Modeling and Analysis of Wireless Networks Using Poisson Hard-Core Process

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***Abstract*—Due to its mathematical tractability, the homogeneous Poisson point process (PPP) has been employed to model wireless networks and analyze their performance. The PPP has the fundamental property that in a network with n nodes, the n nodes are distributed independently from each other. As such the PPP is not a suitable model for many networks where there exists a repulsion among the nodes. In order to address this limitation, in this paper we model the spatial distribution of transmitters in wireless networks as a Poisson hard-core process (PHCP) in which no two nodes can be closer to each other than a given repulsion radius from one another. We first provide an exact expression of the coverage probability of the networks and then introduce the method to efficiently evaluate the derived expression. Additionally, we derive approximations of the coverage probability which have low computational complexities. The accuracy and efficiency of our analytical results are validated by our simulations.**

***Index Terms*—Stochastic geometry, repulsive point process, Poisson hard-core process**

I. Introduction

The spatial distribution of nodes in wireless networks strongly affects the performance of networks. Recently, many researchers have utilized stochastic geometry [1] to model and analyze the wireless networks since conventional methods assuming a regular hexagonal lattice or the Wyner model [2] are unrealistic and difficult to apply. Since the Poisson point process (PPP) has many useful mathematical properties, several works have assumed that the spatial distribution of the nodes in wireless networks are distributed according to a PPP [3]–[6]. The authors in [3] and [4] characterized the transmission capacity of ad hoc wireless networks. In the case of cellular networks, downlink and uplink performances were studied in [5] and [6], respectively. However, in the PPP model, nodes are assumed to be independently distributed, making PPP not a suitable model to reflect the actual node deployment in many wireless networks. In [5], it is shown that modeling the locations of cellular base stations (BSs) as a PPP underestimates the performance of the actual BS deployment.

In practical networks, the locations of transmitters are chosen in order to alleviate interference or extend coverage region, and thus a repulsion among the locations of the transmitters naturally arises [7]–[10]. In this context, hard-core point processes (HCPs) have drawn attention as the models for actual networks which exhibit repulsion. These HCPs aim to model repulsive phenomena, as they are characterized by the property that no two points can be closer to each other than a given repulsion radius from one another. We distinguish two main classes of HCPs which we now introduce.

The Matérn hard-core process (MHCP) type I [11] is obtained by retaining every point of a PPP which is not within a certain distance from another point of the PPP. The MHCP type II model [11] is constructed by assigning an age $t \in [0, 1]$ to each point of a PPP and removing every point which is within a given distance of a younger point of the PPP (a point x with age t_x is said to be younger than y with age t_y if $t_x \leq t_y$). In [12], it was shown that BS locations in wireless cellular networks can be modeled by the MHCPs. The works in [13] and [14] studied mean interference of the MHCPs of types I and II, respectively. The nearest neighbor distribution in the MHCP type II was derived in [15]. However, since the Laplace functional of the MHCPs is unknown, the model is not well suited. As an example, it is intractable to find the exact distribution of the signal-to-noise-plus-interference ratio (SINR), which determines key performance metrics such as achievable rate and coverage probability, for the networks modeled by the MHCP as in [7] and [16].

A better suited HCP turns out to be the Poisson hard-core process (PHCP) [1] (sometimes called Gibbs hard-core process or Strauss hard-core process) which is a PPP conditional on all its points being further than a certain distance from one another. From the experimental results in [8]–[10], it was verified that the PHCP can model the actual deployment of BSs which exhibit a repulsive behavior. However, previous works on the PHCP have only tried to fit the PHCP to real configurations of the nodes. We have found that the performance of networks modeled by a PHCP has not been analytically investigated yet.

In this paper, we model the locations of transmitters in wireless networks as a PHCP, and our focus will be the analysis of some key performance metrics of the network. First, by computing the Laplace functional of pairwise interaction point processes, we derive an exact series expansion of the coverage probability which is the probability that the SINR is larger than a certain target SINR. Then, we introduce a method to compute the derived result which exploits the Quasi-Monte Carlo (QMC) technique. In addition, we provide approximations of the coverage probability which have low computational complexities. Lastly, in order to confirm our analysis, we compare our theoretical results with those given by Monte Carlo simulation.

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II. Preliminaries and System Model

In this section, we first introduce fundamental properties of the pairwise interaction point process which includes the PHCP as a special case. Then, we will present the system model by focusing on the PHCP.

A. Preliminaries

Let $W := \mathcal{B}_0(R) \subset \mathbb{R}^2$ be a circular observation window of radius R centered at the origin $(0,0)$ in $\mathbb{R}^2$, and let Ψ be a point process on W, i.e. a random finite set of points of W. The configuration space (i.e., the space in which Ψ takes its values) is denoted by $\mathcal{X}$. We assume that Ψ has a density f_Ψ with respect to the PPP on W with intensity 1 [17], i.e.,

$$\mathbb{E}\left[\exp\Big(-\sum_{x\in\Psi} g(x)\Big)\right] = \sum_{n\geq 0}\frac{\mathrm{e}^{-\ell(W)}}{n!} \times \int_{W^n} \mathrm{e}^{-\sum_{i=1}^n g(x_i)} f_\Psi(\{x_1,\ldots,x_n\})\,\ell(\mathrm{d}x_1)\cdots\ell(\mathrm{d}x_n), \tag{1}$$

for all non-negative functions $g:\mathbb{R}^2\to[0,\infty)$, where ℓ indicates the Lebesgue measure on $\mathbb{R}^2$. We recall that the Laplace functional of the point process on the left-hand side of (1) characterizes its distribution.

In the following, we introduce the definition of the pairwise interaction point process and Georgii-Nguyen-Zessin formula.

Definition 1 (Pairwise interaction point process)**.** *The point process Ψ is said to be a* pairwise interaction point process *if*

$$f_\Psi(\omega) = c\prod_{x\in\omega}\varphi_1(x)\prod_{\{x,y\}\subset\omega}\varphi_2(\|x-y\|), \quad \omega\in\mathcal{X}, \tag{2}$$

where c is the normalizing constant defined by

$$c^{-1} := \sum_{n\geq 0}\frac{\mathrm{e}^{-\ell(W)}}{n!}\int_{W^n}\prod_{i=1}^n\varphi_1(x_i) \times \prod_{\substack{j,k=1,\ldots,n;\\ j\neq k}}\varphi_2(\|x_j-x_k\|)\,\ell(\mathrm{d}x_1)\cdots\ell(\mathrm{d}x_n), \tag{3}$$

and φ_1 and φ_2 are two non-negative functions such that the right-hand side of (3) *is finite. The* Papangelou conditional intensity *of a pairwise interaction point process is defined as*

$$\pi(x,\omega) := \frac{f_\Psi(\omega\cup\{x\})}{f_\Psi(\omega)}, \quad x\in W,\ \omega\in\mathcal{X}. \tag{4}$$

Definition 2. *A pairwise interaction point process with $\varphi_1(x) = \lambda$ and $\varphi_2(x) = \mathbb{1}_{\{x\geq d\}}$ is called a PHCP with intensity $\lambda > 0$ and radius $d > 0$.*

Proposition 1 (Georgii-Nguyen-Zessin formula)**.** *Assume that Ψ is a pairwise interaction point process. Then, for any $f: W\times\mathcal{X}\to[0,\infty)$, we have*

$$\mathbb{E}\left[\sum_{x\in\Psi} f(x,\Psi\setminus\{x\})\right] = \mathbb{E}\left[\int_{W^2} f(x,\Psi)\pi(x,\Psi)\,\mathrm{d}x\right]. \tag{5}$$

We remark that the Georgii-Nguyen-Zessin formula (5) implies that the (reduced) Palm measure of Ψ at $x\in\mathbb{R}^2$ [18] is

$$\mu_x(\mathrm{d}\xi) := \pi(x,\xi)\lambda^{-1}(x)\,\mathbb{P}_\Psi(\mathrm{d}\xi),$$

where $\mathbb{P}_\Psi$ denotes the distribution of Ψ and λ is the intensity of Ψ. To obtain some heuristics on the Palm measure μ_x, let $\mathrm{d}x$ be an infinitesimally small volume around x, let $A\in\mathcal{A}$, and set $f(x,\omega) := \mathbb{1}_{\{x\in\mathrm{d}x,\ \omega\in A\}}$. Then, by Proposition 1,

$$\mu_x(A) = \lim_{\ell(\mathrm{d}x)\to 0}\mathbb{P}((\Psi\setminus\mathrm{d}x)\in A \mid |\Psi\cap\mathrm{d}x| = 1),$$

or to state things analogously, μ_x is the distribution of the point process obtained by conditioning Ψ on $x\in\Psi$ and removing x from the obtained configuration. When Ψ is the PPP, $\mu_x = \mathbb{P}_\Psi(\mathrm{d}\xi)$, which is known in the literature as the Slivnyak-Mecke theorem [1]. We now characterize the Palm distributions of the pairwise interaction point processes.

Proposition 2. *Let Ψ be a pairwise interaction point process with interaction functions φ_1 and φ_2. Then, μ_x is the law of a pairwise interaction point process with interaction functions given by*

$$\varphi_1^x(y) = \varphi_1(y)\varphi_2(\|x-y\|), \text{ and } \varphi_2^x(y) = \varphi_2(y). \tag{6}$$

Proof: Letting $g:\mathbb{R}^2\to[0,\infty)$, by (4) the Laplace functional of μ_x at g is equal to

$$\int\exp\Big(-\sum_{y\in\xi}g(y)\Big)\mu_x(\mathrm{d}\xi) = \varphi_1(x)\lambda^{-1}(x)\,\mathbb{E}\left[\exp\Big(-\sum_{y\in\Psi}g(y)\Big)\prod_{y\in\Psi}\varphi_2(\|x-y\|)\right],$$

and by (1) we obtain

$$\int\exp\Big(-\sum_{y\in\xi}g(y)\Big)\mu_x(\mathrm{d}\xi) = c\varphi_1(x)\lambda^{-1}(x) \times \sum_{n\geq 0}\frac{\mathrm{e}^{-\ell(W)}}{n!}\int_{W^n}\mathrm{e}^{-\sum_{i=1}^n g(x_i)}\prod_{i=1}^n\varphi_1(x_i)\varphi_2(\|x-x_i\|) \times \prod_{\substack{j,k=1,\ldots,n;\\ j\neq k}}\varphi_2(\|x_j-x_k\|)\,\ell(\mathrm{d}x_1)\cdots\ell(\mathrm{d}x_n). \tag{7}$$

Additionally,

$$\lambda(x) = \mathbb{E}[\pi(x,\Psi)] = c\varphi_1(x)\sum_{n\geq 0}\frac{\mathrm{e}^{-\ell(W)}}{n!}\int_{W^n}\prod_{i=1}^n\varphi_1(x_i)\varphi_2(\|x-x_i\|) \times \prod_{\substack{j,k=1,\ldots,n;\\ j\neq k}}\varphi_2(\|x_j-x_k\|)\,\ell(\mathrm{d}x_1)\cdots\ell(\mathrm{d}x_n),$$

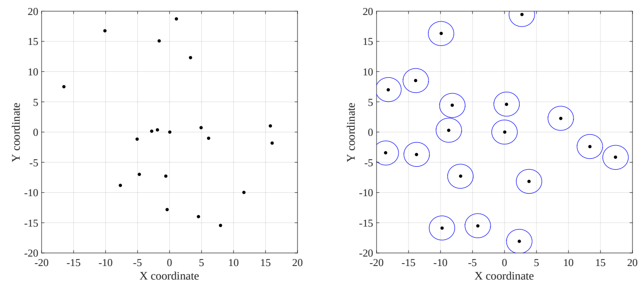
which implies that the normalizing constant appearing in (7) is indeed the one corresponding to the pairwise interaction point process with interaction functions given by (6). We conclude by comparing (7) to (1) and (2). ∎

In the following, we apply the previous proposition to the PHCP.

Remark 1. *By Proposition 2, the Palm measure of the PHCP is the law of a pairwise interaction point process with interaction functions given by*

$$\varphi_1^x(y) = \lambda\mathbb{1}_{\{\|x-y\|\geq d\}}, \text{ and } \varphi_2^x(y) = \mathbb{1}_{\{y\geq d\}}. \tag{8}$$

We note that the Palm measure above is the distribution of a PPP with intensity λ on $W\setminus\mathcal{B}_0(d)$ conditional on all its points lying farther than d from one-another.



(a) a PPP with $\lambda = 0.02$

(b) a PHCP with $\lambda = 0.02$ and $d_{\mathrm{h.c.}} = 4$

Fig. 1. Some realizations of networks distributed according to a PPP and a PHCP, respectively.

B. System model

In this subsection, in order to reflect the repulsion among the locations of transmitters [7]–[10], we model wireless networks utilizing the PHCP defined in Definition 2. Let Ψ denote the point process modeling the locations of the transmitters in wireless networks, and assume that Ψ is a PHCP on the observation window $W = \mathcal{B}_0(R)$ with intensity λ and radius $d_{\mathrm{h.c.}}$ such that $0 \le d_{\mathrm{h.c.}} < R$. Fig. 1 demonstrates realizations of the networks where dots and circles represent the locations of transmitters and balls centered at the transmitters with radius $d_{\mathrm{h.c.}}/2$, respectively. When the transmitters follow a PPP, as shown in Fig. 1-(a), some transmitters are very closely placed. Thus, modeling the spatial distribution of transmitters as a PPP may not be suitable for networks where the points cannot be located close to each other. On the other hand, as seen in Fig. 1-(b), a PHCP with radius $d_{\mathrm{h.c.}}$ can reflect the repulsion by imposing that the transmitters cannot be at a distance less than $d_{\mathrm{h.c.}}$.

In the networks, each transmitter $x \in \Psi$ sends data to its corresponding receiver, located at a distance r from x, in a direction which is assumed to be uniformly distributed. We study the interference at a typical receiver, and its associated transmitter (termed tagged transmitter) is conditioned on being located at the origin. Under this conditioning, the typical receiver is located according to a uniform distribution on $\mathcal{C}_0(r)$, the circle centered at the origin and with radius r. In the remainder, we denote by Ψ^0 a point process whose distribution is the Palm measure of Ψ, i.e., a pairwise interaction point process characterized by (8).

The quantities of interest are the interference from the other transmitters I and the SINR at the typical receiver γ. Denoting the location of the typical receiver as x_1, γ and I can be respectively written as

$$\gamma = \frac{Phr^{-\alpha}}{I + \sigma^2}, \tag{9}$$

and $I = \sum_{y \in \Psi^0} Ph_y \|y - x_1\|^{-\alpha}$ where P, α and σ^2 stand for the transmit power at the transmitters in Ψ, the path loss exponent and the power of additive white Gaussian noise (AWGN), respectively. Here, h and h_y indicate the fading gains of the channel between the typical receiver and the tagged transmitter, and the channel between the receiver and the transmitter positioned at y. We assume that fading channels follow the Rayleigh distribution, and thus h and $\{h_y\}$ are independent exponential random variables with unit mean. Note that x_1 presents the location of the typical receiver whose coordinates are given by $(r\cos(U), r\sin(U))$ where $U \sim \mathcal{U}([0, 2\pi])$ is a uniform random variable on $[0, 2\pi]$ independent from the other random variables.

III. Performance Analysis

In this section, we study the coverage probability, defined as the probability that the SINR γ in (9) is larger than a certain threshold γ_{th}. In the following, we denote by $\mathbb{P}$ the probability on the underlying probability space, and the corresponding expectation is denoted by $\mathbb{E}$. The coverage probability $\mathcal{P}_{cov}$ can be written as

$$\mathcal{P}_{cov} \triangleq \mathbb{P}\left(\gamma \ge \gamma_{th}\right).$$

Since h is an exponential random variable with unit mean,

$$\begin{aligned}\mathcal{P}_{cov} &= \mathbb{P}\left(h \ge \frac{\gamma_{th} r^{\alpha}}{P}(I + \sigma^2)\right)\\ &= \mathbb{E}\left[\exp\left(-\frac{\gamma_{th} r^{\alpha}}{P}(I + \sigma^2)\right)\right]\\ &= \exp\left(-\frac{\gamma_{th} r^{\alpha}\sigma^2}{P}\right)\mathcal{L}_I\left(\frac{\gamma_{th} r^{\alpha}}{P}\right),\end{aligned} \tag{10}$$

where $\mathcal{L}_X(s) = \mathbb{E}\left[\exp(-sX)\right]$ stands for the Laplace transform of a random variable X.

A. Exact performance analysis

In the following theorem, we derive the exact expression for $\mathcal{P}_{cov}$ in (10).

Theorem 1. *For any $\gamma_{th} > 0$, the coverage probability in (10) is given by*

$$\begin{aligned}\mathcal{P}_{cov} &= c\exp\left(-\frac{\gamma_{th} r^{\alpha}\sigma^2}{P}\right)\sum_{n\ge 0}\frac{\lambda^n \mathrm{e}^{-\pi R^2}}{n!} \\ &\times\int_{(W\setminus\mathcal{B}_0(d_{\mathrm{h.c.}}))^n}\prod_{i=1}^{n}\frac{1}{1+\gamma_{th} r^{\alpha}((x_i^{(1)}-r)^2+(x_i^{(2)})^2)^{-\alpha/2}} \\ &\times\prod_{\substack{j,k=1,\dots,n;\\ j\neq k}}\mathbb{1}_{\{\|x_j - x_k\|\ge d_{\mathrm{h.c.}}\}}\,\ell(\mathrm{d}x_1)\cdots\ell(\mathrm{d}x_n),\end{aligned} \tag{11}$$

where $(x^{(1)}, x^{(2)})$ are the coordinates of a point $x \in \mathbb{R}^2$ and c is defined by

$$\begin{aligned}c^{-1} &:= \sum_{n\ge 0}\frac{\lambda^n \mathrm{e}^{-\pi R^2}}{n!}\int_{(W\setminus\mathcal{B}_0(d_{\mathrm{h.c.}}))^n} \\ &\times\prod_{\substack{j,k=1,\dots,n;\\ j\neq k}}\mathbb{1}_{\{\|x_j - x_k\|\ge d_{\mathrm{h.c.}}\}}\,\ell(\mathrm{d}x_1)\cdots\ell(\mathrm{d}x_n).\end{aligned} \tag{12}$$

Proof: See Appendix A. ■

We note that in (11), the expression of the exact coverage probability contains a multi-dimensional integral, and therefore evaluating (11) may require a high computational complexity. As a method to efficiently compute (11), we employ the QMC integration method [19] which approximates a multi-dimensional integral and has a low complexity. For all $f : [0,1]^n \to \mathbb{C}$, the QMC integration method exhibits a deterministic sequence $\mathbf{x}_1, \dots \mathbf{x}_{N_s} \in [0,1]^n$ such that

$$Q_{N_s}(f) := \frac{1}{N_s} \sum_{n=1}^{N_s} f(\mathbf{x}_n) \approx \int_{[0,1]^n} f(\mathbf{x})\, \mathrm{d}\mathbf{x}, \tag{13}$$

when N_s goes to infinity. The advantage of this method compared to the Monte-Carlo method (in which the sequence $\mathbf{x}_n$ is stochastic) is that for high dimensions the QMC approximation converges much faster. In this paper, we choose the Sobol sequence [20] as the deterministic sequence. Since the QMC method is applicable for integrations over the unit square, we rewrite the coverage probability in (11) as shown in the following corollary. Utilizing the results in Corollary 1 and the QMC integration method in (13), we can readily evaluate the coverage probability of the networks.

Corollary 1. *By changing the integral region in (11), we have*

$$\begin{aligned}\mathcal{P}_{cov} &= c \exp\left(-\frac{\gamma_{th} r^\alpha \sigma^2}{P}\right) \sum_{n\geq 0} \frac{\lambda^n \mathrm{e}^{-\pi R^2}}{n!} (2R)^{2n} \\ &\times \int_{([0,1]\times[0,1])^n} \prod_{i=1}^n \mathbb{1}_{\{d_{\text{h.c.}}/(2R) \leq \|z_i - (1/2,1/2)\| \leq 1/2\}} \\ &\times \prod_{i=1}^n \frac{1}{1 + sP((R(2z_i^{(1)}-1)-r)^2 + (R(2z_i^{(2)}-1))^2)^{-\alpha/2}} \\ &\times \prod_{\substack{j,k=1,\dots,n;\\ j\neq k}} \mathbb{1}_{\{\|z_j - z_k\| \geq d_{\text{h.c.}}/(2R)\}}\, \ell(\mathrm{d}z_1)\cdots\ell(\mathrm{d}z_n),\end{aligned} \tag{14}$$

where

$$\begin{aligned}c^{-1} &= \sum_{n\geq 0} \frac{\lambda^n \mathrm{e}^{-\pi R^2} (2R)^{2n}}{n!} \\ &\times \int_{([0,1]\times[0,1])^n} \prod_{i=1}^n \mathbb{1}_{\{d_{\text{h.c.}}/(2R) \leq \|z_i - (1/2,1/2)\| \leq 1/2\}} \\ &\times \prod_{\substack{j,k=1,\dots,n;\\ j\neq k}} \mathbb{1}_{\{\|z_j - z_k\| \geq d_{\text{h.c.}}/(2R)\}}\, \ell(\mathrm{d}z_1)\cdots\ell(\mathrm{d}z_n).\end{aligned} \tag{15}$$

Proof: See Appendix B. ∎

B. Approximations

So far, we have obtained the exact coverage probability $\mathcal{P}_{cov}$ and introduced the method to compute the derived result. Now, we introduce some approximations of $\mathcal{P}_{cov}$ which have simple expressions. The main difficulty in identifying closed-form expressions for the coverage probability $\mathcal{P}_{cov} = \exp\left(-\frac{\gamma_{th} r^\alpha \sigma^2}{P}\right) \mathcal{L}_I\left(\frac{\gamma_{th} r^\alpha}{P}\right)$ in (10) comes from the fact that the Laplace transform with Ψ^0 contains a multi-dimensional integral. Therefore, we approximate Ψ^0 by a PPP in order to obtain simple expressions for the Laplace transform of the interference.

1) As one of the most naive approaches, one may substitute the location of the points in Ψ^0 by a PPP denoted by $\Phi^{(1)}$ with intensity λ. Then, since the PPP is stationary, we can compute the Laplace transform of the interference as [3]

$$\begin{aligned}\mathcal{L}_I^{(1)}(s) &= \mathbb{E}\left[\exp\left(-s \sum_{y\in\Phi^{(1)}} P h_y \|y - x_1\|^{-\alpha}\right)\right] \\ &= A(\lambda, P, s),\end{aligned} \tag{16}$$

where

$$A(\lambda, a, s) \triangleq \exp\left(-\lambda a^{2/\alpha} s^{2/\alpha} \frac{2\pi^2}{\alpha \sin(2\pi/\alpha)}\right). \tag{17}$$

2) The second approximation follows an idea from [21] which we recall. Since the transmitters in Ψ cannot be at a distance less than $d_{\text{h.c.}}$ from one another, and the distance from a transmitter in Ψ to its corresponding receiver is equal to r, the distance between a receiver and its interfering transmitter is always larger than $\tau = \max(d_{\text{h.c.}} - r, 0)$. Thus, by excluding the interference from the points in $\Phi^{(1)}$ within the distance τ, we have [21]

$$\begin{aligned}\mathcal{L}_I^{(2)}(s) &= \mathbb{E}\left[\exp\left(-s \sum_{y\in\Phi^{(1)}} P h_y \|y - x_1\|^{-\alpha} \mathbb{1}_{\{\|y-x_1\|>\tau\}}\right)\right] \\ &= B(\lambda, P, s, \tau),\end{aligned} \tag{18}$$

where

$$\begin{aligned}B(\lambda, a, s, \tau) &\triangleq \exp\Bigg(-\lambda\pi\Bigg(a^{2/\alpha} s^{2/\alpha} \\ &\times \int_0^\infty h^{2/\alpha} \gamma(1 - 2/\alpha, ash\tau^{-\alpha}) e^{-h} \mathrm{d}h - \frac{as\tau^{2-\alpha}}{1 + as\tau^{-\alpha}}\Bigg)\Bigg),\end{aligned} \tag{19}$$

and $\gamma(x, y) \triangleq \int_0^y t^{x-1} e^{-t} \mathrm{d}t$ denotes the lower incomplete gamma function. We refer the reader to [21] for the details of the derivation.

3) Note that when producing points of a PHCP with intensity λ and radius $d_{\text{h.c.}}$, the points are first generated according to a PPP with intensity λ, and then it is checked whether the minimum distance among the points is greater than $d_{\text{h.c.}}$. If the minimum distance is less than $d_{\text{h.c.}}$, the sample is rejected and another set of points is repeatedly generated until the minimum distance becomes larger than $d_{\text{h.c.}}$. Thus, the intensity $\tilde{\lambda}$ of the resulting points of the PHCP (termed scaled intensity) turns out to be less than λ. In the following lemma, we provide the scaled intensity of the PHCP.

Lemma 1. *For the Palm measure of a PHCP with radius $d_{\text{h.c.}}$ and intensity λ, the scaled intensity $\tilde{\lambda}$ is given by*

$$\tilde{\lambda} = \frac{\lambda}{\pi R^2} \frac{\sum_{n\geq 0} \frac{\lambda^n}{n!} \nu_{n+1}}{\sum_{n\geq 0} \frac{\lambda^n}{n!} \nu_n}, \tag{20}$$

where

$$\begin{aligned}\nu_n &\triangleq \int_{(W\setminus\mathcal{B}_0(d_{\text{h.c.}}))^n} \prod_{\substack{j,k=1,\dots,n;\\ j\neq k}} \mathbb{1}_{\{\|x_j - x_k\| \geq d_{\text{h.c.}}\}} \\ &\times \ell(\mathrm{d}x_1)\cdots\ell(\mathrm{d}x_n).\end{aligned} \tag{21}$$

Proof: See Appendix C. ∎

Then, by approximating Ψ^0 as a PPP with the rescaled intensity $\tilde{\lambda}$, in the same way as in (16), we obtain an

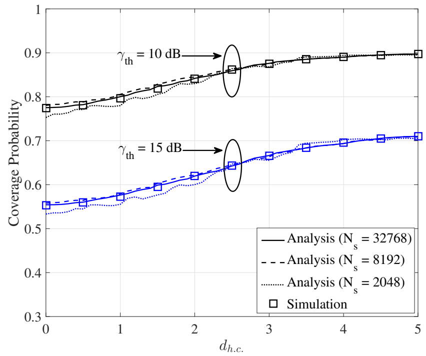


Fig. 2. Coverage probability with different values of N_s

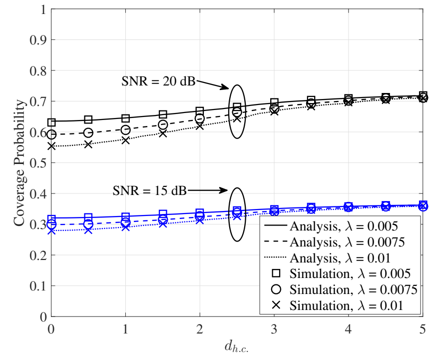


Fig. 3. Coverage probability as a function of $d_{\text{h.c.}}$

approximation of the Laplace transform of interference as

$$\mathcal{L}_I^{(3)}(s) = A(\tilde{\lambda}, P, s). \tag{22}$$

4) Similarly, by (18) we obtain

$$\mathcal{L}_I^{(4)}(s) = B(\tilde{\lambda}, P, s, \tau). \tag{23}$$

IV. Simulation Results

In this section, we provide numerical results to validate our analysis. Let us set SNR $= P/\sigma^2$. We set $\alpha = 4$, $r = 1$ and $\sigma^2 = 1$. The analytical result in (11) is evaluated using the QMC method in (13) and the result in (14). Fig. 2 examines the coverage probability $\mathcal{P}_{cov}$ with various numbers of terms in the Sobol sequence N_s when SNR $= 20$ dB and $\lambda = 0.01$. We see that the analytical results with $N_s = 2^{15}(32768)$ are well matched with the simulated results. Accordingly, in this paper, we set the number of Sobol sequences N_s as $N_s = 2^{15}$.

In Fig. 3, we plot the coverage probability $\mathcal{P}_{cov}$ for different values of SNR and λ in the case of $\gamma_{th} = 15$ dB. We observe that $\mathcal{P}_{cov}$ is an increasing function of $d_{\text{h.c.}}$ and SNR. Since an increase in λ results in additional interference, $\mathcal{P}_{cov}$ decreases as λ becomes larger. In addition, it is observed that the impact of $d_{\text{h.c.}}$ on $\mathcal{P}_{cov}$ is more pronounced when the SNR is high.

Fig. 4 reveals the exact and approximated coverage probabilities of the networks with SNR $= 20$ dB, $\lambda = 0.01$ and $\gamma_{th} = 15$ dB. As the approximation with $\mathcal{L}_I^{(1)}(s)$ does not take $d_{\text{h.c.}}$ into account, the approximation fails to predict the performance of the networks with a PHCP. We see that $\mathcal{L}_I^{(2)}(s)$ enhances the accuracy of the approximations by excluding the interference within the distance τ. Additionally, it is observed that better approximations are obtained by employing the rescaled intensity $\tilde{\lambda}$ in Lemma 1, i.e., $\mathcal{L}_I^{(3)}(s)$ is better than $\mathcal{L}_I^{(1)}(s)$ and $\mathcal{L}_I^{(4)}(s)$ is better than $\mathcal{L}_I^{(2)}(s)$.

V. Conclusion

In this paper, we have modeled wireless networks using the Poisson hard-core process (PHCP) which takes the repulsion among the transmitters into account. We have derived an explicit analytical representation of the coverage probability. In addition, we have introduced the method to compute the derived result by employing the Quasi-Monte Carlo (QMC) method. The approximations on the coverage probability have also been provided. In the simulation results, we have verified that our analysis accurately predicts the performance and confirmed that the coverage probability is an increasing function of the radius of the PHCP.

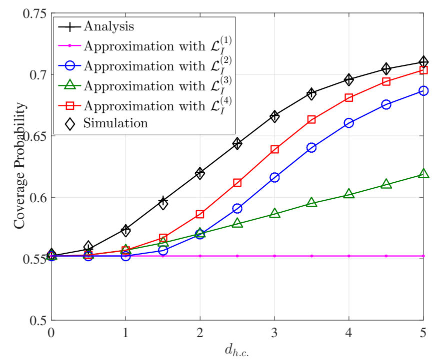


Fig. 4. Approximated coverage probability as a function of $d_{\text{h.c.}}$

Appendix A
Proof of Theorem 1

First, we focus on the computation of the Laplace transform. Let us denote the rotation of angle u by R_u. Then, for any $s > 0$, we have

$$\begin{aligned}
&\mathcal{L}_I(s)\\
&= \frac{1}{2\pi}\int_0^{2\pi} \mathbb{E}\Bigg[\exp\Bigg(-s\sum_{y\in\Psi^0} Ph_y\Big\|y - R_u(r,0)\Big\|^{-\alpha}\Bigg)\Bigg]\,\mathrm{d}u\\
&= \mathbb{E}\Bigg[\exp\Bigg(-s\sum_{y\in\Psi^0} Ph_y\Big\|y-(r,0)\Big\|^{-\alpha}\Bigg)\Bigg],
\end{aligned}$$

since the law of Ψ^0 is invariant with respect to rotations. Hence,

$$\begin{aligned}
&\mathcal{L}_I(s)\\
&= \mathbb{E}\Bigg[\prod_{y\in\Psi^0}\mathbb{E}\big[\exp(-sPh_y((y^{(1)}-r)^2+(y^{(2)})^2)^{-\alpha/2})\mid\Psi^0\big]\Bigg]\\
&\overset{(a)}{=} \mathbb{E}\Bigg[\prod_{y\in\Psi^0}\frac{1}{1+sP((y^{(1)}-r)^2+(y^{(2)})^2)^{-\alpha/2}}\Bigg]\\
&= \mathbb{E}\Bigg[\exp\Bigg(-\sum_{y\in\Psi^0}\ln\Big(1+sP((y^{(1)}-r)^2+(y^{(2)})^2)^{-\alpha/2}\Big)\Bigg)\Bigg]\\
&\overset{(b)}{=} c\sum_{n\geq 0}\frac{\mathrm{e}^{-\pi R^2}}{n!}\int_{W^n}\prod_{i=1}^{n}\frac{\lambda^n}{1+sP((x_i^{(1)}-r)^2+(x_i^{(2)})^2)^{-\alpha/2}}
\end{aligned} \tag{24}$$

$$\times \prod_{i=1}^{n} \mathbb{1}_{\{\|x_i\| \ge d_{\mathrm{h.c.}}\}} \prod_{\substack{j,k=1,\ldots,n;\\ j\neq k}} \mathbb{1}_{\{\|x_j - x_k\| \ge d_{\mathrm{h.c.}}\}} \ell(\mathrm{d}x_1)\cdots\ell(\mathrm{d}x_n)$$

$$= c\sum_{n\ge 0} \frac{\lambda^n \mathrm{e}^{-\pi R^2}}{n!} \int_{(W\setminus \mathcal{B}_0(d_{\mathrm{h.c.}}))^n} \prod_{\substack{j,k=1,\ldots,n;\\ j\neq k}} \mathbb{1}_{\{\|x_j - x_k\| \ge d_{\mathrm{h.c.}}\}}$$

$$\times \prod_{i=1}^{n} \frac{1}{1+sP((x_i^{(1)}-r)^2 + (x_i^{(2)})^2)^{-\alpha/2}} \ell(\mathrm{d}x_1)\cdots\ell(\mathrm{d}x_n),$$

where (a) follows from the independence between $\{h_y\}$ and Ψ^0, and the fact that the moment generating function of h_y is equal to $\mathbb{E}[\exp(th_z)] = (1-t)^{-1}$. In addition, (b) comes from the results in (1), (2) and (8). The result in (11) follows by plugging (24) into (10). ■

Appendix B
Proof of Corollary 1

The Laplace transform in (24) can be rewritten as

$$\mathcal{L}_I(s)$$

$$= c\sum_{n\ge 0} \frac{\mathrm{e}^{-\pi R^2}}{n!} \int_{-R}^{R} \cdots \int_{-R}^{R} \lambda^n \prod_{i=1}^{n} \mathbb{1}_{\{\|x_i\| \ge d_{\mathrm{h.c.}}\}}$$

$$\times \prod_{i=1}^{n} \mathbb{1}_{\{\|x_i\| \le R\}} \prod_{i=1}^{n} \frac{1}{1+sP((x_i^{(1)}-r)^2 + (x_i^{(2)})^2)^{-\alpha/2}}$$

$$\times \prod_{\substack{j,k=1,\ldots,n;\\ j\neq k}} \mathbb{1}_{\{\|x_j - x_k\| \ge d_{\mathrm{h.c.}}\}} \ell(\mathrm{d}x_1)\cdots\ell(\mathrm{d}x_n)$$

$$\overset{\text{(c)}}{=} c\sum_{n\ge 0} \frac{\lambda^n \mathrm{e}^{-\pi R^2}}{n!} (2R)^{2n} \int_{([0,1]\times[0,1])^n}$$

$$\times \prod_{i=1}^{n} \mathbb{1}_{\{d_{\mathrm{h.c.}}/(2R) \le \|z_i - (1/2,1/2)\| \le 1/2\}}$$

$$\times \prod_{i=1}^{n} \frac{1}{1+sP((R(2z_i^{(1)}-1)-r)^2 + (R(2z_i^{(2)}-1))^2)^{-\alpha/2}}$$

$$\times \prod_{\substack{j,k=1,\ldots,n;\\ j\neq k}} \mathbb{1}_{\{\|z_j - z_k\| \ge d_{\mathrm{h.c.}}/(2R)\}} \ell(\mathrm{d}z_1)\cdots\ell(\mathrm{d}z_n),$$

where (c) comes from a change of variables $z_i^{(j)} \equiv x_i^{(j)}/R$ for $i = 1,\ldots,n$ and $j = 1,2$. In a similar fashion, c^{-1} in (12) can be expressed as in (15). ■

Appendix C
Proof of Lemma 1

Let us denote by $\tilde{\lambda}$ the scaled intensity. Then, the equation characterizing the intensity is

$$\mathbb{E}[\Psi(W)] = \tilde{\lambda}\ell(W) = \tilde{\lambda}\pi R^2. \tag{25}$$

From (12) and (21), we have

$$c^{-1} = \sum_{n\ge 0} \frac{\lambda^n \mathrm{e}^{-\pi R^2}}{n!} \nu_n. \tag{26}$$

Additionally, we have

$$\mathbb{E}[\Psi(W)]$$

$$= c\sum_{n\ge 0} \frac{\mathrm{e}^{-\pi R^2}}{n!} \int_{(W\setminus \mathcal{B}_0(d_{\mathrm{h.c.}}))^n} \Psi(\{x_1,\ldots,x_n\})$$

$$\times \ell(\mathrm{d}x_1)\cdots\ell(\mathrm{d}x_n)$$

$$= c\sum_{n\ge 0} \frac{n\lambda^n \mathrm{e}^{-\pi R^2}}{n!} \nu_n = c\lambda \sum_{n\ge 0} \frac{\lambda^n \mathrm{e}^{-\pi R^2}}{n!} \nu_{n+1}. \tag{27}$$

Hence, from (25), (26) and (27), $\tilde{\lambda}$ is obtained as (20). ■

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