}]}(z)\mathbb{P}(X_s \in dz \mid X_t = y, X_0 = x) \\ &\geq ct^{n+d/2}s^{-(2n+d/2)}(t-s)^{-(2n+d/2)}\int_0^{\sqrt{s}} z^{4n+d-1+2\beta n}\, dz \\ &= ct^{n+d/2}(t-s)^{-(2n+d/2)}s^{\beta n}. \end{aligned}$$

□

If $\mu > -1$ and $a = 0$, by (3.12) and the argument of Lemma 4.2 we have

$$\mathbb{E}\left[\mathbf{1}_{[0,\sqrt{t-s}]}(X_s)f^{\beta}_{t-s}(X_s)\Big| X_t = y, X_0 = x\right] \;\geq\; ct^{-\beta d/2} \qquad (4.7)$$

under the same conditions as in (4.4). Next we state an estimate which will be useful in Section 6, and is valid as above under the condition

$$(\mu, a) \in (\mathbb{R}\times(0,\infty)) \cup ((-1,\infty)\times\{0\}).$$

Proposition 4.3 *Assume that* $\phi \geq 0$ *and*

$$\int_0^\infty y^{2n+d-1}\phi(y)\,dy < \infty.$$

Then there exists $t_0 > 0$ *such that for all* $t > t_0$ *we have*

$$\|T_t^a \phi\|_{L^\infty(\mathbb{R})} \leq ct^{-n-d/2}.$$

Proof. From Lemma 3.3 we get

$$\begin{aligned} T_t^a\phi(x) &= \int_0^\infty \phi(y)p_t^a(x,y)\,dy \\ &\leq ct^{-n-d/2}\int_0^\infty y^{2n+d-1}\phi(y)\,dy, \qquad x \in \mathbb{R}. \end{aligned}$$

□

5 Explosion via the Feynman-Kac formula

Let now $g_t(x)$ denote the mild solution to the semilinear equation

$$\begin{cases} \dfrac{\partial g_t}{\partial t}(x) = \dfrac{1}{2}\dfrac{\partial^2 g_t}{\partial x^2}(x) + \dfrac{2\mu+1}{2x}\dfrac{\partial g_t}{\partial x}(x) - \dfrac{a}{x^2}g_t(x) + g_t(x)f_t^\beta(x), \quad t > 0, \\ \\ g_0(x) = \phi(x), \qquad x > 0, \end{cases} \tag{5.1}$$

where $\mu \in \mathbb{R}$, $a \geq 0$ and $f_t(x)$ is defined in (4.1).

Proposition 5.1 *Let* $\mu \in \mathbb{R}$, $a > 0$ *and* $\alpha \in (0, 1]$. *There exists* $c_3, c_4 > 0$ *such that for all* $x > 0$ *and all sufficiently large* $t \geq 1$,

$$g_t(x) \geq c_4 \exp\left(c_3 t^{1-\beta(n+d/2)-n}\right)\mathbf{1}_{[\alpha,1]}(x).$$

Proof. The Feynman-Kac representation of (5.1) yields

$$g_t(x) = \int_0^\infty \phi(y)p_t^a(x,y)\,\mathbb{E}\left[\exp\int_0^t f_{t-s}^\beta(X_s)\,ds \,\middle|\, X_t = y,\ X_0 = x\right]dy.$$

Let $x \in [\alpha, 1]$. Using Jensen's inequality we obtain, for all t large enough,

$$\begin{aligned} g_t(x) &\geq \int_0^\infty \phi(y)p_t^a(x,y)\exp\left(c_1\int_1^{t/2}\mathbb{E}\left[f_{t-s}^\beta(X_s) \mid X_t = y, X_0 = x\right]ds\right)dy \\ &\geq \int_\alpha^1 \phi(y)p_t^a(x,y)\exp\left(c_2 t^{-(n+2\beta n+\beta d/2)}\int_1^{t/2} s^{\beta n}\,ds\right)dy \\ &\geq c_4 t^{-(1+\nu)}\exp\left(c_3 t^{1-\beta(n+d/2)-n}\right), \qquad x \geq 0, \end{aligned}$$

for some positive constants $c_1, c_2, c_3 > 0$, where we used Lemma 4.2 to obtain the second inequality. □

Notice that, when $\mu > -1$ and $a = 0$, the above argument together with (4.7) gives

$$g_t(x) \geq c_4 \exp\left(c_3 t^{1-\beta d/2}\right) \mathbf{1}_{[\alpha,1]}(x), \qquad x \geq 0.$$

As a consequence of the above proposition, g grows to $+\infty$ uniformly on $[\alpha, 1]$ provided that $1-\beta(n+d/2)-n > 0$, and this implies the following result, in which n is defined by (3.3).

Theorem 5.2 *Let $\mu \in \mathbb{R}$, $a > 0$, and assume that*

$$\beta < \frac{1-n}{n+d/2}.$$

Then the mild solution $u_t(x)$ of (1.3) *blows up in finite time.*

Proof. Let $u_t(x)$ denote the solution of (1.3). Since $g_t(x) \leq u_t(x)$, Proposition 5.1 implies that $u_t(x)$ grows to $+\infty$ uniformly on $[\alpha, 1]$ as $t \to \infty$. According to a well-known argument [4], this is sufficient to prove explosion in finite time of $u_t(x)$. Indeed, let $t_0 \geq 1$, $\tilde{u}_t = u_{t+t_0}$ and $K(t_0) = \min_{x\in[\alpha,1]} u_{t_0}(x)$. Then $\tilde{u}_t$ solves

$$\tilde{u}_t(x) = \int p_t^a(x,y) u_{t_0}(y)\, dy + \int_0^t \int p_{t-s}^a(x,y) \left(\tilde{u}_s(y)\right)^{1+\beta} dy\, ds,$$

hence

$$\min_{x\in[\alpha,1]} \tilde{u}_t(x) \geq \xi K(t_0) + \xi \int_0^t \left(\min_{x\in[\alpha,1]} \tilde{u}_s(x)\right)^{1+\beta} ds, \quad t \in [0,1],$$

where

$$\xi := \min_{r\in[0,1]} \min_{x\in[\alpha,1]} \mathbb{P}_x(X_r \in [\alpha, 1]). \tag{5.2}$$

From Lemma 3.1 it follows that the function

$$(r,x) \mapsto P(X_r \in [\alpha,1] \mid X_0 = x) = \int_\alpha^1 p_r^a(x,y)\, dy = x^{2n} \int_\alpha^1 y^{-2n} q_r^\nu(x,y)\, dy$$

is continuous and strictly positive on $[0,1] \times [\alpha, 1]$. Therefore $\xi > 0$, and it suffices to choose $t_0 > 0$ sufficiently large so that the blow-up time ρ_0 of the equation

$$y(t) = \xi K(t_0) + \xi \int_0^t y^{1+\beta}(s)\, ds$$

is smaller than 1 to conclude that $u_t(x)$ blows up at time $t_0 + \rho_0$. □

We remark that in case $\mu > -1$ and $a = 0$, the conclusion of Theorem 5.2 holds for

$$\beta < 2/d.$$

The next result holds for

$$(\mu, a) \in (\mathbb{R} \times (0,\infty)) \cup ((-1,\infty) \times \{0\}).$$

Corollary 5.3 *Let $G : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing and convex, such that*

$$\frac{G(z)}{z} \sim \kappa_1 z^\beta \text{ as } z \to 0, \tag{5.3}$$

for some $\kappa_1 > 0$, and let $w : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ be a measurable function satisfying

$$w_t(x) \geq \kappa_2 t^\zeta \mathbf{1}_{(0,1)}(t^{-1/2}x) \tag{5.4}$$

for all $t \geq 1$ and some $\kappa_2 > 0$. Then any nontrivial positive solution of the semilinear equation

$$\begin{cases} \dfrac{\partial u_t}{\partial t}(x) = \dfrac{1}{2}\dfrac{\partial^2 u_t}{\partial x^2}(x) + \dfrac{2\mu+1}{2x}\dfrac{\partial u_t}{\partial x}(x) - \dfrac{a}{x^2}u_t(x) + w_t(x)G(u_t(x)), \\ \\ u_0(x) = \phi(x) \geq 0, \qquad x > 0, \quad t > 0, \end{cases} \tag{5.5}$$

blows up in finite time provided

$$\beta < \frac{1+\zeta-n}{n+d/2}.$$

Proof. The Feynman-Kac representation of (5.5) yields

$$u_t(x) = \int_0^\infty \phi(y)p_t^a(x,y)\,\mathbb{E}\left[\exp\left(\int_0^t w_{t-s}(X_s)\frac{G(u_{t-s}(X_s))}{u_{t-s}(X_s)}ds\right) \Big| \, X_t = y, \; X_0 = x\right] dy.$$

Since by assumption $w_{t-s}(X_s)G(u_{t-s}(X_s))/u_{t-s}(X_s)$ can be bounded from below by

$$\kappa_1\kappa_2(t-s)^\zeta \mathbf{1}_{[0,\sqrt{t-s}]}(X_s)u_{t-s}^\beta(X_s) \geq \kappa_1\kappa_2(t-s)^\zeta \mathbf{1}_{[0,\sqrt{t-s}]}(X_s)f_{t-s}^\beta(X_s),$$

we get, using Lemma 4.2 as in the proof of Proposition 5.1, that

$$\begin{aligned} u_t(x) \geq \\ & \int_0^\infty \phi(y)p_t^a(x,y)\,\mathbb{E}\left[\exp\left(\kappa_1\kappa_2\int_0^t (t-s)^\zeta \mathbf{1}_{[0,\sqrt{t-s}]}(X_s)f_{t-s}^\beta(X_s)ds\right) \Big| \, X_t = y, \; X_0 = x\right] dy \\ \geq & \int_0^\infty \phi(y)p_t^a(x,y)\exp\left(\kappa_1\kappa_2\int_0^t (t-s)^\zeta\,\mathbb{E}\left[\mathbf{1}_{[0,\sqrt{t-s}]}(X_s)f_{t-s}^\beta(X_s)\Big| \, X_t = y, \; X_0 = x\right] ds\right) dy \\ \geq & \int_\alpha^1 \phi(y)p_t^a(x,y)\exp\left(c_2 t^{-(n+2\beta n+\beta d/2)}\int_1^{t/2} s^{\zeta+\beta n}\,ds\right) dy \\ \geq & \; c_5 t^{-1-\nu}\exp\left(c_6 t^{1+\zeta-\beta(n+d/2)-n}\right) \end{aligned}$$

for all $x \in [\alpha, 1]$, all t large enough, and some constants $c_5, c_6 > 0$. Therefore,

$$\lim_{t\to\infty} \inf_{x\in[\alpha,1]} u_t(x) = \infty$$

due to the condition $1+\zeta > \beta(n+d/2)+n$. The assertions follow in the same way as in the proof of Theorem 5.2. □

Again when $\mu > -1$ and $a = 0$, the conclusion of Corollary 5.3 also holds when

$$\beta < \frac{2(1+\zeta)}{d}.$$

6 Existence of global solutions

The following result gives conditions for existence of a nontrivial positive global solution. Its proof is very similar to that of Theorem 4.1 in [6], and is therefore omitted.

Theorem 6.1 *Let $\mu \in \mathbb{R}$ and $a \geq 0$, and consider the semilinear equation*

$$\begin{cases} \dfrac{\partial u_t}{\partial t}(x) = \dfrac{1}{2}\dfrac{\partial^2 u_t}{\partial x^2}(x) + \dfrac{2\mu+1}{2x}\dfrac{\partial u_t}{\partial x}(x) - \dfrac{a}{x^2}u_t(x) + t^\zeta G(u_t(x)), \quad t > 0, \\ \\ u_0(x) = \phi(x), \quad x > 0, \end{cases} \tag{6.1}$$

where $\zeta \in \mathbb{R}$, ϕ is bounded and measurable, and $G : \mathbb{R}_+ \to \mathbb{R}_+$ is a measurable function satisfying

$$0 \leq \frac{G(z)}{z} \leq \lambda z^\beta, \qquad z > 0, \tag{6.2}$$

for some $\lambda, \beta > 0$. If

$$\lambda\beta \int_0^\infty r^\zeta \|T_r^a \phi\|_{L^\infty(\mathbb{R})}^\beta \, dr < 1, \tag{6.3}$$

then (6.1) admits a global solution.

Notice that by choosing $\|\phi\|_{L^\infty(\mathbb{R})} > 0$ sufficiently small, it is possible to prove existence of a positive global solution of (6.1) under (6.3) and the less restrictive condition

$$0 \leq \frac{G(z)}{z} \leq \lambda z^\beta, \qquad z \in (0, c),$$

for some $\lambda, \beta, c > 0$, see [7], Theorem 4.1.

As a consequence of Theorem 6.1, an existence result can be obtained under an integrability condition on ϕ.

Theorem 6.2 *Given $(\mu, a) \in \mathbb{R} \times (0, \infty) \cup (-1, \infty) \times \{0\}$, let $G : \mathbb{R}_+ \to \mathbb{R}_+$ and $w : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ be measurable functions such that*

$$G(z) \leq \kappa_1 z^{1+\beta}, \qquad z > 0, \quad \textit{and} \quad w_t(x) \leq \kappa_2 t^\zeta, \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{R},$$

where $\beta, \kappa_1, \kappa_2 > 0$ and $\zeta \in \mathbb{R}$. The equation

$$\begin{cases} \dfrac{\partial u_t}{\partial t}(x) = \dfrac{1}{2}\dfrac{\partial^2 u_t}{\partial x^2}(x) + \dfrac{2\mu+1}{2x}\dfrac{\partial u_t}{\partial x}(x) - \dfrac{a}{x^2}u_t(x) + w_t(x)G(u_t(x)), \quad t > 0, \\ \\ u_0(x) = \phi(x), \quad x > 0, \end{cases} \tag{6.4}$$

admits a global solution on $(0, \infty)$ provided

$$\beta > \frac{1+\zeta}{n+d/2}. \tag{6.5}$$

Proof. Clearly, it suffices to consider the semilinear equation

$$\begin{cases} \dfrac{\partial u_t}{\partial t}(x) = \dfrac{1}{2}\dfrac{\partial^2 u_t}{\partial x^2}(x) + \dfrac{2\mu+1}{2x}\dfrac{\partial u_t}{\partial x}(x) - \dfrac{a}{x^2}u_t(x) + \kappa t^{\zeta} u_t^{1+\beta}(x), \quad t > 0, \\ \\ u_0(x) = \phi(x), \quad x > 0, \end{cases} \tag{6.6}$$

for a suitable constant $\kappa > 0$, and to apply Proposition 4.3 and Theorem 6.1. □

Let us remark that from (3.6) one can also show that when $\mu < -1$, $a \geq 0$, and $\mu^2 + 2a < 1$, Equation (6.4) admits a global solution on $(0, \infty)$ provided

$$\beta > \frac{2+2\zeta}{2+\mu-\sqrt{\mu^2+2a}}. \tag{6.7}$$

The above bound (6.7) recovers the critical exponent $(1+\zeta)/(1+\mu)$ when a tends to 0, however it is weaker than (6.5).

References

[1] C. Bandle and H.A. Levine. Fujita type phenomena for reaction-diffusion equations with convection like terms. *Differential Integral Equations*, 7(5-6):1169–1193, 1994.

[2] A. El Hamidi and G. Laptev. Existence and nonexistence results for higher-order semilinear evolution inequalities with critical potential. *J. Math. Anal. Appl.*, 304:451–463, 2005.

[3] J. Kent. Some probabilistic properties of Bessel functions. *Ann. Probab.*, 6(5):760–770, 1978.

[4] K. Kobayashi, T. Sirao, and H. Tanaka. On the growing up problem for semilinear heat equations. *J. Math. Soc. Japan*, 29(3):407–424, 1977.

[5] H.A. Levine and P. Meier. The value of the critical exponent for reaction-diffusion equations in cones. *Arch. Rational Mech. Anal.*, 109(1):73–80, 1990.

[6] J.A. López-Mimbela and N. Privault. Blow-up and stability of semilinear PDE's with Gamma generators. *J. Math. Anal. Appl.*, 370:181–205, 2005.

[7] J.A. López-Mimbela and N. Privault. Critical exponents for semilinear PDE's with bounded potentials. In R. Dalang, M. Dozzi, and F. Russo, editors, *Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 2005)*, volume 59 of *Progress in Probability*, pages 245–262, Basel, 2008. Birkhäuser.

[8] J.A. López-Mimbela and A. Wakolbinger. Length of Galton-Watson trees and blow-up of semilinear systems. *J. Appl. Probab.*, 35(4):802–811, 1998.

[9] M. Nagasawa and T. Sirao. Probabilistic treatment of the blowing up of solutions for a nonlinear integral equation. *Trans. Amer. Math. Soc.*, 139:301–310, 1969.

[10] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion.* 3rd Edition, Springer-Verlag, 1999.

[11] P. Souplet and Q.S. Zhang. Stability for semilinear parabolic equations with decaying potentials in $\mathbb{R}^n$ and dynamical approach to the existence of ground states. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 19(5):683–703, 2002.

[12] M. Yor. Loi de l'indice du lacet brownien, et distribution de Hartman-Watson. *Z. Wahrsch. Verw. Gebiete*, 53(1):71–95, 1980.

[13] M. Yor. *Exponential functionals of Brownian motion and related processes.* Springer Finance. Springer-Verlag, Berlin, 2001.

[14] Q.S. Zhang. Large time behavior of Schrödinger heat kernels and applications. *Comm. Math. Phys.*, 210(2):371–398, 2000.

[15] Q.S. Zhang. The quantizing effect of potentials on the critical number of reaction-diffusion equations. *J. Differential Equations*, 170(1):188–214, 2001.

José Alfredo López-Mimbela
Área de Probabilidad y Estadística
Centro de Investigación en Matemáticas
Apartado Postal 402
36000 Guanajuato, Mexico
jalfredo at cimat.mx

Nicolas Privault
Division of Mathematical Sciences
School of Physical and Mathematical Sciences
Nanyang Technological University
Singapore 637371
nprivault at ntu.edu.sg