Absolute Continuity in Infinite Dimension and Anticipating Stochastic Calculus

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1. Introduction

The problem of the absolute continuity of the Gaussian measure in infinite dimension has been considered in a probabilistic context as an extension of the Girsanov theorem on the Wiener space, cf. (Gir60), (Kus82), (NV90), (Ram74), (ÜZ94). The case of the exponential density is treated in (Pri96) and corresponds to an anticipating Girsanov theorem on the Poisson space, since the interjump times of the standard Poisson process are independent identically distributed exponential random variables. These generalizations of the Girsanov theorem to the anticipative case brought into play an extension of the Itô stochastic integral, called the Skorohod integral and the Carleman-Fredholm determinant. Our aim is to extend the results obtained in the case of the Gaussian or exponential density to the case of the uniform density, using the methods developed in the above references with a unified formulation, and to present in each case a unified probabilistic interpretation in terms of stochastic processes and anticipating stochastic calculus. Results and definitions are valid in general for the Gaussian, exponential and uniform densities, and proofs are given in the case of the uniform density if they differ from the Gaussian or exponential case. Let B be a Banach space of sequences with norm $|\cdot|_B$ and Borel measure P. As shown in Theorem 3.2, necessary conditions for the absolute continuity of a perturbation $I_B + F$ of the identity and for the expression of the density with a divergence operator and a Carleman-Fredholm determinant are that the shift F has to be a.s. continuously differentiable in the direction of $l^2(\mathbf{N})$ on the support of the measure, to leave invariant this support and to "vanish on its boundary" in a sense to be made precise in Theorem 3.2. The change of variables formula for one-dimensional integrals shows the factorization of the density function and the boundary condition that needs to be imposed on the shift F. We work with a probability density of the form $\exp(-h(x))$ with respect to the Lebesgue measure on an interval $]a, b[$.

In such a case, integration by parts shows that the divergence of a smooth function F on $\mathbf{R}$ is given by $div(F) = Fh' - F'$, provided that the boundary condition $F(a) = F(b) = 0$ holds:

$$\int_a^b F(x)G'(x)\exp(-h(x))dx \ = \ \int_a^b G(x)(h'(x)F(x) - F'(x))\exp(-h(x))dx.$$

Now the change of variables formula gives

$$\begin{aligned}
&\int_a^b f(x)\exp(-h(x))dx \\
&= \int_a^b f(x+F(x))(1+F'(x))\exp(-h(x+F(x)))dx \\
&= \int_a^b f(x+F(x))(1+F'(x))\exp(-F'(x))\exp(-div(F)(x)) \\
&\quad \times \exp\left(-\left(h(x+F(x)-h(x)-h'(x)F(x)\right)\right)\exp(-h(x))dx
\end{aligned}$$

if $I_B + F$ is a diffeomorphism of $]a,b[$, which implies $F(a) = F(b) = 0$. The term

$$\exp\left(-\left(h(x+F(x)-h(x)-h'(x)F(x)\right)\right)$$

has a simple expression only if h is a polynomial of degree less than 2:

$$h(x) = \alpha_0 + \alpha_1 x + \frac{1}{2}\alpha_2 x^2.$$

In this case, this term is equal to $\exp\left(-\frac{1}{2}\alpha_2 F(x)^2\right)$ and the factorization

$$(1+F'(x))\exp(-F'(x))\exp\left(-div(F) - \frac{1}{2}\alpha_2 F(x)^2\right)$$

of the Radon-Nykodim density corresponds to the expression (3.1) below which makes use of the Carleman-Fredholm determinant. In the Gaussian case ($\alpha_2 = 1$), $\exp\left(-\frac{1}{2}F(x)^2\right)$ corresponds to the square norm of the perturbation in the Cameron-Martin space. Up to linear transformations, the Gaussian, exponential and uniform densities seem to be the only ones to allow such expressions of the Radon-Nykodim density.

Another common property of the Gaussian, exponential and uniform densities is that they admit orthogonal sequences of polynomials, respectively the Hermite, Laguerre and Legendre polynomials, which satisfy the differential equation

$$\sigma(x)y'' + \tau(x)y'(x) + \alpha y = 0, \tag{1.1}$$

where σ is a polynomial of degree less than 2, τ is a polynomial of degree at most 1, and $\alpha \in \mathbf{N}$. Up to linear transformations, these polynomials are the only ones to satisfy (1.1), cf. (NU88). They are orthogonal on an interval $]a, b[$ with respect to a density ρ such that $(\sigma\rho)' = \tau\rho$ and

$$\sigma(x)\rho(x)x^k \mid_{x=a,b}= 0, \qquad k \in \mathbf{N}.$$

In Section 2, we present a unified framework for the stochastic calculus of variations, the Sobolev spaces and the integration by parts formulas for the Gaussian, exponential and uniform density measures. It shows in particular the necessity of taking into account boundary conditions when constructing test function spaces. Section 3 contains the main theorem (Theorem 3.2), followed by technical lemmata. Section 4 deals with the probabilistic interpretation of the absolute continuity result. Section 5 is devoted to the proof of a generalization of Theorem 3.2. For other approaches to the generalization of the stochastic calculus of variations, we can refer for instance to (BS90), (Smo86).

To end this introduction, we state a more general problem which gives another motivation for this work. Let λ be a probability density on $\mathbf{R}$. If $| \cdot |_B$ is a suitable norm on $\mathbf{R}^\infty$, λ can be extended as a probability measure P on the Borel σ-algebra $\mathcal{F}$ of $B = \{x \in \mathbf{R}^\infty \ : \mid x \mid_B < \infty\}$ from its values on cylinder sets. In this case, can we find

- a stochastic process $(Y_t)_{t\in \mathrm{R}_+}$ on (B, P),
- a filtration $(\mathcal{F}_t)_{t\in \mathrm{R}_+}$ on (B, P),
- a closable gradient operator $\tilde{D} : L^2(B) \to L^2(B) \otimes L^2(\mathbf{R}_+)$ defined by perturbations of the trajectories of Y,
- and an integration by parts formula

$$E\left[(\tilde{D}F, u)_{L^2(\mathrm{R}_+)}\right] = E\left[F\tilde{\delta}(u)\right],$$

  where $\tilde{\delta} : L^2(B) \otimes L^2(\mathbf{R}_+) \to L^2(B)$ is the adjoint of $\tilde{D}$, $F \in Dom(\tilde{D})$, $u \in Dom(\tilde{\delta})$,

such that:

- $\tilde{\delta}$ is an extension of the stochastic integral with respect to the compensated process $\tilde{Y}$ defined from Y, i.e.

$$\tilde{\delta}(u) = \int_0^\infty u(t) d\tilde{Y}_t$$

  for $u \in L^2(B) \otimes L^2(\mathbf{R}_+)$, u being $(\mathcal{F}_t)$-adapted,

– if $G : B \to l^2(\mathbf{N})$ is a mapping satisfying some regularity conditions,
$$E[f(I_B + G) \mid \Lambda_G \mid] = E[f]$$
for f bounded measurable on B, where the Radon-Nykodim $\mid \Lambda_G \mid$ density is expressed with the operators $\tilde{D}$ and $\tilde{\delta}$.

Moreover, one can ask for the spectral decomposition of the operator $\tilde{\delta}\tilde{D}$ and the chaotic decomposition of $L^2(B, P)$.

2. Calculus of variations and integration by parts formula

We consider the separable Banach space $B = \mathbf{R}^\infty$ endowed with a metric d and its associated Borel σ-algebra $\mathcal{F}$, such that a probability P can be defined on $(B, \mathcal{F})$ via its expression on cylinder sets:

$$P(\{x \in B \ : \ (x_0, \ldots, x_n) \in E\}) = \lambda^{\otimes n+1}(E),$$

$n \in \mathbf{N}$ and E Borel set in $\mathbf{R}^{n+1}$, where λ is a Gaussian, exponential or uniform probability measure on an interval $]a, b[$, $a, b \in \mathbf{R} \cup \{\infty\}$. We refer to (BC70), (Kuo75) for the Gaussian case. In the case of the exponential or uniform density, we can choose the metric d to be defined respectively as

$$d(x, y) = \sup_{k \geq 0} \mid x_k - y_k \mid /(k+1),$$

or

$$d(x, y) = \sup_{k \geq 0} \mid x_k - y_k \mid,$$

cf. (Pri97), (Pri94). The coordinate functionals

$$\theta_k : B \longrightarrow \mathbf{R} \quad k \in \mathbf{N},$$

are independent uniformly distributed random variables with distribution λ. As mentioned above, the measure λ and the interval $]a, b[$ can be one of the following:
i) $\lambda(dx) = \exp(-x^2/2)dx/\sqrt{2\pi}$, i.e. $\alpha_0 = \alpha_1 = 0$, $\alpha_2 = 1$, $]a, b[= \mathbf{R}$,
ii) $\lambda(dx) = 1_{[0,\infty[}(x) \exp(-x)dx$, i.e. $\alpha_0 = \alpha_2 = 0$, $\alpha_1 = 1$, $[a, b[= [0, \infty[$,
iii) $\lambda(dx) = 1_{[-1,1]}dx/2$, i.e. $\alpha_0 = \alpha_1 = \alpha_2 = 0$, $[a, b] = [-1, 1]$.
The results obtained depend implicitly on λ, and sometimes explicitly on α_1, α_2, a, b. We denote by $B_{[a,b]}$, $B_{]a,b[}$, $B^c_{[a,b]}$ the subsets of B defined as
$$B_{[a,b]} = \{\omega \in B \ : \ a \leq \omega_k \leq b, \quad k \in \mathbf{N}\},$$

$$B_{]a,b[} = \{\omega \in B \ : \ a < \omega_k < b, \quad k \in \mathbb{N}\},$$
$$B^c_{[a,b]} = \{\omega \in B \ : \ \exists\ k \in \mathbb{N} \textit{ with } \omega_k \notin [a,b]\}.$$

Let $\mathcal{S}$ be the set of functionals on B of the form $f(\theta_{k_1},...,\theta_{k_n})$ on $B_{[a,b]}$ where $n \in \mathbb{N}$, $k_1,...,k_n \in \mathbb{N}$, and f is a polynomial or $f \in \mathcal{C}^\infty_c([a,b]^n)$. It is known that $\mathcal{S}$ is dense in $L^2(B,P)$, cf. (Kuo75), (Pri97), (Pri94). We denote by $(e_k)_{k\geq 0}$ the canonical basis of $H = l^2(\mathbb{N})$. Let X be a real separable Hilbert space with orthonormal basis $(h_i)_{i\in\mathbb{N}}$, and let $H \otimes X$ denote the completed Hilbert-Schmidt tensor product of H with X. Define a set of smooth vector-valued functionals as

$$\mathcal{S}(X) = \left\{ \sum_{i=0}^{i=n} Q_i h_i \ : \ Q_0,\dots,Q_n \in \mathcal{S}, \quad n \in \mathbb{N} \right\}.$$

If $u \in \mathcal{S}(H \otimes X)$, we write $u = \sum_{k=0}^\infty u_k e_k$, $u_k \in \mathcal{S}(X)$, $k \in \mathbb{N}$. Let

$$\mathcal{U}(X) = \left\{ v \in \mathcal{S}(H \otimes X) \ : \ v_k = 0 \textit{ on } \theta_k^{-1}(\{a,b\}), \ k \in \mathbb{N} \right\}.$$

The space $\mathcal{U}(X)$ is the analog of a test function space, in that it consists in smooth functions which vanish at the boundary of the support of a measure. The set $\theta_k^{-1}(\{a,b\})$ is of zero P-measure, but the elements of $\mathcal{U}(X)$ are well-defined since they are continuous. In the Gaussian case, one has simply $\mathcal{U}(X) = \mathcal{S}(X)$. It can be shown in each case that $\mathcal{U}(X)$ is dense in $L^2(B \times \mathbb{N}; X)$, cf. (Kuo75), (Pri97), (Pri94). We let $\mathcal{U}$=$\mathcal{U}(\mathbf{R})$. Only the vector space structure of B is needed in the following definition.

DEFINITION 2.1. *We define a gradient $D : \mathcal{S}(X) \to L^2(B \times \mathbb{N}; X)$ by*

$$(DF(\omega), h)_H = \lim_{\varepsilon \to 0} \frac{F(\omega + \varepsilon h) - F(\omega)}{\varepsilon} \qquad \omega \in B, \ h \in H.$$

The following proposition contains the integration by parts formula (2.1), cf. (NP88), (Pri94), cf. (Pri97).

PROPOSITION 2.2. *The operator $D : L^2(B;X) \to L^2(B \times \mathbb{N}; X)$ is closable and has an adjoint operator $\delta : \mathcal{U}(X) \to L^2(B;X)$. The duality relation is*

$$E\left[(DF,u)_{H\otimes X}\right] = E\left[(\delta(u),F)_X\right], \quad u \in \mathcal{U}(X), F \in \mathcal{S}(X). \qquad (2.1)$$

The expression of δ is

$$\delta(u) = \sum_{k\in\mathbb{N}} (\alpha_1 + \alpha_2\theta_k)u_k - D_k u_k, \quad u \in \mathcal{U}.$$

The ingredients of the proof are the boundary condition imposed on the test functions in $\mathcal{U}(X)$ and the density of $\mathcal{U}(X)$ in the space of X-valued square-integrable functionals. Let $Dom(\delta; X)$ denote the domain of the closed extension of δ for $p = 2$.

DEFINITION 2.3. *We call*

- $D_{2,1}(X)$ *the completion of* $\mathcal{S}(X)$ *with respect to the norm*

$$\| F \|_{D_{2,1}(X)} = \| | F |_X \|_2 + \| | DF |_{H\otimes X} \|_2,$$

- $D^{\mathcal{U}}_{2,1}(H)$ *the completion of* $\mathcal{U}$ *with respect to the norm* $\| \cdot \|_{D_{2,1}(H)}$,

- $D_{\infty,1}(X)$, *resp.* $D^{\mathcal{U}}_{\infty,1}(H)$ *the subset of* $D_{2,1}(X)$, *resp.* $D^{\mathcal{U}}_{2,1}(H)$ *made of the random variables* F *for which* $\| F \|_{D_{\infty,1}(X)}$, *resp.* $\| F \|_{D_{\infty,1}(H)}$ *is bounded.*

In case λ is the Gaussian density, $D^{\mathcal{U}}_{2,1}(H) = D_{2,1}(H)$.

PROPOSITION 2.4. *The norm defined by*

$$\| F \|_{D^{\mathcal{U}}_{2,1}(H)} = \| | DF |_H \|_{L^2(B)} + \alpha_2 \| F \|_{L^2(B)},$$

is equivalent to $\| F \|_{D_{2,1}(H)}$ *on* $D^{\mathcal{U}}_{2,1}(H)$. *More precisely, in the case of the exponential or uniform density,*

$$\| F \|_{L^2(B)} \leq 2 \| | DF |_{H\otimes H} \|_{L^2(B)}, \quad F \in D^{\mathcal{U}}_{2,1}(H).$$

Proof. The proof needs only to be done with $\alpha_2 = 0$, i.e. in the exponential and uniform cases. If λ has the exponential density, it is sufficient to notice that for $u \in \mathcal{C}^\infty_c(\mathbf{R})$ with $u(0) = 0$,

$$\begin{aligned}\int_0^\infty u(x)^2 e^{-x} dx &= 2 \mid \int_0^\infty u(x) u'(x) e^{-x} dx \mid \\ &\leq 2 \left(\int_0^\infty u^2(x) e^{-x} dx \right)^{1/2} \left(\int_0^\infty (u'(x))^2 e^{-x} dx \right)^{1/2},\end{aligned}$$

hence

$$E[u(\theta_k)^2] = \int_0^\infty u(x)^2 e^{-x} dx \leq 4 \int_0^\infty (u'(x))^2 e^{-x} dx = 4E[(D_k u(\theta_k))^2].$$

For the uniform density, let $u \in \mathcal{C}^\infty_c(\mathbf{R})$ with $u(-1) = u(1) = 0$. Then

$$\int_{-1}^1 u(x)^2 dx/2 = \mid \int_{-1}^1 u(x) u'(x) x dx \mid \leq \int_{-1}^1 \mid u(x) u'(x) \mid dx$$

$$\leq \; 2\left(\int_{-1}^{1}(u(x))^2dx/2\right)^{1/2}\left(\int_{-1}^{1}(u'(x))^2dx/2\right)^{1/2},$$

hence

$$E[u(\theta_k)^2] = \int_{-1}^{1} u(x)^2dx/2 \leq 4\int_{-1}^{1}(u'(x))^2dx/2 = 4E[(D_k u(\theta_k))^2].$$

If $F \in \mathcal{U}$, we proceed in both cases by integration with respect to the remaining variables to obtain

$$E[F_k^2] \leq 4E[(D_k F_k)^2],$$

and then by summation on $k \in \mathbf{N}$. □

PROPOSITION 2.5. *The operator δ is continuous from $D_{2,1}^{\mathcal{U}}(H)$ into $L^2(B)$ with*

$$\| \delta(u) \|_{L^2(B)} \leq \| u \|_{D_{2,1}^{\mathcal{U}}(H)}, \quad u \in D_{2,1}^{\mathcal{U}}(H). \tag{2.2}$$

Proof. We only need to interpret the results of cf. (Pri97), (Pri94), (Ram74) with the norm $\| \cdot \|_{D_{2,1}^{\mathcal{U}}(H)}$. □

For the following result, we refer to (BH86) in the Wiener space case. Its proof is identical to the proof of the analog statement in (NP88), (Pri96).

PROPOSITION 2.6. *The operator D is local. More precisely, if $F \in D_{2,1}(X)$, then $DF = 0$ a.s. on $\{F = 0\}$. The operator δ is also local, i.e. if $u \in Dom(\delta; X)$ then $\delta(u) = 0$ a.s. on $\{u = 0\}$.*

DEFINITION 2.7. *Let $1 \leq p \leq \infty$. We say that $F \in D_{p,1}^{loc}(X)$, resp. $D_{p,1}^{\mathcal{U},loc}(H)$ if there is a sequence $(F_n, A_n)_{n\in\mathbf{N}}$ such that*

- *$F_n \in D_{p,1}(X)$, resp. $F_n \in D_{p,1}^{\mathcal{U}}(X)$,*
- *A_n is measurable,*
- *$\bigcup_{n\in\mathbf{N}} A_n = B$ a.s.,*
- *$F_n = F$, a.s. on A_n, $n \in \mathbf{N}$.*

3. Nonlinear transformations of the Gaussian, exponential and uniform measure

Let K be a Hilbert-Schmidt operator with eigenvalues $(\lambda_k)_{k\in\mathbf{N}}$, counted with their multiplicities. The Carleman-Fredholm determinant of $I_H + K$ is defined as

$$\det{}_2(I_H + K) = \prod_{i=0}^{\infty}(1+\lambda_i)\exp(-\lambda_i),$$

cf. (DS57). Note that $\det_2(I_H + \cdot\ ) : H \otimes H \longrightarrow \mathbf{R}$ is continuous, with the bound $\mid \det_2(I_H + K) \mid \leq (1+ \mid K \mid_{H\otimes H}) \exp(1+ \mid K \mid_{H\otimes H}^2)$. Since $D_{2,1}^{\mathcal{U}}(H) \subset Dom(\delta)$ from Proposition 2.5, we can set the following definition.

DEFINITION 3.1. *For $F \in D_{2,1}^{\mathcal{U},loc}(H)$, let*

$$\Lambda_F = \det{}_2(I_H + DF)\exp\left(-\delta(F) - \frac{1}{2}\alpha_2 \mid F \mid_H^2\right). \tag{3.1}$$

In case λ is uniform, Λ_F simplifies to $\Lambda_F = \det(I_H + DF)$ where "det" is the ordinary determinant for trace class operators since the series $\sum_{k\in\mathbf{N}} D_k F(k)$ converges a.s since $F \in Dom(\delta)$, and the Carleman-Fredholm determinant does not have the same renormalization effect. However, even in this case the use of $\det_2$ is essential in order to carry out the uniform integrability argument in Proposition 5.2.

The image measure of P by $I_B + F$ with $F : B \longrightarrow H$ measurable is denoted by $(I_B + F)_* P$. Our main result is the following. A generalization of this theorem to non-invertible transformations is proved in Section 5, following (Pri96), (ÜZ94).

THEOREM 3.2. *Let $F : B \to H$ be such that $h \mapsto F(\omega + h)$ is continuously differentiable on*

$$\{h \in H \ : \ \omega + h \in B_{[a,b]}\},$$

for a.s. ω, i.e. F is $H - C^1$. Assume that

- *$F(k) = 0$ on $\theta_k^{-1}(\{a,b\})$, $k \in \mathbf{N}$,*
- *$I_B + F$ is a.s. bijective,*
- *$I_H + DF$ is a.s. invertible,*
- *$(I_B + F)\left(B_{]a,b[}\right) = B_{]a,b[}$.*

Then

$$E[f] = E[f \circ (I_B + F) \mid \Lambda_F \mid]$$

for f measurable bounded.

More generally, the nonlinear transformations of B that we consider are of the form $I_B + F$ with F locally $H - C^1$, cf. (ÜZ94):

DEFINITION 3.3. *We say that a random variable $F : B \to H$ is $H - C^1_{loc}$ if there is a random variable Q with $Q > 0$ a.s. such that $h \to F(\omega + h)$ is continuously differentiable on*

$$\left\{ h \in H \ : \mid h \mid_H < Q(\omega) \text{ and } \omega + h \in B_{[a,b]} \right\},$$

for any $\omega \in B_{[a,b]}$.

PROPOSITION 3.4. *A sufficient condition for $F \in H - C^1_{loc}$ to be in $D^{\mathcal{U},loc}_{\infty,1}(H)$ is that*

$$F(k) = 0 \text{ on } \theta_k^{-1}(\{a,b\}), \ k > n_0,$$

for some $n_0 \in \mathbf{N}$.

Proof. It suffices to cover B with a countable collection of sets given below in Lemma 3.8. □

DEFINITION 3.5. *If $A \subset B$ is measurable we let for $\omega \in B$*

$$\rho_A(\omega) = \inf_{h \in H} \{\mid h \mid_H : \ \omega + h \in A\}$$

and $\rho_A(\omega) = \infty$ if $\omega \notin A + H$.

The proof of the following result is directly adapted from (Buc92), (Nua93), (Oco88), (Pri96), replacing $W^{2,1}\left(\mathbf{R}^{n+1}, \frac{1}{\sqrt{2\pi}^{n+1}} e^{-(x_0^2 + \cdots + x_n^2)/2}\right)$ with $W^{2,1}(]a,b[^{n+1}, \lambda^{\otimes n+1})$. Let $\mathcal{F}_n$ denote the σ-algebra generated by $\theta_0, \ldots, \theta_n$, and define $\pi_n^* : \mathbf{R}^{n+1} \to H$ by $\pi_n^*(x) = (x_0, \ldots, x_n, 0, \ldots)$.

LEMMA 3.6. *Let $F \in L^2(B; X)$. Then*

- *$F \in D_{2,1}$ if and only if $F_n = E[F \mid \mathcal{F}_n] \in D_{2,1}$ for all $n \in \mathbf{N}$. In this case,*

$$\mid DF_n \mid_H \leq \mid DF \mid_H, \quad a.s., \ n \in \mathbf{N}.$$

- *Moreover, F_n belongs to $D_{2,1}$ if and only if there exists*

$$f \in W^{2,1}(]a,b[^{n+1}, \lambda^{\otimes n+1})$$

such that $F_n = f(\theta_0, \ldots, \theta_n)$. Then $DF_n = (\partial_k f(\theta_0, \ldots, \theta_n))_{k \in \mathbf{N}}$.

– *assuming that for some $c > 0$ and for any $h \in H$,*

$$| F(\omega + h) - F(\omega) |_X \leq c | h |_H$$

for $\omega, \omega + h \in B_{[a,b]}$. Then $F \in D_{2,1}(X)$ and $| DF |_{H\otimes X} \leq c$, a.s.

We notice that as in (Nua93), $\rho_A(\omega) = 0$, $\omega \in A$, and if $\phi \in \mathcal{C}_c^\infty(\mathbf{R})$ with A σ-compact, then

$$| \phi(\rho_A(\omega + h)) - \phi(\rho_A(\omega)) |_H \leq \| \phi' \|_\infty | h |_H, \quad \omega \in B, \ h \in H,$$

hence $\phi(\rho_A) \in D_{\infty,1}$ with $| D\phi(\rho_A) |_H \leq \| \phi' \|_\infty$. Denote by π_n the application $\pi_n : B \longrightarrow H$ defined by $\pi_n(\omega) = \left(\omega_k 1_{\{k\leq n\}}\right)_{k\in\mathbf{N}}$. The following lemma is stated in the general case, but its proof is done in the case of the uniform density.

LEMMA 3.7. *Let $F : B \to H$ measurable with $\| | F |_H \|_\infty < \infty$, such that*

$$F(k) = 0 \text{ on } \theta_k^{-1}(\{a, b\}), \ k \in \mathbf{N},$$

and for some $c > 0$

$$| F(\omega + h) - F(\omega) |_H < c | h |_H$$

$h \in H$, and $\omega, \omega + h \in B_{[a,b]}$. Then $F \in D_{\infty,1}^{\mathcal{U}}$, and there is a sequence $(\Phi_n)_{n\in\mathbf{N}} \subset \mathcal{U}$ that converges to F in $D_{2,1}(H)^{\mathcal{U}}$ with for $n \in \mathbf{N}$:
(i) $\| | \Phi_n |_H \|_\infty \leq \| | F |_H \|_\infty$.
(ii) $\| | D\Phi_n |_{H\otimes H} \|_\infty \leq c$.
Assume moreover that $\theta_k + F(k) \in [a, b]$ a.s., $k > n_0$, for some $n_0 \in \mathbf{N} \bigcup \{\infty\}$. Then the sequence $(\Phi_n)_{n\in\mathbf{N}}$ can be chosen to verify
(iii) $\theta_k + \Phi_n(k) \in [a, b]$, $k > n_0$, $n \in \mathbf{N}$.

Proof. Let $F_n = \pi_n E[F \mid \mathcal{F}_n]$, $n \in \mathbf{N}$. The sequence $(F_n)_{n\in\mathbf{N}}$ converges to F in $D_{2,1}(H)$ and satisfies to (i), (ii). There is a version of $F_n(k)$ which is Lipschitz on $B_{[-1,1]}$ and such that $F_n(k) = 0$ on $\theta_k^{-1}(]-1, 1[^c)$. Let $\omega \in B_{[-1,1]}$, $h \in H$ such that $\pm(\omega_k + h_k) > 1$ and $\tilde{h} = (\pm 1 - \omega_k)1_{\{k\}} + \sum_{i=0}^\infty h_i e_i 1_{\{i\neq k\}}$. Then $F_n(k)(\omega + h) = F_n(k)(\omega + \tilde{h}) = 0$, and

$$\begin{aligned} | F_n(k)(\omega + h) - F_n(k)(\omega) |_H &= | F_n(k)(\omega + \tilde{h}) - F(\omega) |_H \\ &\leq c | \tilde{h} |_H \\ &\leq c \left((\pm 1 - \omega_k)^2 + \sum_{i=0}^\infty 1_{\{i\neq k\}} h_i^2 \right)^{1/2} \leq c | h |_H . \end{aligned}$$

There exists $f_k \in W^{2,1}(\mathbf{R}^{n+1}, dx)$, with $f_k = 0$ a.e. on $[-1, 1]^k \times [-1, 1]^c \times [-1, 1]^{n-k}$, such that $F_n(k) = f_k(\theta_0, \ldots, \theta_n)$ P-a.e., $k =$

$0, \ldots, n$. From the above argument concerning $F_n(k)$, f_k has a Lipschitz version on $[-1,1]^k \times \mathbf{R} \times [-1,1]^{n-k}$ such that $f_k = 0$ on $[-1,1]^k \times]-1,1[^c \times [-1,1]^{n-k}$. Let $\Psi \in \mathcal{C}_c^\infty(\mathbf{R}^{n+1})$ with support in $[-2,0]^k \times [0,2] \times [-2,0]^{n-k}$, $0 \leq \Psi \leq 1$ and $\int_{\mathrm{R}^{n+1}} \Psi(x)dx = 1$. Let for $m \geq 2$

$$\begin{aligned}\phi_{k,m}(y) &= \left(\frac{1}{m}\right)^{n+1} \int_{[-1,1]^k \times \mathrm{R}_+ \times [-1,1]^{n-k}} \Psi(m(y+x)) f_k(x)dx \\ &+ \left(\frac{1}{m}\right)^{n+1} \int_{[-1,1]^k \times \mathrm{R}_- \times [-1,1]^{n-k}} \Psi(m(y-x)) f_k(x)dx,\end{aligned}$$

$y \in [-1,1]^{n+1}$, and $\Phi_m(k) = \phi_{k,m}(\theta_0, \ldots, \theta_n)$, $k = 0, \ldots, n$, $\Phi_m(k) = 0$, $k > n$. Then $(\Phi_m)_{m \geq 2} \subset \mathcal{U}$ converges to F_n in $D_{2,1}$ and satisfies to (i), (ii). If $\theta_k + F(k) \in [-1,1]$ a.s., it can be checked that $\theta_k + \Phi_n(k) \in [-1,1]$ from the definition of Φ_n. □

Let $\phi \in \mathcal{C}_c^\infty(\mathbf{R})$ with $\| \phi \|_\infty \leq 1$, such that $\phi = 0$ on $[2/3, \infty[$, $\phi = 1$ on $[0, 1/3]$ and $\| \phi' \|_\infty < 4$.

LEMMA 3.8. *For $p, q > 0$, let*

$$\begin{aligned} A = \{ \omega \in B_{]a,b[} \ : \ & \omega_k - a, b - \omega_k > 4/p, \ k \leq n_0, \\ & Q(\omega) \geq 4/p, \\ & \sup_{|h|_H \leq 2/p} | F(\omega + h) |_H \leq q/(6p), \\ & \left. \sup_{|h|_H \leq 2/p} | DF(\omega + h) |_{H \otimes H} \leq q/6 \right\} \end{aligned}$$

and $\tilde{F} = \phi(p\rho_G)F$, where G is a σ-compact set contained in A. Then

$$| \tilde{F}(\omega + h) - \tilde{F}(\omega) |_H \leq (5q/6) \mid h \mid_H,$$

for $h \in H$, $\omega, \omega + h \in B_{[a,b]}$, and $\| | \tilde{F} |_H \|_\infty \leq q/(6a)$. Consequently $\tilde{F} \in D_{\infty,1}^{\mathcal{U}}(H)$.

Proof. Identical to the proof of the analog Lemma 4 in (Pri96), replacing the set $\theta_k^{-1}(\{0\})$ with $\theta_k^{-1}(\{-1,1\})$, noticing that for ω in this set, $\rho_A(\omega) \geq 4/p$, hence $\phi(\rho_A(\omega)) = 0$, and $F_n(k) = 0$ on $\theta_k^{-1}(\{-1,1\})$. It remains to use Lemma 3.7 to show that $\tilde{F} \in D_{\infty,1}^{\mathcal{U}}(H)$. □

PROPOSITION 3.9. *Let $F, G \in \mathcal{S}(H)$ and $T = I_B + F$. We have $G \circ T \in Dom(\delta)$ and*

$$\delta(G) \circ T = \delta(G \circ T) + \mathrm{Trace}(DF^*(DG) \circ T) + \alpha_2(F, G \circ T)_H.$$

Proof. cf. (Pri94), (ÜZ92) for the Gaussian and exponential cases. For the uniform density, we have $\delta(G \circ T) \in \mathcal{S}$ and

$$\begin{aligned}\delta(G \circ T) &= -\sum_{k=0}^{\infty} D_k \left(G(k) \circ T\right) = -\sum_{k=0}^{\infty} D_k (I_B + F)^* (DG(k)) \circ T \\ &= \delta(G) \circ T - \sum_{k,l=0}^{\infty} D_k F(l)(D_l G(k)) \circ T.\end{aligned}$$

□

4. Probabilistic interpretation

The aim of this section is to interpret Theorem 3.2 in probabilistic terms. We know that in the Gaussian case, the density (3.1) can be expressed with the Itô integral in the adapted case or with one of its extensions, called the Skorohod integral, in the anticipative case. We are interested in such interpretations in the exponential and uniform cases. The connection between the definitions and the measure λ becomes less explicit than in the previous section. Let $(h_k)_{k \in \mathbf{N}}$ be a Hilbert basis of $L^2(\mathbf{R}_+)$. In connection with the problem stated at the end of the introduction, we are seeking an interpretation of the divergence δ as a stochastic integral of continuous-time processes. For this we need to compose δ with a map $i : L^2(B) \otimes L^2(\mathbf{R}_+) \to L^2(B) \otimes l^2(\mathbf{N})$. We define in each case a linear injection $i : l^2(\mathbf{N}) \to L^2(\mathbf{R}_+)$, a stochastic process $(Y_t)_{t \in \mathbf{R}_+}$, and a filtration $(\mathcal{F}_t)$ in the following way.

DEFINITION 4.1. *If λ is Gaussian, let*

$$i(e_k) = h_k, \ \ k \in \mathbf{N},$$

and denote by $Y = \tilde{Y} = \sum_{k \geq 0} \theta_k \int_0^{\cdot} h_k(s) ds$ the Wiener process on (B, P). Let

$$\mathcal{F}_t = \sigma\left(\tilde{Y}_s \ : \ s \leq t\right), \quad t \in \mathbf{R}_+.$$

In this case, the injection i is actually a Hilbert space isomorphism.

DEFINITION 4.2. *If λ is exponential, let $T_k = \theta_0 + \cdots + \theta_k$,*

$$i(e_k)(t) = -1_{]T_{k-1}, T_k]}(t), \ \ t \in \mathbf{R}_+, \ \ k \in \mathbf{N},$$

and

$$Y_t = \sum_{k \geq 0} 1_{[T_k, \infty[}(t), \quad \tilde{Y}_t = Y_t - t, \quad t \in \mathbf{R}_+.$$

Let $\mathcal{F}_t = \sigma(Y_s \ : \ s < t)$, $t \in \mathbf{R}_+$.

DEFINITION 4.3. *If λ is uniform, let $T_k = k + (1+\theta_k)/2$,*

$$i(e_k)(t) = -\left((1-\theta_k)1_{]k,T_k]}(t) - (1+\theta_k)1_{]T_k,k+1]}(t)\right) \quad t \in \mathbf{R}_+,\ k \in \mathbf{N},$$

and $Y_t = \sum_{k\geq 0} 1_{[T_k,\infty[}(t)$, $\quad \tilde{Y}_t = Y_t - t, \quad t \in \mathbf{R}_+$. *Let*

$$\mathcal{F}_t = \sigma(Y_s \ :\ s \leq [t]), \quad t \in \mathbf{R}_+.$$

($[t]$ denotes the integral part of $x \in \mathbf{R}_+$). Let $\mathcal{W}$ be the dense set in $L^2(B) \otimes L^2(\mathbf{R}_+)$ of continuous-time processes v such that $v(t) = f(t, \theta_0, \ldots, \theta_n)$, $t \in \mathbf{R}_+$, with $f \in \mathcal{C}_c^\infty(\mathbf{R}^{n+2})$, $n \geq -1$.

PROPOSITION 4.4. *The stochastic integral with respect to $(\tilde{Y}_t)_{t\in\mathrm{R}_+}$ can be extended to $(\mathcal{F}_t)$-adapted process $u \in L^2(B) \otimes L^2(\mathbf{R}_+)$, with the bound*

$$E\left[\left(\int_0^\infty u(s)d\tilde{Y}_s\right)^2\right] \leq E\left[\int_0^\infty u(s)^2 ds\right]. \tag{4.1}$$

It is well-known that for normal martingales such as the Wiener and compensated Poisson processes, (4.1) holds as an equality. In the uniform case, it also becomes an equality if $\int_k^{k+1} u(t)dt = 0$, $k \in \mathbf{N}$, cf. (Pri97).

The operator i is easily extended to discrete-time stochastic processes.

DEFINITION 4.5. *Let $j : L^2(\mathbf{R}_+) \to H$ be the adjoint of $i : H \to L^2(\mathbf{R}_+)$, defined as*

$$(i(u), v)_{L^2(\mathrm{R}_+)} = (u, j(v))_H, \quad u \in \mathcal{S}(H),\ v \in \mathcal{W}, \quad P-a.s.$$

We define unbounded operators $\tilde{D} : L^2(B) \longrightarrow L^2(B) \otimes L^2(\mathbf{R}_+)$ and $\tilde{\delta} : L^2(B) \otimes L^2(\mathbf{R}_+) \longrightarrow L^2(B)$ as

$$\tilde{D}F = i \circ DF, \quad F \in \mathcal{S},$$

and

$$\tilde{\delta}(v) = \delta \circ j(v), \quad v \in \mathcal{W}. \tag{4.2}$$

Note that $j(\mathcal{W}) \subset Dom(\delta)$, so that the composition (4.2) is well defined. We have more explicitly:

i) If λ is Gaussian, i is unitary and j is the inverse of i.

$$j_k(v) = \int_0^\infty v(t)h_k(t)dt, \quad k \in \mathbf{N},\ v \in \mathcal{W}.$$

ii) If λ is exponential:

$$j_k(v) = -\int_{T_{k-1}}^{T_k} v(t)dt, \quad k \in \mathbf{N},\ v \in \mathcal{W}.$$

iii) If λ is uniform:

$$j_k(v) = -\left((1-\theta_k)\int_k^{T_k} v(s)ds - (1+\theta_k)\int_{T_k}^{k+1} v(s)ds\right), \quad v \in \mathcal{W},\ k \in \mathbf{N}.$$

PROPOSITION 4.6. *The operators $\tilde{D}$ and $\tilde{\delta}$ are closable adjoint of each other, with*

$$\tilde{\delta}(v) = \int_0^\infty v(s)d\tilde{Y}_s - \int_0^\infty \tilde{D}_s v(s)ds, \qquad v \in \mathcal{W}.$$

If $v \in L^2(B) \otimes L^2(\mathbf{R}_+)$ is $(\mathcal{F}_t)$-adapted, then $v \in Dom(\tilde{\delta})$ and $\tilde{\delta}(v)$ coincides with the compensated integral of v with respect to $(Y_t)_{t\in\mathrm{R}_+}$:

$$\tilde{\delta}(v) = \int_0^\infty v(s)d(Y_s - s).$$

Proof. cf. (CP90), (NP88), (Pri97), (Pri94). □

The eigenvectors of $\tilde{\delta}\tilde{D}$ are given respectively in the Gaussian, exponential and uniform cases by the composition of the Hermite, Laguerre and Legendre polynomials with θ_k, cf. (Pri97), (Pri94), (Wat84). We let

$$L_{2,1} = \{u \in L^2(B) \otimes L^2(\mathbf{R}_+) \ : \ j(u) \in D^{\mathcal{U}}_{2,1}(H)\},$$

endowed with the norm $\| \ u \ \|_{L_{2,1}} = \| \ j(u) \ \|_{D^{\mathcal{U}}_{2,1}(H)}$. This extends the definition of (NP88). The space $L^{loc}_{2,1}$ is defined as in Definition 2.7, and $\tilde{\delta}(u)$ can be locally defined for $u \in L^{loc}_{2,1}$ since $\tilde{\delta}$ is local as δ.

PROPOSITION 4.7. *The mapping $j : L_{2,1} \to D^{\mathcal{U}}_{2,1}(H)$ is an isometric isomorphism. More precisely, for $F \in D^{\mathcal{U}}_{2,1}(H)$, there is $u_F \in L_{2,1}$ such that $F = j(u_F)$, with*

$$E\left[\| \ u_F \ \|^2_{L^2(\mathrm{R}_+)}\right] \leq \| \ F \ \|^2_{D^{\mathcal{U}}_{2,1}(H)} . \tag{4.3}$$

Proof. Let $(F_n)_{n\in\mathrm{N}} \subset \mathcal{U}$ be a sequence converging to F in $D^{\mathcal{U}}_{2,1}(H)$. If λ is exponential, let

$$u_n(t) = \sum_{k\in\mathrm{N}} 1_{]T_k,T_{k+1}]}(t) \left(D_k F_n(k) \mid_{\theta_k = t - T_k}\right).$$

Then $F_n = j(u_n)$, and by integration by parts:

$$E\left[\int_{T_k}^{T_{k+1}} u_n^2(t)dt\right] = E\left[u_n(T_{k+1})^2\right] = E\left[D_k F_n(k)^2\right], \tag{4.4}$$

from the Fubini theorem, since $u_n(t)$ does not depend on θ_k if $t \in]T_k, T_{k+1}[$. Relation (4.3) follows by summation on $k \in \mathbb{N}$. If λ is uniform, let

$$u_n(t) = -\sum_{k\in\mathbb{N}} 1_{]k,k+1]}(t)\left(D_k F_n(k) \mid_{\theta_k=2t-2k-1}\right).$$

We have

$$\begin{aligned}
j_k(u_n) &= (1-\theta_k)\int_k^{T_k} D_k F_n(k) \mid_{\theta_k=2t-2k-1} dt \\
&\quad +(1+\theta_k)\int_{T_k}^{k+1} D_k F_n(k) \mid_{\theta_k=2t-2k-1} dt \\
&= (1-\theta_k)\int_{2k}^{2k+1+\theta_k} D_k F_n(k) \mid_{\theta_k=t-2k-1} dt/2 \\
&\quad +(1+\theta_k)\int_{2k+1+\theta_k}^{2k+2} D_k F_n(k) \mid_{\theta_k=t-2k-1} dt/2 \\
&= (1-\theta_k+1+\theta_k)F_n(k)/2 = F_n(k), \quad k \in \mathbb{N},
\end{aligned}$$

hence $F_n = j(u_n)$ and obviously,

$$\begin{aligned}
E\left[\int_k^{k+1} u_n^2(t)dt\right] &= E\left[\int_{-1}^{1} \left(D_k F_n(k)\right)^2 \mid_{\theta_k=s} ds/2\right] \\
&= E\left[\left(D_k F_n(k)\right)^2\right],
\end{aligned}$$

which implies by summation on $k \in \mathbb{N}$:

$$\begin{aligned}
E\left[\| u_F \|^2_{L^2(\mathbb{R}_+)}\right] &= (1-\alpha_2)E\left[\sum_{k\in\mathbb{N}} D_k F(k)^2\right] + \alpha_2 E\left[\| F \|^2_{l^2(\mathbb{N})}\right] \\
&\leq \| F \|^2_{D^{\mathcal{U}}_{2,1}(H)}.
\end{aligned}$$

In the Gaussian case, j is automatically a Hilbert space isomorphism and (4.3) holds. $\square$

The density function in Theorem 3.2 can now be rewritten after the following proposition.

PROPOSITION 4.8. *let $F \in D^{\mathcal{U},loc}_{2,1}(H)$ satisfy the hypothesis of Theorem 5.1 with $F = j(u)$, $u \in L^{loc}_{2,1}$. The Radon-Nykodim density function is expressed as*

$$\Lambda_F = \det{}_2(I_H + Dj(u))\exp\left(-\tilde{\delta}(u) - \frac{1}{2}\alpha_2 \| u \|^2_{L^2(\mathbb{R}_+)}\right).$$

If u is $(\mathcal{F}_t)$-adapted,

$$\Lambda_F = \det{}_2(I_H + Dj(u)) \exp\left(- \int_0^\infty u(t) d\tilde{Y}_t - \frac{1}{2}\alpha_2 \parallel u \parallel^2_{L^2(\mathrm{R}_+)} \right).$$

Remark. In the uniform case, another definition of i and j can be given so that $\tilde{\delta} = \delta \circ j$ extends the compensated stochastic integral with respect to the natural filtration of $(Y_t)_{t \in \mathrm{R}_+}$, cf. (Pri95). The compensator of $(Y_t)_{t \in \mathrm{R}_+}$ with respect to its natural filtration is given by

$$d\nu(t) = \sum_{k \in \mathrm{N}} \frac{1}{2-t} 1_{[k,T_k[}(t) dt.$$

The linear injection i is then defined as

$$i(e_k)(t) = (1-\theta_k) 1_{[k,T_k[}(t), \quad k \in \mathbf{N},$$

and the dual j of i is taken with respect to $d\nu(t)$:

$$j_k(u) = (1-\theta_k) \int_k^{T_k} u(t) d\nu(t).$$

It satisfies

$$\int_0^\infty i_t(u) v(t) d\nu(t) = (u, j(v))_{l^2(\mathrm{N})}, \quad u \in l^2(\mathbf{N}),\ v \in L^2(\mathbf{R}_+).$$

With those definitions, the operator $\tilde{\delta} = \delta \circ j$ extends the stochastic integral with respect to the compensated process $Y - \nu$, but the eigenvectors of $\tilde{\delta}\tilde{D}$ are no longer given by the Legendre polynomials, cf. (Pri95).

5. Proof of the main result

We will prove the following theorem, which is an extension of Theorem 3.2. This result is also valid on the Wiener and Poisson spaces, cf. (Pri96), (ÜZ94).

THEOREM 5.1. *Let $F \in H - C^1_{loc}$ with $F(k) = 0$ on $\theta_k^{-1}(\{a,b\})$, $k \in \mathbf{N}$. Let $T = I_B + F$ and*

$$M = \left\{ \omega \in B_{[a,b]} \ : \det{}_2(I_H + DF) \neq 0 \right\}.$$

Assume that $T\left(B_{]a,b[}\right) \subset B_{]a,b[}$ and let $N(\omega; M) = card(T^{-1}(\omega) \bigcap M)$. Then $N(\omega; M)$ is at most countably infinite and

$$E\left[f N(\omega; M) \right] = E\left[f \circ T \mid \Lambda_F \mid \right]$$

for $f \in \mathcal{C}_b^+(B)$*. The restriction of* $(I_B + F)_* P$ *to* M *is absolutely continuous with respect to* P*, and*

$$\frac{d(I_B + F)_* P_{|M}}{dP}(\omega) = \sum_{\theta \in (I_B + F)^{-1}(\omega) \cap M} \frac{1}{| \Lambda_F(\theta) |}.$$

Let $\mathcal{K}$ denote the set of finite rank linear operators $K : H \to H$ with rational coefficients such that $I_H + K$ is invertible and let $C_K = \left(\| (I_H + K)^{-1} \|_\infty \right)^{-1}$, $K \in \mathcal{K}$. Let $\mathcal{V}$ denote the subset of H made of sequences with rational coefficients and finite support in $\mathbf{N}$. We start by treating the case of contractive mappings. In the general case, F will be written locally as the composition of a Lipschitz map, a linear map and a translation. The following result extends Prop. 5 of (Pri96).

PROPOSITION 5.2. *Let* $K \in \mathcal{K}$*,* $v \in \mathcal{V}$ *and* $n_0 \in \mathbf{N}$ *such that* $Support(v), Support(Kh) \subset \{0, \ldots, n_0\}$*,* $h \in H$*. Let* A *be a bounded Borel set in* $B_{]a,b[}$*, and let* $F : B \to H$ *be measurable. Let* $T = I_B + F + K + v$*. We make the following assumptions on* (F, K, v, A)*:*

- F *has a bounded support in* B*,*
- $\| | F |_H \|_\infty < \infty$*,*
- $F(k) = 0$ *on* $\theta_k^{-1}(\{a, b\})$*,* $k \in \mathbf{N}$*,*
- *There is* $c \in \mathbf{R}$*,* $0 < c < 1$*, such that*

$$| F(\omega + h) - F(\omega) |_H \leq c C_K \mid h |_H, \tag{5.1}$$

  for $h \in H$*,* $\omega, \omega + h \in B_{[a,b]}$*,*
- $\theta_k + F(k) \in [a, b]$ *a.s.,* $k > n_0$*,*
- $T(A) \subset B_{]a,b[}$*.*

Then T *is injective and*

$$E\left[f 1_{T(A)}\right] = E\left[1_A f \circ T \mid \Lambda_{F+K+v} |\right]$$

for f *bounded measurable on* B*.*

The boundedness assumptions on the set A and the support of F are unnecessary in the case of the uniform density.

Proof. The injectivity of T can be shown as in (Kus82), (Pri96), from (5.1). We modify F with $F = 0$ on $B^c_{[-1,1]}$. Let $(F_n)_{n \geq n_0} \subset \mathcal{U}$ be a sequence given by Lemma 3.7, converging to F in $D_{2,1}(H)$ with $F_n =$

0 on $B^c_{[-1,1]}$, such that $F_n(k) = 0$ if $k > n$, F_n depending only on $\theta_0, \ldots, \theta_n$, and let $T_n = I_B + F_n + K + v$. Then

$$| \, D\left(F_n \circ (I_B + K)^{-1}\right) |_{H\otimes H} \leq c < 1.$$

By a classical argument, cf. (Kus82), (ÜZ94), $I_B + F_n \circ (I_B + K)^{-1} + v$ can be shown to be bijective on B with inverse $I_B + G_n$, where G_n satisfies

$$G_n = -F_n \circ (I_B + K)^{-1} \circ (I_B + G_n) - v, \tag{5.2}$$

and

$$| \, DG_n \, |_{H\otimes H} \leq c/(1-c). \tag{5.3}$$

Moreover,

$$T_n(\{\omega \in B \, : \, \omega_k \in [-1,1], \ \ k > n_0\}) = \{\omega \in B \, : \, \omega_k \in [-1,1], \ \ k > n_0\} \tag{5.4}$$

from Lemma 3.7-*iii)* and (5.2). There is $U, V \in \mathcal{V}$ with

$$\text{Support}(U), \ \text{Support}(V) \subset \{0, \ldots, n_0\}$$

such that $U_k > T_n^{-1}(k) > V_k$ on $B_{]-1,1[}$, $k, n \in \mathbf{N}$, since $(F_n)_{n\geq n_0}$ and $(G_n)_{n\geq n_0}$ are uniformly bounded in n and ω. Let $\mathcal{T}_V : B \longrightarrow B$ denote the application defined as $\mathcal{T}_V(\omega)(k) = \left(\frac{U_k - V_k}{2}\right)\omega_k + \frac{U_k+V_k}{2}$, $k \leq n_0$, $\mathcal{T}_V(\omega)(k) = \omega_k$, $k > n_0$, and let

$$\mu = \left(\prod_{k=0}^{n_0}(U_k - V_k)\right)(\mathcal{T}_V)_* P.$$

There is a function $g \in \mathcal{C}^\infty(\mathbf{R}^{n+1}, \mathbf{R}^{n+1})$ such that $F_n + K + v = \pi_n^* g(\theta_0, \ldots, \theta_n)$, $n \geq n_0$. Let $\pi_n^\perp = I_B - \pi_n$, denote by $P_n^\perp$ the image measure of P by $\pi_n^\perp$ and let $B_n^\perp = \pi_n^\perp(B)$. The Jacobi theorem in dimension $n+1$ gives for $n > n_0$:

$$\begin{aligned}
&\int_B 1_{B_{]-1,1[}} \circ T_n f \circ T_n \mid \Lambda_{F_n+K+v} \mid d\mu \\
&= \int_{B_n^\perp}\int_{\mathrm{R}^{n+1}} 1_{B_{]-1,1[}}(\omega + \pi_n^*(x_0+g_0, \ldots, x_n+g_n)) f(\omega + \pi_n^*(x_0+g_0, \ldots, x_n+g_n)) \\
&\quad \mid \det(I_{\mathrm{R}^{n+1}} + \partial g) \mid dx_0 \cdots dx_n dP_n^\perp(\omega)/2^{n+1} \\
&= \int_{B_n^\perp}\int_{\mathrm{R}^{n+1}} 1_{B_{]-1,1[}}(\omega + \pi_n^* y) f(\omega + \pi_n^* y) dy_0 \cdots dy_n dP_n^\perp(\omega)/2^{n+1} \\
&= E\left[1_{B_{]-1,1[}} f\right], \quad f \in \mathcal{C}_b^+(B).
\end{aligned}$$

The rest of the proof does not differ much from (Pri96), and is given for the sake of completeness. Its consists in an uniform integrability argument as n goes to infinity, using the la Vallée-Poussin lemma. Since $(| DF_n |_{H\otimes H})_{n\in\mathbf{N}}$ is bounded uniformly in n and ω, $(| \det_2 DT_n |)_{n\in\mathbf{N}}$ is uniformly lower and upper bounded, hence instead of $E[| \Lambda_{F_n+K+v} \log | \Lambda_{F_n+K+v} ||]$, we only need to estimate

$$\begin{aligned}
&\int_B | \delta(F_n + K + v)\Lambda_{F_n+K+v} | \, d\mu \\
&= E\left[| \delta(F_n + K + v) \circ T_n^{-1} |\right] \\
&\leq E\left[| \delta(\pi_{n_0} F_n + K + v) \circ T_n^{-1} |\right] \\
&+E\left[| \operatorname{Trace}\left[\left(D\pi_{n_0}^{\perp} F_n\right)^* \circ T_n^{-1} \cdot D\left(-K \circ (I+K)^{-1} + (I+K)^{-1} \circ G_n\right)\right] |\right] \\
&\quad +E\left[| \delta(\pi_{n_0}^{\perp} F_n \circ T_n^{-1}) |\right].
\end{aligned}$$

The first two terms are uniformly bounded in n from (5.3). From the construction of G_n by iterations, cf. (5.2), it can be shown that $\pi_{n_0}^{\perp} G_n \in \mathcal{U}$. We have $\pi_{n_0}^{\perp} G_n = -\pi_{n_0}^{\perp} F_n \circ T_n^{-1}$, hence

$$\begin{aligned}
E\left[| \delta(\pi_{n_0}^{\perp} F_n \circ T_n^{-1}) |\right] &\leq E\left[| \delta(\pi_{n_0}^{\perp} G_n) |\right] \\
&\leq E\left[| D\pi_{n_0}^{\perp} G_n |_{H\otimes H}^2\right] \\
&\leq (c/(1-c))^2,
\end{aligned}$$

$n \in \mathbf{N}$, from (2.2). Choosing a subsequence if necessary and assuming that $g \in \mathcal{C}_b^+(B)$ is zero outside of $B_{]-1,1[}$, we have the μ-a.e. convergence of $(g \circ T_n \mid \Lambda_{F_n+K+v} |)_{n\geq n_0}$ to $g \circ T \mid \Lambda_{F+K+v} |$. Hence

$$\int_B g \circ T \mid \Lambda_{F+K+v} | \, d\mu = E[g]. \tag{5.5}$$

Now (5.5) remains true for $g = f 1_{T(A)}$ where f is measurable and bounded since $T(A) \subset B_{]-1,1[}$. This gives

$$E[f \circ T 1_A \mid \Lambda_{F+K+v} |] = \int_B g \circ T \mid \Lambda_{F+K+v} | \, d\mu = E[g] = E[f 1_{T(A)}].$$

□

Proof of Theorem 5.1. Let $K \in \mathcal{K}$, $v \in \mathcal{V}$ and $n_0 \in \mathbf{N}$ such that

$$\operatorname{Support}(v), \ \operatorname{Support}(Kh) \subset \{0, \ldots, n_0\}, \qquad h \in H.$$

For $n > 8$, let

$$A(n, K, v) = \{ \, \omega \in B_{]-1,1[} : (1 - \omega_k^2) > \frac{8}{n}, \quad k \leq n_0,$$

$$
\left.\begin{aligned}
&Q(\omega) > \frac{4}{n},\\
&\sup_{|h|_H\le 1/n} \mid F(\omega+h)-K(\omega+h)-v\mid_H < C_K/(6n),\\
&\sup_{|h|_H\le 1/n} \mid DF(\omega+h)-K\mid_{H\otimes H} < C_K/6
\end{aligned}\right\},
$$

Let $F_{K,v}=\phi(n\rho_{G(n,K,v)})(F-K-v)$, where $G(n,K,v)$ is a σ-compact modification of $A(n,K,v)\bigcap M$. Then from Lemma 3.8, $F_{K,v}$ and $G(n,K,v)$ satisfy the hypothesis of Proposition 5.2. We can now proceed exactly as in the proof of Theorem 1 of (Pri96), cf. also (ÜZ94). Denote by $(G_k)_{k\in\mathbf{N}}$ the countable family $(G(n,K,v))_{n,K,v}$ and let $M_n = G_n\bigcap\left(\bigcup_{i=0}^{i=n-1}G_i\right)^c$, $n\in\mathbf{N}^*$. We have $\bigcup_{n\in\mathbf{N}^*}M_n = M$, this union being a partition. Now,

$$
\begin{aligned}
E\left[f\circ T \mid \Lambda_F \mid\right] &= \sum_{n=0}^{\infty} E\left[1_{M_n} f\circ T \mid \Lambda_F \mid\right]\\
&= \sum_{n=0}^{\infty} E\left[1_{T(M_n)} f\right] = E\left[fN(\omega;M)\right].
\end{aligned}
$$

We also have

$$
\begin{aligned}
E\left[1_M f\circ T\right] &= \sum_{n=0}^{\infty} E\left[1_{M_n} f\circ T\frac{\Lambda_F}{\Lambda_F\circ T\circ T^{-1}}\right]\\
&= \sum_{n=0}^{\infty} E\left[1_{T(M_n)} f\frac{1}{\Lambda_F\circ T}\right]\\
&= E\left[f\sum_{\theta\in T^{-1}(\omega)\bigcap M}\frac{1}{\Lambda_F(\theta)}\right].
\end{aligned}
$$

□

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