Stochastic SIR Lévy jump model with heavy-tailed increments

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Abstract

This paper considers a general stochastic SIR epidemic model driven by a multidimensional Lévy jump process with heavy-tailed increments and possible correlation between noise components. In this framework, we derive new sufficient conditions for disease extinction and persistence in the mean. Our method differs from previous approaches by the use of Kunita's inequality instead of the Burkholder-Davis-Gundy inequality for continuous processes, and allows for the treatment of infinite Lévy measures by the definition of new threshold $\overline{\mathcal{R}}_0$. An SIR model driven by a tempered stable process is presented as an example of application with the ability to model sudden disease outbreak, illustrated by numerical simulations. The results show that persistence and extinction are dependent not only on the variance of the processes increments, but also on the shapes of their distributions.

Keywords: SIR model; multidimensional Lévy processes; extinction; persistence in the mean; Kunita's inequality; tempered stable process.

Mathematics Subject Classification (2010): 92D25, 92D30, 60H10, 60J60, 60J75, 93E15.

1 Introduction

In this paper, we consider the canonical SIR population dynamics model perturbed by random noise. The model consists in $S_t + I_t + R_t$ individuals submitted to a disease, where S_t, I_t and R_t respectively denote the numbers of susceptible, infected, and recovered individuals

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at time $t \in \mathbb{R}_+$, which are modeled as

$$\begin{cases} dS_t = (\Lambda - \mu S_t - \beta S_t I_t)dt + S_{t^-} dZ_1(t), & \text{(1.1a)} \\ dI_t = (\beta S_t I_t - (\mu + \eta + \varepsilon) I_t)dt + I_{t^-} dZ_2(t), & \text{(1.1b)} \\ dR_t = (\eta I_t - \mu R_t)dt + R_{t^-} dZ_3(t), & \text{(1.1c)} \end{cases}$$

where $Z(t) = (Z_1(t), Z_2(t), Z_3(t))$ is a 3-dimensional stochastic process modeling the intensity of random perturbations of the system. Here, $\Lambda > 0$ denotes the population influx into the susceptible component, $\beta > 0$ reflects the transmission rate from the susceptible group S_t to infected group I_t, $\mu > 0$ represents the nature mortality rate of the three compartments S_t, I_t and R_t, $\varepsilon > 0$ denotes the death rate of infected individuals induced by the disease, and $\eta > 0$ is the recovery rate of the epidemic.

The deterministic version of (1.1a)-(1.1c) with $Z(t) = 0$ has been the object of extensive studies, starting with Kermack and McKendrick (1927) and Anderson and May (1979), where the equilibrium of (1.1a)-(1.1c) in the deterministic case has been characterized by the basic reproduction number

$$\mathcal{R}_0 = \frac{\beta\Lambda}{\mu(\mu + \varepsilon + \eta)}$$

such that when $\mathcal{R}_0 < 1$, the system admits a Globally Asymptotically Stable (GAS) boundary equilibrium $E_0 = (\Lambda/\mu, 0, 0)$ called the disease-free equilibrium, whereas when $\mathcal{R}_0 > 1$ there exists a GAS positive equilibrium

$$E^* = (S^*, I^*, R^*) = \left(\frac{\mu + \varepsilon + \eta}{\beta}, \frac{\mu}{\beta}(\mathcal{R}_0 - 1), \frac{\eta}{\beta}(\mathcal{R}_0 - 1) \right),$$

which is called the endemic equilibrium. In order to model random variations in population numbers, Brownian noise has been added to the deterministic system in e.g. Beddington and May (1977), Tornatore et al. (2005), Gray et al. (2011), to better describe the continuous growth of populations in real ecological systems.

Lévy jump noise has been first incorporated into the stochastic Lotka-Volterra population dynamics model in Bao et al. (2011), Bao and Yuan (2012), where uniform pth moment bounds and asymptotic pathwise estimations have been derived. Driving processes of the form

$$Z_i(t) = \varrho_i B_i(t) + \int_0^t \int_0^\infty \gamma_i(z) \tilde{N}(ds, dz), \qquad i = 1, 2, 3,$$

where $B_1(t)$, $B_2(t)$, $B_3(t)$ are independent standard Brownian motions, $\tilde{N}(ds, dz)$ is a compensated Poisson counting process with intensity $ds\nu(dz)$ on $\mathbb{R}_+ \times [0, \infty)$ and $\nu(dz)$ is a finite Lévy measure on $[0, \infty)$, have been considered in Zhang and Wang (2013), Zhou and Zhang (2016) and Zhang et al. (2018). In this setting, the asymptotic behavior of solutions of (1.1a)-(1.1c) around the equilibrium of the corresponding deterministic system has been studied in Zhang and Wang (2013), and the threshold of this stochastic SIR model has been investigated in Zhou and Zhang (2016). The asymptotic behavior of the stochastic solution of an SIQS epidemic model for quarantine modeling with Lévy jump diffusion term has been analyzed in Zhang et al. (2018).

Previously used jump models, including Zhang and Wang (2013), Zhou and Zhang (2016) and Zhang et al. (2018), share the property of being based on a Poisson counting process $N(dt, dz)$ with finite Lévy measure $\nu(dz)$ on $[0, \infty)$. However, this framework excludes important families of Lévy jump processes having an infinite Lévy measure, as well as flexible correlation between the random noise components of the system (1.1a)-(1.1c). In particular, the increments of jump-diffusion models with finite Lévy measures have exponential tails, see e.g. §4.3 of Cont and Tankov (2004), and they have a limited potential to model extreme events which usually lead to sudden shifts in population numbers.

In this paper, we work in the general setting of finite or infinite Lévy measures ν on $\mathbb{R}$, which allows us to consider heavy-tailed increments having e.g. power law distributions. Indeed, empirical data shows that the jump distribution of population dynamics under sudden environmental shocks such as earthquakes, tsunamis, floods, heatwaves and so on, can follow power law distributions, see e.g. Zhang et al. (2017) and references therein.

We consider a 3-dimensional Lévy noise $Z(t) = (Z_1(t), Z_2(t), Z_3(t))$ with Lévy-Khintchine representation

$$\mathbb{E}\left[e^{iu_1 Z_1(t) + iu_2 Z_2(t) + iu_3 Z_3(t)}\right] = \exp\left(-\frac{t}{2}\langle u, \varrho u\rangle_{\mathbb{R}^3} + t\int_{\mathbb{R}^3\setminus\{0\}} \left(e^{i\langle u, \gamma(z)\rangle_{\mathbb{R}^3}} - i\langle u, \gamma(z)\rangle_{\mathbb{R}^3} - 1\right)\nu(dz)\right),$$

$u = (u_1, u_2, u_3) \in \mathbb{R}^3$, $t \in \mathbb{R}_+$, where $\varrho = (\varrho_{i,j})_{1\le i,j\le 3}$ is a positive definite 3×3 matrix, the functions $\gamma_i : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$ are measurable functions, and $\nu(dz)$ is a σ-finite measure of possibly infinite total mass on $\mathbb{R}^3 \setminus \{0\}$, such that

$$\int_{\mathbb{R}^3\setminus\{0\}} \min(|\gamma_i(z)|^2, 1)\nu(dz) < \infty, \qquad i = 1, 2, 3. \tag{1.2}$$

see e.g. Theorem 1.2.14 in Applebaum (2009). In addition, the process $Z(t) = (Z_1(t), Z_2(t), Z_3(t))$ is known to admit the representation

$$Z_i(t) = B_i^\varrho(t) + \int_0^t \int_{\mathbb{R}^3\setminus\{0\}} \gamma_i(z)\tilde{N}(ds, dz), \quad i = 1, 2, 3, \tag{1.3}$$

where $(B_1^\varrho(t), B_2^\varrho(t), B_3^\varrho(t))$ is a 3-dimensional Gaussian process with independent and stationary increments and covariance matrix $\varrho = (\varrho_{i,j})_{1\le i,j\le 3}$, and $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$ is the compensated Poisson counting process with Lévy measure $\nu(dz)$ on $\mathbb{R}^3 \setminus \{0\}$. All processes are defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbf{P})$, $N(dt, dz)$ is independent of $(B_1^\varrho(t), B_2^\varrho(t), B_3^\varrho(t))$, and the covariances of $(Z_1(t), Z_2(t), Z_3(t))$ are given by

$$\mathbb{E}[Z_i(t)Z_j(t)] = \varrho_{i,j}t + t\int_{\mathbb{R}^3\setminus\{0\}} \gamma_i(z)\gamma_j(z)\nu(dz), \quad t \in \mathbb{R}_+, \quad i, j = 1, 2, 3, \tag{1.4}$$

which allows for the modeling of random interactions between the components (S_t, I_t, R_t) of the model.

In order to investigate the threshold of the stochastic SIR model with finite Lévy measures, Zhou and Zhang (2016) have derived long-term estimates (Lemmas 2.1-2.2 therein) which rely on the finiteness of the quantity

$$\int_0^\infty \big((1 + \overline{\gamma}(z))^p - 1 - \underline{\gamma}(z)\big)\nu(dz), \quad p > 1, \tag{1.5}$$

where

$$\overline{\gamma}(z) := \max(\gamma_1(z), \gamma_2(z), \gamma_3(z)) \;\text{ and }\; \underline{\gamma}(z) := \min(\gamma_1(z), \gamma_2(z), \gamma_3(z)), \quad z \in \mathbb{R}.$$

In our generalized setting (1.3) under (1.2), estimates on solutions are obtained by replacing (1.5) with the expression

$$\lambda(p) := c_p \int_{\mathbb{R}^3\setminus\{0\}} \overline{\gamma}^2(z)\nu(dz) + c_p \int_{\mathbb{R}^3\setminus\{0\}} \overline{\gamma}^p(z)\nu(dz), \quad p > 1. \tag{1.6}$$

where $c_p := p(p-1)\max(2^{p-3}, 1)/2$. In addition, given the jump stochastic integral process

$$K_t := \int_0^t \int_{\mathbb{R}^3\setminus\{0\}} g_s(z)\,\big(N(ds, dz) - \nu(dz)ds\big), \quad t \in \mathbb{R}_+,$$

of the predictable integrand $(g_s(y))_{(s,y)\in\mathbb{R}_+\times\mathbb{R}}$, we use Kunita's inequality

$$\mathbb{E}\left[\sup_{0\le s\le t}|K_s|^p\right] \tag{1.7}$$
$$\le \; C_p\mathbb{E}\left[\int_0^t\int_{\mathbb{R}^3\setminus\{0\}}|g_s(z)|^p\,\nu(dz)ds\right]+C_p\mathbb{E}\left[\left(\int_0^t\int_{\mathbb{R}^3\setminus\{0\}}|g_s(z)|^2\,\nu(dz)ds\right)^{p/2}\right],$$

for all $t\in\mathbb{R}_+$ and $p\ge 2$, where $C_p:=2^p p^{p\log_2 p}\big(2+\big(10e^{\lceil\log_2 p\rceil}\big)^{p/2}\big)$, see Theorem 2.11 of Kunita (2004), Theorem 4.4.23 of Applebaum (2009), and Corollary 2.2 in Breton and Privault (2020). This replaces the Burkholder-Davis-Gundy inequality for continuous martingales

$$\mathbb{E}\left[\sup_{0\le s\le t}|M_s|^p\right]\le C_p\mathbb{E}\big[\langle M,M\rangle_t^{p/2}\big],\qquad p>1, \tag{1.8}$$

which is used in the proof of Lemmas 2.1 and 2.2 in Zhou and Zhang (2016), where $\langle M,M\rangle_t$ is the (predictable) quadratic variation of the continuous martingale $(M_t)_{t\in\mathbb{R}_+}$, see e.g. Theorem 7.3 in Chapter 1 of Mao (2008). Indeed, it is known, see e.g. Remark 357 page 384 of Situ (2005) and Breton and Privault (2020), that (1.8) is invalid for martingales with jumps.

As an example, we consider the tempered stable distribution introduced in Koponen (1995) which belongs to the family of self-decomposable distributions, see § 3.15 in Sato (1999). Given $\alpha\in(0,2)$, the tempered α-stable Lévy measure is defined as

$$\nu(A)=\int_0^\infty\frac{\mathrm{e}^{-r}}{r^{\alpha+1}}\int_{\mathbb{R}^3\setminus\{0\}}\mathbf{1}_A(rx)R_\alpha(dx)dr, \tag{1.9}$$

where $R_\alpha(dx)$ is a measure on $\mathbb{R}^3\setminus\{0\}$ such that

$$\int_{\mathbb{R}^3\setminus\{0\}}\min(\|x\|_{\mathbb{R}^3}^2,\|x\|_{\mathbb{R}^3}^\alpha)R_\alpha(dx)<\infty,$$

see Theorem 2.3 in Rosiński (2007). It has been shown in Sztonyk (2010), Theorem 5, that the increments of tempered stable processes can have (heavy) power tails instead of (semi-heavy) exponential tails, see also Küchler and Tappe (2013). Taking, e.g.

$$R_\alpha(dx)=k_-\lambda_-^\alpha\delta_{(-1/\lambda_-,-1/\lambda_-,-1/\lambda_-)}(dx)+k_+\lambda_+^\alpha\delta_{(1/\lambda_+,1/\lambda_+,1/\lambda_+)}(dx),$$

where $k_-,k_+,\lambda_-,\lambda_+>0$, and δ_y denotes the Dirac measure at $y\in\mathbb{R}^3$, the Lévy measure of the 3-dimensional fully correlated tempered stable process is given by

$$\nu(A)=\int_{\mathbb{R}^3\setminus\{0\}}\int_0^\infty\mathbf{1}_A(rx)\frac{\mathrm{e}^{-r}}{r^{\alpha+1}}dr R_\alpha(dx) \tag{1.10}$$

$$= \quad k_- \int_0^\infty \mathbf{1}_A(-r/\lambda_-, -r/\lambda_-, -r/\lambda_-) \frac{\mathrm{e}^{-r}}{r^{\alpha+1}} dr + k_+ \int_0^\infty \mathbf{1}_A(r/\lambda_+, r/\lambda_+, r/\lambda_+) \frac{\mathrm{e}^{-r}}{r^{\alpha+1}} dr,$$

with $\nu(\mathbb{R}) = +\infty$ for all $\alpha \in (0, 2)$. We note that (1.5) is infinite when $\alpha \in [1, 2)$ and $p > 1$, whereas $\lambda(p)$ given by (1.6) remains finite whenever $p > \alpha$.

This paper is organised as follows. After stating preliminary results on the existence and uniqueness of solutions to (1.1a)-(1.1c) in Proposition 2.1, in the key Lemmas 2.2 and 2.3 we derive new solution estimates by respectively using $\lambda(p)$ defined in (1.6) and Kunita's inequality (1.7) for jump processes. Then in Theorems 3.1 and 3.2 we respectively deal with disease extinction and persistence in the mean for the system (1.1a)-(1.1c). We show that the threshold behavior of the stochastic SIR system (1.1a)-(1.1c) is determined by the basic reproduction number

$$\overline{\mathcal{R}}_0 = \mathcal{R}_0 - \frac{\beta_2}{\mu + \varepsilon + \eta} = \frac{\beta \Lambda / \mu - \beta_2}{\mu + \varepsilon + \eta} \tag{1.11}$$

which differs from Zhou and Zhang (2016) due to the quantity

$$\beta_2 := \frac{1}{2} \varrho_{2,2} + \int_{\mathbb{R}^3 \setminus \{0\}} (\gamma_2(z) - \log(1 + \gamma_2(z))) \, \nu(dz).$$

We note that the values of $\overline{\mathcal{R}}_0$ and β_2 depend on $\varrho_{2,2}$ and not on the Brownian (off-diagonal) correlation coefficients in the matrix ϱ. However they may depend on jump process correlations through the integral of $\gamma_2(z)$ with respect to the Lévy measure $\nu(dz)$ in (1.4), see the proof of Theorem 3.1.

In Section 4 we present numerical simulations based on tempered stable processes with parameter $\alpha \in (0, 1)$. We show in particular that the addition of a jump component to the system (1.1a)-(1.1c) may result into the extinction of the infected and recovered populations as $\alpha \in (0, 1)$ becomes large enough and the variance of random fluctuations increases, which is consistent with related observations in the literature, see e.g. Cai et al. (2015). In addition, we note that this phenomenon can be observed when the noise variances are normalized to identical values, showing that the shape of the distribution alone can affect the long term behavior of the system. The proofs which are similar to the literature, see Zhou and Zhang (2016), are presented in the Appendix for completeness.

2 Large time estimates

For f an integrable function on $[0,t]$, we denote

$$\langle f\rangle_t = \frac{1}{t}\int_0^t f(s)ds, \quad \langle f\rangle^* = \limsup_{t\to\infty}\frac{1}{t}\int_0^t f(s)ds, \quad \langle f\rangle_* = \liminf_{t\to\infty}\frac{1}{t}\int_0^t f(s)ds, \quad t>0.$$

In addition, we assume that the jump coefficients $\gamma_i(z)$ in (1.3) satisfy

(H_1) $\displaystyle\int_{\mathbb{R}^3\setminus\{0\}} |\gamma_i(z)|^2\nu(dz) < \infty$, $i=1,2,3$,

together with the condition:

(H_2) $\gamma_i(z) > -1$, $\nu(dz)$-*a.e.*, and $\displaystyle\int_{\mathbb{R}^3\setminus\{0\}} \big(\gamma_i(z) - \log(1+\gamma_i(z))\big)\nu(dz) < \infty$, $i=1,2,3.$

Proposition 2.1 *Under* (H_1)-(H_2), *for any given initial data* $(S_0, I_0, R_0) \in \mathbb{R}_+^3$, *the system* (1.1a)-(1.1c) *admits a unique positive solution* $(S_t, I_t, R_t)_{t\in\mathbb{R}_+}$ *which exists in* $(0,\infty)^3$ *for all* $t\geq 0$, *almost surely.*

Proof. By Theorem 6.2.11 in Applebaum (2009) or Theorem 2.1 in Bao et al. (2011), the system (1.1a)-(1.1c) admits a unique local solution $(S_t, I_t, R_t)_{t\in(0,\tau_e]}$ up to the explosion time τ_e for any initial data $(S_0, I_0, R_0) \in \mathbb{R}_+^3$, since it has affine coefficients by (H_1). In addition, by (H_2) we have $\gamma_i(z) > -1$, $\nu(dz)$-*a.e.*, $i = 1,2,3$, hence the solution is positive. The remaining of the proof follows the lines of proof of Theorem 1 in Zhang and Wang (2013) by noting that the condition (H_2) page 868 therein can be replaced by (H_2) above. □

Next, given $\lambda(p)$ defined in (1.6) we let

$$\|\varrho\|_\infty := \max_{i=1,2,3}\sum_{j=1}^3 |\varrho_{i,j}|,$$

and we consider the following condition:

$$(H_3^{(p)}) \quad \mu > \frac{p-1}{2}\|\varrho\|_\infty + \frac{\lambda(p)}{p}, \qquad p>1.$$

The proofs of Lemmas 2.2-2.3 below present several significant differences from the arguments of Zhao and Jiang (2014) and Zhou and Zhang (2016). First, our arguments do not require the finiteness of the Lévy measure $\nu(dz)$, and in the proof of our Lemma 2.2 we replace the Burkholder-Davis-Gundy inequality (1.8) for continuous processes used in Zhou and Zhang

(2016) with the simpler bound (2.4) below, as (1.8) is not valid for jump processes and the inequality used at the beginning of the proof of Lemma 2.1 in Zhou and Zhang (2016) may not hold in general because the compensated Poisson process $\tilde{N}(t)$ can have a negative drift. Second, the proof of our Lemma 2.3 uses Kunita's inequality (1.7) for jump processes instead of relying on the Burkholder-Davis-Gundy inequality (1.8) for continuous processes.

In the sequel, we consider the condition

$$(H_4^{(p)}) \quad \int_{\mathbb{R}^3\setminus\{0\}} |(1+\overline{\gamma}(z))^p - 1|\nu(dz) < \infty, \qquad p > 1.$$

Lemma 2.2 *Assume that* (H_1)*-*(H_2) *and* $(H_3^{(p)})$*-*$(H_4^{(p)})$ *hold for some* $p > 1$*, and let* (S_t, I_t, R_t) *be the solution of the system* (1.1a)*-*(1.1c) *with initial condition* $(S_0, I_0, R_0) \in \mathbb{R}_+^3$*. Then we have*

$$\lim_{t\to\infty} \frac{S_t}{t} = 0, \quad \lim_{t\to\infty} \frac{I_t}{t} = 0, \quad \lim_{t\to\infty} \frac{R_t}{t} = 0, \quad \mathbb{P}\text{-}a.s.$$

Proof. We let $U_t := S_t + I_t + R_t$ and

$$H_t(z) := \gamma_1(z)S_t + \gamma_2(z)I_t + \gamma_3(z)R_t, \qquad z \in \mathbb{R}^3, \quad t \in \mathbb{R}_+,$$

with the inequality $H_t(z) \le \overline{\gamma}(z)U_t$, $z \in \mathbb{R}^3$. Applying the Itô formula with jumps (see e.g. Theorem 1.16 in Øksendal and Sulem (2005)) to the function $x \mapsto V(x) = (1+x)^p$, we obtain

$$\begin{aligned}
dV(U_t) &= p(1+U_t)^{p-1}(\Lambda - \mu U_t - \varepsilon I_t)dt \\
&\quad + \frac{p(p-1)}{2}(1+U_t)^{p-2}\big(\varrho_{1,1}S_t^2 + \varrho_{2,2}I_t^2 + \varrho_{3,3}R_t^2 + 2\varrho_{1,2}S_tI_t + 2\varrho_{1,3}S_tR_t + 2\varrho_{2,3}I_tR_t\big)dt \\
&\quad + p(1+U_t)^{p-1}(S_t dB_1^\varrho(t) + I_t dB_2^\varrho(t) + R_t dB_3^\varrho(t)) \\
&\quad + \int_{\mathbb{R}^3\setminus\{0\}} ((1+U_t+H_t(z))^p - (1+U_t)^p - p(1+U_t)^{p-1}H_t(z))\nu(dz)dt \\
&\quad + \int_{\mathbb{R}^3\setminus\{0\}} ((1+U_{t^-}+H_{t^-}(z))^p - (1+U_{t^-})^p)\tilde{N}(dt,dz) \\
&= LV(U_t)dt + p(1+U_t)^{p-1}(S_t dB_1^\varrho(t) + I_t dB_2^\varrho(t) + R_t dB_3^\varrho(t)) \\
&\quad + \int_{\mathbb{R}^3\setminus\{0\}} \big((1+U_{t^-}+H_{t^-}(z))^p - (1+U_{t^-})^p\big)\tilde{N}(dt,dz), \qquad (2.1)
\end{aligned}$$

where we let

$$LV(U_t) := p(1+U_t)^{p-1}(\Lambda - \mu U_t - \varepsilon I_t)$$

$$+\frac{p(p-1)}{2}(1+U_t)^{p-2}\big(\varrho_{1,1}S_t^2+\varrho_{2,2}I_t^2+\varrho_{3,3}R_t^2+2\varrho_{1,2}S_tI_t+2\varrho_{1,3}S_tR_t+2\varrho_{2,3}I_tR_t\big)$$
$$+\int_{\mathbb{R}^3\setminus\{0\}}((1+U_t+H_t(z))^p\;-(1+U_t)^p-p(1+U_t)^{p-1}H_t(z))\nu(dz).$$

We note that for all $z\in\mathbb{R}^3$ and $t\in\mathbb{R}_+$ there exists $\theta\in(0,1)$ such that

$$\begin{aligned}
&(1+U_t+H_t(z))^p-(1+U_t)^p-p(1+U_t)^{p-1}H_t(z)\\
&\quad= (1+U_t)^p+p(1+U_t)^{p-1}H_t(z)+\frac{p(p-1)}{2}(1+U_t+\theta H_t(z))^{p-2}H_t^2(z)\\
&\qquad -(1+U_t)^p-p(1+U_t)^{p-1}H_t(z)\\
&\quad= \frac{p(p-1)}{2}(1+U_t+\theta H_t(z))^{p-2}H_t^2(z)\\
&\quad\leq \frac{p(p-1)}{2}\max(2^{p-3},1)((1+U_t)^{p-2}+\theta H_t^{p-2}(z))H_t^2(z)\\
&\quad\leq c_p(1+U_t)^{p-2}H_t^2(z)+c_pH_t^p(z)\\
&\quad\leq c_p(1+U_t)^{p-2}U_t^2(\overline{\gamma}^2(z)+\overline{\gamma}^p(z)),\quad z\in\mathbb{R}^3,\quad t\in\mathbb{R}_+,
\end{aligned}$$

where we used the bound $H_t(z)\leq\overline{\gamma}(z)U_t$, with $c_p:=p(p-1)\max(2^{p-3},1)/2$. It then follows that

$$\begin{aligned}
LV(U_t) &\leq p(1+U_t)^{p-2}((1+U_t)(\Lambda-\mu U_t)-\varepsilon(1+U_t)I_t)\\
&\quad+\frac{p(p-1)}{2}(1+U_t)^{p-2}\big(\varrho_{1,1}S_t^2+\varrho_{2,2}I_t^2+\varrho_{3,3}R_t^2+2\varrho_{1,2}S_tI_t+2\varrho_{1,3}S_tR_t+2\varrho_{2,3}I_tR_t\big)\\
&\quad+c_p(1+U_t)^{p-2}U_t^2\int_{\mathbb{R}^3\setminus\{0\}}\overline{\gamma}^2(z)\nu(dz)+c_p(1+U_t)^{p-2}U_t^2\int_{\mathbb{R}^3\setminus\{0\}}\overline{\gamma}^p(z)\nu(dz)\\
&\leq p(1+U_t)^{p-2}(-\mu U_t^2+(\Lambda-\mu)U_t+\Lambda)+\frac{p(p-1)}{2}(1+U_t)^{p-2}\|\varrho\|_\infty U_t^2\\
&\quad+\frac{pc_p}{p}(1+U_t)^{p-2}U_t^2\int_{\mathbb{R}^3\setminus\{0\}}(\overline{\gamma}^2(z)+\overline{\gamma}^p(z))\nu(dz)\\
&= p(1+U_t)^{p-2}(-bU_t^2+(\Lambda-\mu)U_t+\Lambda),
\end{aligned}\tag{2.2}$$

where

$$b:=\mu-\frac{p-1}{2}\|\varrho\|_\infty-\frac{\lambda(p)}{p}>0$$

by $(H_3^{(p)})$. Next, for any $k\in\mathbb{R}$ it holds that

$$\begin{aligned}
e^{kt}(1+U_t)^p &= (1+U_0)^p+\int_0^t e^{ks}(k(1+U_s)^p+LV(U_s))ds\\
&\quad+p\int_0^t e^{ks}(1+U_s)^{p-1}(S_sdB_1^\varrho(s)+I_sdB_2^\varrho(s)+R_sdB_3^\varrho(s))
\end{aligned}$$

$$+\int_0^t e^{ks}\int_{\mathbb{R}^3\setminus\{0\}}\big((1+U_{s^-}+H_{s^-}(z))^p-(1+U_{s^-})^p\big)\tilde{N}(ds,dz),$$

hence by taking expectations in (2.1) and in view of (2.2), for any $k<bp$ we obtain

$$\begin{aligned}
e^{kt}\mathbb{E}[(1+U_t)^p] &= (1+U_0)^p+\mathbb{E}\left[\int_0^t e^{ks}\big(k(1+U_s)^p+LV(U_s)\big)ds\right].\\
&\le (1+U_0)^p+\mathbb{E}\left[\int_0^t e^{ks}\left(k(1+U_s)^p+p(1+U_s)^{p-2}(\Lambda+(\Lambda-\mu)U_s-bU_s^2)\right)ds\right]\\
&= (1+U_0)^p+p\mathbb{E}\left[\int_0^t e^{ks}U_s^{p-2}\left(-\left(b-\frac{k}{p}\right)U_s^2+\left(\Lambda-\mu+\frac{2k}{p}\right)U_s+\Lambda+\frac{k}{p}\right)ds\right]\\
&\le (1+U_0)^p+pM\int_0^t e^{ks}ds\\
&= (1+U_0)^p+\frac{pM}{k}e^{kt},\quad t\in\mathbb{R}_+,
\end{aligned}$$

where

$$0<M:=1+\sup_{x\in\mathbb{R}_+}x^{p-2}\left(-\left(b-\frac{k}{p}\right)x^2+\left(\Lambda-\mu+\frac{2k}{p}\right)x+\Lambda+\frac{k}{p}\right)<\infty.$$

Hence, for any $k\in(0,bp)$ we have

$$\limsup_{t\to\infty}\mathbb{E}[(1+U_t)^p]\le\frac{pM}{k},$$

which implies that there exists $M_0>0$ such that

$$\mathbb{E}[(1+U_t)^p]\le M_0,\qquad t\in\mathbb{R}_+. \tag{2.3}$$

Now, by (2.1) and (2.2) there will be

$$\begin{aligned}
(1+U_t)^p-(1+U_{k\delta})^p &\le p\int_{k\delta}^t(1+U_s)^{p-2}(\Lambda+(\Lambda-\mu)U_s-bU_s^2)ds\\
&+p\int_{k\delta}^t(1+U_s)^{p-1}(S_sdB_1^\varrho(s)+I_sdB_2^\varrho(s)+R_sdB_3^\varrho(s))\\
&+\int_{k\delta}^t\int_{\mathbb{R}^3\setminus\{0\}}\big((1+U_{s^-}+H_{s^-}(z))^p-(1+U_{s^-})^p\big)\tilde{N}(ds,dz),\qquad t\ge k\delta,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\sup_{k\delta\le t\le(k+1)\delta}(1+U_t)^p &\le (1+U_{k\delta})^p+p\sup_{k\delta\le t\le(k+1)\delta}\left|\int_{k\delta}^t(1+U_s)^{p-2}(\Lambda+(\Lambda-\mu)U_s-bU_s^2)ds\right|\\
&+p\sup_{k\delta\le t\le(k+1)\delta}\left|\int_{k\delta}^t(1+U_s)^{p-1}(S_sdB_1^\varrho(s)+I_sdB_2^\varrho(s)+R_sdB_3^\varrho(s))\right|
\end{aligned}$$

$$+\sup_{k\delta\le t\le (k+1)\delta}\left|\int_{k\delta}^t\int_{\mathbb{R}^3\setminus\{0\}}\big((1+U_{s^-}+H_{s^-}(z))^p-(1+U_{s^-})^p\big)\tilde{N}(ds,dz)\right|.$$

Taking expectation on both sides, we obtain

$$\mathbb{E}\bigg[\sup_{k\delta\le t\le (k+1)\delta}(1+U_t)^p\bigg]\le \mathbb{E}[(1+U_{k\delta})^p]+J_1+J_2+J_3\le M_0+J_1+J_2+J_3,$$

where, for some $c_3>0$,

$$\begin{aligned}
J_1 &:= p\mathbb{E}\bigg[\sup_{k\delta\le t\le (k+1)\delta}\left|\int_{k\delta}^t(1+U_s)^{p-2}(\Lambda+(\Lambda-\mu)U_s-bU_s^2)ds\right|\bigg]\\
&\le c_3\mathbb{E}\bigg[\sup_{k\delta\le t\le (k+1)\delta}\int_{k\delta}^t(1+U_s)^p ds\bigg]\\
&= c_3\mathbb{E}\bigg[\int_{k\delta}^{(k+1)\delta}(1+U_s)^p ds\bigg]\\
&\le c_3\delta\mathbb{E}\bigg[\sup_{k\delta\le t\le (k+1)\delta}(1+U_t)^p\bigg],
\end{aligned}$$

and, for some $c_4>0$,

$$\begin{aligned}
J_2 &:= p\mathbb{E}\bigg[\sup_{k\delta\le t\le (k+1)\delta}\left|\int_{k\delta}^t(1+U_s)^{p-1}\big(S_s dB_1^{\varrho}(s)+I_s dB_2^{\varrho}(s)+R_s dB_3^{\varrho}(s)\big)\right|\bigg]\\
&\le p\sqrt{32}\mathbb{E}\bigg[\bigg(\int_{k\delta}^{(k+1)\delta}(1+U_s)^{2(p-1)}\big(\varrho_{1,1}S_s^2+\varrho_{2,2}I_s^2+\varrho_{3,3}R_s^2\big)ds\bigg)^{1/2}\bigg]\\
&\le p\sqrt{32\delta\|\varrho\|_\infty}\mathbb{E}\bigg[\bigg(\sup_{k\delta\le t\le (k+1)\delta}(1+U_t)^{2p}\bigg)^{1/2}\bigg]\\
&= c_4\sqrt{\delta}\mathbb{E}\bigg[\sup_{k\delta\le t\le (k+1)\delta}(1+U_t)^p\bigg],
\end{aligned}$$

where we used the Burkholder-Davis-Gundy inequality (1.8) for continuous martingales, see Theorem IV.4.48 page 193 of Protter (2004), or Theorem 7.3 in Chapter 1 of Mao (2008). Furthermore, we have

$$\begin{aligned}
J_3 &= \mathbb{E}\left[\sup_{k\delta\le t\le (k+1)\delta}\left|\int_{k\delta}^t\int_{\mathbb{R}^3\setminus\{0\}}\big((1+U_{s^-}+H_{s^-}(z))^p-(1+U_{s^-})^p\big)\tilde{N}(ds,dz)\right|\right]\\
&\le \mathbb{E}\left[\left|\int_{k\delta}^{(k+1)\delta}\int_{\mathbb{R}^3\setminus\{0\}}\big((1+U_{s^-}+H_{s^-}(z))^p-(1+U_{s^-})^p\big)N(ds,dz)\right|\right]\\
&\quad+\mathbb{E}\left[\left|\int_{k\delta}^{(k+1)\delta}\int_{\mathbb{R}^3\setminus\{0\}}\big((1+U_{s^-}+H_{s^-}(z))^p-(1+U_{s^-})^p\big)ds\nu(dz)\right|\right]
\end{aligned}$$

$$
\begin{aligned}
&= 2\mathbb{E}\left[\left|\int_{k\delta}^{(k+1)\delta}\int_{\mathbb{R}^3\setminus\{0\}}\big((1+U_{s^-}+H_{s^-}(z))^p-(1+U_{s^-})^p\big)ds\nu(dz)\right|\right] \\
&\le 2\mathbb{E}\left[\int_{k\delta}^{(k+1)\delta}\int_{\mathbb{R}^3\setminus\{0\}}(1+U_{s^-})^p|(1+\overline{\gamma}(z))^p-1|ds\nu(dz)\right] \\
&= 2\mathbb{E}\left[\int_{k\delta}^{(k+1)\delta}(1+U_s)^p ds\right]\int_{\mathbb{R}^3\setminus\{0\}}|(1+\overline{\gamma}(z))^p-1|\nu(dz) \\
&\le 2\delta\mathbb{E}\left[\sup_{k\delta\le t\le(k+1)\delta}(1+U_t)^p\right]\int_{\mathbb{R}^3\setminus\{0\}}|(1+\overline{\gamma}(z))^p-1|\nu(dz). \qquad (2.4)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{k\delta\le t\le(k+1)\delta}(1+U_t)^p\right] \qquad (2.5)\\
&\le \mathbb{E}[(1+U_{k\delta})^p]+\left(c_3\delta+c_4\sqrt{\delta}+2\delta\int_{\mathbb{R}^3\setminus\{0\}}|(1+\overline{\gamma}(z))^p-1|\nu(dz)\right)\mathbb{E}\left[\sup_{k\delta\le t\le(k+1)\delta}(1+U_t)^p\right].
\end{aligned}
$$

Furthermore, from $(H_4^{(p)})$ we can choose $\delta>0$ such that

$$
c_3\delta+c_4\sqrt{\delta}+2\delta\int_{\mathbb{R}^3\setminus\{0\}}|(1+\overline{\gamma}(z))^p-1|\nu(dz)<\frac{1}{2},
$$

and, combining (2.3) with (2.5), we obtain

$$
\mathbb{E}\left[\sup_{k\delta\le t\le(k+1)\delta}(1+U_t)^p\right]\le 2\mathbb{E}[(1+U_{k\delta})^p]\le 2M_0. \qquad (2.6)
$$

Let now $\varepsilon>0$ be arbitrary. By Chebyshev's inequality, we get

$$
\mathbb{P}\left(\sup_{k\delta\le t\le(k+1)\delta}(1+U_t)^p>(k\delta)^{1+\varepsilon}\right)\le\frac{1}{(k\delta)^{1+\varepsilon}}\mathbb{E}\left[\sup_{k\delta\le t\le(k+1)\delta}(1+U_t)^p\right]\le\frac{2M_0}{(k\delta)^{1+\varepsilon}}
$$

for all $k\ge 1$. Then, by the Borel-Cantelli lemma (see Lemma 2.4 in Chapter 1 of Mao (2008)) it follows that for almost all $\omega\in\Omega$, the bound

$$
\sup_{k\delta\le t\le(k+1)\delta}(1+U_t)^p\le(k\delta)^{1+\varepsilon},
$$

holds for all but finitely many k. Thus, for almost all $\omega\in\Omega$ there exists $k_0(\omega)$ such that whenever $k\ge k_0(\omega)$ we have

$$
\frac{\log(1+U_t)^p}{\log t}\le 1+\varepsilon, \qquad \varepsilon>0, \quad k\delta\le t\le(k+1)\delta,
$$

hence

$$\limsup_{t\to\infty} \frac{\log U_t}{\log t} \leq \limsup_{t\to\infty} \frac{\log(1+U_t)}{\log t} \leq \frac{1}{p}, \quad \mathbb{P}\text{-}a.s., \quad p > 1.$$

In other words, for any $\xi \in (0, 1-1/p)$ there exists an a.s. finite random time $\overline{T}(\omega)$ such that

$$\log U_t \leq \left(\frac{1}{p} + \xi\right) \log t, \qquad t \geq \overline{T}.$$

It follows that

$$\limsup_{t\to\infty} \frac{U_t}{t} \leq \limsup_{t\to\infty} \frac{t^{\xi+1/p}}{t} = 0,$$

therefore we have

$$\limsup_{t\to\infty} \frac{S_t}{t} \leq 0, \quad \limsup_{t\to\infty} \frac{I_t}{t} \leq 0, \quad \limsup_{t\to\infty} \frac{R_t}{t} \leq 0, \quad \mathbb{P}\text{-}a.s.$$

This, together with the positivity of the solution, implies that

$$\lim_{t\to\infty} \frac{S_t}{t} = 0, \quad \lim_{t\to\infty} \frac{I_t}{t} = 0, \quad \lim_{t\to\infty} \frac{R_t}{t} = 0, \quad \mathbb{P}\text{-}a.s.$$

□

The next Lemma 2.3 is proved by using Kunita's inequality (1.7) for jump processes instead of the Burkholder-Davis-Gundy inequality (1.8) for continuous martingales.

Lemma 2.3 *Assume that* (H_1)-(H_2) *and* $(H_3^{(p)})$-$(H_4^{(p)})$ *hold for some* $p > 2$, *and let* (S_t, I_t, R_t) *be the solution of the system* (1.1a)-(1.1c) *with initial condition* $(S_0, I_0, R_0) \in \mathbb{R}_+^3$. *Then we have*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t S_{r^-} \int_{\mathbb{R}^3\setminus\{0\}} \gamma_1(z)\tilde{N}(dr, dz) = 0, \quad \lim_{t\to\infty} \frac{1}{t} \int_0^t I_{r^-} \int_{\mathbb{R}^3\setminus\{0\}} \gamma_2(z)\tilde{N}(dr, dz) = 0,$$

and

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t R_{r^-} \int_{\mathbb{R}^3\setminus\{0\}} \gamma_3(z)\tilde{N}(dr, dz) = 0, \quad \mathbb{P}\text{-}a.s.$$

Proof. Denote

$$X_1(t) := \int_0^t S_{r^-} \int_{\mathbb{R}^3\setminus\{0\}} \gamma_1(z)\tilde{N}(dr, dz), \quad X_2(t) := \int_0^t I_{r^-} \int_{\mathbb{R}^3\setminus\{0\}} \gamma_2(z)\tilde{N}(dr, dz),$$

and

$$X_3(t) := \int_0^t R_{r^-} \int_{\mathbb{R}^3\setminus\{0\}} \gamma_3(z)\tilde{N}(dr, dz), \quad t \in \mathbb{R}_+.$$

By Kunita's inequality (1.7), for any $p \geq 2$ there exists a positive constant C_p such that

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{0<r\leq t}|X_1(r)|^p\Big] \\
&\leq C_p\mathbb{E}\Big[\Big(\int_0^t |S_r|^2\int_{\mathbb{R}^3\setminus\{0\}}|\gamma_1(z)|^2\nu(dz)dr\Big)^{p/2}\Big] + C_p\mathbb{E}\Big[\int_0^t |S_r|^p\int_{\mathbb{R}^3\setminus\{0\}}|\gamma_1(z)|^p\nu(dz)dr\Big] \\
&= C_p\Big(\int_{\mathbb{R}^3\setminus\{0\}}\gamma_1^2(z)\nu(dz)\Big)^{p/2}\mathbb{E}\Big[\Big(\int_0^t|S_r|^2dr\Big)^{p/2}\Big] + C_p\left(\int_{\mathbb{R}^3\setminus\{0\}}\gamma_1^p(z)\nu(dz)\right)\mathbb{E}\Big[\int_0^t|S_r|^pdr\Big] \\
&\leq C_pt^{p/2}\left(\int_{\mathbb{R}^3\setminus\{0\}}\gamma_1^2(z)\nu(dz)\right)^{p/2}\mathbb{E}\Big[\left(\sup_{0<r\leq t}|S_r|^2\right)^{p/2}\Big] \\
&\quad + C_p\int_{\mathbb{R}^3\setminus\{0\}}\gamma_1^p(z)\nu(dz)\int_0^t\mathbb{E}[|S_r|^p]dr \\
&\leq C_pt^{p/2}\left(\int_{\mathbb{R}^3\setminus\{0\}}\gamma_1^2(z)\nu(dz)\right)^{p/2}\mathbb{E}\left[\sup_{0<r\leq t}|S_r|^p\right] + C_pM_0t\int_{\mathbb{R}^3\setminus\{0\}}\gamma_1^p(z)\nu(dz),
\end{aligned}
$$

where the bound (2.3) has been used in the last inequality. Combining the above inequality with (2.6) yields

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{k\delta\leq t\leq(k+1)\delta}|X_1(t)|^p\Big] \\
&\leq \quad C_p((k+1)\delta)^{p/2}\left(\int_{\mathbb{R}^3\setminus\{0\}}\gamma_1^2(z)\nu(dz)\right)^{p/2}\mathbb{E}\Big[\sup_{k\delta\leq t\leq(k+1)\delta}|S_r|^p\Big] \\
&\quad\quad +C_pM_0(k+1)\delta\int_{\mathbb{R}^3\setminus\{0\}}\gamma_1^p(z)\nu(dz) \\
&\leq \quad C_p2M_0((k+1)\delta)^{p/2}\left(\int_{\mathbb{R}^3\setminus\{0\}}\gamma_1^2(z)\nu(dz)\right)^{p/2} + C_p\delta M_0(k+1)\int_{\mathbb{R}^3\setminus\{0\}}\gamma_1^p(z)\nu(dz).
\end{aligned}
$$

Let $\varepsilon > 0$ be arbitrary. It follows from Doob's martingale inequality (see e.g. Theorem 3.8 in Chapter 1 of Mao (2008)) that

$$
\begin{aligned}
&\mathbb{P}\Big(\sup_{k\delta\leq t\leq(k+1)\delta}|X_1(t)|^p > (k\delta)^{1+\varepsilon+p/2}\Big) \leq (k\delta)^{-1-\varepsilon-p/2}\mathbb{E}\Big[\sup_{k\delta\leq t\leq(k+1)\delta}|X_1(t)|^p\Big] \\
&\leq \quad \frac{C_p2M_0((k+1)\delta)^{p/2}}{(k\delta)^{1+\varepsilon+p/2}}\left(\int_{\mathbb{R}^3\setminus\{0\}}\gamma_1^2(z)\nu(dz)\right)^{p/2} + \frac{C_pM_0(k+1)\delta}{(k\delta)^{1+\varepsilon+p/2}}\int_{\mathbb{R}^3\setminus\{0\}}\gamma_1^p(z)\nu(dz).
\end{aligned}
$$

By the Borel-Cantelli lemma it follows that for almost all $\omega \in \Omega$ the bound

$$\sup_{k\delta\leq t\leq(k+1)\delta}|X_1(t)|^p \leq (k\delta)^{1+\varepsilon+p/2}$$

holds for all but finitely many k. Thus, for almost all $\omega \in \Omega$ there exists $k_0(\omega)$ such that for all $k \geq k_0(\omega)$ we have

$$\frac{\log |X_1(t)|}{\log t} \leq \frac{1}{2} + \frac{1+\varepsilon}{p}, \qquad \varepsilon > 0, \quad k\delta \leq t \leq (k+1)\delta,$$

hence

$$\limsup_{t\to\infty} \frac{\log |X_1(t)|}{\log t} \leq \frac{1}{2} + \frac{1}{p}, \qquad p > 2,$$

and as in the proof of Lemma 2.2, this yields

$$\limsup_{t\to\infty} \frac{|X_1(t)|}{t} \leq \limsup_{t\to\infty} \frac{t^{1/2+1/p}}{t} = 0$$

since $p > 2$, which shows that

$$\lim_{t\to\infty} \frac{|X_1(t)|}{t} = 0, \quad \mathbb{P}\text{-}a.s.$$

By similar arguments, we also obtain

$$\lim_{t\to\infty} \frac{X_2(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{X_3(t)}{t} = 0, \quad \mathbb{P}\text{-}a.s.$$

□

Remark 2.4 *We note that by the continuity of $p \mapsto \lambda(p)$ in (1.6), in order for $(H_3^{(p)})$ to hold for some $p > 2$ it suffices that $(H_3^{(2)})$ be satisfied, i.e.*

$$(H_3^{(2)}): \quad \mu > \frac{\|\varrho\|_\infty}{2} + \frac{\lambda(2)}{2}.$$

The next Lemma 2.5 can be proved on (2.3), by noting that the argument of Lemma 2.2 in Zhou and Zhang (2016) is valid for correlated Brownian motions $(B_1^\varrho(t), B_2^\varrho(t), B_3^\varrho(t))$, without requiring the continuity of $(S_t, I_t, R_t)_{t\in\mathbb{R}_+}$.

Lemma 2.5 *Assume that (H_1)-(H_2) and $(H_3^{(p)})$-$(H_4^{(p)})$ hold for some $p > 1$, and let (S_t, I_t, R_t) be the solution of (1.1a)-(1.1c) with initial condition $(S_0, I_0, R_0) \in \mathbb{R}_+^3$. Then, $\mathbb{P}$-a.s, we have*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t S_r dB_1^\varrho(r) = 0, \ \lim_{t\to\infty} \frac{1}{t} \int_0^t I_r dB_2^\varrho(r) = 0, \ \lim_{t\to\infty} \frac{1}{t} \int_0^t R_r dB_3^\varrho(r) = 0.$$

3 Extinction and persistence

By virtue of the large time estimates for the solution of (1.1a)-(1.1c) and its diffusion and jump components obtained in Section 2, in this section we determine the threshold behavior of the stochastic SIR epidemic model.

In Theorems 3.1 and 3.2 below, the extinction and persistence of the disease is characterized by means of the critical reproduction number $\overline{\mathcal{R}}_0$ in (1.11) which becomes lower in the presence of jumps, showing that the additional environmental noise induced by Lévy jumps can limit the outbreak of the disease. In the sequel, we let

$$0 < \beta_i := \frac{1}{2}\varrho_{i,i} + \int_{\mathbb{R}^3\setminus\{0\}} (\gamma_i(z) - \log(1+\gamma_i(z)))\,\nu(dz), \quad i = 1,2,3,$$

which is finite under (H_1), and we consider the following condition:

(H_5) $\displaystyle\int_{\mathbb{R}^3\setminus\{0\}} \big(\log(1+\gamma_i(z))\big)^2 \nu(dz) < \infty.$

Theorem 3.1 (Extinction). *Assume that (H_1)-(H_2), $(H_3^{(p)})$-$(H_4^{(p)})$ and (H_5) hold for some $p > 2$. If in addition*

$$\overline{\mathcal{R}}_0 := \mathcal{R}_0 - \frac{\beta_2}{\mu+\varepsilon+\eta} < 1,$$

then for any initial condition $(S_0, I_0, R_0) \in \mathbb{R}_+^3$, the disease vanishes with probability one in large time, i.e. the solution (S_t, I_t, R_t) of (1.1a)-(1.1c) satisfies

$$\lim_{t\to\infty} \langle S \rangle_t = \frac{\Lambda}{\mu}, \quad \lim_{t\to\infty} I_t = 0 \quad and \quad \lim_{t\to\infty} R_t = 0, \quad \mathbb{P}\text{-}a.s.$$

The proof of Theorem 3.1 follows the lines of the proof of Theorem 2.1 in Zhou and Zhang (2016), up to the new Condition $(H_3^{(p)})$ which allows for infinite Lévy measures in Lemma 2.2. For reference, the proof of Theorem 3.1 is stated in Appendix. By the bound (5.7) below, it also shows that for any sufficiently small ε there exists a random variable $T(\omega) > 0$ and an event Ω_ε such that $\mathbb{P}(\Omega_\varepsilon) > 1 - \varepsilon$ and $I_t \leq \mathrm{e}^{(\mu+\varepsilon+\eta)(\overline{\mathcal{R}}_0-1)t/2}$, $t \geq T_1(\omega)$, $\omega \in \Omega_\varepsilon$, as in Remark 2.1 of Ji and Jiang (2014).

Next, we consider the persistence of the system (1.1a)-(1.1c). We recall that the system (1.1a)-(1.1c) is said to be persistent in the mean if

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t S_r dr > 0, \quad \liminf_{t\to\infty} \frac{1}{t}\int_0^t I_r dr > 0, \quad \liminf_{t\to\infty} \frac{1}{t}\int_0^t R_r dr > 0, \quad \mathbb{P}\text{-}a.s.$$

In Theorem 3.2 we explore the conditions for the disease to be endemic, in other words, sufficient conditions for the persistence of the infected population I_t. Treating the critical case $\overline{\mathcal{R}}_0 = 1$ involves significant additional difficulties linked to bifurcation theory, which are outside the scope of this paper.

Theorem 3.2 (Persistence). *Assume that* (H_1)-(H_2), $(H_3^{(p)})$-$(H_4^{(p)})$ *and* (H_5) *hold for some* $p > 2$. *If in addition*

$$\overline{\mathcal{R}}_0 := \mathcal{R}_0 - \frac{\beta_2}{\mu + \varepsilon + \eta} > 1,$$

then for any initial condition $(S_0, I_0, R_0) \in \mathbb{R}_+^3$, *the solution* (S_t, I_t, R_t) *of* (1.1a)-(1.1c) *satisfies*

$$\lim_{t\to\infty} \langle S\rangle_t = S^* + \frac{\beta_2}{\beta}, \quad \lim_{t\to\infty} \langle I\rangle_t = \frac{\mu}{\beta}(\overline{\mathcal{R}}_0 - 1), \quad \lim_{t\to\infty} \langle R\rangle_t = \frac{\eta}{\beta}(\overline{\mathcal{R}}_0 - 1), \quad \mathbb{P}\text{-}a.s.,$$

where $S^* := (\mu + \varepsilon + \eta)/\beta$ *is the equilibrium value for the susceptible population* S_t *in the corresponding deterministic SIR model.*

For reference, the proof of Theorem 3.2 is stated in Appendix. It follows the lines of the proof of Theorem 3.1 in Zhou and Zhang (2016), up to the use of Lemma 5.1 (in Appendix) which extends Lemma 2 of Liu and Wang (2014) to discontinuous functions.

4 Numerical experiments

In this section, we provide numerical simulations for the behavior of (1.1a)-(1.1c) using tempered stable processes. The (compensated) one-dimensional tempered stable Lévy process

$$Y(t) = \int_0^t \int_{\mathbb{R}\setminus\{0\}} z\tilde{N}(ds, dz), \qquad t \in \mathbb{R}_+,$$

is defined by its Lévy measure (1.10) on $\mathbb{R} \setminus \{0\}$ where $k_-, k_+, \lambda_-, \lambda_+ > 0$ and $\alpha \in (0, 2)$, i.e.

$$\nu(dz) = \frac{k_-}{z^{\alpha+1}} \mathrm{e}^{-\lambda_- z} dz + \frac{k_+}{z^{\alpha+1}} \mathrm{e}^{-\lambda_+ z} dz. \tag{4.1}$$

As $\nu(\mathbb{R}) = \infty$, the tempered stable process $(Y(t))_{t\in\mathbb{R}_+}$ is not covered by the proof arguments of Zhang and Wang (2013), Zhou and Zhang (2016) and Zhang et al. (2018), in particular the quantity defined in (1.5) is not finite in this case.

Random simulations

The 3-dimensional correlated Brownian motion $(B_1^\varrho(t), B_2^\varrho(t), B_3^\varrho(t))_{t\in\mathbb{R}_+}$ can be generated as $B_1^\varrho(t) = a_{1,1}B_1(t)$, $B_2^\varrho(t) = a_{2,1}B_1(t) + a_{2,2}B_2(t)$, $B_3^\varrho(t) = a_{3,1}B_1(t) + a_{3,2}B_2(t) + a_{3,3}B_3(t)$, $t \in \mathbb{R}_+$, where $B_1(t)$, $B_2(t)$, $B_3(t)$ are independent standard one-dimensional Brownian motions and

$$A = \begin{pmatrix} a_{1,1} & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

is the lower diagonal matrix obtained from the Cholesky decomposition $\varrho = AA^\top$ of the covariance matrix $\varrho = (\varrho_{i,j})_{1\le i,j\le 3}$.

Regarding stable processes we use the simulation algorithm of Rosiński (2007) for the tempered stable process with Lévy measure (1.9). Consider $(\epsilon_j)_{j\ge1}$ an independent and identically distributed (i.i.d.) Bernoulli $(-\lambda_-, \lambda_+)$-valued random sequence with distribution $(k_-/(k_-+k_+), k_+/(k_-+k_+))$, $(\xi_j)_{j\ge1}$ an i.i.d. uniform $U(0,1)$ random sequence, and $(\eta_j)_{j\ge1}$, $(\eta'_j)_{j\ge1}$ i.i.d. exponentially distributed random sequences with parameter 1, with $\Gamma_j := \eta'_1 + \cdots + \eta'_j$, $j \ge 1$. We also let $(u_j)_{j\ge1}$ denote an i.i.d. sequence of uniform random variables on $[0,T]$, where $T > 0$, and assume that the sequence $(\epsilon_j)_{j\ge1}$, $(\xi_j)_{j\ge1}$, $(\eta_j)_{j\ge1}$, $(\eta'_j)_{j\ge1}$, and $(u_j)_{j\ge1}$ are mutually independent. By Theorem 5.3 in Rosiński (2007), the tempered stable process $Y(t)$ with Lévy measure (1.9) admits the following series representations which converge a.s. uniformly in $t \in [0,T]$.

(i) If $\alpha \in (0,1)$, set

$$Y(t) = \sum_{j=1}^\infty \mathbb{I}_{(0,t]}(u_j) \min\left\{ \left(\frac{T(k_-+k_+)}{\alpha\Gamma_j}\right)^{1/\alpha}, \frac{\eta_j}{|\epsilon_j|}\xi_j^{1/\alpha} \right\} \frac{\epsilon_j}{|\epsilon_j|}, \qquad t \in [0,T]. \tag{4.2}$$

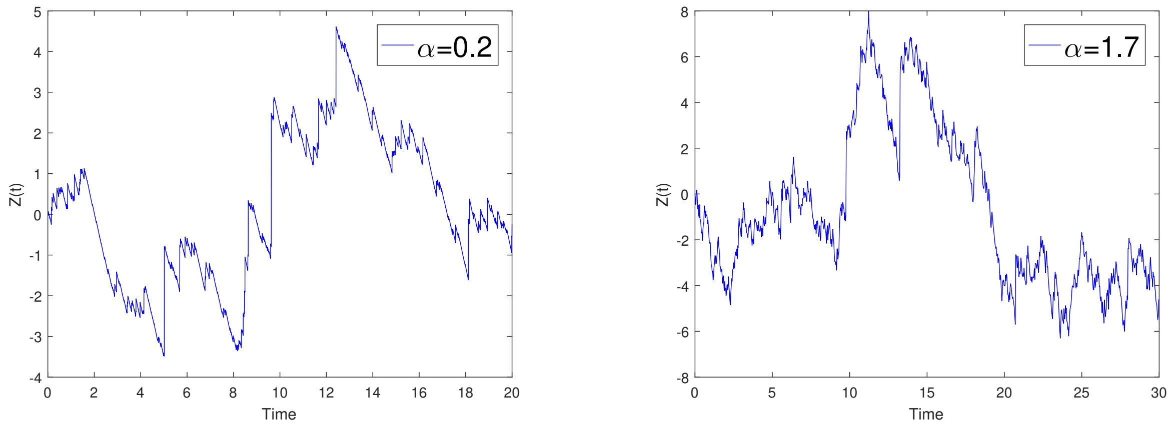
(ii) If $\alpha \in (1,2)$, set

$$Y(t) = \sum_{j=1}^\infty \left(\mathbb{I}_{(0,t]}(u_j) \min\left\{ \left(\frac{k_-+k_+}{\alpha\Gamma_j/T}\right)^{1/\alpha}, \frac{\eta_j}{|\epsilon_j|}\xi_j^{1/\alpha} \right\} \frac{\epsilon_j}{|\epsilon_j|} - x_0\frac{t}{T}\left(\frac{k_-+k_+}{\alpha j/T}\right)^{1/\alpha} \right) + tb_T, \tag{4.3}$$

$t \in [0,T]$, with $x_0 = (k_- - k_+)/(k_- + k_+)$, $x_1 = k_+\lambda_+^{-1-\alpha} - k_-\lambda_-^{-1-\alpha}$, and

$$b_T := \frac{x_0}{T}\zeta\left(\frac{1}{\alpha}\right)\left(\frac{T(k_-+k_+)}{\alpha}\right)^{1/\alpha} - x_1\Gamma(1-\alpha),$$

where ζ is the Riemann zeta function. The following simulations use truncations of (4.2)-(4.3) with 1000 terms and $10,000$ time steps.



(a) Tempered stable process with $\alpha = 0.2$ (b) Tempered stable process with $\alpha = 1.7$

Figure 1: Simulated sample paths of the tempered stable process.

Next, we take $\gamma_i(z) := \sigma_i z_i$ with $\sigma_i > 0$, $i = 1, 2, 3$, and we consider the system

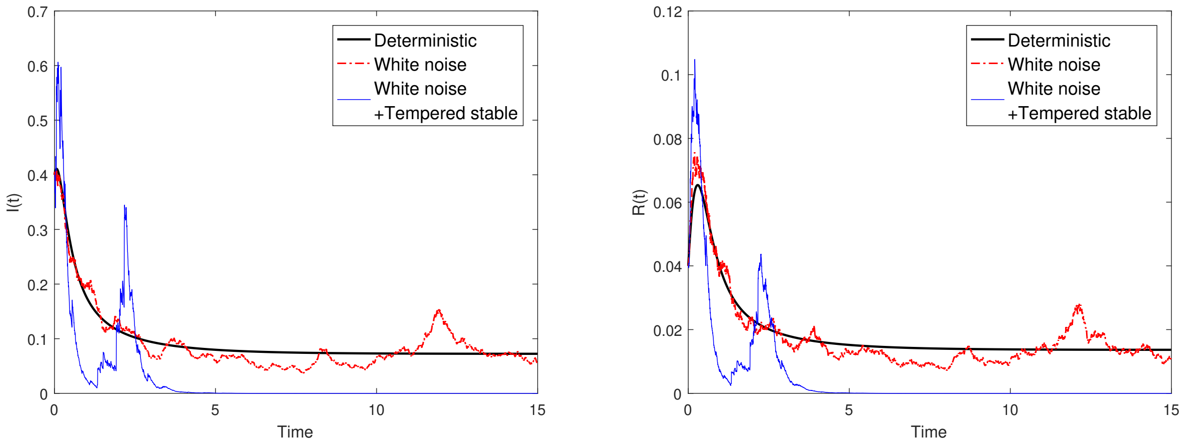
$$\begin{cases} dS_t = (\Lambda - \beta S_t I_t - \mu S_t)dt + S_t dB_1^{\varrho}(t) + \sigma_1 S_t dY(t), \\ dI_t = (\beta S_t I_t - (\mu + \varepsilon + \eta)I_t)dt + I_t dB_2^{\varrho}(t) + \sigma_2 I_t dY(t), \\ dR_t = (\eta I_t - \mu R_t)dt + R_t dB_3^{\varrho}(t) + \sigma_3 R_t dY(t), \end{cases}$$

i.e. (1.3) reads $Z_i(t) = B_i^{\varrho}(t) + Y(t)$, $i = 1, 2, 3$, and $(Y(t))_{t\in\mathbb{R}_+}$ is a one-sided tempered stable process with $k_- = 0$ in (1.9). We note that (H_1)-(H_2), (H_5) are satisfied, and that $(H_4^{(p)})$ holds for all $\alpha \in (0, 1)$ and $p > 1$. In addition, letting $\overline{\sigma} := \max(\sigma_1, \sigma_2, \sigma_3)$, the quantity

$$\begin{aligned} \lambda(p) &= c_p \overline{\sigma}^2 \int_{\mathbb{R}^3\setminus\{0\}} z^2 \nu(dz) + c_p \overline{\sigma}^p \int_{\mathbb{R}^3\setminus\{0\}} z^p \nu(dz) \\ &= c_p k_+ \overline{\sigma}^2 \frac{\Gamma(2-\alpha)}{\lambda_+^{2-\alpha}} + c_p k_+ \overline{\sigma}^2 \frac{\Gamma(p-\alpha)}{\lambda_+^{p-\alpha}} \end{aligned}$$

in (1.6) is finite when $p > \alpha$, where $\Gamma(\cdot)$ is the Gamma function. We note that the variance $k_+ t\Gamma(2-\alpha)/\lambda_+^{2-\alpha}$ of the one-sided tempered stable process $Y(t)$ is an increasing function of $\alpha \in (0, 1)$ when $\lambda_+ \geq 1$.

First, we take $\alpha = 0.7$, $\lambda_+ = 1.2$, $k_+ = 2.8$ with the initial condition $(S_0, I_0, R_0) = (1.6, 0.4, 0.04)$ and the parameters $\Lambda = 8$, $\mu = 5.3$, $\beta = 4.8$, $\eta = 1$ and $\varepsilon = 0.5$. The covariance matrix is set at $\varrho = 10^{-2}\begin{pmatrix} 4 & 3.2 & 3.0 \\ 3.2 & 4 & 3.84 \\ 3 & 3.84 & 4.69 \end{pmatrix}$, with $\sigma_1 = 0.2$, $\sigma_2 = 0.8$ and $\sigma_3 = 0.5$, in which case Condition $(H_3^{(2)})$ is also satisfied by Remark 2.4.



(a) Extinction of the infected population

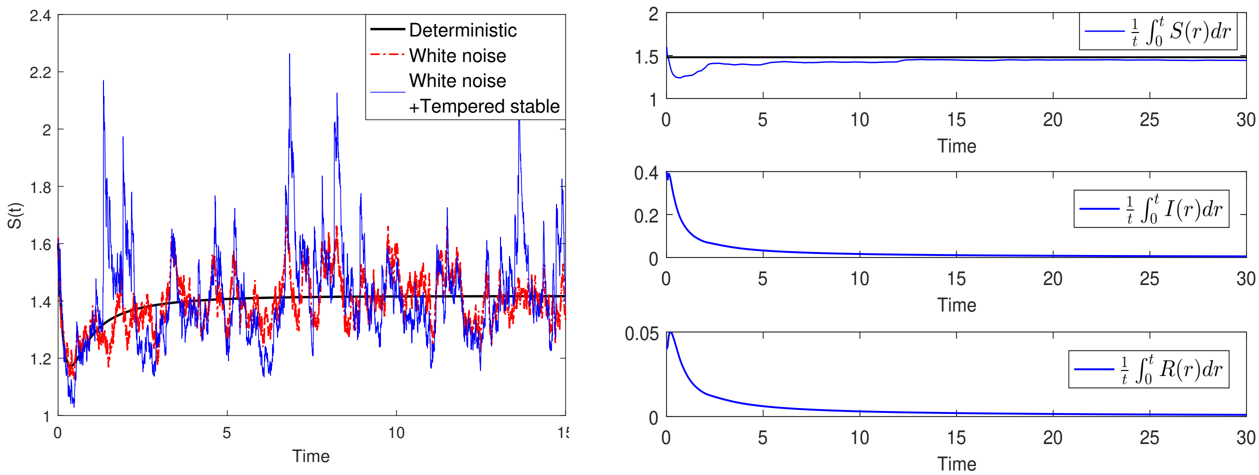
(b) Extinction of the recovered population

Figure 2: Disease extinction in the epidemic population dynamics model with $\alpha = 0.7$.

We note that the deterministic system is persistent as $\mathcal{R}_0 = 1.0655 > 1$, with the positive equilibrium value $E^* = (S^*, I^*, R^*) = (1.417, 0.0723, 0.0136)$. On the other hand, for the stochastic system with $\alpha = 0.7$ we have $\overline{\mathcal{R}}_0 = 0.9976 < 1$, and disease extinction is induced by the jump noise with

$$\lim_{t\to\infty} \langle S \rangle_t = \frac{\Lambda}{\mu} = 1.5, \quad \lim_{t\to\infty} I_t = 0, \quad \lim_{t\to\infty} R_t = 0, \quad \mathbb{P}\text{-}a.s.$$

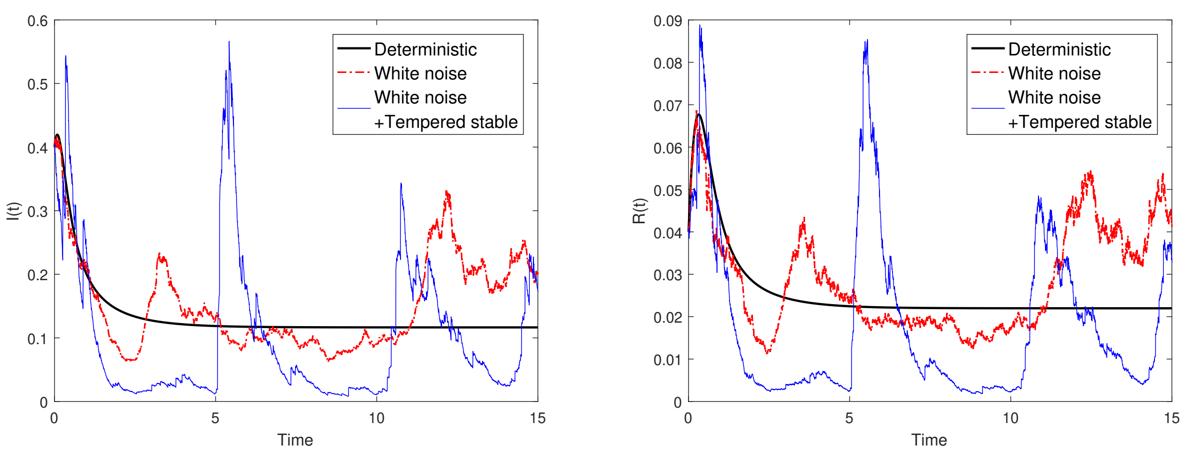
according to Theorem 3.1, see Figures 2-3.



(a) Dynamical behavior of $S(t)$ (b) Time averages of $S(t)$, $I(t)$ and $R(t)$

Figure 3: Disease extinction in the epidemic population dynamics model with $\alpha = 0.7$.

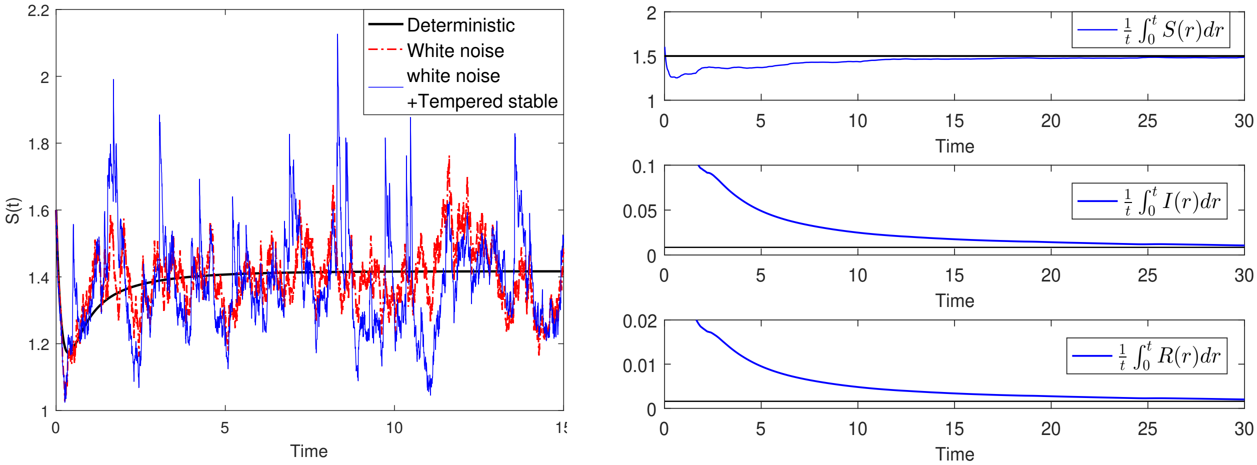
We also note that the tempered stable model generates jumps of large size which can model sudden disease outbreak. Next, we decrease the value of the index to $\alpha = 0.2$ and keep the initial value and other parametric values unchanged, in which case Condition $(H_3^{(2)})$ still holds true and $\overline{\mathcal{R}}_0 = 1.00767 > 1$.



(a) Persistence of the infected population (b) Persistence of the recovered population

Figure 4: Persistence in the epidemic population dynamics model with $\alpha = 0.2$.

Based on Theorem 3.2, the solution (S_t, I_t, R_t) of stochastic system (1.1a)-(1.1c) satisfies $\lim_{t\to\infty} \langle S \rangle_t = 1.49$, $\lim_{t\to\infty} \langle I \rangle_t = 0.0085$, $\lim_{t\to\infty} \langle R \rangle_t = 0.0016$. The system is persistent and the disease becomes endemic, as illustrated in Figures 4-5.



(a) Dynamical behavior of $S(t)$

(b) Time averages of $S(t)$, $I(t)$ and $R(t)$

Figure 5: Persistence in the epidemic population dynamics model with $\alpha = 0.2$.

Finally, we consider a pure jump model with two different values $\alpha^{(1)}$ and $\alpha^{(2)}$ and Lévy measures $\nu^{(1)}(dz)$ and $\nu^{(2)}(dz)$ given by (4.1) as

$$\nu^{(j)}(dz) = \frac{k_+}{z^{\alpha^{(j)}+1}} \mathrm{e}^{-\lambda_+ z} dz, \qquad j = 1, 2,$$

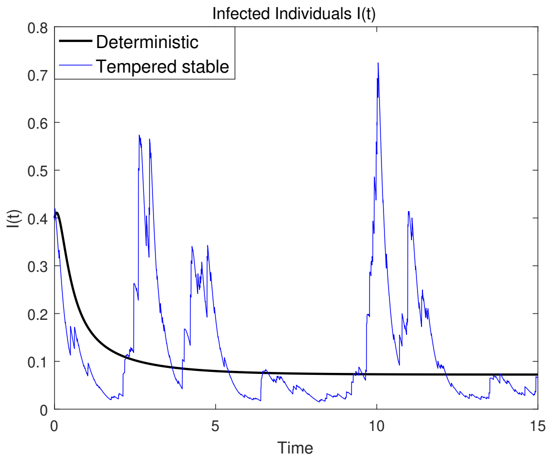
while normalizing the jump size variances

$$\left(\sigma_i^{(1)}\right)^2 \int_{\mathbb{R}^3 \setminus \{0\}} z^2 \nu(dz) = \left(\sigma_i^{(2)}\right)^2 \int_{\mathbb{R}^3 \setminus \{0\}} z^2 \nu(dz)$$

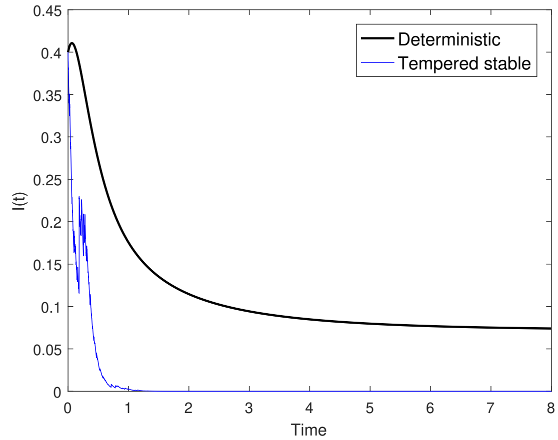
to the same level in both cases, i.e.

$$k_+ \left(\sigma_i^{(1)}\right)^2 \frac{\Gamma(2 - \alpha^{(1)})}{\lambda_+^{2-\alpha^{(1)}}} = k_+ \left(\sigma_i^{(2)}\right)^2 \frac{\Gamma(2 - \alpha^{(2)})}{\lambda_+^{2-\alpha^{(2)}}}$$

with $k_+ = 2.8$, $\lambda_+ = 1.2$. When $\alpha^{(1)} = 0.2$ we take $\sigma_1^{(1)} = 0.2$, $\sigma_2^{(1)} = 0.8$ and $\sigma_3^{(1)} = 0.5$, in which case we have $\overline{\mathcal{R}}_0 = 1.01 > 1$ and both $I(t)$ and $R(t)$ are persistent.



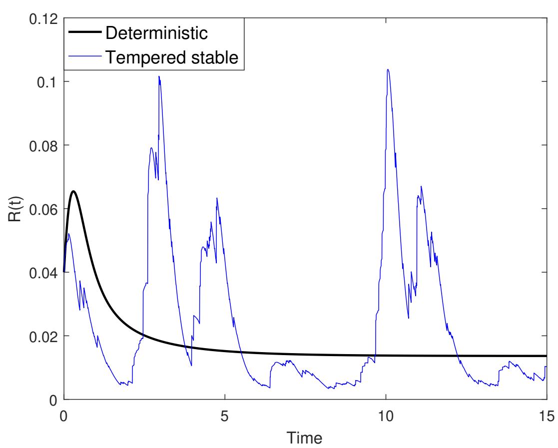
(a) Persistence of $I(t)$ for $\alpha^{(1)} = 0.2$



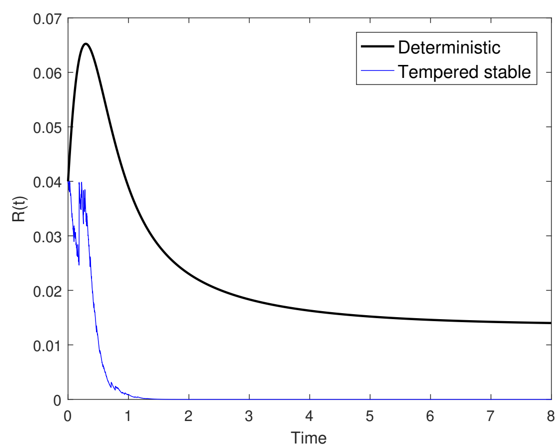
(b) Extinction of $I(t)$ for $\alpha^{(2)} = 0.9$

Figure 6: Behavior of the infected population for two different values of α.

When $\alpha^{(2)} = 0.9$ we take $\sigma_1^{(2)} = 0.1857$, $\sigma_2^{(2)} = 0.7426$ and $\sigma_3^{(2)} = 0.4641$, in which case we have $\overline{\mathcal{R}}_0 = 0.99 < 1$, and both $I(t)$ and $R(t)$ become extinct, showing that persistence and extinction can depend on the shape of the jump size distribution for a given variance level, see Figures 6-7. In particular, the presence of larger positive jumps for small values of α can result into persistence of the disease.



(a) Persistence of $R(t)$ for $\alpha^{(1)} = 0.2$



(b) Extinction of $R(t)$ for $\alpha^{(2)} = 0.7$

Figure 7: Behavior of the recovered population for two different values of α.

5 Appendix

This section is devoted to proof arguments which are similar to the literature, see Zhou and Zhang (2016). The next Lemma 5.1, which extends Lemma 2 of Liu and Wang (2014) to possibly discontinuous functions f, is needed for the proof of Theorem 3.2.

Lemma 5.1 *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function integrable on any interval $[0, t]$, $t > 0$, and consider $\Phi : \mathbb{R}_+ \to \mathbb{R}$ a function such that $\lim_{t\to\infty} (\Phi(t)/t) = 0$.*

i) Assume that there exist nonnegative constants $\rho_0 \geq 0$, $T \geq 0$ such that

$$\log f(t) \leq \rho t - \rho_0 \int_0^t f(r)dr + \Phi(t), \quad \mathbb{P}\text{-}a.s.$$

for all $t \geq T$, where $\rho \in \mathbb{R}$. Then we have

$$\begin{cases} \limsup_{t\to\infty} \frac{1}{t} \int_0^t f(r)dr \leq \frac{\rho}{\rho_0}, \quad \mathbb{P}\text{-}a.s. \quad \text{if} \quad \rho \geq 0; \\ \lim_{t\to\infty} f(t) = 0, \quad \mathbb{P}\text{-}a.s. \quad \text{if} \quad \rho < 0. \end{cases}$$

ii) Assume that there exists positive constants ρ, ρ_0, and $T \geq 0$ such that

$$\log f(t) \geq \rho t - \rho_0 \int_0^t f(r)dr + \Phi(t), \quad \mathbb{P}\text{-}a.s.$$

for all $t \geq T$. Then we have

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t f(r)dr \geq \frac{\rho}{\rho_0}, \quad \mathbb{P}\text{-}a.s.$$

Proof. Define

$$F(t) = \int_0^t f(r)dr.$$

By the integrability of $f(t)$, the Lebesgue differentiation theorem (see e.g. Theorem 1.6.11 in Tao (2011)) shows that $F(t)$ is continuous and almost everywhere differentiable, with

$$\frac{dF(t)}{dt} = f(t)$$

for almost every $t \geq 0$. The rest of the proof is the same as in Lemma 2 of Liu and Wang (2014), and is omitted. □

Proof of Theorem 3.1. The proof of existence and uniqueness of solutions for stochastic differential equation driven by Lévy processes in Applebaum (2009) ensure the integrability of S_t, I_t and R_t on any bounded interval $[0,T]$. In view of (1.1a)-(1.1c), we deduce

$$\frac{S_t - S_0}{t} + \frac{I_t - I_0}{t} = \Lambda - \mu\langle S\rangle_t - (\mu+\varepsilon+\eta)\langle I\rangle_t + \frac{1}{t}\int_0^t S_r dB_1^{\varrho}(r) + \frac{1}{t}\int_0^t I_r dB_2^{\varrho}(r)$$
$$+\frac{1}{t}\int_0^t S_{r^-}\int_{\mathbb{R}^3\setminus\{0\}}\gamma_1(z)\tilde{N}(dr,dz) + \frac{1}{t}\int_0^t I_{r^-}\int_{\mathbb{R}^3\setminus\{0\}}\gamma_2(z)\tilde{N}(dr,dz),$$

which yields

$$\mu\langle S\rangle_t + (\mu+\varepsilon+\eta)\langle I\rangle_t = \Lambda - \varphi(t), \tag{5.1}$$

where

$$\begin{aligned}\varphi(t) \;:=\; & \frac{S_t - S_0}{t} + \frac{I_t - I_0}{t} - \frac{1}{t}\int_0^t S_r dB_1^{\varrho}(r) - \frac{1}{t}\int_0^t I_r dB_2^{\varrho}(r) \\ & -\frac{1}{t}\int_0^t S_{r^-}\int_{\mathbb{R}^3\setminus\{0\}}\gamma_1(z)\tilde{N}(dr,dz) - \frac{1}{t}\int_0^t I_{r^-}\int_{\mathbb{R}^3\setminus\{0\}}\gamma_2(z)\tilde{N}(dr,dz).\end{aligned}$$

By the Itô formula for Lévy-type stochastic integrals (see Theorem 1.16 in Øksendal and Sulem (2005)) and (5.1), letting

$$M_2(t) := \int_0^t\int_{\mathbb{R}^3\setminus\{0\}}\log(1+\gamma_2(z))\tilde{N}(ds,dz) \tag{5.2}$$

we have

$$\begin{aligned}\log I_t \;=\; & \log I_0 + \beta\int_0^t S_r dr - (\mu+\varepsilon+\eta)t - \beta_2 t + B_2^{\varrho}(t) + M_2(t) \\ =\; & \log I_0 + \left(\beta\frac{\Lambda}{\mu} - (\mu+\varepsilon+\eta+\beta_2)\right)t - \frac{\beta(\mu+\varepsilon+\eta)}{\mu}\int_0^t I_r dr \\ & -\frac{\beta}{\mu}t\varphi(t) + B_2^{\varrho}(t) + M_2(t) \\ \leq\; & \log I_0 + (\mu+\varepsilon+\eta)(\overline{\mathcal{R}}_0 - 1)t - \frac{\beta}{\mu}t\varphi(t) + B_2^{\varrho}(t) + M_2(t).\end{aligned} \tag{5.3}$$

We deduce from Lemmas 2.2, 2.5 and 2.3 that

$$\lim_{t\to\infty}\varphi(t) = 0, \quad \mathbb{P}\text{-}a.s. \tag{5.4}$$

In addition, under (H_5) we have

$$\int_0^t \frac{d\langle M_2, M_2\rangle(r)}{(1+r)^2}dr \;=\; \int_0^t \frac{1}{(1+r)^2}dr\int_{\mathbb{R}^3\setminus\{0\}}\big(\log(1+\gamma_2(z))\big)^2\nu(dz)$$

$$= \quad \frac{t}{1+t} \int_{\mathbb{R}^3 \setminus \{0\}} \big(\log(1+\gamma_2(z)) \big)^2 \nu(dz) < +\infty, \quad t \in \mathbb{R}_+,$$

hence by the law of large numbers for local martingales (see Theorem 1 in Liptser (1980)) we have

$$\lim_{t \to \infty} \frac{M_2(t)}{t} = 0, \quad \mathbb{P}\text{-}a.s. \tag{5.5}$$

By the law of large numbers (see Theorem 3.4 in Chapter 1 of Mao (2008)) we also get

$$\lim_{t \to \infty} \frac{B_2^{\varrho}(t)}{t} = 0, \quad \mathbb{P}\text{-}a.s. \tag{5.6}$$

Therefore, by (5.3), if $\overline{\mathcal{R}}_0 < 1$ we have

$$\limsup_{t \to \infty} \frac{\log I_t}{t} \leq (\mu + \varepsilon + \eta)(\overline{\mathcal{R}}_0 - 1) < 0, \quad \mathbb{P}\text{-}a.s., \tag{5.7}$$

which, together with the positivity of I_t, implies

$$\lim_{t \to \infty} I_t = 0, \quad \mathbb{P}\text{-}a.s. \tag{5.8}$$

In other words, the disease goes to extinction with probability one. We note that extinction and the value of $\overline{\mathcal{R}}_0$ are governed by the behavior of I_t, which uses S_t only in its drift term. The influence of this drift term disappears when $\varphi(t)$ tends to zero, and its behavior does not affect the value of β_2 and $\overline{\mathcal{R}}_0$ which are given by second order terms in the Itô formula. As a result, β_2 and $\overline{\mathcal{R}}_0$ depend on $\varrho_{2,2}$ and γ_2, and not on (off-diagonal) correlation coefficients in the matrix ϱ.

Furthermore, from (5.1) we obtain

$$\lim_{t \to \infty} \langle S \rangle_t = \frac{\Lambda}{\mu}, \quad \mathbb{P}\text{-}a.s.$$

We derive from (1.1c) that

$$\frac{R_t - R_0}{t} = -\frac{\mu}{t} \int_0^t R_r dr + \frac{\eta}{t} \int_0^t I_r dr + \frac{1}{t} \int_0^t R_r dB_3^{\varrho}(r) + \frac{1}{t} \int_0^t R_{r^-} \int_{\mathbb{R}^3 \setminus \{0\}} \gamma_3(z) \tilde{N}(dr, dz),$$

and taking limits on both sides yields

$$\begin{aligned} \mu \lim_{t \to \infty} \frac{1}{t} \int_0^t R_r dr \quad &= \quad \eta \lim_{t \to \infty} \frac{1}{t} \int_0^t I_r dr - \lim_{t \to \infty} \frac{R_t}{t} + \lim_{t \to \infty} \frac{1}{t} \int_0^t R_r dB_3^{\varrho}(r) \\ &\quad + \lim_{t \to \infty} \frac{1}{t} \int_0^t R_{r^-} \int_{\mathbb{R}^3 \setminus \{0\}} \gamma_3(z) \tilde{N}(dr, dz), \quad \mathbb{P}\text{-}a.s. \end{aligned} \tag{5.9}$$

Together with (5.8) and the conclusions in Lemmas 2.2, 2.5 and 2.3, we conclude to

$$\lim_{t\to\infty} R_t = 0, \quad \mathbb{P}\text{-}a.s.$$

□

Proof of Theorem 3.2. By (5.1) we deduce that

$$\beta\langle S\rangle_t = \frac{\beta\Lambda}{\mu} - \frac{\beta}{\mu}\varphi(t) - \frac{\beta(\mu+\varepsilon+\eta)}{\mu}\langle I\rangle_t.$$

It then follows from (5.3) that

$$\begin{aligned}
\log I_t &= \log I_0 + \beta\int_0^t S_r dr - (\mu+\varepsilon+\eta+\beta_2)t \\
&\quad + B_2^{\varrho}(t) + \int_0^t\int_{\mathbb{R}^3\setminus\{0\}} \log(1+\gamma_2(z))\tilde{N}(ds,dz) \\
&= \left(\beta\frac{\Lambda}{\mu} - (\mu+\varepsilon+\eta) - \beta_2\right)t - \frac{\beta(\mu+\varepsilon+\eta)}{\mu}\int_0^t I_r dr + \Psi(t),
\end{aligned}$$

where we denote

$$\Psi(t) := \log I_0 - \frac{\beta}{\mu}t\varphi(t) + B_2^{\varrho}(t) + M_2(t), \qquad t\in\mathbb{R}_+,$$

and $M_2(t)$ is defined as in (5.2). From (5.4), (5.5) and (5.6) it follows that $\lim_{t\to\infty}(\Psi(t)/t) = 0$, hence applying Lemma 5.1 to the function $f(t) := I_t$ which is a.s. integrable over $[0,T]$, $T>0$, we obtain

$$\lim_{t\to\infty}\langle I\rangle_t = \frac{\mu\left(\beta\Lambda/\mu - (\mu+\varepsilon+\eta) - \beta_2\right)}{\beta(\mu+\varepsilon+\eta)} = \frac{\mu}{\beta}(\overline{\mathcal{R}}_0 - 1), \quad \mathbb{P}\text{-}a.s.$$

Consequently, on account of (5.1) and (5.4) we get

$$\lim_{t\to\infty}\langle S\rangle_t = \frac{\Lambda}{\mu} - \frac{(\mu+\varepsilon+\eta)}{\beta}(\overline{\mathcal{R}}_0 - 1) = S^* + \frac{\beta_2}{\beta}, \quad \mathbb{P}\text{-}a.s.,$$

and it follows from (5.9) and Lemmas 2.2, 2.5 and 2.3 that

$$\lim_{t\to\infty}\langle R\rangle_t = \frac{\eta}{\beta}(\overline{\mathcal{R}}_0 - 1), \quad \mathbb{P}\text{-}a.s.$$

□

Conclusion

In this paper, we consider a stochastic version of the SIR epidemic model (1.1a)-(1.1c), driven by correlated Brownian and Lévy jump components with heavy-tailed increments. We present new solution estimates using the parameter $\lambda(p)$ defined in (1.6) and Kunita's inequality for jump processes in the key Lemmas 2.2 and 2.3. Our approach relaxes the restriction on the finiteness of the Lévy measure $\nu(dz)$ imposed in Zhang and Wang (2013) and Zhou and Zhang (2016), and our definition of the parameter $\lambda(p)$ in (1.6) applies to a wider range of Lévy measures. In Theorems 3.1 and 3.2 we derive the basic reproduction number $\overline{\mathcal{R}}_0$ which characterizes the extinction and persistence properties of the stochastic epidemic system (1.1a)-(1.1c). As an illustration we present numerical simulations based on tempered stable processes, showing that the additional presence of jumps and the level of the index $\alpha \in (0, 1)$ can have a significant influence on the dynamical behavior of the epidemic system.

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