Large deviations for Bernstein bridges

Nicolas Privault* Xiangfeng Yang† Jean-Claude Zambrini‡

October 7, 2015

Abstract

Bernstein processes over a finite time interval are simultaneously forward and backward Markov processes with arbitrarily fixed initial and terminal probability distributions. In this paper, a large deviation principle is proved for a family of Bernstein processes (depending on a small parameter $\hbar$ which is called the Planck constant) arising naturally in Euclidean quantum physics. The method consists in nontrivial Girsanov transformations of integral forms, suitable equivalence forms for large deviations and the (local and global) estimates on the parabolic kernel of the Schrödinger operator.

Keywords and phrases: Bernstein process, large deviation principle, Girsanov transformation, rate function, Schrödinger operator.

AMS 2010 subject classifications: Primary 60F10, 35Q40, 60J65

1 Introduction

1.1 Bernstein processes

Consider $(X_t)_{0\le t\le 1}$ a one-dimensional stochastic process in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the forward, resp. backward filtrations

$$\mathcal{F}_t = \sigma(X_v,\ t \le v \le 1), \qquad 0 \le t \le 1, \quad \text{(backward filtration)},$$

*nprivault@ntu.edu.sg; School of Physical and Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, 637371, Singapore

†xiangfeng.yang@liu.se; Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden

‡jczambrini@gmail.com; Group of Mathematical Physics, University of Lisbon, PT-1649 003 Lisbon, Portugal

resp.

$$\mathcal{P}_t = \sigma(X_s,\ 0 \le s \le t), \qquad 0 \le t \le 1, \qquad \text{(forward filtration)}.$$

A stochastic process $(X_t)_{0\le t\le 1}$ is called a *Bernstein process* if for any $0 \le s_1 < t < t_1 \le 1$ and any f bounded measurable,

$$\mathbb{E}\left[f(X_t)\ \middle|\ \mathcal{P}_{s_1} \vee \mathcal{F}_{t_1}\right] = \mathbb{E}\left[f(X_t)\ \middle|\ X_{s_1},\ X_{t_1}\right].$$

Stochastic processes satisfying this time-symmetric property have been introduced as *reciprocal* processes by Bernstein [3] (1932). Bernstein processes are also referred to as *local Markov* or *two-sided Markov* processes; see [5] and [13]. It has been shown [5] that Bernstein processes provide new tools for further analysis of Feynman's path integral as well as of a number of fundamental issues of quantum physics and its probabilistic content.

In this paper we consider a special family of Bernstein processes $(X_t^{\hbar,a,b})_{0\le t\le 1}$ related to a system with Hamiltonian $H = -\frac{\hbar^2}{2}\triangle + V$, where $\hbar$ is Planck constant and the potential $V(x) : \mathbb{R} \to \mathbb{R}$ depends only on the space variable x. The definition of the Bernstein process for every value of $\hbar > 0$ is as follows (cf. Section 5.1 of [5]).

(I). The distribution of $X_t^{\hbar,a,b}$ is given by

$$\mathbb{P}\left(X_t^{\hbar,a,b} \in A\right) = \frac{1}{\eta_\hbar(0,a)} \int_A \eta_\hbar^*(t,x)\eta_\hbar(t,x)dx \tag{1.1}$$

for A measurable, where $\eta_\hbar^*$ and $\eta_\hbar$ are two positive fundamental solutions (parabolic kernels) of adjoint partial differential equations

$$\begin{cases} -\hbar\dfrac{\partial \eta_\hbar^*}{\partial s}(s,x) = H\eta_\hbar^*(s,x), \\ \\ \eta_\hbar^*(0,x) = \delta_a(x), \end{cases} \qquad \text{and} \qquad \begin{cases} \hbar\dfrac{\partial \eta_\hbar}{\partial s}(s,x) = H\eta_\hbar(s,x), \\ \\ \eta_\hbar(1,x) = \delta_b(x), \end{cases} \tag{1.2}$$

$0 < s \le 1$, where δ_c denotes the Dirac delta distribution at $c \in \mathbb{R}$.

(II). For any $0 \le t < 1$, the process $(X_t^{\hbar,a,b})_{0\le t\le 1}$ solves the following $\mathcal{P}_t$-forward stochastic differential equation

$$dX_t^{\hbar,a,b} = \sqrt{\hbar}dW_t + \hbar\nabla \log \eta_\hbar(t, X_t^{\hbar,a,b})dt, \quad X_0^{\hbar,a,b} = a, \tag{1.3}$$

where $(W_t)_{0\leq t\leq 1}$ stands for a standard Wiener process. We remark that the process $(X_t^{\hbar,a,b})_{0\leq t\leq 1}$ solves as well a backward stochastic differential equation with respect to $(\mathcal{F}_t)_{0\leq t\leq 1}$ (which is a kind of reciprocal property; see [5]), but only the forward stochastic differential equation will be used throughout the paper, which is sufficient for our purpose.

For every $\hbar > 0$ the Bernstein process $(X_t^{\hbar,a,b})_{0\leq t\leq 1}$ is a bridge starting from $a \in \mathbb{R}$ and ending at $b \in \mathbb{R}$, i.e. $X_0^{\hbar,a,b} = a$ and $X_1^{\hbar,a,b} = b$. Such a Bernstein process is generally regarded as a version of the *killed* process with infinitesimal generator $-H$, conditioned to fixed initial and terminal positions; see Section 3 of [13], however this construction by killing is not natural in our time symmetric context, as the potential V has the same interpretation as in classical (and quantum) mechanics.

This paper investigates the limiting behavior of the Bernstein processes $(X_t^{\hbar,a,b})_{0\leq t\leq 1}$ as $\hbar \to 0$. Such asymptotics are the basis of the "quasi" or "semi"-classical analysis of quantum physics. In this context the results can be represented in terms of quantities characterizing the underlying classical dynamical system, in particular its classical action functional. Feynman's original path integral, for a system of Hamiltonian H as before, is an oscillatory integral. To study its asymptotic when $\hbar$ tends to zero requires to use the method of stationary phase. As is well known, however, the Feynman path integral does not have any probabilistic content and it is not possible to construct well defined measures compatible with it on path spaces. If the time parameter of the Schrödinger equation for this system becomes purely imaginary, however, the path integral becomes rigorous, and this is called the "Euclidean" approach in mathematical physics. As a matter of fact, the two adjoint PDEs (1.2) constitute one way to present this Euclidean approach. Although it is not the traditional one, let us observe that, in contrast with this one, and together with (1.1), it manifestly preserves a fundamental invariance under time reversal (for any time-independent potential V). We refer the reader to [5] and [26] for more about the origin and meaning of our Euclidean approach.

1.2 Main results

We study the large deviations of the family of processes $(X_t^{\hbar,a,b})_{0\le t\le 1}$ over the space

$$\mathbb{C}^{a,b}([0,1]) = \{\phi(\cdot)\in\mathbb{C}([0,1]) \ : \ \phi(0)=a \text{ and } \phi(1)=b\}$$

equipped with the topology of uniform convergence and Borel σ-algebra. We will prove that the family obeys a large deviation principle with a *rate function* (or *action functional* in the quantum context) defined by

$$S_X^{a,b}(\phi) = \frac{1}{2}\int_0^1 \phi'(t)^2 dt + \int_0^1 V(\phi(t))dt - 2\rho(a,b,1) \tag{1.4}$$

for absolutely continuous ϕ (otherwise $S_X^{a,b}(\phi)=\infty$), where $\rho(a,b,1)$ represents the distance between a and b introduced by Li and Yau in [18] as

$$\rho(a,b,1) = \frac{1}{2}\inf\left\{\frac{1}{2}\int_0^1 \phi'(t)^2 dt + \int_0^1 V(\phi(t))dt\right\},$$

where the infimum is taken over functions $\phi:[0,1]\to\mathbb{R}$ with $\phi(0)=a$ and $\phi(1)=b$. To achieve large deviations for $(X_t^{\hbar,a,b})_{0\le t\le 1}$, we list the following technical assumptions.

(A1) For some $\alpha>1$ it holds that

$$\lim_{t\to 1^-}\limsup_{\hbar\to 0^+}\hbar\log\mathbb{E}\left[\exp\left(-\frac{\alpha}{\hbar}\int_0^{1-t} V\left(a+\sqrt{\hbar}W_s\right)ds\right)\right]<\infty, \tag{\textbf{A1.1}}$$

and

$$\lim_{t\to 1^-}\limsup_{\hbar\to 0^+}\hbar\log\mathbb{E}\left[\exp\left(-\frac{\alpha}{\hbar}\int_0^{t} V\left(a+\sqrt{\hbar}W_s\right)ds\right)\right]<\infty. \tag{\textbf{A1.2}}$$

(A2) We have the bounds

$$\eta_\hbar(t,x)\le g(\hbar,t) \quad \text{and} \quad \eta_\hbar(t,x)\ge f_0(\hbar,t)\exp\left(\sum_{i=1}^k f_i(\hbar,t)\cdot g_i(b,x)\right)$$

for some integer $k>0$, where $f_i\ge 0$, g and g_i are functions that satisfy

(i) $\lim_{t\to 1^-}\limsup_{\hbar\to 0^+}\hbar\log g(\hbar,t)<\infty$,

(ii) $g_i(b,x)\ge\min\{g_i(b,x_0+1/n),g_i(b,x_0-1/n)\}$ for any fixed $x_0\in\mathbb{R}$ and fixed integer $n\ge 1$ with $|x-x_0|<1/n$, and

(iii) $\lim_{t\to 1^-}\lim_{n\to\infty}\liminf_{\hbar\to 0^+}\Big(\hbar\log f_0(\hbar,t)$

$$+\hbar\cdot\sum_{i=1}^{k} f_i(\hbar,t)\cdot\min\{g_i(b,b+(1-t)+1/n),g_i(b,b+(1-t)-1/n)\}\Big)=0.$$

In Section 1.3 below we will provide examples that satisfy the above Conditions **(A1)** and **(A2)** such as *bounded* potentials $\sup_{x\in\mathbb{R}}|V(x)|<\infty$ and the *quadratic* potential $V(x)=x^2$. The main result of this paper is formulated as follows.

Theorem 1.1. *Assume that* V *is twice continuously differentiable and satisfies Conditions* **(A1)** *and* **(A2)**. *Then*
(1) for any open set $O\subseteq\mathbb{C}^{a,b}([0,1])$,

$$\liminf_{\hbar\to 0^+}\ \hbar\,\log\mathbb{P}\left(X^{\hbar,a,b}\in O\right)\geq-\inf_{\phi\in O}S_X^{a,b}(\phi);\tag{1.5}$$

(2) for any closed set $F\subseteq\mathbb{C}^{a,b}([0,1])$,

$$\limsup_{\hbar\to 0^+}\ \hbar\,\log\mathbb{P}\left(X^{\hbar,a,b}\in F\right)\leq-\inf_{\phi\in F}S_X^{a,b}(\phi).\tag{1.6}$$

Theorem 1.1 suggests that the most probable trajectories of $(X_t^{\hbar,a,b})_{0\leq t\leq 1}$ as $\hbar\to 0$ are contained in the set of ϕ such that $S_X^{a,b}(\phi)=0$, and this is precisely the content of the Principle of Least Action in a classical mechanical system; see Section 1.2 in [16]. The existence of ϕ with $S_X^{a,b}(\phi)=0$ can be easily seen from the lower semi-continuity of $S_X^{a,b}$, while the uniqueness can also hold under more restrictions on V. Furthermore, if the initial and terminal probability distributions are more regular than Dirac measures, then the problem of searching the most probable trajectories has also interesting connections with Monge's problem; see [20], [23], [17] and references therein.

When $V\equiv 0$ and $H=-\frac{\hbar^2}{2}\triangle$ the Bernstein process $(X_t^{\hbar,a,b})_{0\leq t\leq 1}$ reduces to the well-known one-dimensional Brownian bridge from a to b, by comparing (1.3) and (2.1) below, in which case we will denote $(X_t^{\hbar,a,b})_{0\leq t\leq 1}$ by $(B_t^{\hbar,a,b})_{0\leq t\leq 1}$. A large deviation principle for $(B_t^{\hbar,a,b})_{0\leq t\leq 1}$ on a Riemannian manifold has been derived by Hsu in [12] based on a Girsanov transformation involving minimal heat kernels on the manifold by a direct proof of (1.5) and (1.6) which involves a number of technical difficulties, cf. the elaborate proof of Lemma 2.4 in [12]. There are also related discussions on local uniform large deviation bound for Brownian bridges on a Riemannian manifold in [24]. Here our approach is to prove (1.5) and (1.6) indirectly based on suitable equivalents (cf. Remark 2.2 in Section 2), which is more tractable than the one in [12]. Because of the singularity at $t = 1$ in (1.3), it is then natural to split the interval $[0, 1]$ as $[0, t] \cup [t, 1]$ and investigate on each separated interval.

Another possibility to get rid of the singularity at $t = 1$ is to first approximate the boundary condition $\delta_b(x)$ by a smooth function ψ, in which case the law of $(X_t^{\hbar,a,b})_{0\leq t\leq 1}$ becomes absolutely continuous and large deviations arguments for $(B_t^{\hbar,a,b})_{0\leq t\leq 1}$ can be applied. However this method requires *exponential* convergence of the smoothed process to the original processes, which in turn also involves the above splitting of the interval. We thus do not proceed in this direction.

The method proposed in this paper has also been successfully applied in [25] to a family of Lévy bridges. In order to explain our method we start by applying it in Section 2 to the large deviations of one-dimensional Brownian bridges. We also notice that from the *contraction principle* it is trivial to prove the large deviations of one-dimensional Brownian bridges based on (2.3) below, however this approach does not apply to our Bernstein processes. Although the method proposed in this paper is applied only to one-dimensional processes for the sake of notational and computational simplicity, it can be equally applied to multi-dimensional processes.

Note that for the time-homogeneous diffusion bridges constructed in [22] and [21], direct proofs of (1.5) and (1.6) were provided via the analysis of arbitrarily small partitions of $[0, 1]$. Besides, large deviations for Brownian bridges in Hölder norm were presented in [1] based on arguments of abstract Wiener spaces. In contrast, our method requires only a direct analysis over $\mathbb{C}^{a,b}([0, 1])$.

We also remark that the Girsanov transformation in integral form for a Brownian bridge with respect to a Brownian motion, cf. (2.6) in Section 2, is well-known. It is expected that such Girsanov transformation holds for Bernstein processes $(X_t^{\hbar,a,b})_{0\le t\le 1}$. The feasibility (3.1) of such Girsanov transformation is verified in Section 4.

1.3 Examples

Example 1. Bounded potential.

We show that any bounded potential V satisfies Conditions **(A1)** and **(A2)**. First, it is trivial to verify **(A1)**. In order to achieve **(A2)**, we take into account the following lower and upper global bounds for the kernel $\eta_\hbar(t,x)$:

$$\eta_\hbar(t,x) \ge \frac{c}{\sqrt{(1-t)\hbar}} \exp\left(-(1-t)\frac{\|V\|}{\hbar} - \frac{(b-x)^2}{2\hbar(1-t)}\right)$$

and

$$\eta_\hbar(t,x) \le \frac{c}{\sqrt{(1-t)\hbar}} \exp\left((1-t)\frac{\|V\|}{\hbar} - \frac{(b-x)^2}{2\hbar(1-t)}\right)$$

for some $c>0$ (cf. [27]), where $\|V\| = \sup_{x\in\mathbb{R}} |V(x)|$. Therefore in **(A2)** we can take

$$g(\hbar,t) = \frac{c}{\sqrt{(1-t)\hbar}} e^{(1-t)\|V\|/\hbar}, \quad f_0(\hbar,t) = \frac{c}{\sqrt{(1-t)\hbar}} e^{-(1-t)\|V\|/\hbar},$$

$f_1(\hbar,t) = \frac{1}{2\hbar(1-t)}$ and $g_1(\hbar,t) = -(b-x)^2$, $t\in[0,1]$. It is then straightforward to verify each requirement in **(A2)**, for instance we have

$$\lim_{t\to 1^-} \limsup_{\hbar\to 0^+} \hbar \log g(\hbar,t) = \lim_{t\to 1^-} \|V\|(1-t) = 0.$$

The next example will give an explicit illustration of a particular bounded potential.

Example 2. Constant potential.

In this case $(X_t^{\hbar,a,b})_{0\le t\le 1}$ can be regarded as a Brownian motion killed at an independent random time τ, and conditioned to fixed initial and terminal positions. Since the random time τ is independent of $(X_t^{\hbar,a,b})_{0\le t\le 1}$, under the condition $X_1^{\hbar,a,b}=b$ the process $(X_t^{\hbar,a,b})_{0\le t\le 1}$ is not killed as it does not end at the "cemetery" state. Therefore the limiting behavior of $(X_t^{\hbar,a,b})_{0\le t\le 1}$ will be identical to that of the classical Brownian bridge with $V=0$, and this can be also easily verified from the rate function $S_X^{a,b}$

defined in (1.4). More precisely we have

$$\lim_{\hbar\to 0^+} \hbar \log \mathbb{P}\left(\sup_{0\le t\le 1} X_t^{\hbar,a,b} > c\right) = \begin{cases} -2(c-a)(c-b) & \text{if } c > \max\{a,b\}, \\ 0 & \text{otherwise,} \end{cases}$$

which is a fact for Brownian bridges (cf. [4]). When $\hbar \to 0^+$ the limiting trajectory of $(X_t^{\hbar,a,b})_{0\le t\le 1}$ is the straight line $a(1-t)+bt$, thus we can also study the convergence rate of $(X_t^{\hbar,a,b})_{0\le t\le 1}$ to this line, i.e.

$$\lim_{\hbar\to 0^+} \hbar \log \mathbb{P}\left(\sup_{0\le t\le 1} \left|X_t^{\hbar,a,b} - a(1-t) - bt\right| > c\right) = -2\cdot c^2, \tag{1.7}$$

for any fixed $c>0$, which follows from Fernique's theorem on the tail probabilities of Gaussian measures. For the reader's convenience we recall Fernique's theorem as stated in e.g. [19]: for any (a.s) bounded Gaussian process $(X_t)_{t\in T}$ with supremum $\|X\| := \sup_{t\in T} |X_t|$ we have

$$\lim_{v\to\infty} \frac{1}{v^2} \log \mathbb{P}\left(\|X\| > v\right) = -\frac{1}{2\sigma^2},$$

where $\sigma^2 := \sup_{t\in[0,T]} \mathrm{Var}[X_t]$ is the supremum of the variances of the individual X_t's.

Example 3. Quadratic potential.
We let $\omega > 0$ and check that the potential

$$V(x) := \frac{1}{2}\omega^2 x^2.$$

satisfies Condition **(A1)** and **(A2)**. Clearly, Condition **(A1)** is satisfied. Regarding **(A2)** we note that the parabolic kernel $\eta_\hbar(t,x)$ has the explicit form

$$\eta_\hbar(t,x) = \left(\frac{\sqrt{2}\omega}{4\pi\hbar\sinh(\sqrt{2}\omega(1-t))}\right)^{1/2} \times \exp\left(-\frac{\sqrt{2}\omega}{4\hbar}\cdot(x^2+b^2)\cdot\coth(\sqrt{2}\omega(1-t)) + \frac{\omega x b}{\sqrt{2}\hbar\sinh(\sqrt{2}\omega(1-t))}\right),$$

cf. § 3.3 of [7], and we take

$$f_0(\hbar,t) = g(\hbar,t) = \left(\frac{\sqrt{2}\omega}{4\pi\hbar\sinh(\sqrt{2}\omega(1-t))}\right)^{1/2}$$

with $\coth(x) \geq 1/\sinh(x)$. Next we choose

$$f_1(\hbar, t) = \frac{\sqrt{2}\omega}{4\hbar} \cdot \coth(\sqrt{2}\omega(1-t)), \quad f_2(\hbar, t) = \frac{\sqrt{2}\omega}{4\hbar \sinh(\sqrt{2}\omega(1-t))},$$

and $g_1(\hbar, t) = -(x^2 + b^2)$, $g_2(\hbar, t) = 2xb$.

In this case $(X_t^{\hbar,a,b})_{0\leq t\leq 1}$ is a Gaussian process with drifts given by

$$B^\hbar(x,t) = \frac{\omega}{\hbar} \cdot \frac{b - x\cosh(\omega(1-t))}{\sinh(\omega(1-t))} \quad \text{and} \quad B_*^\hbar(x,t) = \frac{\omega}{\hbar} \cdot \frac{x\cosh(\omega t) - a}{\sinh(\omega t)},$$

cf. page 1308 of [15], expectation

$$\mathbb{E}[X_t^{\hbar,a,b}] = a \cdot \frac{\sinh(\omega(1-t))}{\sinh(\omega)} + b \cdot \frac{\sinh(\omega t)}{\sinh(\omega)},$$

and variance

$$\operatorname{Var}(X_t^{\hbar,a,b}) = \hbar \frac{\sinh(\omega(1-t))\sinh(\omega t)}{\omega \sinh \omega}.$$

Its covariance is

$$\operatorname{Cov}(X_t^{\hbar,a,b}, X_r^{\hbar,a,b}) = \hbar \frac{\sinh^2(\omega(1-r))}{\sinh(\omega(1-t))} \frac{\sinh(\omega t)}{\omega \sinh \omega} + \int_t^r \frac{\sinh^2(\omega(1-r))}{\sinh^2(\omega(1-u))} du,$$

$0 < t < r < 1$, cf. page 1323 of [15]. We note that in this case, the most probable trajectory of $(X_t^{\hbar,a,b})_{0\leq t\leq 1}$ as $\hbar \to 0^+$ is, as expected,

$$\phi_*(t) := \mathbb{E}[X_t^{\hbar,a,b}] = a \cdot \frac{\sinh(\omega(1-t))}{\sinh(\omega)} + b \cdot \frac{\sinh(\omega t)}{\sinh(\omega)},$$

which can be checked from $S_X^{a,b}(\phi_*(t)) = 0$. In addition $\phi_*(t)$ solves indeed the ordinary differential equation

$$\phi''(t) = \omega^2 \cdot \phi(t), \text{ with } \phi(0) = a \text{ and } \phi(1) = b.$$

We now study the convergence speed of $(X_t^{\hbar,a,b})_{0\leq t\leq 1}$ to the most probable trajectory $\phi_*(t)$ as $\hbar \to 0^+$ via the asymptotics

$$\lim_{\hbar\to 0^+} \hbar \log \mathbb{P}\left(\sup_{0\leq t\leq 1} \left|X_t^{\hbar,a,b} - \phi_*(t)\right| > 1\right).$$

To this end, we note that

$$Y_t := \frac{1}{\sqrt{\hbar}}(X_t^{\hbar,a,b} - \phi_*(t))$$

is a centered Gaussian process with variance

$$\mathrm{Var}(Y_t) = \frac{\sinh(\omega(1-t))\sinh(\omega t)}{\omega \sinh \omega}.$$

It is easy to see the maximal variance occurs uniquely at $t = 1/2$, therefore from Fernique's theorem it follows that

$$\lim_{\hbar \to 0^+} \hbar \log \mathbb{P}\left(\sup_{0 \le t \le 1} \left| X_t^{\hbar,a,b} - \phi_*(t) \right| > c \right) = -\frac{\omega c^2}{\tanh(\omega/2)} \tag{1.8}$$

for any fixed $c > 0$, and this recovers (1.7) in the free case when $\omega \to 0$.

The limiting behavior (1.8) might have potential applications in related variational problems. Here we consider the simplest case $a = b = 0$. More precisely, according to Theorem 1.1 the left hand side of (1.8) should be

$$-\frac{1}{2} \cdot \inf_{\phi \in A} \int_0^1 \left(\phi'(t)^2 + \omega^2 \phi^2(t) \right) dt \tag{1.9}$$

where the set A consists of absolutely continuous functions $\phi(t)$ on $[0, 1]$ such that $\phi(0) = \phi(1) = 0$ and $\max_{0 \le t \le 1} |\phi(t)| > c$. The variational problem (1.9) is less easy to solve, while (1.8) is much easier.

2 Brownian bridges - an illustration of the method

In this section we illustrate the method of the proof used in this paper by applying it first to the derivation of the standard large deviation result for Brownian bridges, cf. Theorem 2.1 below. The following are three equivalent constructions in distribution sense of the one-dimensional Brownian bridge $(B_t^{\varepsilon,a,b})_{0 \le t \le 1}$ parametrized by the scale parameter $\varepsilon > 0$ with fixed initial and terminal conditions $B_0^{\varepsilon,a,b} = a$ and $B_1^{\varepsilon,a,b} = b$; see e.g. Section 5.6 of [14], [10], and [2].

$$dB_t^{\varepsilon,a,b} = \sqrt{\varepsilon} dW_t + \frac{b - B_t^{\varepsilon,a,b}}{1-t} dt, \quad t \in [0, 1), \;\; B_0^{\varepsilon,a,b} = a, \tag{2.1}$$

$$B_t^{\varepsilon,a,b} = a + (b-a)t + \sqrt{\varepsilon}\int_0^t \frac{1-t}{1-s} dW_s, \quad t \in [0,1), \tag{2.2}$$

$$\widetilde{B}_t^{\varepsilon,a,b} = a(1-t) + bt + \sqrt{\varepsilon}(W_t - tW_1), \quad t \in [0,1], \tag{2.3}$$

where $(W_t)_{0\le t\le 1}$ is a standard Wiener process started at zero. The representation (2.3) is generally said to be anticipative due to the fact that $(\widetilde{B}_t^{\varepsilon,a,b})_{0\le t\le 1}$ is not adapted to the natural filtration of W. We will use the functional

$$S^{a,b}(\phi) = \frac{1}{2}\left(\int_0^1 \phi'(t)^2 dt - (b-a)^2\right)$$

for absolutely continuous ϕ.

Theorem 2.1. *(1) For any open set $O \subseteq \mathbb{C}^{a,b}([0,1])$,*

$$\liminf_{\varepsilon\to 0^+} \varepsilon \log \mathbb{P}\left(B^{\varepsilon,a,b} \in O\right) \ge -\inf_{\phi\in O} S^{a,b}(\phi); \tag{2.4}$$

(2) For any closed set $F \subseteq \mathbb{C}^{a,b}([0,1])$,

$$\limsup_{\varepsilon\to 0^+} \varepsilon \log \mathbb{P}\left(B^{\varepsilon,a,b} \in F\right) \le -\inf_{\phi\in F} S^{a,b}(\phi). \tag{2.5}$$

Our first step is, as usual, to split $[0,1]$ into $[0,1] = [0,t] \cup [t,1]$ due to the singularity at time 1, to change the bridge on $[0,t]$ into a Wiener process on $[0,t]$ by a Girsanov transformation, and then to let t go to one. Letting $X_s^\varepsilon = a + \sqrt{\varepsilon}W_s$, the distribution $\mu_{B_{[0,t]}^{\varepsilon,a,b}}$ of $(B_t^{\varepsilon,a,b})_{0\le t\le 1}$ restricted to $[0,t]$, $t<1$, is absolutely continuous with respect to the distribution $\mu_{X_{[0,t]}^\varepsilon}$ of X^ε restricted to $[0,t]$ over the space $\mathbb{C}^{a,b}([0,t])$ of restrictions to $[0,t]$ of functions in $\mathbb{C}^{a,b}([0,1])$, with Girsanov-Radon-Nikodym density

$$\frac{d\mu_{B_{[0,t]}^{\varepsilon,a,b}}}{d\mu_{X_{[0,t]}^\varepsilon}}(x.) = \exp\left(\frac{1}{\varepsilon}\int_0^t \frac{b-x_s}{1-s} dx_s - \frac{1}{2\varepsilon}\int_0^t \left(\frac{b-x_s}{1-s}\right)^2 ds\right), \tag{2.6}$$

cf. Section 3.2 in [11] and Section 4 below for details.

2.1 Proof of the lower bound (2.4)

For fixed $\delta > 0, \gamma > 0$ and $\phi(\cdot) \in \mathbb{C}^{a,b}([0,1])$, from (2.6) we have

$$\mathbb{P}\left(\max_{0\le s\le t}\left|B_s^{\varepsilon,a,b} - \phi(s)\right| < \delta\right)$$

$$
\begin{aligned}
&= \int_{\left\{\max_{0\le s\le t}|X_s^\varepsilon-\phi(s)|<\delta\right\}} \exp\left(\frac{1}{\varepsilon}\int_0^t \frac{b-X_s^\varepsilon}{1-s}dX_s^\varepsilon - \frac{1}{2\varepsilon}\int_0^t\left(\frac{b-X_s^\varepsilon}{1-s}\right)^2 ds\right) d\mathbb{P} \\
&= (1-t)^{-1/2}\cdot\exp\left(\frac{(b-a)^2}{2\varepsilon}\right)\cdot\int_{\left\{\max_{0\le s\le t}|X_s^\varepsilon-\phi(s)|<\delta\right\}}\exp\left(-\frac{(b-X_t^\varepsilon)^2}{2\varepsilon(1-t)}\right)d\mathbb{P} \\
&\ge (1-t)^{-1/2}\exp\left(\frac{1}{2\varepsilon}\left((b-a)^2-\frac{\max\{(b-\phi(t)-\delta)^2,(b-\phi(t)+\delta)^2\}}{1-t}\right)\right) \qquad (2.7)\\
&\quad \times\mathbb{P}\left(\max_{0\le s\le t}|X_s^\varepsilon-\phi(s)|<\delta\right) \\
&\ge (1-t)^{-1/2}\cdot\exp\left(-\frac{1}{\varepsilon}\left(S^{a,b}(\phi[0,t])+\gamma+\frac{\max\{(b-\phi(t)-\delta)^2,(b-\phi(t)+\delta)^2\}}{2(1-t)}\right)\right)
\end{aligned}
$$

with

$$S^{a,b}(\phi[0,t]) = \frac{1}{2}\int_0^t \phi'(t)^2dt - \frac{(b-a)^2}{2},$$

where the last inequality comes from the lower bound of large deviations for Brownian motions; see Section 3.2 in [9]. Now for any open set $O \subseteq \mathbb{C}^{a,b}([0,1])$, $x^*(\cdot)\in O$ with $S^{a,b}(x^*)<\infty$ and $\mathrm{Ball}_\delta(x^*)\subseteq O$ for some $\delta>0$, we define

$$O_t := \left\{\phi\in\mathbb{C}^{a,b}([0,1]) \;:\; \max_{0\le s\le t}|\phi(s)-x^*(s)|<\delta\right\}$$

and

$$O^t := \left\{\phi\in\mathbb{C}^{a,b}([0,1]) \;:\; \max_{t\le s\le 1}|\phi(s)-x^*(s)|\ge\delta\right\}.$$

Then we have $O_t \subseteq O\cup O^t$ and

$$\mathbb{P}\left(B^{\varepsilon,a,b}\in O\right)+\mathbb{P}\left(B^{\varepsilon,a,b}\in O^t\right)\ge\mathbb{P}\left(B^{\varepsilon,a,b}\in O_t\right). \qquad (2.8)$$

We first deal with $\mathbb{P}\left(B^{\varepsilon,a,b}\in O_t\right)$. It follows from (2.7) that for large n and $x^*\in O_t$,

$$
\begin{aligned}
&\liminf_{\varepsilon\to0^+}\varepsilon\,\log\mathbb{P}\left(B^{\varepsilon,a,b}\in O_t\right)\ge\liminf_{\varepsilon\to0^+}\varepsilon\,\log\mathbb{P}\left(\max_{0\le s\le t}\left|B_s^{\varepsilon,a,b}-x^*(s)\right|<1/n\right)\\
&\ge-\left(S^{a,b}(x^*[0,t])+\gamma+\frac{\max\{(b-x^*(t)-1/n)^2,(b-x^*(t)+1/n)^2\}}{2(1-t)}\right)\\
&=-\left(S^{a,b}(x^*[0,t])+\frac{(b-x^*(t))^2}{2(1-t)}\right)\qquad\text{(after the limit } n\to\infty,\gamma\to0).
\end{aligned}
$$

Thus

$$\lim_{t\to1}\liminf_{\varepsilon\to0^+}\varepsilon\,\log\mathbb{P}\left(B^{\varepsilon,a,b}\in O_t\right)\ge -S^{a,b}(x^*). \qquad (2.9)$$

Now for the term $\mathbb{P}\left(B^{\varepsilon,a,b} \in O^t\right)$, we apply Fernique's theorem to get

$$\begin{aligned}
&\lim_{t\to 1}\liminf_{\varepsilon\to 0^+} \varepsilon\, \log \mathbb{P}\left(B^{\varepsilon,a,b} \in O^t\right) = \lim_{t\to 1}\liminf_{\varepsilon\to 0^+} \varepsilon\, \log \mathbb{P}\left(\max_{t\le s\le 1}\left|B^{\varepsilon,a,b}_s - x^*(s)\right| \ge \delta\right) \\
&= \lim_{t\to 1}\liminf_{\varepsilon\to 0^+} \varepsilon\, \log \mathbb{P}\left(\max_{t\le s\le 1}|W_s - sW_1| \ge \delta/(2\sqrt{\varepsilon})\right) \\
&= -\lim_{t\to 1}\frac{\delta^2}{8t(1-t)} = -\infty.
\end{aligned} \tag{2.10}$$

The lower bound (2.4) is thus proved by combining (2.8), (2.9) and (2.10).

2.2 Proof of the upper bound (2.5)

For any $s > 0$, we define the compact set

$$\Phi(s) := \{\phi \in \mathbb{C}^{a,b}([0,1]) \;:\; S^{a,b}(\phi) \le s\},$$

and denoted by $\mathrm{dist}(\phi, A)$ the distance between a point ϕ and a set A in the continuous path space. Due to the singularity at time 1, for any $\delta > 0$ we write

$$\begin{aligned}
&\left\{\mathrm{dist}\left(B^{\varepsilon,a,b}, \Phi(s)\right) \ge \delta\right\} \\
&\qquad = \left\{\mathrm{dist}\left(B^{\varepsilon,a,b}_{[0,t]}, \Phi_{[0,t]}(s)\right) \ge \delta\right\} \cup \left\{\mathrm{dist}\left(B^{\varepsilon,a,b}_{[t,1]}, \Phi_{[t,1]}(s)\right) \ge \delta\right\}
\end{aligned} \tag{2.11}$$

where $\Phi_{[0,t]}(s)$, resp. $\Phi_{[t,1]}(s)$, denotes the collection of all elements in $\Phi(s)$ restricted on $[0,t]$ (resp. $[t,1]$). Thus we have

$$\begin{aligned}
&\mathbb{P}\left(\mathrm{dist}\left(B^{\varepsilon,a,b}, \Phi(s)\right) \ge \delta\right) \\
&\qquad \le \mathbb{P}\left(\mathrm{dist}\left(B^{\varepsilon,a,b}_{[0,t]}, \Phi_{[0,t]}(s)\right) \ge \delta\right) + \mathbb{P}\left(\mathrm{dist}\left(B^{\varepsilon,a,b}_{[t,1]}, \Phi_{[t,1]}(s)\right) \ge \delta\right).
\end{aligned}$$

Again, from (2.6) we estimate

$$\begin{aligned}
&\mathbb{P}\left(\mathrm{dist}\left(B^{\varepsilon,a,b}_{[0,t]}, \Phi_{[0,t]}(s)\right) \ge \delta\right) \\
&= (1-t)^{-1/2}\cdot \exp\left(\frac{(b-a)^2}{2\varepsilon}\right)\cdot \int_{\left\{\mathrm{dist}\left(X^{\varepsilon}_{[0,t]}, \Phi_{[0,t]}(s)\right)\ge\delta\right\}} \exp\left(-\frac{(b-X^{\varepsilon}_t)^2}{2\varepsilon(1-t)}\right) d\mathbb{P} \\
&\le (1-t)^{-1/2}\cdot \exp\left(\frac{(b-a)^2}{2\varepsilon}\right)\cdot \mathbb{P}\left(\mathrm{dist}\left(X^{\varepsilon}_{[0,t]}, \Phi_{[0,t]}(s)\right) \ge \delta\right).
\end{aligned} \tag{2.12}$$

Defining $s := \inf_{\phi \in F} S^{a,b}(\phi) - \gamma$ for any closed set F and arbitrarily small positive γ, we have

$$\limsup_{\varepsilon \to 0^+} \ \varepsilon \, \log \mathbb{P}\left(B^{\varepsilon,a,b} \in F\right) \leq \lim_{t \to 1} \limsup_{\varepsilon \to 0^+} \ \varepsilon \, \log \mathbb{P}\left(\operatorname{dist}\left(B^{\varepsilon,a,b}_{[0,t]}, \Phi_{[0,t]}(s)\right) \geq \delta\right) \tag{2.13}$$

for some $\delta > 0$, since by Fernique's theorem we have

$$\lim_{t \to 1} \limsup_{\varepsilon \to 0^+} \ \varepsilon \, \log \mathbb{P}\left(\operatorname{dist}\left(B^{\varepsilon,a,b}_{[t,1]}, \Phi_{[t,1]}(s)\right) \geq \delta\right) = -\infty.$$

By combining (2.12) and (2.13), we obtain

$$\begin{aligned}
&\limsup_{\varepsilon \to 0^+} \ \varepsilon \, \log \mathbb{P}\left(B^{\varepsilon,a,b} \in F\right) \\
&\leq \ \frac{1}{2}(b-a)^2 + \lim_{t \to 1} \limsup_{\varepsilon \to 0^+} \ \varepsilon \, \log \mathbb{P}\left(\operatorname{dist}\left(X^{\varepsilon}_{[0,t]}, \Phi_{[0,t]}(s)\right) \geq \delta\right) \\
&\leq \ \frac{1}{2}(b-a)^2 + \limsup_{\varepsilon \to 0^+} \ \varepsilon \, \log \mathbb{P}\left(\operatorname{dist}\left(X^{\varepsilon}, \Phi(s)\right) \geq \delta\right) \\
&\leq \ \frac{1}{2}(b-a)^2 - \frac{1}{2} \inf_{\phi \in A} \int_0^1 \phi'(t)^2 dt
\end{aligned}$$

from the upper bound of large deviations for Brownian motions, where A is the closed set $A = \{\phi(\cdot) : \operatorname{dist}(\phi, \Phi(s)) \geq \delta\}$. We conclude by noting that

$$\frac{1}{2} \int_0^1 \phi'(t)^2 \geq s + \frac{1}{2}(b-a)^2$$

for any $\phi \in A$.

Remark 2.1. In the case of general Bernstein processes, several technical difficulties arise from the presence of a potential V, as can be seen in the three lemmas at the beginning of Section 3. We point out in particular that Fernique's theorem will not be applicable to Bernstein processes since they can not be written as linear combinations of elementary Gaussian processes; see (2.10) for Brownian bridges built as linear combinations of Brownian motions.

Remark 2.2. The Girsanov transformation (2.6) played an important role in our proof. Analogous transformations will be used for Bernstein processes in Section 4. On the other hand, for the proof of the upper bound, we used upper probability estimates involving compact sets $\Phi(s)$, which are equivalent to the standard upper bounds of large deviations for closed sets; see Section 3.3 in [9]. The advantage of this method is to split an event into two parts on $[0,t]$ and $[t,1]$ as in (2.11), which is not obvious for closed sets. Lastly, the transition density of the Wiener process which has been implicitly used throughout the proof, cf. e.g. (2.7), will be replaced for Bernstein processes by various local and global estimates on the parabolic kernel of the Schrödinger operator in Section 3.

3 Proof of Theorem 1.1

The following three technical lemmas will be needed for the proof of Theorem 1.1, and their proofs are given at the end of this section. The first lemma will be used in both proofs of the lower and the upper bounds.

Lemma 3.1. *Under the conditions of Theorem 1.1, for any $\delta > 0$ and $\phi(\cdot) \in \mathbb{C}^{a,b}([0,1])$, we have*

$$\lim_{t\to 1^-}\lim_{\hbar\to 0^+} \hbar \log \mathbb{P}\left(\max_{t\le s\le 1}\left|X_s^{\hbar,a,b}-\phi(s)\right|\ge\delta\right) = -\infty.$$

The second and third lemmas play important parts in the proof of the upper bound.

Lemma 3.2. *For the rate function $S_X^{a,b}$ defined by* (1.4)*, we define the compact sets*

$$\Phi^X(s) = \{\phi\in\mathbb{C}^{a,b}([0,1])\ :\ S_X^{a,b}(\phi)\le s\},\qquad s>0,$$

and let $\Phi^X_{[0,t]}(s)$ denote the collection of all elements in $\Phi^X(s)$ restricted to $[0,t]$, $0\le t<1$. Then under the conditions of Theorem 1.1, we have

$$\lim_{t\to 1^-}\limsup_{\hbar\to 0^+} \hbar \log \int_{\left\{\mathrm{dist}\left(a+\sqrt{\hbar}W_{[0,t]},\Phi^X_{[0,t]}(s)\right)\ge\delta\right\}} \exp\left(-\frac{1}{\hbar}\int_0^t V\left(a+\sqrt{\hbar}W_u\right)du\right)d\mathbb{P}$$
$$\le -s-2\rho(a,b,1),$$

for all $\delta>0$ and $s>0$.

Lemma 3.3. *Under the conditions of Theorem 1.1, the following holds*

$$\lim_{t\to 1^-}\limsup_{\hbar\to 0^+} \hbar \log \mathbb{P}\left(\mathrm{dist}\left(X^{\hbar,a,b}_{[t,1]},\Phi^X_{[t,1]}(s)\right)\ge\delta\right) = -\infty,$$

where $\Phi^X_{[t,1]}(s)$ denotes the collection of all elements in $\Phi^X(s)$ restricted on $[t,1]$.

In view of (1.3) the Girsanov transformation is given by

$$\frac{d\mu_{X^{\hbar,a,b}_{[0,t]}}}{d\mu_{a+\sqrt{\hbar}W_{[0,t]}}}(y.) = \exp\left(\int_0^t \nabla \log \eta_\hbar(s,y_s)dy_s - \frac{\hbar}{2}\int_0^t \left(\nabla \log \eta_\hbar(s,y_s)\right)^2 ds\right), \quad (3.1)$$

$t \in [0,1)$, cf. Section 4. It follows that for any measurable $A \subseteq \mathbb{C}^{a,b}([0,t])$ we have

$$\begin{aligned}\mu_{X^{\hbar,a,b}_{[0,t]}}(A) &= \int_{\{a+\sqrt{\hbar}W_{[0,t]}\in A\}} \exp\left(\sqrt{\hbar}\int_0^t \nabla \log \eta_\hbar(s,a+\sqrt{\hbar}W_s)dW_s\right. \\ &\qquad\left. -\frac{\hbar}{2}\int_0^t \left(\nabla \log \eta_\hbar(s,a+\sqrt{\hbar}W_s)\right)^2 ds\right) d\mathbb{P} \\ &= \int_{\{a+\sqrt{\hbar}W_{[0,t]}\in A\}} \frac{\eta_\hbar(t,a+\sqrt{\hbar}W_t)}{\eta_\hbar(0,a)} \exp\left(-\frac{1}{\hbar}\int_0^t V\left(a+\sqrt{\hbar}W_s\right)ds\right) d\mathbb{P}\end{aligned} \quad (3.2)$$

where the last equality comes from (1.2) and the Itô formula applied to $d\log \eta_\hbar(s,a+\sqrt{\hbar}W_s)$.

For simplicity of exposition, some steps in the proof of Theorem 1.1 are derived using simpler bounds that hold under the boundedness of the potential V. Indeed, in this case the estimates on the kernel have relatively simpler forms as seen in Example 1. As noted within the proof, those arguments extend to potentials satisfying Conditions **(A1)** and **(A2)**.

3.1 Proof of the lower bound (1.5)

For any open set $O \subseteq \mathbb{C}^{a,b}([0,1])$ and $x^*(\cdot) \in O$ with $S^{a,b}_X(x^*) < \infty$ and $\mathrm{Ball}_\delta(x^*) \subseteq O$ for some $\delta > 0$, we define

$$O_t = \left\{\phi \in \mathbb{C}^{a,b}([0,1]) \ : \ \max_{0\le s\le t} |\phi(s) - x^*(s)| < \delta\right\},$$

and

$$O^t = \left\{\phi \in \mathbb{C}^{a,b}([0,1]) \ : \ \max_{t\le s\le 1} |\phi(s) - x^*(s)| \ge \delta\right\}.$$

We note that $O_t \subseteq O \cup O^t$, and

$$\mathbb{P}\left(X^{\hbar,a,b} \in O\right) + \mathbb{P}\left(X^{\hbar,a,b} \in O^t\right) \ge \mathbb{P}\left(X^{\hbar,a,b} \in O_t\right), \qquad t \in [0,1). \quad (3.3)$$

By (3.2) we have, for large enough n,

$$
\begin{aligned}
\liminf_{\hbar\to 0^+} \ \hbar \log \mathbb{P}\left(X^{\hbar,a,b}\in O_t\right) &\geq \liminf_{\hbar\to 0^+} \ \hbar \log \mathbb{P}\left(\max_{0\leq s\leq t}\left|X_s^{\varepsilon,a,b}-x^*(s)\right|<1/n\right)\\
\geq \liminf_{\hbar\to 0^+} \ \hbar \log \int_{\left\{\max_{0\leq s\leq t}\left|a+\sqrt{\hbar}W_s-x^*(s)\right|<1/n\right\}} & \frac{\eta_\hbar(t,a+\sqrt{\hbar}W_t)}{\eta_\hbar(0,a)}\\
&\cdot \exp\left(-\frac{1}{\hbar}\int_0^t V\left(a+\sqrt{\hbar}W_s\right)ds\right)d\mathbb{P} \qquad (3.4)\\
\geq -\int_0^t V(x^*(s))ds-\alpha(n)-\liminf_{\hbar\to 0^+} \ \hbar \log \eta_\hbar(0,a)&\\
+\liminf_{\hbar\to 0^+} \ \hbar \log \int_{\left\{\max_{0\leq s\leq t}|a+\sqrt{\hbar}W_s-x^*(s)|<1/n\right\}} & \eta_\hbar(t,a+\sqrt{\hbar}W_t)d\mathbb{P}
\end{aligned}
$$

where $\alpha(n)\to 0$ as $n\to\infty$ from the fact that V is continuous. We now analyze each term in the last step of (3.4).

First, we clearly have

$$
\int_0^t V(x^*(s))ds+\alpha(n)\to\int_0^t V(x^*(s))ds
$$

when $n\to\infty$. Second, from the asymptotic behavior for the parabolic kernel of the operator $\frac{\hbar}{2}\triangle-\frac{1}{\hbar}V-\frac{\partial}{\partial t}$ developed in Theorem 6.1 of [18] for twice continuously differentiable V it follows that

$$
\liminf_{\hbar\to 0^+} \ \hbar \log \eta_\hbar(0,a)=-2\rho(a,b,1). \tag{3.5}
$$

Lastly, in case the potential V is bounded the function $\eta_\hbar(t,x)$ admits the lower bound

$$
\eta_\hbar(t,x)\geq\frac{c}{\sqrt{(1-t)\hbar}}\exp\left(-(1-t)\frac{\|V\|}{\hbar}-\frac{(b-x)^2}{2\hbar(1-t)}\right) \tag{3.6}
$$

for some $c>0$, cf. for instance [27] where we can make the change of variable $\widetilde{\eta}_\hbar(t,x)=\eta_\hbar(1-2t/\hbar,x)$. Thus on the set $\left\{\max_{0\leq s\leq t}\left|a-x^*(s)+\sqrt{\hbar}W_s\right|<1/n\right\}$ we get

$$
\begin{aligned}
\eta_\hbar(t,a+\sqrt{\hbar}W_t)\geq{}&\frac{c}{\sqrt{(1-t)\hbar}}\exp\left(-(1-t)\frac{\|V\|}{\hbar}\right) \qquad (3.7)\\
&\times\exp\left(-\frac{(b-x^*(t)+1/n)^2+(b-x^*(t)-1/n)^2}{2\hbar(1-t)}\right),
\end{aligned}
$$

when V is bounded. By (3.4) and the above inequality we find

$$\lim_{t\to 1^-}\lim_{n\to\infty}\liminf_{\hbar\to 0^+}\ \hbar\log\int_{\{\max_{0\le s\le t}\left|a-x^*(s)+\sqrt{\hbar}W_s\right|<1/n\}}\eta_\hbar(t,a+\sqrt{\hbar}W_t)d\mathbb{P} \tag{3.8}$$
$$= \lim_{t\to 1^-}\lim_{n\to\infty}\liminf_{\hbar\to 0^+}\ \hbar\log\mathbb{P}\left(\max_{0\le s\le t}\left|a-x^*(s)+\sqrt{\hbar}W_s\right|<1/n\right)$$
$$\ge -\lim_{t\to 1^-}\lim_{n\to\infty}\frac{1}{2}\int_0^t (dx^*(s)/ds)^2 ds = -\frac{1}{2}\int_0^1 (dx^*(s)/ds)^2 ds.$$

The arguments between (3.4) and (3.8) imply

$$\lim_{t\to 1^-}\liminf_{\hbar\to 0^+}\ \hbar\log\mathbb{P}\left(X^{\hbar,a,b}\in O_t\right)\ge -S_X^{a,b}(x^*).$$

For the term $\mathbb{P}\left(X^{\hbar,a,b}\in O^t\right)$, Lemma 3.1 implies

$$\lim_{t\to 1^-}\liminf_{\hbar\to 0^+}\ \hbar\log\mathbb{P}\left(X^{\hbar,a,b}\in O^t\right)$$
$$= \lim_{t\to 1^-}\liminf_{\hbar\to 0^+}\hbar\log\mathbb{P}\left(\max_{t\le s\le 1}\left|X_s^{\hbar,a,b}-x^*(s)\right|\ge\delta\right)=-\infty,$$

which completes the proof of the lower bound. If V is not bounded we can conclude similarly by replacing (3.7) with **(A2)** from (3.6) to (3.8).

3.2 Proof of the upper bound (1.6)

Let us recall the rate function

$$S_X^{a,b}(\phi)=\frac{1}{2}\int_0^1\phi'(t)^2dt+\int_0^1 V(\phi(t))dt-2\rho(a,b,1)$$

and the definition of the compact sets

$$\Phi^X(s)=\{\phi\in\mathbb{C}^{a,b}([0,1])\ :\ S_X^{a,b}(\phi)\le s\},\quad \text{for } s>0.$$

For $\delta>0$ and $0\le t<1$ we now write

$$\left\{\mathrm{dist}\left(X^{\hbar,a,b},\Phi^X(s)\right)\ge\delta\right\}$$
$$= \left\{\mathrm{dist}\left(X_{[0,t]}^{\hbar,a,b},\Phi_{[0,t]}^X(s)\right)\ge\delta\right\}\cup\left\{\mathrm{dist}\left(X_{[t,1]}^{\hbar,a,b},\Phi_{[t,1]}^X(s)\right)\ge\delta\right\}$$

where $\Phi_{[0,t]}^X(s)$, resp. $\Phi_{[t,1]}^X(s)$, denote again the restrictions to $[0,t]$, resp. $[t,1]$, of elements in $\Phi^X(s)$, hence we have

$$\mathbb{P}\left(\mathrm{dist}\left(X^{\hbar,a,b},\Phi^X(s)\right)\ge\delta\right)$$

$$\leq \mathbb{P}\left(\mathrm{dist}\left(X^{\hbar,a,b}_{[0,t]}, \Phi^X_{[0,t]}(s)\right) \geq \delta\right) + \mathbb{P}\left(\mathrm{dist}\left(X^{\hbar,a,b}_{[t,1]}, \Phi^X_{[t,1]}(s)\right) \geq \delta\right).$$

On the other hand the Girsanov transformation (3.2) shows that

$$\begin{aligned}
&\mathbb{P}\left(\mathrm{dist}\left(X^{\hbar,a,b}_{[0,t]}, \Phi^X_{[0,t]}(s)\right) \geq \delta\right) \\
&= \int_{\left\{\mathrm{dist}\left(a+\sqrt{\hbar}W_{[0,t]}, \Phi^X_{[0,t]}(s)\right)\geq\delta\right\}} \frac{\eta_\hbar(t, a+\sqrt{\hbar}W_t)}{\eta_\hbar(0,a)} \exp\left(-\frac{1}{\hbar}\int_0^t V\left(a+\sqrt{\hbar}W_s\right) ds\right) d\mathbb{P}.
\end{aligned}$$

If V is bounded, we then use an upper bound estimate for $\eta_\hbar(t,x)$ as

$$\eta_\hbar(t,x) \leq \frac{c}{\sqrt{(1-t)\hbar}} \exp\left((1-t)\frac{\|V\|}{\hbar} - \frac{(b-x)^2}{2\hbar(1-t)}\right), \tag{3.9}$$

cf. [27]. Thus, using (3.5) we have

$$\begin{aligned}
&\lim_{t\to1^-}\limsup_{\hbar\to0^+} \ \hbar\log\mathbb{P}\left(\mathrm{dist}\left(X^{\hbar,a,b}_{[0,t]}, \Phi^X_{[0,t]}(s)\right) \geq \delta\right) \leq 2\rho(a,b,1) + \|V\|(1-t) \\
&+ \lim_{t\to1^-}\limsup_{\hbar\to0^+}\hbar\log\int_{\left\{\mathrm{dist}\left(a+\sqrt{\hbar}W_{[0,t]}, \Phi^X_{[0,t]}(s)\right)\geq\delta\right\}} \exp\left(-\frac{1}{\hbar}\int_0^t V\left(a+\sqrt{\hbar}W_s\right) ds\right) d\mathbb{P} \\
&\leq -s,
\end{aligned} \tag{3.10}$$

where the last inequality follows from Lemma 3.2. Now for F a closed set and $\gamma > 0$ arbitrarily small we let $s := -\gamma + \inf_{\phi\in F} S^{a,b}_X(\phi)$ and get

$$\lim_{t\to1^-}\limsup_{\hbar\to0^+} \ \hbar\log\mathbb{P}\left(\mathrm{dist}\left(X^{\hbar,a,b}_{[t,1]}, \Phi^X_{[t,1]}(s)\right) \geq \delta\right) = -\infty,$$

for some $\delta > 0$ by Lemma 3.3, hence

$$\limsup_{\hbar\to0^+} \ \hbar\log\mathbb{P}\left(X^{\hbar,a,b}\in F\right) \leq \lim_{t\to1^-}\limsup_{\hbar\to0^+} \ \hbar\log\mathbb{P}\left(\mathrm{dist}\left(X^{\hbar,a,b}_{[0,t]}, \Phi^X_{[0,t]}(s)\right) \geq \delta\right). \tag{3.11}$$

By combining (3.10) and (3.11), we obtain

$$\limsup_{\hbar\to0^+} \ \hbar\log\mathbb{P}\left(X^{\hbar,a,b}\in F\right) \leq -s = \gamma - \inf_{\phi\in F} S^{a,b}_X(\phi),$$

which completes the proof by sending γ to 0. In case V is not bounded we replace (3.9) by **(A2)**.

3.3 Proofs of the lemmas

Proof of Lemma 3.1. From the distribution of $X_t^{\hbar,a,b}$ in (1.1) and (1.2), it can be seen that the distribution of the Bernstein process $(X_t^{\hbar,a,b})_{0\le t\le 1}$ is equal to the distribution of its time reversed process $(X_{1-t}^{\hbar,b,a})_{0\le t\le 1}$, as this property is constitutive of all Bernstein processes and not only of bridge processes. This yields

$$\mathbb{P}\left(\max_{t\le s\le 1}\left|X_s^{\hbar,a,b}-\phi(s)\right|\ge\delta\right)=\mathbb{P}\left(\max_{0\le s\le 1-t}\left|X_s^{\hbar,b,a}-\phi(1-s)\right|\ge\delta\right).$$

Noting that the new kernel $\eta_\hbar$ has the new terminal condition $\eta_\hbar(1,x)=\delta_a(x)$, by the Girsanov transformation (3.2) we find

$$\begin{aligned}&\mathbb{P}\left(\max_{0\le s\le 1-t}\left|X_s^{\hbar,b,a}-\phi(1-s)\right|\ge\delta\right)\\ &=\int_{\left\{\max_{0\le s\le 1-t}\left|\sqrt{\hbar}W_s+b-\phi(1-s)\right|\ge\delta\right\}}\frac{\eta_\hbar(1-t,\sqrt{\hbar}W_{1-t}+b)}{\eta_\hbar(0,b)}\\ &\qquad\qquad\times\exp\left(-\frac{1}{\hbar}\int_0^{1-t}V\left(\sqrt{\hbar}W_s+b\right)ds\right)d\mathbb{P}.\end{aligned}\tag{3.12}$$

In case the potential V is bounded we have the upper estimate

$$\eta_\hbar(t,x)\le\frac{c}{\sqrt{(1-t)\hbar}}\exp\left((1-t)\frac{\|V\|}{\hbar}-\frac{(x-a)^2}{2\hbar(1-t)}\right),\tag{3.13}$$

which implies

$$\begin{aligned}&\lim_{\hbar\to0^+}\hbar\log\mathbb{P}\left(\max_{0\le s\le 1-t}\left|X_s^{\hbar,b,a}-\phi(1-s)\right|\ge\delta\right)\\ &\le 2\rho(a,b,1)+2\|V\|+\lim_{\hbar\to0^+}\hbar\log\mathbb{P}\left(\max_{0\le s\le 1-t}\left|\sqrt{\hbar}W_s+b-\phi(1-s)\right|\ge\delta\right)\\ &\le 2\rho(a,b,1)+2\|V\|+\lim_{\hbar\to0^+}\hbar\log\mathbb{P}\left(\max_{0\le s\le 1-t}\left|\sqrt{\hbar}W_s\right|\ge\delta/2\right),\ \text{ for } t \text{ near } 1,\\ &=2\rho(a,b,1)+2\|V\|-\frac{\delta^2}{8(1-t)},\end{aligned}$$

from (3.5) and Fernique's theorem. By taking $t\to1^-$ we conclude that

$$\lim_{t\to1^-}\lim_{\hbar\to0^+}\hbar\log\mathbb{P}\left(\max_{t\le s\le 1}\left|X_s^{\hbar,a,b}-\phi(s)\right|\ge\delta\right)=-\infty.$$

In case V is not bounded, we replace (3.13) by **(A2)**, and apply a Hölder inequality in (3.12) together with the following fact

$$\lim_{t\to 1^-}\limsup_{\hbar\to 0^+}\hbar\log\mathbb{E}\left[\exp\left(-\frac{\alpha}{\hbar}\int_0^{1-t}V\left(a+\sqrt{\hbar}W_s\right)ds\right)\right]<\infty,$$

which is Condition **(A1.1)**. $\square$

Proof of Lemma 3.2. We start by noting the upper estimate

$$\begin{aligned}
&\limsup_{\hbar\to 0^+}\ \hbar\ \log\int_{\left\{\mathrm{dist}\left(a+\sqrt{\hbar}W_{[0,t]},\Phi^X_{[0,t]}(s)\right)\geq\delta\right\}}\exp\left(-\frac{1}{\hbar}\int_0^t V\left(a+\sqrt{\hbar}W_s\right)ds\right)d\mathbb{P}\\
&\leq\ \limsup_{\hbar\to 0^+}\ \hbar\ \log\int_{\left\{\mathrm{dist}\left(a+\sqrt{\hbar}W,\Phi^X(s)\right)\geq\delta\right\}}\exp\left(-\frac{1}{\hbar}\int_0^t V\left(a+\sqrt{\hbar}W_s\right)ds\right)d\mathbb{P},
\end{aligned}\tag{3.14}$$

and inspired by the proof of Varadhan's Integral Lemma in Section 4.3 of [6], we analyze the last limit in (3.14). New difficulties arise in this case since the domain of the integration is not the whole function space.

From the lower semi-continuity of the functional $\int_0^1\phi'(s)^2ds$ and the continuity of V, it follows that for a fixed $\varrho>0$ and every point ϕ in the compact set $\Phi^X(n)$, $n\geq 1$, there exists an open neighborhood O_ϕ of ϕ such that

$$\inf_{\varphi\in\overline{O_\phi}}\int_0^1\varphi'(s)^2ds\geq\int_0^1\phi'(s)^2ds-\varrho$$

and

$$\sup_{\varphi\in\overline{O_\phi}}\left(-\int_0^t V(\varphi(s))ds\right)\leq-\int_0^t V(\phi(s))ds+\varrho,\ \text{for all } t\in[0,1].$$

For our purpose we use the neighborhood $O_\phi:=\mathrm{Ball}_\varepsilon(\phi)$ with a sufficiently small $\varepsilon(\varrho,\phi)$ depending on ϱ and on ϕ, such that $\varepsilon(\varrho,\phi)<\delta/2$, and from the compactness of $\Phi^X(n)$, we choose a finite cover of $\Phi^X(n)$ as $\cup_{k=1}^N\mathrm{Ball}_\varepsilon(\phi_k)$. Next, introducing the notation

$$A_\kappa=\left\{\phi(\cdot):\mathrm{dist}(\phi,\Phi^X(s))\geq\kappa\right\},$$

for any constant $\kappa>0$ we have

$$\limsup_{\hbar\to 0^+}\ \hbar\ \log\int_{\left\{\mathrm{dist}\left(a+\sqrt{\hbar}W,\Phi^X(s)\right)\geq\delta\right\}}\exp\left(-\frac{1}{\hbar}\int_0^t V\left(a+\sqrt{\hbar}W_s\right)ds\right)d\mathbb{P}$$

$$
\begin{aligned}
&= \limsup_{\hbar\to 0^+} \ \hbar \log \left(\int_{\{x.\in A_\delta\}\cap\left\{x.\in \cup_{k=1}^N \mathrm{Ball}_\varepsilon(\phi_k)\right\}} \exp\left(-\frac{1}{\hbar}\int_0^t V(x(s))ds\right) d\mu_{a+\sqrt{\hbar}W}(x.) \right. \\
&\quad \left. + \int_{\{x.\in A_\delta\}\cap\left\{x.\in \left(\cup_{k=1}^N \mathrm{Ball}_\varepsilon(\phi_k)\right)^c\right\}} \exp\left(-\frac{1}{\hbar}\int_0^t V(x(s))ds\right) d\mu_{a+\sqrt{\hbar}W}(x.) \right) \\
&=: \limsup_{\hbar\to 0^+} \ \hbar \log \left(\Gamma_1^{\hbar,t} + \Gamma_2^{\hbar,t} \right).
\end{aligned}
$$

From the fact that $\varepsilon(\varrho,\phi) < \delta/2$ we see that if $A_\delta \cap \mathrm{Ball}_\varepsilon(\phi_k) \neq \emptyset$ then $\phi_k \in A_{\delta/2}$. Therefore, denoting by $(\phi_{k_i})_{i=1}^M$ the collection of all points in $(\phi_k)_{k=1}^N$ which belong to $A_{\delta/2}$ we have

$$
\Gamma_1^{\hbar,t} \leq \sum_{i=1}^M \exp\left(-\frac{1}{\hbar}\left(\int_0^t V(\phi_{k_i}(s))ds - \varrho\right)\right) \cdot \mu_{a+\sqrt{\hbar}W}\left(\overline{\mathrm{Ball}_\varepsilon(\phi_{k_i})}\right).
$$

Thus by the large deviation upper bounds for $\{a+\sqrt{\hbar}W\}$ applied to the closed sets $\overline{\mathrm{Ball}_\varepsilon(\phi_{k_i})}$ we obtain

$$
\begin{aligned}
\limsup_{\hbar\to 0^+} \ \hbar \log \Gamma_1^{\hbar,t} &\leq \max_{i=1,\ldots,M} \left\{ -\int_0^t V(\phi_{k_i}(s))ds + \varrho - \inf_{\phi\in\overline{\mathrm{Ball}_\varepsilon(\phi_{k_i})}} \frac{1}{2}\int_0^1 \phi'(s)^2 ds \right\} \\
&\leq \max_{i=1,\ldots,M} \left\{ -\int_0^t V(\phi_{k_i}(s))ds + 2\varrho - \frac{1}{2}\int_0^1 \phi_{k_i}'(s)^2 ds \right\} \\
&\leq 2\varrho - (s + 2\rho(a,b,1)) + \max_{i=1,\ldots,M} \left\{ \int_t^1 V(\phi_{k_i}(s))ds \right\}.
\end{aligned}
$$

Therefore

$$
\lim_{t\to 1} \limsup_{\hbar\to 0^+} \ \hbar \log \Gamma_1^{\hbar,t} \leq 2\varrho - (s + 2\rho(a,b,1)).
$$

For $\Gamma_2^{\hbar,t}$, we apply Hölder inequality to get

$$
\Gamma_2^{\hbar,t} \leq \left(\mathbb{E}\left[\exp\left(-\frac{\alpha}{\hbar}\int_0^t V\left(a+\sqrt{\hbar}W_s\right)ds\right)\right]\right)^{1/\alpha} \cdot \left(\mu_{a+\sqrt{\hbar}W}\left(\left(\bigcup_{k=1}^N \mathrm{Ball}_\varepsilon(\phi_k)\right)^c\right)\right)^{1/\beta}
$$

with $\alpha > 1$ and $1/\alpha + 1/\beta = 1$. Thus

$$
\begin{aligned}
\limsup_{\hbar\to 0^+} \ \hbar \log \Gamma_2^{\hbar,t} \leq \frac{1}{\alpha} \limsup_{\hbar\to 0^+} \hbar \log \mathbb{E}\left[\exp\left(-\frac{\alpha}{\hbar}\int_0^t V\left(a+\sqrt{\hbar}W_s\right)ds\right)\right] \\
- \frac{1}{2\beta} \inf_{\phi\in(\Phi^X(n))^c} \int_0^1 \phi'(s)^2 ds
\end{aligned}
$$

by noting that $\left(\cup_{k=1}^N \mathrm{Ball}_\varepsilon(\phi_k)\right)^c \subseteq \left(\Phi^X(n)\right)^c$. It follows from Condition **(A1.2)** that the constant

$$C := \frac{1}{\alpha} \lim_{t\to 1} \limsup_{\hbar\to 0^+} \hbar \log \mathbb{E}\left[\exp\left(-\frac{\alpha}{\hbar}\int_0^t V\left(a+\sqrt{\hbar}W_s\right)ds\right)\right] < \infty$$

is finite. To summarize, we obtain

$$\begin{aligned}
&\lim_{t\to 1}\limsup_{\hbar\to 0^+}\ \hbar\ \log\left(\Gamma_1^{\hbar,t}+\Gamma_2^{\hbar,t}\right) \qquad (3.15)\\
&\le\ \max\left\{2\varrho - s - 2\rho(a,b,1), C - \frac{1}{2\beta}\inf_{\phi\in(\Phi^X(n))^c}\int_0^1 \phi'(s)^2 ds\right\}\\
&=\ -s-2\rho(a,b,1),
\end{aligned}$$

where the last equality of (3.15) follows from sending $\varrho \to 0$ and using the fact that

$$\lim_{n\to\infty}\inf_{\phi\in(\Phi^X(n))^c}\int_0^1 \phi'(s)^2 ds = \infty. \qquad (3.16)$$

If (3.16) were not true, then there would be a constant $D > 0$ and a sequence $\{\phi_n \in \left(\Phi^X(n)\right)^c\}_{n\ge 1}$ such that

$$\frac{1}{2}\int_0^1 \phi_n'(s)^2 ds \le D.$$

Then $\|\phi_n\|^2 \le 2D$. This in turn implies that

$$\begin{aligned}
S_X^{a,b}(\phi_n) &= \frac{1}{2}\int_0^1 \phi_n'(t)^2 dt + \int_0^1 V(\phi_n(t))dt - 2\rho(a,b,1)\\
&\le D + \max_{-\sqrt{2D}\le x\le\sqrt{2D}} |V(x)| - 2\rho(a,b,1),
\end{aligned}$$

which is a contradiction with the fact that $\lim_{n\to\infty} S_X^{a,b}(\phi_n) \ge \lim_{n\to\infty} n = \infty$. □

Proof of Lemma 3.3. From the compactness of

$$\left\{\phi \in \mathbb{C}^{a,b}([0,1])\ :\ \phi \text{ is absolutely continuous and } \frac{1}{2}\int_0^1 \phi'(s)^2 ds \le \alpha\right\}$$

for any $\alpha > 0$, we easily see that the infimum

$$\rho(a,b,1) = \inf\left\{\frac{1}{4}\int_0^1 \left(\phi'(t)^2 + 2V(\phi(t))\right)dt\right\},$$

over $\phi : [0,1] \to \mathbb{R}$ with $\phi(0) = a$ and $\phi(1) = b$, is reached at some ϕ_0, i.e.

$$\rho(a,b,1) = \frac{1}{4}\int_0^1 \left(\phi_0'(t)^2 + 2V(\phi_0(t))\right)dt.$$

In this case $S_X^{a,b}(\phi_0) = 0$. Therefore we have

$$\begin{aligned}
&\lim_{t\to 1^-} \limsup_{\hbar\to 0^+} \ \hbar \log \mathbb{P}\left(\operatorname{dist}\left(X_{[t,1]}^{\hbar,a,b}, \Phi_{[t,1]}^X(s)\right) \geq \delta\right) \\
&= \lim_{t\to 1^-} \limsup_{\hbar\to 0^+} \ \hbar \log \mathbb{P}\left(\operatorname{dist}\left(X_{[t,1]}^{\hbar,a,b}, \Phi_{[t,1]}^X(s)\right) \geq \delta, \max_{t\leq s\leq 1} \left|X_s^{\hbar,a,b} - \phi_0(s)\right| \geq \delta\right) \\
&\leq \lim_{t\to 1^-} \limsup_{\hbar\to 0^+} \ \hbar \log \mathbb{P}\left(\max_{t\leq s\leq 1} \left|X_s^{\hbar,a,b} - \phi_0(s)\right| \geq \delta\right) = -\infty,
\end{aligned}$$

where the last step follows from Lemma 3.1. □

Remark 3.4. The most probable trajectories of $(X_t^{\hbar,a,b})_{0\leq t\leq 1}$ as $\hbar \to 0$ are contained in the set of ϕ such that $S_X^{a,b}(\phi) = 0$. If we further assume that the infimum of $\frac{1}{4}\int_0^1 (\phi'(t)^2 + 2V(\phi(t)))\,dt$, over all absolutely continuous functions $\phi : [0,1] \to \mathbb{R}$ with $\phi(0) = a$ and $\phi(1) = b$, is attained uniquely, then the (unique) most probable trajectory is the minimizer of $\rho(a,b,1)$. This is also the conclusion of the Least Action Principle in a mechanical system; see Section 1.2 in [16]. See [26] for a recent overview of "stochastic deformation" of classical mechanics founded on Bernstein processes.

4 On the Girsanov transformation

In this section, we study the feasibility of the Girsanov transformation

$$\frac{d\mu_{X_{[0,t]}^{\hbar,a,b}}}{d\mu_{a+\sqrt{\hbar}W_{[0,t]}}}(y.) = \exp\left(\int_0^t \nabla \log \eta_\hbar(s,y_s)dy_s - \frac{1}{2}\int_0^t \hbar \left(\nabla \log \eta_\hbar(s,y_s)\right)^2 ds\right)$$

for Bernstein processes. Using Itô's formula, it becomes equivalent to prove that

$$\mathbb{E}\left[\frac{\eta_\hbar\left(t, a+\sqrt{\hbar}W_t\right)}{\eta_\hbar(0,a)} \exp\left(-\frac{1}{\hbar}\int_0^t V\left(a+\sqrt{\hbar}W_s\right)ds\right)\right] = 1. \tag{4.1}$$

To this end, let us consider the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial s}(s,y) = \dfrac{\hbar}{2}\triangle u(s,y) - \dfrac{1}{\hbar}V(y)u(s,y), \quad 0 < s \leq 1, \\ \\ u(0,y) = f(y) \end{cases} \tag{4.2}$$

for a continuous and bounded function $f(y)$. On one hand, from the theory of partial differential equations, the solution $u(s,y)$ can be given in terms of the fundamental solution

$$u(t,y) = \int_{\mathbb{R}} \eta_{\hbar,y}(1-t,z) f(z) dz$$

where $\eta_{\hbar,y}$ is the fundamental solution to (1.2) with $\eta_{\hbar,y}(1,z) = \delta_y(z)$. On the other hand, this solution $u(s,y)$ has a Feynman-Kac type representation (see Theorem 4.4.2 in [14] and [8])

$$u(t,y) = \mathbb{E}\left[f(\sqrt{\hbar}B_t + y) \exp\left(-\frac{1}{\hbar} \int_0^t V\left(\sqrt{\hbar}B_s + y\right) ds \right) \right].$$

For a fixed $0 \le t < 1$, we now choose a special continuous and bounded (positive) function $f(y) = \eta_\hbar(t,y)/\eta_\hbar(0,a)$. By equating these two representations of the solution, it follows that

$$\int_{\mathbb{R}} \eta_{\hbar,a}(1-t,z) \frac{\eta_\hbar(t,z)}{\eta_\hbar(0,a)} dz = \mathbb{E}\left[\frac{\eta_\hbar\left(t, a + \sqrt{\hbar}W_t\right)}{\eta_\hbar(0,a)} \exp\left(-\frac{1}{\hbar} \int_0^t V\left(a + \sqrt{\hbar}W_s\right) ds \right) \right] \tag{4.3}$$

where $\eta_\hbar$ is defined by (1.2). From the construction of a Bernstein process, the left hand side of (4.3) is equal to $\mathbb{P}\left(X_t^{\hbar,a,b} \in \mathbb{R}\right) = 1$ by observing that $\eta_{\hbar,a}(1-t,z) = \eta_\hbar^*(t,z)$ which is defined in (1.2). This proves (4.1).

References

[1] P. Baldi, G. Ben Arous, and G. Kerkyacharian. Large deviation and Strassen law in Hölder norm. *Stochastic Process. Appl.*, 42:171–180, 1992.

[2] M. Barczy and P. Kern. Representations of multidimensional linear process bridges. *Random Oper. Stoch. Equ.*, 21(2):159–189, 2013.

[3] S. Bernstein. Sur les liaisons entre les grandeurs aléatoires. In *Verh. des intern. Mathematikerkongr.*, volume 1, Zürich, 1932.

[4] B. Boukai. An explicit expression for the distribution of the supremum of Brownian motion with a change point. *Comm. Statist. Theory Methods*, 19(1):31–40, 1990.

[5] K.L. Chung and J.-C. Zambrini. *Introduction to Random Time and Quantum Randomness.* World Scientific, 2003.

[6] A. Dembo and O. Zeitouni. *Large deviations techniques and applications*, volume 38 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2010. Corrected reprint of the second (1998) edition.

[7] S.Y. Feng. Heat kernel analysis for Ornstein-Uhlenbeck operators with quadratic potentials. Preprint arXiv:1208.4264, 2012.

[8] M. Freidlin. *Functional integration and partial differential equations*, volume 109 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.

[9] M.I. Freidlin and A. D. Wentzell. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, second edition, 1998.

[10] D. Gasbarra, T. Sottinen, and E. Valkeila. Gaussian bridges. In *Stochastic analysis and applications*, Abel Symp. Springer, 2007.

[11] I.I. Gihman and A.V. Skorokhod. *The Theory of Stochastic Processes*, volume III. Springer-Verlag, 2007.

[12] E.P. Hsu. Brownian bridges on Riemannian manifolds. *Probab. Theory Related Fields*, 84:103–118, 1990.

[13] B. Jamison. Reciprocal processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 30:65–86, 1974.

[14] I. Karatzas and S. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[15] T. Kolsrud and J.-C. Zambrini. An introduction to the semiclassical limit of Euclidean quantum mechanics. *J. Math. Phys.*, 33(4):1301–1334, 1992.

[16] E.M. Lifshitz L.D. Landau. *Mechanics*, volume I of *Course of Theoretical Physics*. Addison-Wesley, 1976.

[17] C. Léonard. From the Schrödinger problem to the Monge-Kantorovich problem. *J. Funct. Anal.*, 262(4):1879–1920, 2012.

[18] P. Li and S.T. Yau. On the parabolic kernel of the Schrödinger operator. *Acta Mathematica*, 156:153–201, 1986.

[19] M.B. Marcus and L.A. Shepp. Sample behavior of Gaussian processes. In *Proc. Sixth Berkeley Sympos. Mathematical Statistics and Probability, Vol. II*, pages 423–441. Univ. California Press, Berkeley, Calif., 1972.

[20] T. Mikami. Monge's problem with a quadratic cost by the zero-noise limit of h-path processes. *Probab. Theory Related Fields*, 129(2):245–260, 2004.

[21] Z. Qian and G. Wei. Aronson's estimates and conditional diffusion processes. *Acta Mathematicae Aplicatae Sinica*, 10(2), 1994.

[22] Z. Qian and G. Wei. Conditional diffusion processes in a small time interval. *Chinese Journal of Applied Probability*, 10(2), 1994.

[23] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[24] O. Wittich. An explicit local uniform large deviation bound for brownian bridges. *Statistics & Probability Letters*, 73:51–56, 2005.

[25] X.F. Yang. Large deviations for Markov bridges with jumps. *J. Math. Anal. Appl.*, 416(1):1–12, 2014.

[26] J.-C. Zambrini. The research program of stochastic deformation (with a view towards geometric mechanics). Preprint arXiv:1212.4186, to appear in Stochastic analysis, a series of lectures, Birkhäuser Verlag 2014.

[27] Q.S. Zhang. Large time behavior of Schrödinger heat kernels and applications. *Comm. Math. Phys.*, 210:371–398, 2000.