G-expectation approach to stochastic ordering

Sel Ly   Nicolas Privault

School of Physical and Mathematical Sciences
Division of Mathematical Sciences
Nanyang Technological University
Singapore 637371

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Abstract

This paper studies stochastic ordering under nonlinear expectations $\mathcal{E}_{\mathcal{G}}$ generated by solutions of G-Backward Stochastic Differential Equations (G-BSDEs) defined on G-expectation spaces. We derive sufficient conditions for the convex, increasing convex, and monotonic G-stochastic orderings of G-diffusion processes at terminal time. Our approach relies on comparison properties for G-Forward-Backward Stochastic Differential Equations (G-FBSDEs) and on relevant extensions of convexity, monotonicity and continuous dependence properties for the solutions of associated Hamilton-Jacobi-Bellman (HJB) equations. Applications of G-stochastic ordering to contingent claim superhedging price comparison under ambiguous coefficients are provided.

Keywords: Stochastic ordering, sublinear expectation, G-expectation, nonlinear expectation, G-Brownian motion, G-Forward-Backward SDE, increasing order, convex order, convexity, monotonicity.
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1 Introduction

Partial orderings of probability distributions have various applications in risk management, reliability, economics, finance, actual sciences, operation research, biology, option evaluation, etc., see e.g. Müller and Stoyan (2002), Denuit et al. (2005), Shaked and Shanthikumar (2007), Sriboonchita et al. (2009), Levy (2015), Belzunce et al. (2015), Perrakis (2019). For example, in utility theory, a portfolio return X is said to be dominated by another portfolio return Y in the convex order if $\mathbb{E}[\varphi(X)] \leq \mathbb{E}[\varphi(Y)]$ holds for all convex utility functions $\varphi : \mathbb{R} \to \mathbb{R}$. When X and Y are modeled as the terminal values of continuous diffusion processes, such comparison bounds have been established in El Karoui et al. (1998) by a stochastic calculus approach, which has been generalized to discontinuous processes in e.g.

Gushchin and Mordecki (2002), Bergenthum and Rüschendorf (2006; 2007), Klein et al. (2006), Arnaudon et al. (2008). With the aim of dealing with financial problems under uncertain volatility, stochastic ordering has been studied in Ly and Privault (2021) under the nonlinear g-expectations and g-evaluations introduced in Peng (2004).

On the other hand, a new form of stochastic calculus has been developed in Peng (2007b) on sublinear expectation spaces, which generalize usual probability spaces with the construction of a sublinear expectation $\mathbb{E}_G$ called the G-expectation and the corresponding G-Brownian motion, G-Itô's formula, G-Itô stochastic integral, and G-stochastic differential equations. Sublinear expectations have been used to establish a central limit theorem for G-normal distributions Peng (2008), and for the study of dynamic risk measures Peng (2007a), Peng et al. (2018). Those results have been applied to contingent claim pricing in financial markets with uncertain volatility Hu and Ji (2013), Vorbrink (2014), as well as to stochastic control Xu (2010), Hu et al. (2014c) and to robust mean-variance hedging in Biagini et al. (2019). Recently, Stein's method has extended to G-normal approximation under sublinear expectations in Song (2019a;b).

In this paper, we consider stochastic ordering in the framework of nonlinear $\mathcal{E}_\mathcal{G}$-expectations and $\mathcal{E}_\mathcal{G}$-evaluations generated by G-BSDEs defined on G-expectation spaces, which are not sublinear in general. In comparison with the standard linear expectation setting, this leads to more general ordering concepts which are suitable for the modeling of volatility uncertainty in finance. See also Tian and Jiang (2016) for the construction of uncertainty orders on the sublinear expectation space, and Grigorova (2014b;a) for the construction of monotonic and increasing convex stochastic orders using Choquet's expectation, and application to financial optimization.

Namely, we use the $\mathcal{E}_\mathcal{G}$-expectation and $\mathcal{E}_\mathcal{G}$-evaluation $\mathcal{E}_\mathcal{G}[\xi]$ of a random variable ξ, defined by Hu et al. (2014b), as the initial value Y_0 for a triple $(Y_t, Z_t, K_t)_{t\in[0,T]}$ of adapted processes which solve a G-BSDE of the form

$$-dY_t = f(t, X_t, Y_t, Z_t)dt + g(t, X_t, Y_t, Z_t)d\langle B\rangle_t - Z_t dB_t - dK_t, \qquad 0 \le t \le T, \tag{1.1}$$

with coefficients $\mathcal{G} = \{f, g\}$, terminal condition $Y_T = \xi$, $(B_t)_{t\in\mathbb{R}_+}$ is a G-Brownian motion, $(X_t)_{t\in\mathbb{R}_+}$ is a G-diffusion process driven by $(B_t)_{t\in\mathbb{R}_+}$, and $(K_t)_{t\in\mathbb{R}_+}$ is a decreasing G-martingale, see Section 2 for details.

In comparison with the g-stochastic ordering setting of Ly and Privault (2021), treating the G-stochastic ordering involves additional ingredients from G-stochastic calculus, and we need to assume under additional boundedness conditions.

In Theorems 3.1 and 3.7 we derive sufficient conditions on two G-BSDE generators $f_i(t,x,y,z)$, $g_i(t,x,y,z)$, $i=1,2$ for the convex ordering

$$\mathcal{E}_{\mathcal{G}_1}\big[\varphi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{\mathcal{G}_2}\big[\varphi\big(X_T^{(2)}\big)\big], \tag{1.2}$$

where $\varphi(x)$ is convex with polynomial growth and $X_T^{(1)}$ and $X_T^{(2)}$ are the terminal values of the solutions of the G-Forward Stochastic Differential Equations (G-FSDEs)

$$\begin{cases} dX_t^{(1)} = b_1\big(t, X_t^{(1)}\big)dt + h_1\big(t, X_t^{(1)}\big)d\langle B\rangle_t + \sigma_1\big(t, X_t^{(1)}\big)dB_t, \\ \\ dX_t^{(2)} = b_2\big(t, X_t^{(2)}\big)dt + h_2\big(t, X_t^{(2)}\big)d\langle B\rangle_t + \sigma_2\big(t, X_t^{(2)}\big)dB_t, \end{cases}$$

with $X_0^{(1)} = X_0^{(2)}$, under the bound $0 < \sigma_1(t,x) \leq \sigma_2(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}$. In an option pricing setting, the forward processes $X_t^{(i)}$, $i=1,2$, represents the prices of risky assets driven by G-Brownian motion under uncertain volatility, and the $Y_t^{(i)}$ in (1.1), $i=1,2$, represent the portfolio wealth processes, or superhedging prices, of a contingent claim. The processes $Z_t^{(i)}$ in (1.1), $i=1,2$, play a crucial role in hedging the claim with payoff $\varphi\big(X_T^{(i)}\big)$ at maturity time T, as $\pi_t^{(i)} := Z_t^{(i)}/\sigma_i\big(t, X_t^{(i)}\big)$ is the amount invested in the risky asset $X_t^{(i)}$ at time $t \in [0,T]$, see e.g. Section 4 and also Vorbrink (2014). In addition, the quantity $\mathcal{E}_{\mathcal{G}_i}\big[-\varphi\big(X_T^{(i)}\big)\big]$, makes sense as a risk measure. Here, the choice of generator functions f_i and g_i determines the investor's portfolio strategy and the corresponding risk measures, and inequalities in *G-stochastic ordering* (1.2) can be interpreted as comparisons of portfolios values, option prices, and risk measures relating to underlying assets with uncertain volatility.

The G-stochastic ordering can be used for the comparison of expected utilities when φ represents a utility function, $X_t^{(i)}$ represents a state process, and $Y_t^{(i)}$ is used as the value function of a stochastic control problem, see Touzi (2004), Xu (2010), Hu et al. (2014c), and as such they are also applicable to risk management. Precisely, they enable one to study the behavior of risk seeking (resp. risk averse) investors based on the convexity (resp. concavity) of φ, see e.g. Sriboonchita et al. (2009).

First, in Theorem 3.1 we show that the convex ordering (1.2) can be derived as a consequence of the comparison Theorem C.2.5 in Peng (2019), provided that

$$\begin{cases} zb_1(t,x) + f_1(t,x,y,z\sigma_1(t,x)) \leq zb_2(t,x) + f_2(t,x,y,z\sigma_2(t,x)), \quad x,y,z \in \mathbb{R},\ t \in [0,T], \\ \\ zh_1(t,x) + g_1(t,x,y,z\sigma_1(t,x)) \leq zh_2(t,x) + g_2(t,x,y,z\sigma_2(t,x)), \quad x,y,z \in \mathbb{R},\ t \in [0,T], \end{cases}$$

and both functions $(x, y, z) \mapsto zb_i(t, x) + f_i(t, x, y, z\sigma_i(t, x))$ and $(x, y, z) \mapsto zh_i(t, x) + g_i(t, x, y, z\sigma_i(t, x))$ are convex in (x, y) and in (y, z) on $\mathbb{R}^2$ for both $i = 1, 2$ and $t \in [0, T]$. Then in Theorem 3.7 we show, using G-stochastic calculus, that this condition can be relaxed into a single convexity assumption in (x, y) and in (y, z) for the functions

$$\begin{cases} (x, y, z) \mapsto M_i(t, x, y, z) := zb_i(t, x) + f_i(t, x, y, z\sigma_i(t, x)), & i = 1 \text{ or } i = 2, \\ (x, y, z) \mapsto N_i(t, x, y, z) := zh_i(t, x) + g_i(t, x, y, z\sigma_i(t, x)), & i = 1 \text{ or } i = 2. \end{cases}$$

Increasing convex ordering is dealt with in Theorems 3.2, and in Theorem 3.4 under ordering conditions on the drifts $b_i(t, x)$, $h_i(t, x)$, $f_i(t, x, y, z)$ and $g_i(t, x, y, z)$. In Theorem 3.8, increasing convex ordering is considered under weaker convexity conditions on the functions $M_i(t, x, y, z)$ and $N_i(t, x, y, z)$ which only need to be convex for $i = 1$ or $i = 2$. Monotonic ordering is considered in Corollary 3.3, and the particular cases of equal drifts and equal volatilities are treated in Corollaries 3.5 and 3.6 for the convex and monotonic orderings. Applications of the G-stochastic ordering to superhedging price comparison of the contingent claim under ambiguous coefficients are provided in Section 4.

The proofs of Theorems 3.1-3.2 and 3.7-3.8 rely on an extension of convexity property of the solutions of associated HJB equations types proved in Appendix A, see Theorem 2.9. The convexity properties of solutions of nonlinear PDEs have been studied by several authors, see e.g. Theorem 3.1 in Lions and Musiela (2006), Theorem 2.1 in Giga et al. (1991), and Theorem 1.1 in Bian and Guan (2008), see also Theorem 1 in Alvarez et al. (1997) in the elliptic case. Those works typically require global convexity of nonlinear drifts in all state variables (x, y, z), a condition which is too strong for even the Black-Scholes-Barenblatt equation to be satisfied. For this reason, in Theorem 2.9 below we extend Theorem 1.1 of Bian and Guan (2008) in dimension one, by replacing the global convexity of nonlinear drifts with convexity in (x, y) and (y, z), $i = 1, 2$.

Appendices B and C deal with monotonicity properties and continuous dependence results for the solutions of G-FBSDEs and associated HJB equations, which are used in the proofs of Theorems 3.1-3.8 and Corollaries 3.3-3.6. Due to the lack of a full dominated convergence theorem without monotonic continuity property, see Theorem 3.2 in Hu and Zhou (2019) and Cohen et al. (2011), we solely rely on L_G^p convergence arguments based on Proposition 4.1 in Hu et al. (2014b) and Proposition 6.1.22 in Peng (2019).

2 Preliminaries

2.1 Sublinear expectations

Let Ω be a given sample space, and let $\mathcal{H}$ be a linear space of real-valued functions defined on Ω, such that $c \in \mathcal{H}$ for all constant $c \in \mathbb{R}$, and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. Furthermore, if $X_1, X_2, \ldots, X_n \in \mathcal{H}$, we assume that $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$, for each $\varphi \in \mathcal{C}_{\text{l-Lip}}(\mathbb{R}^n)$, where $\mathcal{C}_{\text{l-Lip}}(\mathbb{R}^n)$ denotes the linear space of locally Lipschitz functions satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \tag{2.1}$$

for some constant C and m depending only on φ.

Definition 2.1 (Sublinear expectation) *A sublinear expectation $\widehat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\widehat{\mathbb{E}} : \mathcal{H} :\to \mathbb{R}$ satisfying the following properties for all $X, Y \in \mathcal{H}$:*

(i) Monotonicity: If $X \leq Y$, then $\widehat{\mathbb{E}}[X] \leq \widehat{\mathbb{E}}[Y]$.

(ii) Constant preserving: $\widehat{\mathbb{E}}[c] = c$, $c \in \mathbb{R}$.

(iii) Sub-additivity: $\widehat{\mathbb{E}}[X + Y] \leq \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$.

(iv) Positive homogeneity: $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X]$, $\lambda \geq 0$.

The triple $\big(\Omega, \mathcal{H}, \widehat{\mathbb{E}}\big)$ is called a sublinear expectation space.

In case $\widehat{\mathbb{E}}$ satisfies only the monotonicity and the constant preserving properties, $\widehat{\mathbb{E}}$ is called a *nonlinear expectation*.

Theorem 2.2 (Peng (2019), Theorem 1.2.1.) *Let $\big(\Omega, \mathcal{H}, \widehat{\mathbb{E}}\big)$ be a sublinear expectation space. Then there exists a family $\{\mathbb{E}_\theta : \theta \in \Theta\}$ of linear expectations defined on $\mathcal{H}$, such that $\widehat{\mathbb{E}}[X] = \sup_{\theta \in \Theta} \mathbb{E}_\theta[X]$, $X \in \mathcal{H}$, and for each $X \in \mathcal{H}$, there exist $\theta_X \in \Theta$ such that $\widehat{\mathbb{E}}[X] = \mathbb{E}_{\theta_X}[X]$.*

Given $X \in \mathcal{H}$, the distribution of X is the function $\mathbb{F}_X$ defined by

$$\mathbb{F}_X[\varphi] := \widehat{\mathbb{E}}[\varphi(X)], \quad \varphi \in \mathcal{C}_{\text{l-Lip}}(\mathbb{R}), \tag{2.2}$$

and $(\mathbb{R}, \mathcal{C}_{\text{l-Lip}}(\mathbb{R}), \mathbb{F}_X)$ is also a sublinear expectation space and by Theorem 2.2, $\mathbb{F}_X$ admits the representation

$$\mathbb{F}_X[\varphi] = \sup_{\theta \in \Theta} \int_{\mathbb{R}} \varphi(x) F_X(\theta, dx), \tag{2.3}$$

where $(F_X(\theta, dx))_{\theta\in\Theta}$ is a family of probability distributions in the usual sense. The distribution of $X \in \mathcal{H}$ defines the parameters

$$\underline{\mu} := -\widehat{\mathbb{E}}[-X], \quad \bar{\mu} := \widehat{\mathbb{E}}[X], \quad \text{and} \quad \underline{\sigma}^2 := -\widehat{\mathbb{E}}[-X^2], \quad \bar{\sigma}^2 := \widehat{\mathbb{E}}[X^2], \tag{2.4}$$

and the intervals $[\underline{\mu}, \bar{\mu}]$ and $[\underline{\sigma}^2, \bar{\sigma}^2]$ characterize the mean-uncertainty and the variance-uncertainty of X, respectively. Two random variables X_1, X_2 defined on sublinear expectation spaces $\big(\Omega_1, \mathcal{H}_1, \widehat{\mathbb{E}}_1\big)$ and $\big(\Omega_2, \mathcal{H}_2, \widehat{\mathbb{E}}_2\big)$ are said to be identically distributed, i.e. $X_1 \overset{d}{=} X_2$, if $\widehat{\mathbb{E}}_1[\varphi(X_1)] = \widehat{\mathbb{E}}_2[\varphi(X_2)]$ for all $\varphi \in \mathcal{C}_{\text{l-Lip}}(\mathbb{R})$. If in addition $\widehat{\mathbb{E}}_1[\varphi(X_1)] \leq \widehat{\mathbb{E}}_2[\varphi(X_2)]$, for all $\varphi \in \mathcal{C}_{l.Lip}(\mathbb{R})$ we say that the distribution of X_2 is *stronger* than the distribution of X_1, in which case we have $\{F_{X_1}(\theta_1, \cdot) \ : \ \theta_1 \in \Theta_1\} \subset \{F_{X_2}(\theta_2, \cdot) \ : \ \theta_2 \in \Theta_2\}$. Given two random variables X and Y on the sublinear expectation space, we say that Y is independent from X under $\widehat{\mathbb{E}}$ if for all test functions $\varphi \in \mathcal{C}_{\text{l-Lip}}(\mathbb{R}^2)$, we have

$$\widehat{\mathbb{E}}[\varphi(X, Y)] = \widehat{\mathbb{E}}\big[\widehat{\mathbb{E}}[\varphi(x, Y)]_{|x=X}\big].$$

A random variable X in the sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is said to be G-normally distributed if $aX + b\overline{X} \overset{d}{=} \sqrt{a^2 + b^2}X$, $a, b \geq 0$, where $\overline{X}$ is an independent copy of X and, $G : \mathbb{R} \to \mathbb{R}$ is the function defined by $G(a) := \widehat{\mathbb{E}}[aX^2]/2$, $a \in \mathbb{R}$. By Peng (2007b; 2008), letting $\underline{\sigma}^2 := -\widehat{\mathbb{E}}\big[-\widehat{X}^2\big]$ and $\overline{\sigma}^2 := \widehat{\mathbb{E}}\big[\widehat{X}^2\big]$, $\widehat{X}$ is G-normally distributed if and only if for each $\varphi \in \mathcal{C}_{\text{l-Lip}}(\mathbb{R})$, the function $u_\varphi(t, x) := \widehat{\mathbb{E}}\big[\varphi\big(x + \sqrt{t}\widehat{X}\big)\big]$ solves the G-heat equation

$$\frac{\partial u_\varphi}{\partial t}(t, x) - G\left(\frac{\partial^2 u_\varphi}{\partial x^2}(t, x)\right) = 0, \quad u_\varphi(0, x) = \varphi(x), \tag{2.5}$$

where

$$G(a) := \frac{1}{2} \sup_{\underline{\sigma}^2 \leq \sigma^2 \leq \overline{\sigma}^2} (\sigma^2 a) = \frac{1}{2}\big(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-\big) = \frac{1}{2} \max\big\{a\underline{\sigma}^2, a\overline{\sigma}^2\big\}, \quad a \in \mathbb{R}, \tag{2.6}$$

with $a^+ = \max(a, 0)$ and $a^- = -\min(a, 0)$. In particular, we have $\widehat{\mathbb{E}}\big[\varphi\big(\widehat{X}\big)\big] = P_1^G(\varphi) := u_\varphi(1, 0)$, and the function $G : \mathbb{R} \to \mathbb{R}$ in (2.6) is sublinear and monotonic, with $-\underline{\sigma}^2|a|/2 \leq G(a) \leq \overline{\sigma}^2|a|/2$, $a \in \mathbb{R}$. In the sequel we only deal with non-degenerate G-normal distributions $\mathcal{N}\big(\{0\}, [\underline{\sigma}^2, \overline{\sigma}^2]\big)$ for which $\underline{\sigma}^2 > 0$, in which case G defined is (2.6) and non-degenerate, i.e.

$$G(b) - G(a) \geq \frac{\underline{\sigma}^2}{2}(b - a), \qquad 0 \leq a \leq b,$$

and (2.5) admits a unique classical solution $u(t, x) \in C^{1,2}([0, T) \times \mathbb{R})$, see e.g. Peng (2019).

2.2 G-Expectation, G-Brownian motion and G-Itô integral

Let $\Omega := C_0(\mathbb{R}_+)$ denote the space of $\mathbb{R}$-valued continuous paths $(\omega_t)_{t\in\mathbb{R}_+}$ with $\omega_0 = 0$, equipped with the distance $\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} \max\left(1, \max_{t\in[0,i]} \left|\omega_t^1 - \omega_t^2\right|\right)$. For each $t \in [0, \infty)$, we set

$$\mathbf{W}_t := \{\omega_{\cdot\wedge t} \ : \ \omega \in \Omega\}, \quad \mathcal{F}_t := \mathcal{B}_t(\mathbf{W}) = \mathcal{B}(\mathbf{W}_t), \quad \mathcal{F}_{t+} := \mathcal{B}_{t+}(\mathbf{W}) = \bigcap_{s>t} \mathcal{B}_s(\mathbf{W}),$$

and $\mathcal{F} := \bigvee_{s>t} \mathcal{F}_s$, where $(\Omega, \mathcal{F})$ is the canonical space equipped with its natural filtration $(\mathcal{F}_t)_{t\geq 0}$, and $B_t(\omega) = (\omega_t)_{t\geq 0}$ is the corresponding canonical process. For each fixed $T \geq 0$, we also let $L^0_{\text{ip}}(\mathcal{F}) := \bigcup_{n=1}^{\infty} L^0_{\text{ip}}(\mathcal{F}_n)$, where

$$L^0_{\text{ip}}(\mathcal{F}_T) := \left\{X(\omega) = \varphi(\omega_{t_1}, \ldots, \omega_{t_m}) \ : \ m \geq 1, \ t_1, \ldots, t_m \in [0, T], \ \varphi \in \mathcal{C}_{\text{l-Lip}}(\mathbb{R}^m)\right\}, \quad T > 0.$$

Definition 2.3 *(Peng (2019), §3.2)*

i) The G-expectation is the sublinear expectation defined on $L^0_{\text{ip}}(\mathcal{F}_T)$ by

$$\mathbb{E}_G[X] = \widehat{\mathbb{E}}\left[\varphi\left(\sqrt{t_1 - t_0}\xi_1, \ldots, \sqrt{t_m - t_{m-1}}\xi_m\right)\right],$$

for X of the form $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})$, where $\xi_1, \ldots, \xi_m$ are identically G-normally distributed random variables in a sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, such that ξ_{i+1} is independent from $(\xi_1, \ldots, \xi_i), i = 1, 2, \ldots, m-1$. The corresponding canonical process $B_t(\omega) := \omega_t$, $\omega \in \Omega$, $t \geq 0$, is called a G-Brownian motion.

ii) The conditional G-expectation of $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})$ given $\mathcal{F}_{t_j}$ is defined by

$$\mathbb{E}_G[X \mid \mathcal{F}_{t_j}] := \mathbb{E}_G[\varphi(B_{t_1} - B_{t_0}, \ldots, B_{t_m} - B_{t_{m-1}}) \mid \mathcal{F}_{t_j}] = \widetilde{\varphi}(B_{t_1} - B_{t_0}, \ldots, B_{t_j} - B_{t_{j-1}}),$$

where $\widetilde{\varphi}(x_1, \ldots, x_{t_j}) := \widehat{\mathbb{E}}\left[\varphi\left(x_1, \ldots, x_{t_j}, \sqrt{t_{t_{j+1}} - t_j}\xi_j, \ldots, \sqrt{t_m - t_{m-1}}\xi_m\right)\right]$.

For $0 \leq s \leq t$, we have $\mathbb{E}_G[B_t - B_s|\mathcal{F}_s] = 0$ and $\mathbb{E}_G[(B_t - B_s)^2 \mid \mathcal{F}_s] = \overline{\sigma}^2(t - s)$. Peng (2007b) proved that $\mathbb{E}_G[\,\cdot\,]$ consistently defines a sublinear expectation on the vector lattice $L^0_{\text{ip}}(\mathcal{F}_T)$ as well as on $L^0_{\text{ip}}(\mathcal{F})$. These spaces can be continuously extended to Banach spaces denoted respectively by $L^1_G(\mathcal{F}_T)$ and $L^1_G(\mathcal{F})$, under the norm $\mathbb{E}_G[|X|]$, $X \in L^0_{\text{ip}}(\mathcal{F}_T)$, *resp.* $L^0_{\text{ip}}(\mathcal{F})$. For $p > 1$ we let $L^P_G(\mathcal{F}) := \left\{X \in L^1_G(\mathcal{F}) \ : \ |X|^p \in L^1_G(\mathcal{F})\right\}$, which is a Banach space under the norm $\|X\|_p := (\mathbb{E}_G[|X|^p])^{1/p}$.

We note that by e.g. Theorem 6.2.5 in Peng (2019), the G-expectation $\mathbb{E}_G$ can be represented as

$$\mathbb{E}_G[\xi] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[\xi], \qquad \xi \in L^1_G(\mathcal{F}), \tag{2.7}$$

where $\mathcal{P}$ is a weakly compact family of probability measures on $(\Omega, \mathcal{F})$.

Given $\pi_T^N := \{t_0 = 0, t_1, \ldots, t_N = T\}$ a subdivision of the interval of $[0,T]$, denote by $\mathcal{M}_G^{p,0}(0,T)$ the collection of simple processes of the form $\eta_t(\omega) = \sum_{j=0}^{N} \xi_j(\omega)\mathbf{I}_{[t_j,t_{j+1})}(t)$ where $\xi_j \in L^p_G(\mathcal{F}_{t_i})$, $i = 0,1,2,\ldots,N-1$, $p \geq 1$. For $\eta_t \in \mathcal{M}_G^{p,0}(0,T)$, the G-Itô integral is defined as

$$I(\eta) = \int_0^T \eta_s dB_s := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}).$$

Let $\mathcal{M}_G^p(0,T)$, $H_G^p(0,T)$ and $S_G^p(0,T)$ denote the respective completions of $\mathcal{M}_G^{p,0}(0,T)$ under the norms $\|\cdot\|_{\mathcal{M}^p}$, $\|\cdot\|_{H^p}$, $\|\cdot\|_{S^p}$ defined by

$$\|\eta_t\|^p_{\mathcal{M}^p} := \mathbb{E}_G\left[\int_0^T |\eta_t|^p dt\right], \ \|\eta_t\|^p_{H^p} := \mathbb{E}_G\left[\left(\int_0^T |\eta_t|^p dt\right)^{p/2}\right], \ \|\eta_t\|^p_{S^p} := \mathbb{E}_G\left[\sup_{t\in[0,T]} |\eta_t|^p\right].$$

Lemma 2.4 *(Peng (2019)) The linear mapping $I : \mathcal{M}_G^{2,0}(0,T) \to L^2_G(\mathcal{F}_T)$ can be continuously extended to $\mathcal{M}_G^2(0,T) \to L^2_G(\mathcal{F}_T)$ with*

$$\mathbb{E}_G\left[\int_0^T \eta_s dB_s\right] = 0 \ \ \text{and} \ \ \underline{\sigma}^2 \mathbb{E}_G\left[\int_0^T \eta_s^2 ds\right] \leq \mathbb{E}_G\left[\left(\int_0^T \eta_s dB_s\right)^2\right] \leq \overline{\sigma}^2 \mathbb{E}_G\left[\int_0^T \eta_s^2 ds\right].$$

The following property of the G-Itô integral will be useful, see Proposition 3.3.6-(iii) in Peng (2019):

$$\mathbb{E}_G\left[X + \int_r^T \eta_t dB_t \,\middle|\, \mathcal{F}_s\right] = \mathbb{E}_G[X], \quad X \in L^1_G(\mathcal{F}), \ 0 \leq s \leq r \leq t \leq T. \tag{2.8}$$

Moreover, by Proposition 4.1.4 of Peng (2019), the process

$$M_t := \int_0^t \eta_s d\langle B \rangle_s - 2\int_0^t G(\eta_s) ds$$

is a G-martingale, which will be used in the proof of Theorem 3.7 below, where the

$$\langle B \rangle_t := B_t^2 - 2\int_0^t B_s dB_s, \qquad t \in [0,T],$$

is called the quadratic variation of G-Brownian motion.

2.3 G-forward-backward stochastic differential equations

Let $(B_t)_{t\geq 0}$ be a G-Brownian motion with $-\mathbb{E}_G[-B_1^2] = \underline{\sigma}^2$ and $\mathbb{E}_G[B_1^2] = \overline{\sigma}^2$, where G is given by (2.6) in dimension one. We consider G-FSDEs of the form

$$dX_s^{t,x} = b(s, X_s^{t,x})ds + h(s, X_s^{t,x})d\langle B\rangle_s + \sigma(s, X_s^{t,x})dB_s, \qquad 0 \leq t \leq s \leq T, \tag{2.9}$$

with $X_t^{t,x} = x$, and the associated G-BSDEs are defined by

$$\begin{aligned} Y_s^{t,x} = \varphi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})d\langle B\rangle_r \\ - \int_s^T Z_r^{t,x} dB_r - (K_T^{t,x} - K_s^{t,x}), \qquad 0 \leq t \leq s \leq T, \end{aligned} \tag{2.10}$$

where $K_s^{t,x}$ is a decreasing G-martingale and $b, h, \sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$, $\varphi : \mathbb{R} \to \mathbb{R}$, $f, g : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ are deterministic functions satisfying the following conditions:

(A_1) The functions $b(t,x)$, $h(t,x)$ are continuous and bounded on $[0,T] \times \mathbb{R}$, the function $\sigma(t,x)$ is strictly positive and continuous in $t \in [0,T]$ for all $x \in \mathbb{R}$, and they satisfy the uniform Lipschitz condition

$$|b(t,x) - b(t,y)| + |h(t,x) - h(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq C\,|x-y|\,, \quad x, y \in \mathbb{R},\ t \in [0,T],$$

for some constant $C > 0$.

(A_2) The functions $f(t,x,y,z)$ and $g(t,x,y,z)$ are continuous in $t \in [0,T]$ for all $(x,y,z) \in \mathbb{R}^3$, and there exist an integer $m \geq 0$ and a constant $C > 0$ such that

(i) $|f(t,x,y,z) - f(t,x',y',z')| \leq C\big((1 + |x|^m + |x'|^m)|x-x'| + |y-y'| + |z-z'|\big)$,

(ii) $|g(t,x,y,z) - g(t,x',y',z')| \leq C\big((1 + |x|^m + |x'|^m)|x-x'| + |y-y'| + |z-z'|\big)$,

(iii) $|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)\,|x-y|$,

$x, x', y, y', z, z' \in \mathbb{R}$.

(A_3) Furthermore, we assume that $f(t,x,0,0) = g(t,x,0,0) = 0$, $(t,x) \in [0,T] \times \mathbb{R}$.

Under (A_1), the G-FSDE (2.9) admits a unique solution $(X_t)_{t\in[0,T]} \in \mathcal{M}_G^2(0,T)$ by Theorem 5.2.2 of Peng (2019), while by Theorem 4.1 in Hu et al. (2014a), under (A_2)-(A_3) the G-BSDE (2.10) admits a unique solution $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{s\in[t,T]}$ such that $(Y_s^{t,x}, Z_s^{t,x})_{s\in[t,T]} \in S_G^{\vartheta}(0,T) \times H_G^{\vartheta}(0,T)$ and $K_s^{t,x} \in L_G^{\vartheta}(\mathcal{F}_T)$, $s \in [t,T]$, for $\varphi(X_T^{t,x}) \in L_G^2(\mathcal{F}_T)$ and $\vartheta \in (1,2)$. Condition (A_3) is needed for the later definition of G-evaluations, however in general it is sufficient to assume that $f(\cdot,x,0,0)$ and $g(\cdot,x,0,0)$ are in $\mathcal{M}_G^2(0,T)$ for all $x \in \mathbb{R}$.

Definition 2.5 *Given* $\xi \in L^2_G(\mathcal{F}_T)$ *and the G-Backward SDE*

$$\begin{cases} dY^{0,x}_t = -f(t, X^{0,x}_t, Y^{0,x}_t, Z^{0,x}_t)dt - g(t, X^{0,x}_t, Y^{0,x}_t, Z^{0,x}_t)d\langle B\rangle_t + Z^{0,x}_t dB_t + dK^{0,x}_t, \ 0 \le t \le T, \\ Y^{0,x}_T = \xi, \end{cases}$$

we respectively call $\mathcal{E}_{\mathcal{G}}[\xi] := Y^{0,x}_0$ *and* $\mathcal{E}_{\mathcal{G}}[\xi \mid \mathcal{F}_t] := Y^{0,x}_t$ *the* $\mathcal{E}_{\mathcal{G}}$*-evaluation and the* $\mathcal{F}_t$*-conditional* $\mathcal{E}_{\mathcal{G}}$*-evaluation of* ξ, $t \in [0, T]$, *where* $\mathcal{G} := \{f, g\}$ *denotes the generators of the G-BSDE.*

Under (A_3) it can be shown that the map $\xi \mapsto \mathcal{E}_{\mathcal{G}}[\xi]$ preserves all properties of the expectation $\mathbb{E}_G$, except for sublinearity and the constant preserving property. Under the (stronger than (A_3)) condition

(A'_3) $f(t, x, y, 0) = g(t, x, y, 0) = 0$ for all $t \in [0, T]$, $x, y \in \mathbb{R}$,

the $\mathcal{E}_{\mathcal{G}}$-evaluation satisfies the property $\mathcal{E}_{\mathcal{G}}[c] = c$ for constant $c \in \mathbb{R}$. The results of Sections 3.1 and 3.2 remain valid for $\mathcal{E}_{\mathcal{G}}$-expectations if we assume (A'_3) instead of (A_3).

Definition 2.6 *Assume (*A_2*)-(*A_3*). For any* $X^{(1)}, X^{(2)} \in L^2_G(\mathcal{F}_T)$*, we say that*

i) $X^{(1)}$ *is dominated by* $X^{(2)}$ *in the monotonic G-ordering* $X^{(1)} \le^{\text{mon}}_{\mathcal{G}_1,\mathcal{G}_2} X^{(2)}$*, if*

$$\mathcal{E}_{\mathcal{G}_1}\big[\varphi\big(X^{(1)}\big)\big] \le \mathcal{E}_{\mathcal{G}_2}\big[\varphi\big(X^{(2)}\big)\big], \tag{2.11}$$

for non-decreasing $\varphi \in \mathcal{C}_{\text{l-Lip}}(\mathbb{R})$*.*

ii) $X^{(1)}$ *is dominated by* $X^{(2)}$ *in the convex G-ordering* $X^{(1)} \le^{\text{conv}}_{\mathcal{G}_1,\mathcal{G}_2} X^{(2)}$*, if* (2.11) *holds for* φ *convex on* $\mathbb{R}$*.*

iii) $X^{(1)}$ *is dominated by* $X^{(2)}$ *in the increasing convex G-ordering* $X^{(1)} \le^{\text{icon}}_{\mathcal{G}_1,\mathcal{G}_2} X^{(2)}$*, if* (2.11) *holds for* φ *non-decreasing convex on* $\mathbb{R}$*.*

2.4 Associated G-PDE

Under (A_1)-(A_2), let $(Y^{t,x}_s)_{s\in[t,T]}$ denote the solution of the G-BSDE (2.10). Then, according to the nonlinear Feynman-Kac formula of Hu et al. (2014b), the function u defined as $u(t, x) := Y^{t,x}_t$, $(t, x) \in [0, T] \times \mathbb{R}$, is the unique viscosity solution of the HJB equation

$$\frac{\partial u}{\partial t}(t, x) + F\left(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), \frac{\partial^2 u}{\partial x^2}(t, x)\right) = 0, \tag{2.12}$$

with $u(T,x) = \varphi(x)$, where

$$F(t,x,y,z,w) := 2G(H(t,x,y,z,w)) + M(t,x,y,z),$$

and

$$H(t,x,y,z,w) := \frac{w}{2}\sigma^2(t,x) + N(t,x,y,z),$$

where

$$M(t,x,y,z) := zb(t,x) + f(t,x,y,z\sigma(t,x)), \quad N(t,x,y,z) := zh(t,x) + g(t,x,y,z\sigma(t,x)). \tag{2.13}$$

We refer to Douglas et al. (1996) for the following definition.

Definition 2.7 *i) Let $\mathcal{C}^{p,q}([0,T]\times\mathbb{R})$ denote the space of functions $v(t,x)$ which are p times continuously differentiable in $t \in [0,T]$ and q times differentiable in $x \in \mathbb{R}$, for $p,q \geq 1$.*

ii) Let $\mathcal{C}_b^{p,q}([0,T]\times\mathbb{R}^n)$ denote the subspace of functions in $\mathcal{C}^{p,q}([0,T]\times\mathbb{R})$ whose partial derivatives with respect to x (resp. t) of orders 1 to q (resp. 1 to p) are bounded on $\mathbb{R}^n$ (resp. on $[0,T]$).

In addition to (A_1)-(A_3), we consider the following conditions.

(A_4) The functions $f(t,x,y,z)$ and $g(t,x,y,z)$ are in $\mathcal{C}_b^{1,2}([0,T]\times\mathbb{R}^3)$ and they satisfy the homogeneity condition

$$f(t,x,y,z) = \delta f(t,x,y/\delta,z/\delta), \quad g(t,x,y,z) = \delta g(t,x,y/\delta,z/\delta), \quad (t,x,y,z) \in [0,T]\times\mathbb{R}^3,$$

for every $\delta > 0$.

(A_5) The functions $f(t,x,y,z)$ and $g(t,x,y,z)$ satisfy the bounds

$$\left|\frac{\partial^2 f}{\partial z^2}(t,x,y,z)\right| \leq \frac{C}{(1+x^2)^{(m+1+\varrho)/2+1}}, \text{ and } \left|\frac{\partial^2 g}{\partial z^2}(t,x,y,z)\right| \leq \frac{C}{(1+x^2)^{(m+1+\varrho)/2+1}},$$

for some positive constants $C > 0$ and $\varrho > 0$, $(t,x) \in [0,T]\times\mathbb{R}$.

The next result provides $\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ solutions of (2.12) with polynomial growth, based on Krylov (1983).

Proposition 2.8 *In addition to (A_1)-(A_4), assume that*

(H_1) *φ is in $\mathcal{C}_b^3(\mathbb{R})$, b, h, σ are in $\mathcal{C}_b^{1,3}([0,T]\times\mathbb{R})$, and f, g are in $\mathcal{C}_b^{1,3}([0,T]\times\mathbb{R}^3)$,*

(H_2) *$\sigma(t,x)$ is lower and upper bounded on $[0,T]\times\mathbb{R}$ by positive constants.*

Then the PDE (2.12) admits a unique solution $u(t,x)\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$, such that $u(t,x)$ and its partial derivatives have polynomial growth of order at most $m+1$ in $x\in\mathbb{R}$, for all $t\in[0,T]$.

Proof. When φ is bounded in $\mathcal{C}_b^3(\mathbb{R})$, the existence of a bounded solution with bounded derivatives in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ is a consequence of Theorem 1.1 in Krylov (1983). If φ has polynomial growth in $\mathcal{C}_b^3(\mathbb{R})$, we apply the above to $\widetilde{\varphi}(x):=\varphi(x)/(1+x^2)^{(m+1)/2}$ by proceeding as in Case 3 of the proof of Theorem C.3.4 pages 193-194 of Peng (2019). □

In Theorem 2.9 we extend the convexity result Theorem 4.1 in Bian and Guan (2008) to HJB equations of the form (2.12) by only assuming convexity in (x,y) and in (y,z) of the function $M(t,x,y,z)$ and $N(t,x,y,z)$ in (2.13), instead of the joint convexity of $(x,y,z)\mapsto M(t,x,y,z)$ and $(x,y,z)\mapsto N(t,x,y,z)$.

Theorem 2.9 *Assume that Conditions (A_1)-(A_5) hold. Suppose that $u(t,x)$ is a $\mathcal{C}^{1,2}([0,T)\times\mathbb{R})$ solution of (2.12) with terminal condition $u(T,x)=\varphi(x)$, together with the conditions*

(H_3) *$(x,y)\mapsto M(t,x,y,z)$ and $(x,y)\mapsto N(t,x,y,z)$ are convex on $\mathbb{R}^2$, $(t,z)\in[0,T]\times\mathbb{R}$,*

(H_4) *$(y,z)\mapsto M(t,x,y,z)$ and $(y,z)\mapsto N(t,x,y,z)$ are convex on $\mathbb{R}^2$, $(t,x)\in[0,T]\times\mathbb{R}$.*

Then the function $x\mapsto u(t,x)$ is convex on $\mathbb{R}$ for all $t\in[0,T]$, provided that $u(T,x)=\varphi(x)$ is convex in $x\in\mathbb{R}$.

Theorem 2.9 is proved in Appendix A and will be used in the proofs of Theorems 3.1-3.2, Corollaries 3.4-3.5, and Theorems 3.7-3.8. The next result is proved in Appendix B.

Proposition 2.10 *Assume that (A_1)-(A_3) hold. If $\varphi(x)$, $f(t,x,y,z)$, and $g(t,x,y,z)$ are non-decreasing in $x\in\mathbb{R}$ for all $t\in[0,T]$ and $y,z\in\mathbb{R}$, then the solution $Y_s^{t,x}$ of (2.10) is q.s. non-decreasing in $x\in\mathbb{R}$ for all $s\in[t,T]$. As a consequence, if $u(t,x)$ solves the backward PDE (2.12), then $u(t,x)$ is also a non-decreasing function of $x\in\mathbb{R}$ for all $t\in[0,T]$.*

3 Comparison in G-stochastic ordering

In this section, under (A_1)-(A_5) we derive comparison results $\mathcal{E}_{\mathcal{G}}$-expectation by considering two systems of G-FBSDEs given by

$$dX_s^{(i)} = b_i\big(s, X_s^{(i)}\big)ds + h_i\big(s, X_s^{(i)}\big)d\langle B\rangle_s + \sigma_i\big(s, X_s^{(i)}\big)dB_s, \qquad X_0^{(i)} = x_0^{(i)}, \tag{3.1}$$

$$\begin{aligned} Y_s^{(i)} &= \varphi\big(X_T^{(i)}\big) + \int_s^T f_i\big(r, X_r^{(i)}, Y_r^{(i)}, Z_r^{(i)}\big)dr + \int_s^T g_i\big(r, X_r^{(i)}, Y_r^{(i)}, Z_r^{(i)}\big)d\langle B\rangle_r \\ &\quad - \int_s^T Z_r^{(i)}dB_r - \big(K_T^{(i)} - K_s^{(i)}\big), \qquad i = 1, 2. \end{aligned} \tag{3.2}$$

3.1 Ordering with convex drifts

In the sequel, we will use the notation $\mathcal{G}_i = \{f_i, g_i\}$, $i = 1, 2$, and the functions

$$\begin{cases} M_i(t, x, y, z) := zb_i(t, x) + f_i(t, x, y, z\sigma_i(t, x)), \quad i = 1, 2, \\ \\ N_i(t, x, y, z) := zh_i(t, x) + g_i(t, x, y, z\sigma_i(t, x)), \quad i = 1, 2. \end{cases}$$

Theorem 3.1 (Convex order). *Assume that $X_0^{(1)} = X_0^{(2)}$ and that $0 < \sigma_1(t, x) \le \sigma_2(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}$, together with the conditions*

(B_1) $M_1(t, x, y, z) \le M_2(t, x, y, z)$, $\quad x, y, z \in \mathbb{R}$, $t \in [0, T]$,

(B_2) $N_1(t, x, y, z) \le N_2(t, x, y, z)$, $\quad x, y, z \in \mathbb{R}$, $t \in [0, T]$,

(B_3) $(x, y, z) \mapsto M_i(t, x, y, z)$ *and* $(x, y, z) \mapsto N_i(t, x, y, z)$ *are both convex in* $(x, y) \in \mathbb{R}^2$ *and in* $(y, z) \in \mathbb{R}^2$, $t \in [0, T]$, $i = 1, 2$.

Then we have $X_T^{(1)} \le_{\mathcal{G}_1, \mathcal{G}_2}^{conv} X_T^{(2)}$, i.e.,

$$\mathcal{E}_{\mathcal{G}_1}\big[\varphi\big(X_T^{(1)}\big)\big] \le \mathcal{E}_{\mathcal{G}_2}\big[\varphi\big(X_T^{(2)}\big)\big],$$

for φ convex on $\mathbb{R}$.

Proof. We start by assuming that (H_1)-(H_2) hold. By Proposition 2.8, the functions $u_1(t, x) := Y_t^{(1),t,x}$ and $u_2(t, x) := Y_t^{(2),t,x}$ are continuous in t and x, and they solve the backward PDEs

$$\frac{\partial u_i}{\partial \tau}(\tau, x) = F_i\left(\tau, x, u_i(\tau, x), \frac{\partial u_i}{\partial x}(\tau, x), \frac{\partial^2 u_i}{\partial x^2}(\tau, x)\right) \text{ with } u_i(0, x) = \varphi(x), \quad i = 1, 2, \tag{3.4}$$

by setting $\tau := T - t$, where $F_i(\tau, x, y, z, w) := 2G(H_i(\tau, x, y, z, w) + M_i(\tau, x, y, z))$ and

$$H_i(\tau, x, y, z, w) := \frac{w}{2}\sigma_i^2(\tau, x) + N_i(\tau, x, y, z). \tag{3.5}$$

In addition, under (B_3), both solutions $u_1(t, x)$ and $u_2(t, x)$ of (3.4) are convex functions of x by Theorem 2.9, hence we have $\dfrac{\partial^2 u_i}{\partial x^2}(\tau, x) \geq 0$, $(\tau, x) \in [0, T] \times \mathbb{R}$. Therefore, in (3.4) we can replace $H_i(\tau, x, y, z, w)$ with

$$\widetilde{H}_i(\tau, x, y, z, w) := N_i(\tau, x, y, z) + \frac{1}{2}w^+\sigma_i^2(t, x), \qquad i = 1, 2, \tag{3.6}$$

where $w^+ = \max(w, 0)$, and rewrite the backward PDEs (3.4) as

$$\frac{\partial u_i}{\partial \tau}(\tau, x) = \widetilde{F}_i\left(\tau, x, u_i(\tau, x), \frac{\partial u_i}{\partial x}(\tau, x), \frac{\partial^2 u_i}{\partial x^2}(\tau, x)\right) \text{ with } u_i(0, x) = \varphi(x), \quad i = 1, 2.$$

where

$$\widetilde{F}_i(t, x, y, z, w) := 2G\big(\widetilde{H}_i(t, x, y, z, w)\big) + M_i(t, x, y, z).$$

By the conditions $0 < \sigma_1(t, x) \leq \sigma_2(t, x)$, and ($B_1$)-($B_2$), we check that

$$\begin{aligned}
&\widetilde{F}_2(\tau, x, y, z, w) - \widetilde{F}_1(\tau, x, y, z, w)\\
&\quad \geq \underline{\sigma}^2\big(N_2(\tau, x, y, z) - N_1(\tau, x, y, z)\big) + \frac{\underline{\sigma}^2}{2}w^+\big(\sigma_2^2(\tau, x) - \sigma_1^2(\tau, x)\big)\\
&\qquad + M_2(\tau, x, y, z) - M_1(\tau, x, y, z) \geq 0, \qquad x, y, z, w \in \mathbb{R}, \quad \tau \in [0, T].
\end{aligned}$$

Moreover, by Conditions (A_1) and (H_2) the coefficients b_i, h_i and σ_i, $i = 1, 2$ are bounded, hence we have

$$\begin{aligned}
&\big|\widetilde{F}_i(\tau, x, y_1, z_1, w_1) - \widetilde{F}_i(\tau, x, y_2, z_2, w_2)\big| \leq \frac{\overline{\sigma}^2}{2}\sigma_i^2(\tau, x)|w_1 - w_2|\\
&\quad + \overline{\sigma}^2|N_i(\tau, x, y_1, z_1) - N_i(\tau, x, y_2, z_2)| + |M_i(\tau, x, y_1, z_1) - M_i(\tau, x, y_2, z_2)|\\
&\leq C'(|y_1 - y_2| + |z_1 - z_2| + |w_1 - w_2|), \quad (\tau, x) \in [0, T] \times \mathbb{R}, \ (y_1, z_1, w_1), \ (y_2, z_2, w_2) \in \mathbb{R}^3,
\end{aligned}$$

hence $\widetilde{F}_i(t, x, y, z, w)$ is Lipschitz in $(y, z, w) \in \mathbb{R}^2$, uniformly in $(\tau, x) \in [0, T] \times \mathbb{R}$. In addition, $\widetilde{F}_i(\tau, x, y, z, w)$, $i = 1, 2$ is positive homogeneous in (y, z, w) by Condition (A_4) and pages 188-189 of Peng (2019), and $\widetilde{F}_i(\tau, x, y, z, w)$ satisfies Condition (G') therein. Therefore, by the Comparison Theorem C.2.5 under (G'), see page 188 of Peng (2019), it follows that $u_1(t, x) \leq u_2(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}$, from which we conclude to

$$Y_0^{(1)} = u_1\big(0, X_0^{(1)}\big) \leq Y_0^{(2)} = u_2\big(0, X_0^{(1)}\big) = u_2\big(0, X_0^{(2)}\big),$$

hence we have $\mathcal{E}_{\mathcal{G}_1}\big[\varphi(X_T^{(1))}\big] \le \mathcal{E}_{\mathcal{G}_2}\big[\varphi\big(X_T^{(2)}\big)\big]$ for convex $\varphi \in \mathcal{C}_b^3(\mathbb{R})$. In order to relax Condition (H_1) we apply Theorem C.1 under Conditions (A_1)-(A_2) after regularizing the functions φ, b, h, σ, f, g, as in e.g. Problem 1.4.14 in Zhang (2017). For example, when mollifying φ into φ_n by convolution with $\rho_n(y) := n\rho(ny)$ where ρ is a mollifier on $\mathbb{R}$, $n \ge 1$, and using the representation (2.7), Condition (A_2) and Proposition 4.1 in Hu et al. (2014b), for $p \ge 1$ we have

$$\begin{aligned}
\mathbb{E}_G[|\varphi(X_T) - \varphi_n(X_T)|^p] &= \sup_{P\in\mathcal{P}} \mathbb{E}_P\left[\left|\int_{-\infty}^{\infty}(\varphi(X_T) - \varphi(X_T - y))\rho_n(y)dy\right|^p\right] \\
&\le \sup_{P\in\mathcal{P}} \int_{-\infty}^{\infty} \mathbb{E}_P\big[|\varphi(X_T) - \varphi(X_T - y/n)|^p\big]\rho(y)dy \\
&\le \frac{C}{n^p}\int_{-\infty}^{\infty} \sup_{P\in\mathcal{P}} \mathbb{E}_P\big[(1 + |X_T|^m + |X_T - y/n|^m)^p\big]|y|^p\rho(y)dy \\
&\le 2^{p-1}\frac{C'}{n^p}\int_{-\infty}^{\infty}\big(1 + |x_0|^{mp} + 2^{(m-1)p}(|x_0|^{mp} + |y/n|^{mp})\big)|y|^p\rho(y)dy,
\end{aligned}$$

which tends to zero as n tends to infinity. Finally, to relax Condition (H_2) using Theorem C.1, we approximate $\sigma(t,x)$ by $\sigma_n(t,x) := \min(\max(\sigma(t,x),1/n),n)$, $n \ge 1$, and note that by sublinearity we have

$$\mathbb{E}_G\big[|\sigma(t,X_t) - \sigma_n(t,X_t)|^{2p}\big] \le \frac{1}{n^{2p}} + \mathbb{E}_G[(|\sigma(t,X_t)|^{2p} - n)^+], \quad t \in [0,T],$$

which tends to zero as n tends to infinity by Proposition 6.1.22 of Peng (2019), since $\sigma(t,X_t) \in L_b^{2p}$ therein for $p \ge 1$, as by (A_1) we have

$$\begin{aligned}
&\mathbb{E}_G\big[|\sigma(t,X_t)|^{2p}\mathbf{1}_{\{|\sigma(t,X_t)|^{2p}>n\}}\big] \le C\mathbb{E}_G\big[(1 + |X_t|^{2p})\mathbf{1}_{\{(1+|X_t|^{2p})>n\}}\big] \\
&\le C\mathbb{E}_G\big[\mathbf{1}_{\{(1+|X_t|^{2p})>n\}}\big] + C\big(\mathbb{E}_G\big[|X_t|^{4p}\big]\big)^{1/2}\big(\mathbb{E}_G\big[\mathbf{1}_{\{(1+|X_t|^{2p})>n\}}\big]\big)^{1/2} \\
&\le C\mathbb{E}_G\big[\mathbf{1}_{\{(1+|X_t|^{2p})>n\}}\big] + C'\big(1 + x_0^{4p}\big)^{1/2}\big(\mathbb{E}_G\big[\mathbf{1}_{\{(1+|X_t|^{2p})>n\}}\big]\big)^{1/2}
\end{aligned}$$

which tends to 0 as n tends to infinity by Proposition 4.1 in Hu et al. (2014b). □

Theorem 3.2 (Increasing convex order). *Assume that $X_0^{(1)} \le X_0^{(2)}$ and $0 < \sigma_1(t,x) \le \sigma_2(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}$, together with the conditions*

(B_1') *$M_1(t,x,y,z) \le M_2(t,x,y,z)$, $t \in [0,T]$, $x,y \in \mathbb{R}$, $z \in \mathbb{R}_+$,*

(B_2') *$N_1(t,x,y,z) \le N_2(t,x,y,z)$, $t \in [0,T]$, $x,y \in \mathbb{R}$, $z \in \mathbb{R}_+$,*

(B_3') *$(x,y,z) \mapsto M_i(t,x,y,z)$ and $(x,y,z) \mapsto N_i(t,x,y,z)$ are both convex in $(x,y) \in \mathbb{R}^2$ and in $(y,z) \in \mathbb{R} \times \mathbb{R}_+$, for $i = 1,2$, $t \in [0,T]$,*

(B_4') *$x \mapsto f_i(t,x,y,z)$ and $x \mapsto g_i(t,x,y,z)$ are non-decreasing on $\mathbb{R}$, for $i=1,2$, $y \in \mathbb{R}$, $z \in \mathbb{R}_+$, $t \in [0,T]$.*

Then we have $X_T^{(1)} \leq^{\rm icon}_{\mathcal{G}_1,\mathcal{G}_2} X_T^{(2)}$, i.e., $\mathcal{E}_{\mathcal{G}_1}\big[\varphi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{\mathcal{G}_2}\big[\varphi\big(X_T^{(2)}\big)\big]$, for non-decreasing convex φ on $\mathbb{R}$.

Proof. Under (B_4'), when $\varphi(x)$, $f_i(t,x,y,z)$ and $g_i(t,x,y,z)$, $i=1,2$, are non-decreasing in x, Proposition 2.10 states that the PDE solutions $u_1(t,x)$ and $u_2(t,x)$satisfy

$$\frac{\partial u_1}{\partial x}(t,x) \geq 0 \quad \text{and} \quad \frac{\partial u_2}{\partial x}(t,x) \geq 0, \qquad (t,x) \in [0,T] \times \mathbb{R},$$

hence Conditions (B_1)-(B_3) only need to hold for $z \geq 0$, and the conclusion follows by repeating the arguments in the proof of Theorem 3.1. □

We note that in case $\sigma_1(t,x) = \sigma_2(t,x)$, the convexity of $u_i(t,x)$, $i=1,2$, is no longer required in the proofs of Theorems 3.1-3.2, and one can then remove Condition (B_3') to obtain a result for the monotonic order.

Corollary 3.3 (Monotonic order with equal volatilities). *Assume that $X_0^{(1)} \leq X_0^{(2)}$ and $0 < \sigma(t,x) := \sigma_1(t,x) = \sigma_2(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}$, together with the conditions*

(B_1'') *$M_1(t,x,y,z) \leq M_2(t,x,y,z)$, $t \in [0,T]$, $x,y \in \mathbb{R}$, $z \in \mathbb{R}_+$,*

(B_2'') *$N_1(t,x,y,z) \leq N_2(t,x,y,z)$, $t \in [0,T]$, $x,y \in \mathbb{R}$, $z \in \mathbb{R}_+$,*

(B_3'') *$x \mapsto f_i(t,x,y,z)$ and $x \mapsto g_i(t,x,y,z)$ are non-decreasing on $\mathbb{R}$, for $i=1,2$, $y \in \mathbb{R}$, $z \in \mathbb{R}_+$, $t \in [0,T]$.*

Then we have $X_T^{(1)} \leq^{\rm mon}_{\mathcal{G}_1,\mathcal{G}_2} X_T^{(2)}$, i.e., $\mathcal{E}_{\mathcal{G}_1}\big[\varphi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{\mathcal{G}_2}\big[\varphi\big(X_T^{(2)}\big)\big]$, for non-decreasing φ on $\mathbb{R}$.

Proof. When $\sigma_1(t,x) = \sigma_2(t,x)$, we can repeat the proof of Theorem 3.1 by using H_i in (3.5), without defining $\widetilde{H}_i$ in (3.6) and without assuming (B_3), and then follow the proof argument of Theorem 3.2 without requiring the convexity of $u_i(t,x)$, $i=1,2$. □

Theorem 3.4 (Increasing convex order). *Assume that $X_0^{(1)} \leq X_0^{(2)}$ and*

$$b_1(t,x) \leq b_2(t,x), \quad h_1(t,x) \leq h_2(t,x) \quad \text{and} \quad 0 < \sigma_1(t,x) \leq \sigma_2(t,x), \quad (t,x) \in [0,T] \times \mathbb{R},$$

together with the conditions

(C_1) $f_1(t,x,y,z) \leq f_2(t,x,y,z)$, $t \in [0,T]$, $x,y \in \mathbb{R}$, $z \in \mathbb{R}_+$,

(C_2) $g_1(t,x,y,z) \leq g_2(t,x,y,z)$, $t \in [0,T]$, $x,y \in \mathbb{R}$, $z \in \mathbb{R}_+$,

(C_3) $z \mapsto f_i(t,x,y,z)$ *and* $z \mapsto g_i(t,x,y,z)$ *are both non-decreasing on* $\mathbb{R}_+$, *for* $i = 1$ *or* $i = 2$, $x,y \in \mathbb{R}$, $t \in [0,T]$,

(C_4) $x \mapsto f_i(t,x,y,z)$ *and* $x \mapsto g_i(t,x,y,z)$ *are both non-decreasing on* $\mathbb{R}$, *for* $i = 1,2$, $x,y \in \mathbb{R}$, $z \in \mathbb{R}_+$, $t \in [0,T]$,

(C_5) $(x,y,z) \mapsto M_i(t,x,y,z)$ *and* $(x,y,z) \mapsto N_i(t,x,y,z)$ *are both convex in* $(x,y) \in \mathbb{R}^2$ *and in* $(y,z) \in \mathbb{R} \times \mathbb{R}_+$, *for* $i = 1,2$, $x,y \in \mathbb{R}$, $(t,z) \in [0,T] \times \mathbb{R}_+$.

Then we have $X_T^{(1)} \leq^{\mathrm{icon}}_{\mathcal{G}_1,\mathcal{G}_2} X_T^{(2)}$, *i.e.,* $\mathcal{E}_{\mathcal{G}_1}\big[\varphi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{\mathcal{G}_2}\big[\varphi\big(X_T^{(2)}\big)\big]$, *for non-decreasing convex* φ *on* $\mathbb{R}$.

Proof. Under (C_4), since $\varphi(x)$, $f_i(t,x,y,z)$ and $g_i(t,x,y,z)$, $i = 1,2$, are non-decreasing in x, by Proposition 2.10 the solutions $u_1(t,x)$ and $u_2(t,x)$ of (3.4) are non-decreasing in x and, as in the proof of Theorem 3.2, one can take $z \geq 0$ since $\dfrac{\partial u_i}{\partial x}(t,x) \geq 0$. Assuming that e.g. $f_1(t,x,y,z)$ and $g_1(t,x,y,z)$ are non-decreasing in z under (C_3), then by $z\sigma_1(t,x) \leq z\sigma_2(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}$, $z \in \mathbb{R}_+$, and (C_1)-(C_2), we have

$$f_1(t,x,y,z\sigma_1(t,x)) \leq f_1(t,x,y,z\sigma_2(t,x)) \leq f_2(t,x,y,z\sigma_2(t,x)),$$

and

$$g_1(t,x,y,z\sigma_1(t,x)) \leq g_1(t,x,y,z\sigma_2(t,x)) \leq g_2(t,x,y,z\sigma_2(t,x)).$$

Combining the above with the inequality $zb_1(t,x) \leq zb_2(t,x)$ and $zh_1(t,x) \leq zh_2(t,x)$, for $(t,x) \in [0,T] \times \mathbb{R}$, $z \in \mathbb{R}_+$, one finds

$$M_1(t,x,y,z) \leq M_2(t,x,y,z) \quad \text{and} \quad N_1(t,x,y,z) \leq N_2(t,x,y,z),$$

and by Theorem 3.2 we conclude that $\mathcal{E}_{\mathcal{G}_1}\big[\varphi(X_T^{(1))}\big] \leq \mathcal{E}_{\mathcal{G}_2}\big[\varphi\big(X_T^{(2)}\big)\big]$, for non-decreasing convex φ on $\mathbb{R}$. □

When the drift coefficients, $b(t,x) := b_1(t,x) = b_2(t,x)$ and $h(t,x) := h_1(t,x) = h_2(t,x)$, are equal and $f_i(t,x,y,z)$ and $g_i(t,x,y,z)$ are independent of z, $i = 1,2$, the following corollary can be proved for the convex G-ordering similarly to Theorem 3.4, by applying Theorem 3.1 which deals with convex ordering, instead of Theorem 3.2.

Corollary 3.5 (Convex order with equal drifts). *Assume that* $X_0^{(1)} = X_0^{(2)}$ *and*

$$b_1(t,x) = b_2(t,x),\ h_1(t,x) = h_2(t,x),\ \text{and } 0 < \sigma_1(t,x) \le \sigma_2(t,x),\ (t,x) \in [0,T] \times \mathbb{R},$$

together with the conditions

(C_1') $f_i(t,x,y,z) = f_i(t,x,y)$ *and* $g_i(t,x,y,z) = g_i(t,x,y)$ *are independent of* $z \in \mathbb{R}$, $t \in [0,T]$, $x,y \in \mathbb{R}$, $i = 1,2$,

(C_2') $f_1(t,x,y) \le f_2(t,x,y)$ *and* $g_1(t,x,y) \le g_2(t,x,y)$, $t \in [0,T]$, $x,y \in \mathbb{R}$,

(C_3') $(x,y,z) \mapsto M_i(t,x,y,z)$ *and* $(x,y,z) \mapsto N_i(t,x,y,z)$ *are both convex in* $(x,y) \in \mathbb{R}^2$ *and in* $(y,z) \in \mathbb{R}^2$, $t \in [0,T]$, $i = 1,2$.

Then we have $X_T^{(1)} \le^{\text{conv}}_{\mathcal{G}_1,\mathcal{G}_2} X_T^{(2)}$, *i.e.* $\mathcal{E}_{\mathcal{G}_1}\big[\varphi\big(X_T^{(1)}\big)\big] \le \mathcal{E}_{\mathcal{G}_2}\big[\varphi\big(X_T^{(2)}\big)\big]$, *for non-decreasing convex* φ *on* $\mathbb{R}$.

We note that the convexity of $u_1(t,x)$ and $u_2(t,x)$ is not needed in the proof of Theorem 3.1 when $\sigma_1(t,x) = \sigma_2(t,x)$, and in this case we can remove Condition (B_3') in Theorem 3.2 as in the next corollary.

Corollary 3.6 (Monotonic order with equal volatilities). *Assume that* $X_0^{(1)} \le X_0^{(2)}$ *and* $0 < \sigma(t,x) := \sigma_1(t,x) = \sigma_2(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}$, *together with the conditions*

(D_1) $b_1(t,x) \le b_2(t,x)$ *and* $h_1(t,x) \le h_2(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}$,

(D_2) $f_1(t,x,y,z) \le f_2(t,x,y,z)$ *and* $g_1(t,x,y,z) \le g_2(t,x,y,z)$, *for all* $x,y \in \mathbb{R}$ *and* $z \in \mathbb{R}_+$, $t \in [0,T]$,

(D_3) $x \mapsto f_i(t,x,y,z)$ *and* $x \mapsto g_i(t,x,y,z)$ *are non-decreasing on* $\mathbb{R}$, $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}_+$, $i = 1,2$.

Then we have $X_T^{(1)} \le^{\text{mon}}_{\mathcal{G}_1,\mathcal{G}_2} X_T^{(2)}$, *i.e.,* $\mathcal{E}_{\mathcal{G}_1}\big[\varphi\big(X_T^{(1)}\big)\big] \le \mathcal{E}_{\mathcal{G}_2}\big[\varphi\big(X_T^{(2)}\big)\big]$, *for non-decreasing* φ *on* $\mathbb{R}$.

Proof. Similarly to the proof of Corollary 3.3, under the condition $\sigma_1(t,x) = \sigma_2(t,x)$ the convexity of $u_i(t,x)$ and the non-decreasing property of $f_i(t,x,y,z)$ and $g_i(t,x,y,z)$ with respect to z, $i = 1$ or $i = 2$, are no longer required. In addition, the conditions

$$M_1(t,x,y,z) \le M_2(t,x,y,z),\ N_1(t,x,y,z) \le N_2(t,x,y,z),\ x,y \in \mathbb{R},\ z \in \mathbb{R}_+,\ t \in [0,T],$$

clearly hold from (D_1)-(D_2), and we can conclude as in the proof of Theorem 3.4. □

3.2 Ordering with partially convex drifts

Theorems 3.1 and 3.2 require the convexity assumptions (B_2) and (B_2') on

$$(t,x,y,z) \mapsto M_i(t,x,y,z) := zb_i(t,x) + f_i(t,x,y,z\sigma_i(t,x))$$

and

$$(t,x,y,z) \mapsto N_i(t,x,y,z) := zh_i(t,x) + g_i(t,x,y,z\sigma_i(t,x))$$

in (x,y) and (y,z) to hold for both $i=1,2$. In this section, we develop different convex G-ordering results under weaker convexity conditions, based on a measurable functions $\zeta(t,x)$ and $\eta(t,x)$ such that

$$\theta_i\big(t,X_t^{(i)}\big) := \frac{b_i\big(t,X_t^{(i)}\big) - \zeta\big(t,X_t^{(i)}\big)}{\sigma_i\big(t,X_t^{(i)}\big)} \quad \text{and} \quad \lambda_i\big(t,X_t^{(i)}\big) := \frac{h_i\big(t,X_t^{(i)}\big) - \eta\big(t,X_t^{(i)}\big)}{\sigma_i\big(t,X_t^{(i)}\big)}, \quad i=1,2,$$

are bounded processes.

Theorem 3.7 (Convex order). *Assume that $X_0^{(1)} = X_0^{(2)}$ and $0 < \sigma_1(t,x) \leq \sigma_2(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}$, together with the conditions*

(E_1) $M_1(t,x,y,z) \leq z\zeta(t,x) \leq M_2(t,x,y,z)$, *$t \in [0,T]$, $x,y,z \in \mathbb{R}$,*

(E_2) $N_1(t,x,y,z) \leq z\eta(t,x) \leq N_2(t,x,y,z)$, *$t \in [0,T]$, $x,y,z \in \mathbb{R}$,*

(E_3) *$(x,y,z) \mapsto M_i(t,x,y,z)$ and $(x,y,z) \mapsto N_i(t,x,y,z)$ are convex in (x,y) and in $(y,z) \in \mathbb{R}^2$ for $i=1$ or $i=2$, $t \in [0,T]$.*

Then we have $X_T^{(1)} \leq^{\text{conv}}_{\mathcal{G}_1,\mathcal{G}_2} X_T^{(2)}$, i.e.,

$$\mathcal{E}_{\mathcal{G}_1}\big[\varphi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{\mathcal{G}_2}\big[\varphi\big(X_T^{(2)}\big)\big],$$

for non-decreasing convex φ on $\mathbb{R}$.

Proof. (i) We assume that (E_3) holds with $i=1$. Let

$$\theta_2(t,x) := \frac{b_2(t,x) - \zeta(t,x)}{\sigma_2(t,x)} \quad \text{and} \quad \lambda_2(t,x) := \frac{h_2(t,x) - \eta(t,x)}{\sigma_2(t,x)}, \quad (t,x) \in [0,T] \times \mathbb{R}.$$

are bounded functions. By the G-Girsanov transformation, see e.g. Theorems 3.2 and 5.2 in Hu et al. (2014b), the process

$$\widetilde{B}_t^G := B_t + \int_0^t \theta_2\big(s,X_s^{(2)}\big)ds + \int_0^t \lambda_2\big(s,X_s^{(2)}\big)d\langle B\rangle_s, \qquad t \in [0,T],$$

is a G-Brownian motion under the G-expectation $\widetilde{\mathbb{E}}_2$ defined by

$$\widetilde{\mathbb{E}}_2[X] := \widetilde{\mathbb{E}}_G\bigg[X\exp\Big(-\int_0^T \lambda_2\big(s,X_s^{(2)}\big)dB_s - \frac{1}{2}\int_0^T \big(\lambda_2\big(s,X_s^{(2)}\big)\big)^2 d\langle B\rangle_s$$
$$-\int_0^T \theta_2\big(s,X_s^{(2)}\big)\lambda_2\big(s,X_s^{(2)}\big)ds - \int_0^T \theta_2\big(s,X_s^{(2)}\big)d\widetilde{B}_s - \frac{1}{2}\int_0^T \big(\theta_2\big(s,X_s^{(2)}\big)\big)^2 d\langle\widetilde{B}\rangle_s\Big)\bigg],$$

where $\big(B,\widetilde{B}\big)$ is an auxiliary extended $\widetilde{G}$-Brownian motion, and

$$\widetilde{G}(A) := \frac{1}{2}\sup_{\underline{\sigma}^2\le\nu\le\overline{\sigma}^2} \operatorname{tr}\left[A\begin{pmatrix}\nu & 1\\ 1 & \nu^{-1}\end{pmatrix}\right],$$

for A in the set $\mathbb{S}_2$ of 2×2 symmetric matrices. The forward SDEs (3.1) can be rewritten as

$$\begin{cases} dX_t^{(1)} &= \big(b_1\big(t,X_t^{(1)}\big) - \theta_2\big(t,X_t^{(2)}\big)\sigma_1\big(t,X_t^{(1)}\big)\big)dt \\ &\quad + \big(h_1\big(t,X_t^{(1)}\big) - \lambda_2\big(t,X_t^{(2)}\big)\sigma_1\big(t,X_t^{(1)}\big)\big)d\langle B\rangle_t + \sigma_1\big(t,X_t^{(1)}\big)d\widetilde{B}_t^G, \\ dX_t^{(2)} &= \zeta\big(t,X_t^{(2)}\big)dt + \eta\big(t,X_t^{(2)}\big)d\langle B\rangle_t + \sigma_2\big(t,X_t^{(2)}\big)d\widetilde{B}_t^G, \end{cases}$$

with the associated BSDEs

$$\begin{cases} dY_t^{(1)} = -\big(f_1\big(t,X_t^{(1)},Y_t^{(1)},Z_t^{(1)}\big) + Z_t^{(1)}\theta_2\big(t,X_t^{(2)}\big)\big)dt \\ \qquad - \big(g_1\big(t,X_t^{(1)},Y_t^{(1)},Z_t^{(1)}\big) + Z_t^{(1)}\lambda_2\big(t,X_t^{(2)}\big)\big)d\langle B\rangle_t + Z_t^{(1)}d\widetilde{B}_t^G + dK_t^{(1)}, \\ dY_t^{(2)} = -\big(f_2\big(t,X_t^{(2)},Y_t^{(2)},Z_t^{(2)}\big) + Z_t^{(2)}\theta_2\big(t,X_t^{(2)}\big)\big)dt \\ \qquad - \big(g_2\big(t,X_t^{(2)},Y_t^{(2)},Z_t^{(2)}\big) + Z_t^{(2)}\lambda_2\big(t,X_t^{(2)}\big)\big)d\langle B\rangle_t + Z_t^{(2)}d\widetilde{B}_t^G + dK_t^{(2)}, \end{cases} \tag{3.7a}$$

where $Y_T^{(1)} = \varphi\big(X_T^{(1)}\big)$ and $Y_T^{(2)} = \varphi\big(X_T^{(2)}\big)$ at terminal time T. By Proposition 2.8 we have $Y_t^{(1)} = u_1\big(t,X_t^{(1)}\big)$ and $Y_t^{(2)} = u_2\big(t,X_t^{(2)}\big)$, where the functions $u_1(t,x)$ and $u_2(t,x)$ are in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ and solve the PDEs

$$\frac{\partial u_i}{\partial t}(t,x) + 2G\Big(\frac{1}{2}\sigma_i^2(t,x)\frac{\partial^2 u_i}{\partial x^2}(t,x) + N_i\Big(t,x,u_i(t,x),\frac{\partial u_i}{\partial x}(t,x)\Big)\Big)$$
$$+ M_i\Big(t,x,u_i(t,x),\frac{\partial u_i}{\partial x}(t,x)\Big) = 0, \tag{3.8}$$

with $u_i(T,x) = \varphi(x)$, $i = 1,2$ and

$$G(a) = \frac{1}{2}\sup_{\underline{\sigma}^2\le\sigma^2\le\overline{\sigma}^2}(\sigma^2 a) = \frac{1}{2}\big(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-\big), \qquad a\in\mathbb{R}.$$

Applying the G-Itô formula to $u_1\big(t,X_t^{(2)}\big)$ and using (3.8), we have

$$u_1\big(t,X_t^{(2)}\big) = u_1\big(0,X_0^{(2)}\big) + \int_0^t \frac{\partial u_1}{\partial s}\big(s,X_s^{(2)}\big)ds + \int_0^t \zeta\big(s,X_s^{(2)}\big)\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)ds$$

$$
\begin{aligned}
&+\int_0^t \left(\eta\big(s,X_s^{(2)}\big)\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)+\frac{1}{2}\sigma_2^2\big(s,X_s^{(2)}\big)\frac{\partial^2 u_1}{\partial x^2}\big(s,X_s^{(2)}\big)\right)d\langle B\rangle_s+\\
&+\int_0^t \sigma_2\big(s,X_s^{(2)}\big)\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)d\widetilde{B}_s^G\\
&=u_1\big(0,X_0^{(2)}\big)+\int_0^t \zeta\big(s,X_s^{(2)}\big)\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)ds-\int_0^t M_1\Big(s,X_s^{(2)},u_1\big(s,X_s^{(2)}\big),\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)\Big)ds\\
&-2\int_0^t G\Big(\frac{1}{2}\sigma_1^2\big(s,X_s^{(2)}\big)\frac{\partial^2 u_1}{\partial x^2}\big(s,X_s^{(2)}\big)+N_1\Big(s,X_s^{(2)},u_1\big(s,X_s^{(2)}\big),\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)\Big)\Big)ds\\
&+\int_0^t \left(\eta(s,X_s^{(2)})\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)+\frac{1}{2}\sigma_2^2\big(s,X_s^{(2)}\big)\frac{\partial^2 u_1}{\partial x^2}\big(s,X_s^{(2)}\big)\right)d\langle B\rangle_s\\
&+\int_0^t \sigma_2\big(s,X_s^{(2)}\big)\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)d\widetilde{B}_s^G\\
&\geq u_1\big(0,X_0^{(2)}\big)-2\int_0^t G\Big(\frac{1}{2}\sigma_1^2\big(s,X_s^{(2)}\big)\frac{\partial^2 u_1}{\partial x^2}\big(s,X_s^{(2)}\big)+N_1\Big(s,X_s^{(2)},u_1\big(s,X_s^{(2)}\big),\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)\Big)\Big)ds\\
&+\int_0^t \left(\eta(s,X_s^{(2)})\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)+\frac{1}{2}\sigma_2^2\big(s,X_s^{(2)}\big)\frac{\partial^2 u_1}{\partial x^2}\big(s,X_s^{(2)}\big)\right)d\langle B\rangle_s\\
&+\int_0^t \sigma_2\big(s,X_s^{(2)}\big)\frac{\partial u_1}{\partial x}\big(s,X_s^{(2)}\big)d\widetilde{B}_s^G,
\end{aligned}
$$

where we used the first part of (E_2). Since $u_2(t,x)$ is a solution of the equation (3.8), we know that $Y_s^{(2)}=u_2\big(s,X_s^{(2)}\big)$ and $Z_s^{(2)}=\sigma_2\big(s,X_s^{(2)}\big)\frac{\partial u_2}{\partial x}\big(s,X_s^{(2)}\big)$, hence, plugging those terms in the second G-BSDE (3.7a) we find

$$
\begin{aligned}
u_2\big(t,X_t^{(2)}\big)&=u_2\big(0,X_0^{(2)}\big)-\int_0^t f_2\left(s,X_s^{(2)},u_2\big(s,X_s^{(2)}\big),\sigma_2\big(s,X_s^{(2)}\big)\frac{\partial u_2}{\partial x}\big(s,X_s^{(2)}\big)\right)ds\\
&-\int_0^t \sigma_2\big(s,X_s^{(2)}\big)\frac{\partial u_2}{\partial x}\big(s,X_s^{(2)}\big)\theta_2\big(s,X_s^{(2)}\big)\big)ds\\
&-\int_0^t g_2\left(s,X_s^{(2)},u_2\big(s,X_s^{(2)}\big),\sigma_2\big(s,X_s^{(2)}\big)\frac{\partial u_2}{\partial x}\big(s,X_s^{(2)}\big)\right)d\langle B\rangle_s\\
&-\int_0^t \sigma_2\big(s,X_s^{(2)}\big)\frac{\partial u_2}{\partial x}\big(s,X_s^{(2)}\big)\lambda_2\big(s,X_s^{(2)}\big)\big)d\langle B\rangle_s+\int_0^t \sigma_2\big(s,X_s^{(2)}\big)\frac{\partial u_2}{\partial x}\big(s,X_s^{(2)}\big)d\widetilde{B}_s^G+K_t^{(2)}\\
&=u_2\big(0,X_0^{(2)}\big)+\int_0^t\left(\zeta(s,X_s^{(2)})\frac{\partial u_2}{\partial x}(s,X_s^{(2)})-M_2\left(s,X_s^{(2)},u_2(s,X_s^{(2)}),\frac{\partial u_2}{\partial x}(s,X_s^{(2)})\right)\right)ds\\
&+\int_0^t\left(\eta(s,X_s^{(2)})\frac{\partial u_2}{\partial x}(s,X_s^{(2)})-N_2\Big(s,X_s^{(2)},u_2\big(s,X_s^{(2)}\big),\frac{\partial u_2}{\partial x}\big(s,X_s^{(2)}\big)\Big)\right)d\langle B\rangle_s\\
&+\int_0^t \sigma_2\big(s,X_s^{(2)}\big)\frac{\partial u_2}{\partial x}\big(s,X_s^{(2)}\big)d\widetilde{B}_s^G+K_t^{(2)}.
\end{aligned}
$$

Next, given that $u_1\big(T,X_T^{(2)}\big)=u_2\big(T,X_T^{(2)}\big)=\varphi\big(X_T^{(2)}\big)$ a.s. at the terminal time T, we have

$$
u_2\big(0,X_0^{(2)}\big)-u_1\big(0,X_0^{(2)}\big)\geq\int_0^T\left(M_2\Big(s,X_s^{(2)},u_2(s,X_s^{(2)}),\frac{\partial u_2}{\partial x}(s,X_s^{(2)})\Big)-\zeta(s,X_s^{(2)})\frac{\partial u_2}{\partial x}(s,X_s^{(2)})\right)ds
$$

$$
\begin{aligned}
&+\int_0^T \left(N_2\big(s, X_s^{(2)}, u_2\big(s, X_s^{(2)}\big), \frac{\partial u_2}{\partial x}\big(s, X_s^{(2)}\big)\big) - \eta(s, X_s^{(2)})\frac{\partial u_2}{\partial x}(s, X_s^{(2)}) \right) d\langle B\rangle_s \\
&-2\int_0^T G\left(\frac{1}{2}\sigma_1^2\big(s, X_s^{(2)}\big)\frac{\partial^2 u_1}{\partial x^2}\big(s, X_s^{(2)}\big) + N_1\Big(s, X_s^{(2)}, u_1\big(s, X_s^{(2)}\big), \frac{\partial u_1}{\partial x}\big(s, X_s^{(2)}\big)\Big) \right) ds \\
&+\int_0^t \left(\eta(s, X_s^{(2)})\frac{\partial u_1}{\partial x}(s, X_s^{(2)}) + \frac{1}{2}\sigma_2^2(s, X_s^{(2)})\frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) \right) d\langle B\rangle_s \\
&+\int_0^T \sigma_1\big(s, X_s^{(2)}\big)\frac{\partial u_1}{\partial x}\big(s, X_s^{(2)}\big) d\widetilde{B}_s^G - \int_0^T \sigma_2\big(s, X_s^{(2)}\big)\frac{\partial u_2}{\partial x}\big(s, X_s^{(2)}\big) d\widetilde{B}_s^G - K_T^{(2)} \\
\geq &-2\int_0^T G\left(\frac{1}{2}\sigma_1^2\big(s, X_s^{(2)}\big)\frac{\partial^2 u_1}{\partial x^2}\big(s, X_s^{(2)}\big) + N_1\Big(s, X_s^{(2)}, u_1\big(s, X_s^{(2)}\big), \frac{\partial u_1}{\partial x}\big(s, X_s^{(2)}\big)\Big) \right) ds \\
&+\int_0^t \left(N_1\Big(s, X_s^{(2)}, u_1\big(s, X_s^{(2)}\big), \frac{\partial u_1}{\partial x}\big(s, X_s^{(2)}\big)\Big) + \frac{1}{2}\sigma_1^2(s, X_s^{(2)})\frac{\partial^2 u_1}{\partial x^2}(s, X_s^{(2)}) \right) d\langle B\rangle_s \\
&+\int_0^T \sigma_1(s, X_s^{(2)})\frac{\partial u_1}{\partial x}(s, X_s^{(2)}) d\widetilde{B}_s^G - \int_0^T \sigma_2(s, X_s^{(2)})\frac{\partial u_2}{\partial x}(s, X_s^{(2)}) d\widetilde{B}_s^G - K_T^{(2)} \qquad (3.9)
\end{aligned}
$$

where we applied the second parts of (E_1)-(E_2) and the first part of (E_1), with the conditions $|\sigma_1(t,x)| \leq |\sigma_2(t,x)|$ and $\dfrac{\partial^2 u_1(t,x)}{\partial x^2} \geq 0$. Finally, taking expectations under $\widetilde{\mathbb{E}}_2$ using (2.8) and the facts that $K_T^{(2)} \leq 0$, a.s., and that $\int_0^t \eta_s d\langle B\rangle_s - 2\int_0^t G(\eta_s) ds$ is a G-martingale, $\eta_s \in \mathcal{M}_G^1(0,T)$, see Proposition 4.1.4 of Peng (2019), we obtain $u_2\big(0, X_0^{(2)}\big) - u_1\big(0, X_0^{(2)}\big) \geq 0$. Since $X_0^{(2)} = X_0^{(1)}$ and $Y_0^{(i)} = u_i\big(0, X_0^{(i)}\big)$, $i = 1, 2$, we get $Y_0^{(1)} \leq Y_0^{(2)}$. Therefore, by definition, we have $\mathcal{E}_{\mathcal{G}_1}\big[\varphi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{\mathcal{G}_2}\big[\varphi\big(X_T^{(2)}\big)\big]$, for φ convex on $\mathbb{R}$. (ii) The case $i = 2$ in Assumption (E_3) can be proved similarly. □

The next theorem deals with the increasing convex order, for which only the conditions (E_1')-(E_3') and $X_0^{(1)} \leq X_0^{(2)}$ are required in addition to (3.10) and Condition (E_4') below.

Theorem 3.8 (Increasing convex order). *Assume that* $X_0^{(1)} \leq X_0^{(2)}$ *and*

$$
0 < \sigma_1(t,x) \leq \sigma_2(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}, \qquad (3.10)
$$

together with the conditions

(E_1') $M_1(t,x,y,z) \leq z\zeta(t,x) \leq M_2(t,x,y,z)$, $t \in [0,T]$, $x, y \in \mathbb{R}$, $z \in \mathbb{R}_+$,

(E_2') $N_1(t,x,y,z) \leq z\eta(t,x) \leq N_2(t,x,y,z)$, $t \in [0,T]$, $x, y \in \mathbb{R}$, $z \in \mathbb{R}_+$,

(E_3') $(x,y) \mapsto M_i(t,x,y,z)$ *and* $(x,y) \mapsto N_i(t,x,y,z)$ *are convex on* $\mathbb{R}^2$*;* $(y,z) \mapsto M_i(t,x,y,z)$ *and* $(y,z) \mapsto N_i(t,x,y,z)$ *are convex on* $\mathbb{R} \times \mathbb{R}_+$ *for* $i = 1$ *or* $i = 2$,

(E_4') $x \mapsto f_i(t,x,y,z)$ *and* $x \mapsto g_i(t,x,y,z)$ *are non-decreasing on* $\mathbb{R}$ *for* $i = 1, 2$, $y \in \mathbb{R}$, $z \in \mathbb{R}_+$, $t \in [0,T]$.

Then we have $X_T^{(1)} \leq^{\rm icon}_{\mathcal{G}_1,\mathcal{G}_2} X_T^{(2)}$, *i.e.*, $\mathcal{E}_{\mathcal{G}_1}\big[\varphi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{\mathcal{G}_2}\big[\varphi\big(X_T^{(2)}\big)\big]$, *for non-decreasing convex* φ *on* $\mathbb{R}$.

Proof. If $\varphi(x)$, $f_i(t,x,y,z)$ and $g_i(t,x,y,z)$ are non-decreasing in x by (E_4'), $i=1,2$, then by Proposition 2.10 the solutions $u_1(t,x)$ and $u_2(t,x)$ of the PDE (3.8) are non-decreasing in x and satisfy

$$\frac{\partial u_1}{\partial x}\big(t,X_t^{(2)}\big) \geq 0 \quad \text{and} \quad \frac{\partial u_2}{\partial x}\big(t,X_t^{(2)}\big) \geq 0,$$

a.s., $t\in[0,T]$, hence conditions (E_1)-(E_3) only need to hold for $z\geq 0$, showing the sufficiency of (E_i'), $i=1,2,3$. In addition, we have $Y_0^{(1)} = u_1\big(0,X_0^{(1)}\big) \leq u_1\big(0,X_0^{(2)}\big)$ by the assumption $X_0^{(1)} \leq X_0^{(2)}$, hence by repeating arguments in the proof of Theorem 3.7 for $i=1$ we find by (3.9) that $Y_0^{(2)} - Y_0^{(1)} \geq u_2\big(0,X_0^{(2)}\big) - u_1\big(0,X_0^{(2)}\big) \geq 0$ under Assumption (E_3') for $i=1$. The case $i=2$ is treated similarly according to the proof of Theorem 3.7. □

Corollary 3.9 (Increasing order). *Assume that* $X_0^{(1)} \leq X_0^{(2)}$ *and* $0<\sigma_1(t,x)=\sigma_2(t,x)$, $(t,x)\in[0,T]\times\mathbb{R}$, *together with the conditions*

(E_1'') $M_1(t,x,y,z) \leq z\zeta(t,x) \leq M_2(t,x,y,z)$, $t\in[0,T]$, $x,y\in\mathbb{R}$, $z\in\mathbb{R}_+$,

(E_2'') $N_1(t,x,y,z) \leq z\eta(t,x) \leq N_2(t,x,y,z)$, $t\in[0,T]$, $x,y\in\mathbb{R}$, $z\in\mathbb{R}_+$,

(E_3'') $x\mapsto f_i(t,x,y,z)$ *and* $x\mapsto g_i(t,x,y,z)$ *are non-decreasing on* $\mathbb{R}$ *for* $i=1,2$, $y\in\mathbb{R}$, $z\in\mathbb{R}_+$, $t\in[0,T]$.

Then we have $X_T^{(1)} \leq^{\rm mon}_{\mathcal{G}_1,\mathcal{G}_2} X_T^{(2)}$, *i.e.*, $\mathcal{E}_{\mathcal{G}_1}\big[\varphi\big(X_T^{(1)}\big)\big] \leq \mathcal{E}_{\mathcal{G}_2}\big[\varphi\big(X_T^{(2)}\big)\big]$, *for non-decreasing* φ *on* $\mathbb{R}$.

Proof. If $\sigma_1(t,x)=\sigma_2(t,x)$, we can check that the convexity of $u_i(t,x)$, $i=1,2$ is no longer required, so one can remove Condition (E_3'). Then this result is a directly consequence of Theorem 3.8. □

4 Superhedging with ambiguous coefficients

In this section we study the effects of ambiguous drift and volatility coefficients of G-FSDEs on associated contingent claim superhedging prices and portfolio values. Consider a risk-free asset priced $E_t := E_0 e^{rt}$, and a risky asset with non-negative prices modeled by the G-FSDE

$$dX_s = X_s\alpha(s,X_s)ds + X_s\beta(s,X_s)d\langle B\rangle_s + X_s\gamma(s,X_s)dB_s, \quad X_0=x_0, \quad 0\leq s\leq T, \quad (4.1)$$

where the coefficients $b(t,x) := x\alpha(t,x)$, $h(t,x) := x\beta(t,x)$ and $\sigma(t,x) := x\gamma(t,x)$ satisfy Condition (A_1). Denoting by $(p_t, q_t)_{t\in[0,T]}$ the quantities (strategy process) respectively invested in the risk-free asset and in the risky asset at time $t \in [0,T]$, we consider the wealth process $Y_t := p_t E_t + q_t X_t - C_t$, where $(C_t)_{t\geq 0}$ is a cumulative consumption process with $C_0 = 0$, such that

$$\begin{aligned} dY_t &= rp_tE_tdt + q_tdX_t - dC_t \\ &= r(Y_t - q_tX_t)dt + q_t\big(X_t\alpha(t,X_t)dt + X_t\beta(t,X_t)d\langle B\rangle_t + X_t\gamma(t,X_t)dB_t\big) - dC_t \\ &= rY_tdt + \frac{\alpha(t,X_t)-r}{\gamma(t,X_t)}q_tX_t\gamma(t,X_t)dt + \frac{\beta(t,X_t)}{\gamma(t,X_t)}q_tX_t\gamma(t,X_t)d\langle B\rangle_t + q_tX_t\gamma(t,X_t)dB_t - dC_t. \end{aligned}$$

Setting

$$\theta(t,x) := \frac{\alpha(t,x)-r}{\gamma(t,x)}, \quad \lambda(t,x) := \frac{\beta(t,x)}{\gamma(t,x)}, \quad Z_t := q_tX_t\gamma(t,X_t), \quad \text{and} \quad K_t := -C_t,$$

for $0 \leq t \leq T$, the wealth process can be rewritten as the G-BSDE

$$Y_t = Y_T - \int_t^T \big(rY_s + Z_s\theta(s,X_s)\big)ds - \int_t^T Z_s\lambda(s,X_s)d\langle B\rangle_s - \int_t^T Z_sdB_s - (K_T - K_t), \tag{4.2}$$

for $t \in [0,T]$, or, by the G-Itô formula, as

$$\widetilde{Y}_t = \widetilde{Y}_T - \int_t^T \theta(s,X_s)\widetilde{Z}_sds - \int_t^T \lambda(s,X_s)\widetilde{Z}_sd\langle B\rangle_s - \int_t^T \widetilde{Z}_sdB_s - \big(\widetilde{K}_T - \widetilde{K}_t\big), \tag{4.3}$$

$t \in [0,T]$, using the discounted processes $\widetilde{Y}_t := e^{-rt}Y_t$, $\widetilde{Z}_t := e^{-rt}Z_t$ and $\widetilde{K}_t = e^{-rt}K_t$, and the generator functions $\widetilde{f}(t,x,y,z) := -z\theta(t,x)$ and $\widetilde{g}(t,x,y,z) := -z\lambda(t,x)$, $(t,x) \in [0,T]\times\mathbb{R}_+$, $y,z \in \mathbb{R}$.

Next, we apply our G-stochastic ordering results to a model misspecification problem where we estimate the impact of uncertain or ambiguous bounded drift coefficients $\alpha(t,x)$, $\beta(t,x)$ and volatility coefficient $\gamma(t,x)$ on the superhedging prices of a contingent call claim with convex payoff $\widetilde{Y}_T = \varphi(X_T)$.

Scenario 1: Suppose that the volatility $\gamma(t,x)$ is ambiguous or uncertain, i.e., $\underline{\gamma}(t) \leq \gamma(t,x) \leq \overline{\gamma}(t)$, and assume for simplicity that the drift coefficients $\alpha(t,x)$ and $\beta(t,x)$ are precisely known to the hedgers. In this case, the ambiguity of $\gamma(t,x)$ yields a bound on the superhedging price of the contingent claim by Theorems 3.1 and 3.7. Letting

$$\theta_{\underline{\gamma}}(t,x) := \frac{\alpha(t,x)-r}{\underline{\gamma}(t)}, \quad \theta_{\overline{\gamma}}(t,x) := \frac{\alpha(t,x)-r}{\overline{\gamma}(t)}, \quad \lambda_{\underline{\gamma}}(t,x) := \frac{\beta(t,x)}{\underline{\gamma}(t)}, \quad \lambda^{\overline{\gamma}}(t,x) := \frac{\beta(t,x)}{\overline{\gamma}(t)},$$

by the relations $M(t,x,y,z) = xz\alpha(t,x) - xz\theta(t,x)\gamma(t,x) = rxz$ and

$$M_{\underline{\gamma}}(t,x,y,z) = xz\alpha(t,x) - xz\underline{\gamma}(t)\theta_{\underline{\gamma}}(t,x) = rxz,\ M_{\overline{\gamma}}(t,x,y,z) = xz\alpha(t,x) - xz\overline{\gamma}(t)\theta_{\overline{\gamma}}(t,x) = rxz,$$

we check that the function $M(t,x,y,z) = M_{\underline{\gamma}}(t,x,y,z) = M_{\overline{\gamma}}(t,x,y,z)$ is convex in $x \in \mathbb{R}$ and in $z \in \mathbb{R}$ separately. Similarly, we have $N(t,x,y,z) = xz\beta(t,x) - xz\gamma(t,x)\lambda(t,x) = 0$, and

$$N_{\underline{\gamma}}(t,x,y,z) = xz\beta(t,x) - xz\underline{\gamma}(t)\lambda_{\underline{\gamma}}(t,x) = 0,\ N_{\overline{\gamma}}(t,x,y,z) = xz\beta(t,x) - xz\overline{\gamma}(t)\lambda_{\overline{\gamma}}(t,x) = 0,$$

hence $N(t,x,y,z) = N_{\underline{\gamma}}(t,x,y,z) = N_{\overline{\gamma}}(t,x,y,z)$, $t \in [0,T]$, $x,y,z \in \mathbb{R}$, and $x\underline{\gamma}(t) \leq x\gamma(t,x) \leq x\overline{\gamma}(t)$, $(t,x) \in [0,T] \times \mathbb{R}_+$. Hence, all conditions in Theorem 3.1 hold true and Conditions (E_1)-(E_2) of Theorem 3.7 are satisfied with $\eta(t,x) = 0$ and $\zeta(t,x) = rx$. We deduce the portfolio value inequalities $\mathcal{E}_{\mathcal{G}_{\underline{\gamma}}}\big[\varphi\big(X_T^{\underline{\gamma}}\big)\big] \leq \mathcal{E}_{\mathcal{G}}[\varphi(X_T)] \leq \mathcal{E}_{\mathcal{G}_{\overline{\gamma}}}\big[\varphi\big(X_T^{\overline{\gamma}}\big)\big]$ for φ convex on $\mathbb{R}$, where $\big(X_t^{\underline{\gamma}}\big)_{t\in[0,T]}$ and $\big(X_t^{\overline{\gamma}}\big)_{t\in[0,T]}$ represent the misspecified risky price processes defined using $\underline{\gamma}(t)$ and $\overline{\gamma}(t)$ in (4.1), respectively.

Scenario 2: Suppose that the ambiguous or uncertain drift $\alpha(t,x)$ satisfies $\underline{\alpha}(t) \leq \alpha(t,x) \leq \overline{\alpha}(t)$, while the coefficients $\beta(t,x)$, and $\gamma(t,x)$ are precisely known to the hedgers. Here, in contrast to the first scenario, the superhedging price of the contingent claim are not affected by the ambiguity of the drift $\alpha(t,x)$. Let $(X_t^{\underline{\alpha}})_{t\in[0,T]}$ and $(X_t^{\overline{\alpha}})_{t\in[0,T]}$ denote the risky asset price processes given by

$$\begin{aligned} dX_s^{\underline{\alpha}} &= \underline{\alpha}(t)X_s^{\underline{\alpha}}ds + X_s^{\underline{\alpha}}\beta(s,X_s^{\underline{\alpha}})d\langle B\rangle_s + X_s^{\underline{\alpha}}\gamma(s,X_s^{\underline{\alpha}})dB_s, \\ dX_s^{\overline{\alpha}} &= \overline{\alpha}(t)X_s^{\overline{\alpha}}ds + X_s^{\overline{\alpha}}\beta(s,X_s^{\overline{\alpha}})d\langle B\rangle_s + X_s^{\overline{\alpha}}\gamma(s,X_s^{\overline{\alpha}})dB_s, \end{aligned}$$

where $X_0^{\underline{\alpha}} = X_0^{\overline{\alpha}} = x_0$, and let

$$\begin{aligned} \widetilde{Y}_t^{\underline{\alpha}} &= \varphi(X_T^{\underline{\alpha}}) - \int_t^T \theta^{\underline{\alpha}}(s,X_s^{\underline{\alpha}})\widetilde{Z}_s^{\underline{\alpha}}ds - \int_t^T \lambda(s,X_s^{\underline{\alpha}})\widetilde{Z}_s^{\underline{\alpha}}d\langle B\rangle_s - \int_s^T \widetilde{Z}_s^{\underline{\alpha}}dB_s - d\widetilde{K}_s^{\underline{\alpha}}, \\ \widetilde{Y}_t^{\overline{\alpha}} &= \varphi\big(X_T^{\overline{\alpha}}\big) - \int_t^T \theta^{\overline{\alpha}}(s,X_s^{\overline{\alpha}})\widetilde{Z}_s^{\overline{\alpha}}ds - \int_t^T \lambda(s,X_s^{\overline{\alpha}})\widetilde{Z}_s^{\overline{\alpha}}d\langle B\rangle_s - \int_s^T \widetilde{Z}_s^{\overline{\alpha}}dB_s - d\widetilde{K}_s^{\overline{\alpha}}, \end{aligned} \tag{4.4}$$

denote the corresponding discounted wealth processes, where $\theta^{\underline{\alpha}}(t,x) := (\underline{\alpha}(t) - r)/\gamma(t,x)$ and $\theta^{\overline{\alpha}}(t,x) := (\overline{\alpha}(t) - r)/\gamma(t,x)$. The true superhedging price of the contingent claim at $t = 0$, is defined by $\widetilde{Y}_0 := \mathcal{E}_{\mathcal{G}}[\varphi(X_T)]$, where $\big(\widetilde{Y}_t\big)_{t\in[0,T]}$ solves the G-BSDE (4.3). We check that the functions $M(t,x,y,z) := xz\alpha(t,x) - xz\theta(t,x)\gamma(t,x)$ and

$$M_{\underline{\alpha}}(t,x,y,z) := xz\underline{\alpha}(t)(t,x) - xz\theta_{\underline{\alpha}}(t,x)\gamma(t,x), \quad M_{\overline{\alpha}}(t,x,y,z) := xz\overline{\alpha}(t) - xz\theta_{\overline{\alpha}}(t,x)\gamma(t,x),$$

are separately convex in $x \in \mathbb{R}$ and in $z \in \mathbb{R}$, as they satisfy

$$M(t,x,y,z) = M_{\underline{\alpha}}(t,x,y,z) = M_{\overline{\alpha}(t)}(t,x,y,z) = rxz,$$

$t \in [0,T]$, $x,y,z \in \mathbb{R}$. We also have

$$N(t,x,y,z) = N_{\underline{\alpha}}(t,x,y,z) = N_{\overline{\alpha}}(t,x,y,z) := xz\beta(t,x) - xz\gamma(t,x)\lambda(t,x) = 0,$$

$t \in [0,T]$, $x,y,z \in \mathbb{R}$. Therefore (B_1)-(B_2) hold as equalities and, since $X_0^{(1)} = X_0^{(2)}$ and the volatility $\gamma(t,x)$ is precisely known, the G-PDE (3.4) is the same for $\underline{\alpha}(t)$ and $\overline{\alpha}(t)$, hence we have $\mathcal{E}_{\mathcal{G}}[\varphi(X_T)] = \mathcal{E}_{\mathcal{G}_{\underline{\alpha}}}\big[\varphi\big(X_T^{\underline{\alpha}}\big)\big] = \mathcal{E}_{\mathcal{G}_{\overline{\alpha}}}\big[\varphi\big(X_T^{\overline{\alpha}}\big)\big]$ for φ convex on $\mathbb{R}$.

Similarly, if the second drift $\beta(t,x)$ is ambiguous or uncertain, i.e., $\underline{\beta}(t) \le \beta(t,x) \le \overline{\beta}(t)$, while $\alpha(t,x)$ and $\gamma(t,x)$ are precisely determined by the hedgers. As in Scenario 2, we find that the ambiguity of $\beta(t,x)$ does not affect the superhedging price of the contingent claim. In fact, letting $\big(X_t^{\underline{\beta}}\big)_{t\in[0,T]}$ and $\big(X_t^{\overline{\beta}}\big)_{t\in[0,T]}$ represent the risky price processes corresponding to $\beta(t,x) = \underline{\beta}(t)$ and $\beta(t,x) = \overline{\beta}(t)$, respectively, then $\mathcal{E}_{\mathcal{G}}\big[\varphi(X_T)\big] = \mathcal{E}_{\mathcal{G}_{\underline{\beta}}}\big[\varphi\big(X_T^{\underline{\beta}}\big)\big] = \mathcal{E}_{\mathcal{G}_{\overline{\beta}}}\big[\varphi\big(X_T^{\overline{\beta}}\big)\big]$, for φ convex on $\mathbb{R}$.

A Convexity of nonlinear G-PDE solutions

In the sequel, for $\delta > 0$ we consider the mollification of G defined by

$$G_\delta(a) := \frac{1}{\delta}\int_{-\infty}^{\infty} \rho\left(\frac{a-b}{\delta}\right) G(b)db, \quad a \in \mathbb{R}, \tag{A.1}$$

where ρ is a mollifier on $\mathbb{R}$, and we let

$$F_\delta(t,x,y,z,w) = 2G_\delta(H(t,x,y,z,w)) + M(t,x,y,z), \tag{A.2}$$

and $H(t,x,y,z,w) := w\sigma^2(t,x)/2 + N(t,x,y,z)$. Denote by $u_\delta(t,x)$ is the solution of the following G-PDE

$$\frac{\partial u_\delta}{\partial t}(t,x) + F_\delta\left(t, x, u_\delta(t,x), \frac{\partial u_\delta}{\partial x}(t,x), \frac{\partial^2 u_\delta}{\partial x^2}(t,x)\right) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}, \tag{A.3}$$

with terminal condition $u_\delta(T,x) = \varphi(x)$.

Theorem 2.9 is proved by adapting arguments of Bian and Guan (2008), using the constant rank Theorem 2.3 therein and a new Lemma A.1. In order to deal with the lack of smoothness of $G(a)$ we use a family of smooth approximations G_δ of G as in (A.1), which admit derivatives up to the desired orders and still yield the convexity of $u(t,x)$ after taking limits of the corresponding solutions $u_\delta(t,x)$. We note that, in the one-dimensional setting, the constant rank Theorem 2.3 in Bian and Guan (2008), see also Theorem 1.2 in Bian and Guan (2009), only requires convexity of the nonlinear drift $M(t,x,y,z)$ and $N(t,x,y,z)$ in $(x,y) \in \mathbb{R}^2$ for all $(t,z) \in [0,T] \times \mathbb{R}$, instead of global convexity in (x,y,z). Namely, Condition (2.6) in Theorem 2.3 of Bian and Guan (2008) reduces to the condition

$$\begin{aligned}\frac{\partial^2 F}{\partial x^2}\Big(t,x,u(t,x),\frac{\partial u}{\partial x}(t,x),0\Big)+2b\frac{\partial^2 F}{\partial x\partial y}\Big(t,x,u(t,x),\frac{\partial u}{\partial x}(t,x),0\Big)\\ +b^2\frac{\partial^2 F}{\partial y^2}\Big(t,x,u(t,x),\frac{\partial u}{\partial x}(t,x),0\Big)\geq 0,\quad t\in[0,T],\quad b,x\in\mathbb{R}.\end{aligned}\tag{A.4}$$

In the sequel, we let $h_{\varrho,m}(x) := (1+x^2)^{(m+1+\varrho)/2+1}$ with $\varrho > 0$ given in Condition (A_5), and for any $K \in \mathbb{R}$ and $\varepsilon > 0$ we define

$$v_K(t,x) := e^{-Kt}h_{\varrho,m}(x) \quad \text{and} \quad u_{\delta,\varepsilon}(t,x) := u_\delta(t,x) + \varepsilon v_K(t,x), \quad (t,x) \in [0,T]\times\mathbb{R},$$

and

$$\begin{aligned}F_{K,\delta,\varepsilon}(t,x,y,z,w) &:= -\varepsilon\frac{\partial v_K}{\partial t}(t,x)+M\Big(t,x,y-\varepsilon v_K(t,x),z-\varepsilon\frac{\partial v_K}{\partial x}(t,x)\Big)\\ &+2G_\delta\bigg(H\Big(t,x,y-\varepsilon v_K(t,x),z-\varepsilon\frac{\partial v_K}{\partial x}(t,x),w-\varepsilon\frac{\partial^2 v_K}{\partial x^2}(t,x)\Big)\bigg),\quad (t,x,y,z,w)\in[0,T]\times\mathbb{R}^4.\end{aligned}\tag{A.5}$$

Lemma A.1 *Under Conditions (A_1)-(A_5) and (H_1)-(H_4), for $T = T(\varepsilon,\varrho,\delta)$ small enough we can choose $K \in \mathbb{R}$ such that $(x,y) \mapsto F_{K,\delta,\varepsilon}(t,x,y,z,0)$ defined in (A.5) satisfies (A.4).*

Proof. We need to show that

$$\begin{aligned}S(b) \ := \ &\frac{\partial^2 F_{K,\delta,\varepsilon}}{\partial x^2}\Big(t,x,u_{\delta,\varepsilon}(t,x),\frac{\partial u_{\delta,\varepsilon}}{\partial x}(t,x),0\Big)+2b\frac{\partial^2 F_{K,\delta,\varepsilon}}{\partial x\partial y}\Big(t,x,u_{\delta,\varepsilon}(t,x),\frac{\partial u_{\delta,\varepsilon}}{\partial x}(t,x),0\Big)\\ &+b^2\frac{\partial^2 F_{K,\delta,\varepsilon}}{\partial y^2}\Big(t,x,u_{\delta,\varepsilon}(t,x),\frac{\partial u_{\delta,\varepsilon}}{\partial x}(t,x),0\Big)\geq 0,\quad b\in\mathbb{R}.\end{aligned}$$

By (A.5), we have

$$\begin{aligned}\frac{\partial F_{K,\delta,\epsilon}}{\partial x}(t,x,y,z,0) &= -\varepsilon\frac{\partial^2 v_K}{\partial x\partial t}(t,x)+\frac{\partial M}{\partial x}\left(t,x,y-\varepsilon v_K(t,x),z-\varepsilon\frac{\partial v_K}{\partial x}(t,x)\right)\\ &+2\frac{\partial H}{\partial x}\left(t,x,y-\varepsilon v_K(t,x),z-\varepsilon\frac{\partial v_K}{\partial x}(t,x),-\varepsilon\frac{\partial^2 v_K}{\partial x^2}(t,x)\right)G'_\delta,\end{aligned}$$

and

$$\frac{\partial F_{K,\delta,\epsilon}}{\partial y}(t,x,y,z,0) = \frac{\partial M}{\partial y}\left(t,x,y-\varepsilon v_K(t,x),z-\varepsilon\frac{\partial v_K}{\partial x}(t,x)\right)$$

$$+\,2\frac{\partial H}{\partial y}\left(t,x,y-\varepsilon v_K(t,x),z-\varepsilon\frac{\partial v_K}{\partial x}(t,x),-\varepsilon\frac{\partial^2 v_K}{\partial x^2}(t,x)\right)G'_\delta,$$

where we write for short $G'_\delta := G'_\delta\Big(H\big(t,x,y-\varepsilon v_K(t,x),z-\varepsilon\frac{\partial v_K}{\partial x}(t,x),-\varepsilon\frac{\partial^2 v_K}{\partial x^2}(t,x)\big)\Big)$. Similarly, we have

$$\begin{aligned}\frac{\partial^2 F_{K,\delta,\epsilon}}{\partial x^2}\big(t,x,y,z,0\big) = &-\varepsilon\frac{\partial^3 v_K}{\partial x^2\partial t}(t,x)+\frac{\partial^2 M}{\partial x^2}\left(t,x,y-\varepsilon v_K(t,x),z-\varepsilon\frac{\partial v_K}{\partial x}(t,x)\right)\\ &+2\frac{\partial^2 H}{\partial x^2}\left(t,x,y-\varepsilon v_K(t,x),z-\varepsilon\frac{\partial v_K}{\partial x}(t,x),-\varepsilon\frac{\partial^2 v_K}{\partial x^2}(t,x)\right)G'_\delta\\ &+2\left(\frac{\partial H}{\partial x}\left(t,x,y-\varepsilon v_K(t,x),z-\varepsilon\frac{\partial v_K}{\partial x}(t,x),-\varepsilon\frac{\partial^2 v_K}{\partial x^2}(t,x)\right)\right)^2 G''_\delta,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 F_{K,\delta,\epsilon}}{\partial x\partial y}(t,x,y,z,0) = &\frac{\partial^2 M}{\partial x\partial y}\left(t,x,y-\varepsilon v_K(t,x),z-\varepsilon\frac{\partial v_K}{\partial x}(t,x)\right)\\ &+2\frac{\partial H}{\partial x}\left(t,x,y-\varepsilon v_K(t,x),z-\frac{\partial v_K}{\partial x}(t,x),-\varepsilon\frac{\partial^2 v_K}{\partial x^2}(t,x)\right)\\ &\times\frac{\partial H}{\partial y}\left(t,x,y-\varepsilon v_K(t,x),z-\varepsilon\frac{\partial v_K}{\partial x}(t,x),-\varepsilon\frac{\partial^2 v_K}{\partial x^2}(t,x)\right)G''_\delta,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 F_{K,\delta,\epsilon}}{\partial y^2}(t,x,y,z,0) = &\frac{\partial^2 M}{\partial y^2}\left(t,x,y-\varepsilon v_K(t,x),z-\varepsilon\frac{\partial v_K}{\partial x}(t,x)\right)\\ &+2\frac{\partial^2 H}{\partial y^2}\left(t,x,y-\varepsilon v_K(t,x),z-\frac{\partial v_K}{\partial x}(t,x),-\varepsilon\frac{\partial^2 v_K}{\partial x^2}(t,x)\right)G'_\delta\\ &+2\left(\frac{\partial H}{\partial y}\left(t,x,y-\varepsilon v_K(t,x),z-\varepsilon\frac{\partial v_K}{\partial x}(t,x),-\varepsilon\frac{\partial^2 v_K}{\partial x^2}(t,x)\right)\right)^2 G''_\delta.\end{aligned}$$

Therefore, we can write $S(b)=-\varepsilon\dfrac{\partial^3 v_K}{\partial x^2\partial t}(t,x)+S_1(b)+2S_2(b)G'_\delta+2S_3(b)G''_\delta$, where

$$\begin{aligned}S_1(b)=&\frac{\partial^2 M}{\partial x^2}\Big(t,x,u_{\delta,\varepsilon}(t,x),\frac{\partial u_{\delta,\varepsilon}}{\partial x}(t,x)\Big)+2b\frac{\partial^2 M}{\partial x\partial y}\Big(t,x,u_{\delta,\varepsilon}(t,x),\frac{\partial u_{\delta,\varepsilon}}{\partial x}(t,x)\Big)\\ &+b^2\frac{\partial^2 M}{\partial y^2}\Big(t,x,u_{\delta,\varepsilon}(t,x),\frac{\partial u_{\delta,\varepsilon}}{\partial x}(t,x)\Big),\end{aligned}$$

and

$$\begin{aligned}S_2(b)\;=\;&\frac{\partial^2 H}{\partial x^2}\Big(t,x,u_{\delta,\varepsilon}(t,x),\frac{\partial u_{\delta,\varepsilon}}{\partial x}(t,x),-\varepsilon\frac{\partial^2 v_K}{\partial x^2}(t,x)\Big)\\ &+2b\frac{\partial^2 H}{\partial x\partial y}\Big(t,x,u_{\delta,\varepsilon}(t,x),\frac{\partial u_{\delta,\varepsilon}}{\partial x}(t,x),-\varepsilon\frac{\partial^2 v_K}{\partial x^2}(t,x)\Big)\end{aligned}$$

$$+b^2\frac{\partial^2 H}{\partial y^2}\Big(t,x,u_{\delta,\varepsilon}(t,x),\frac{\partial u_{\delta,\varepsilon}}{\partial x}(t,x),-\varepsilon\frac{\partial^2 v_K}{\partial x^2}(t,x)\Big),$$

and

$$S_3(b)=\left(\frac{\partial H}{\partial x}\Big(t,x,u_{\delta,\varepsilon}(t,x),\frac{\partial u_{\delta,\varepsilon}}{\partial x}(t,x),-\varepsilon\frac{\partial^2 v_K}{\partial x^2}(t,x)\Big)\right.$$
$$\left.+b\frac{\partial H}{\partial y}\Big(t,x,u_{\delta,\varepsilon}(t,x),\frac{\partial u_{\delta,\varepsilon}}{\partial x}(t,x),-\varepsilon\frac{\partial^2 v_K}{\partial x^2}(t,x)\Big)\right)^2,$$

for all $b\in\mathbb{R}$. Note that since $G(a)$ is a convex function on $\mathbb{R}$, so is $G_\delta(a)$. Thus, we have $S_3(b)G''_\delta\geq 0$, $b\in\mathbb{R}$. Besides, we have $0\leq G'_\delta(a)\leq\underline{\sigma}^2+\overline{\sigma}^2$, $a\in\mathbb{R}$. Hence, proceeding as in the proof of Lemma 7.3 in Ly and Privault (2021), under (A_1)-(A_5) and (H_3)-(H_4), for $T=T(\varepsilon,\varrho,\delta)$ small enough we can choose a positive constant $K=K(\varepsilon,\varrho,\delta)$ such that

$$S_1(b)-\frac{\varepsilon}{2}\frac{\partial^3 v_K}{\partial x^2\partial t}(t,x)\geq 0,\quad 2S_2(b)\frac{\partial G_\delta}{\partial a}-\frac{\varepsilon}{2}\frac{\partial^3 v_K}{\partial x^2\partial t}(t,x)\geq 0,\quad b\in\mathbb{R},$$

we conclude to $S(b)\geq 0$ for all $b\in\mathbb{R}$. □

Proof of Theorem 2.9. We extend the proof argument of Theorem 4.1 in Bian and Guan (2008) by applying Theorem 2.3 therein, see also Theorem 7.1 in Ly and Privault (2021), to mollified equations.

We start by proving the convexity of $u_\delta(t,x)$ under (H_1)-(H_2), with φ in $\mathcal{C}_b^5(\mathbb{R})$. Since the function $F_\delta(t,x,y,z,w)$ defined by (A.2) is in $\mathcal{C}^{1,2}([0,T)\times\mathbb{R}^4)$, the solution $u_\delta(t,x)$ of (A.3) can be shown to be in $\mathcal{C}^{2,4}([0,T)\times\mathbb{R})$ as in Theorem 2.2 of Douglas et al. (1996), by successively applying Schauder interior estimates to the difference quotients used to approximate the derivatives of $u_\delta(t,x)$, see e.g. the inequality (4.15) in Theorem 4.5 of Urbas (1996), and Theorem 6.17 of D. Gilbarg and Trudinger (2001), and Theorem 5.1 in Krylov (1983). Besides, we note that by Proposition 2.8 there exists $C>0$ such that

$$\left|\frac{\partial^2 u_\delta}{\partial x^2}(t,x)\right|\leq C(1+x^2)^{(m+1)/2},\qquad (t,x)\in[0,T]\times\mathbb{R}. \tag{A.6}$$

Next, we let $E_{\delta,\varepsilon}:=\left\{(t,x)\in[0,T]\times\mathbb{R}\ :\ \frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(t,x)\leq 0\right\}$ and suppose that $E_{\delta,\varepsilon}\neq\emptyset$. From the relation $h''_{\varrho,m}(x)\geq(1+x^2)^{(m+\varrho+1)/2}$ and the bound (A.6) we get

$$\frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(t,x)\geq\varepsilon e^{-Kt}(1+x^2)^{(m+\varrho+1)/2}-C(1+x^2)^{(m+1)/2},$$

therefore there exists $R_\varepsilon > 0$ such that $\frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(t,x) > 0$ for all $|x| \geq R_\varepsilon$, and we have $E_{\delta,\varepsilon} \subseteq [0,T] \times B(0,R_\varepsilon)$, where $B(0,R_\varepsilon)$ is the centered open ball with radius R_ε, so that $E_{\delta,\varepsilon}$ is compact. Consequently, the supremum

$$\tau_0 := \sup \big\{ t \in [0,T] \ : \ (t,x) \in E_{\delta,\varepsilon} \text{ for some } x \in \mathbb{R} \big\}$$

is attained at some $(\tau_0, x_0) \in E_{\delta,\varepsilon}$ with $x_0 \in B(0,R_\varepsilon)$, such that $\frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(\tau_0,x_0) \leq 0$. In addition, by the convexity assumption on $x \mapsto u_\delta(T,x)$ we have

$$\frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(T,x) = \frac{\partial^2 u_\delta}{\partial x^2}(T,x) + \varepsilon \frac{\partial^2 v_K}{\partial x^2}(T,x) \geq \varepsilon e^{-KT} h''_{\varrho,m}(x) > 0, \qquad x \in \mathbb{R},$$

hence $\tau_0 < T$ and by the continuity of $u_{\delta,\varepsilon}$ we have $\frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(\tau_0,x) \geq 0$, $x \in \mathbb{R}$, since $\frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(t,x) > 0$ for all $t \in (\tau_0,T)$ and $x \in \mathbb{R}$. Consequently, the function $u_{\delta,\varepsilon}(t,x)$ is convex in x on $[\tau_0,T] \times B(0,R_\varepsilon)$. On the other hand, we note that $\frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(\tau_0,x_0) = 0$ for $x_0 \in B(0,R_\varepsilon)$, and that $u_{\delta,\varepsilon}(t,x)$ satisfies the equation

$$\frac{\partial u_{\delta,\varepsilon}}{\partial t}(t,x) + F_{K,\delta,\varepsilon}\left(t, x, u_{\delta,\varepsilon}(t,x), \frac{\partial u_{\delta,\varepsilon}}{\partial x}(t,x), \frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(t,x) \right) = 0,$$

where $F_{K,\delta,\varepsilon}(t,x,y,z,w)$ is defined in (A.5). By the constant rank Theorem 2.3 in Bian and Guan (2008), see also Theorem 7.1 in Ly and Privault (2021) and Lemma A.1, we deduce that for small enough $T = T(\varepsilon,\varrho,\delta)$ the second derivative $\frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(t,x)$ vanishes on $[\tau_0,T) \times B(0,R_\varepsilon)$ hence $\tau_0 = T$, which is a contradiction showing that $E_{\delta,\varepsilon} = \emptyset$. Therefore, we have

$$\frac{\partial^2 u_\delta}{\partial x^2}(t,x) + \varepsilon \frac{\partial^2 v_K}{\partial x^2}(t,x) = \frac{\partial^2 u_{\delta,\varepsilon}}{\partial x^2}(t,x) > 0, \qquad (t,x) \in [0,T] \times \mathbb{R},$$

and after letting ε tend to 0, we conclude that $\frac{\partial^2 u_\delta}{\partial x^2}(t,x) \geq 0$, $(t,x) \in [0,T] \times \mathbb{R}$ for small enough $T = T(\varepsilon,\varrho,\delta)$. This conclusion extends to all $T > 0$ by decomposing $[0,T]$ into subintervals of lengths at most $T(\varepsilon,\varrho,\delta)$. The convexity of the original solution $u(t,x)$ follows by taking the limit as $\delta \to 0$, as in D. Gilbarg and Trudinger (2001) or page 49 of Bian and Guan (2008), see also Theorem 3.1 in Jakobsen and Karlsen (2002). Finally, in order to relax Conditions (H_1)-(H_2) and φ in $\mathcal{C}^5_b(\mathbb{R})$ we apply the continuous dependence Theorem C.1 after mollifying φ, b, h, σ, f, g, and approximating σ by $\min(\max(\sigma(t,x),1/n),n)$, $n \geq 1$, as in the proof of Theorem 3.1. □

B Monotonicity of G-FBSDEs solutions

Lemma B.1 *Assume that (A_1)-(A_3) hold. The solution $(X_s^{t,x})_{t\in[s,T]}$ of the G-forward diffusion equation (2.9) is q.s. non-decreasing in x for all $s \in [t,T]$ and $t \in [0,T]$.*

Proof. Letting $\widehat{X}_s^{t,x,y} := X_s^{t,y} - X_s^{t,x}$ for $y \geq x$, $s \in [t,T]$, the processes,

$$\widehat{b}_s := \frac{b(s,X_s^{t,y}) - b(s,X_s^{t,x})}{X_s^{t,y} - X_s^{t,x}} \mathbf{1}_{\{X_s^{t,y} \neq X_s^{t,x}\}}, \quad \widehat{h}_s := \frac{h(s,X_s^{t,y}) - h(s,X_s^{t,x})}{X_s^{t,y} - X_s^{t,x}} \mathbf{1}_{\{X_s^{t,y} \neq X_s^{t,x}\}},$$

and $\widehat{\sigma}_s := \dfrac{\sigma(s,X_s^{t,y}) - \sigma(s,X_s^{t,x})}{X_s^{t,y} - X_s^{t,x}} \mathbf{1}_{\{X_s^{t,y} \neq X_s^{t,x}\}}$ are bounded in $\mathcal{M}_G^p(0,T)$, $p > 1$, and $\widehat{X}_s^{t,x,y}$ satisfies the equation

$$\widehat{X}_s^{t,x,y} = (y-x) + \int_t^s \widehat{b}_r \widehat{X}_r dr + \int_t^s \widehat{h}_r \widehat{X}_r d\langle B\rangle_r + \int_t^s \widehat{\sigma}_r \widehat{X}_r dB_r.$$

which by (3.3) in Hu et al. (2014b) yields

$$\widehat{X}_s^{t,x,y} = (y-x)\exp\Big(\int_t^s \widehat{b}_r dr + \int_t^s \widehat{h}_r d\langle B\rangle_r + \int_t^s \widehat{\sigma}_r dB_r - \frac{1}{2}\int_t^s \widehat{\sigma}_r^2 d\langle B\rangle_r\Big) \geq 0.$$

□

Following Hu et al. (2014b), we will construct an auxiliary extended $\widetilde{G}$-expectation space $\big(\widetilde{\Omega}_T, L^1_{\widetilde{G}}(\widetilde{\Omega}_T), \widetilde{\mathbb{E}}_G\big)$ with $\widetilde{\Omega}_T := C_0([0,T],\mathbb{R}^2)$, where for A in the space $\mathbb{S}_2$ of 2×2 symmetric matrices we let

$$\widetilde{G}(A) := \frac{1}{2}\sup_{\underline{\sigma}^2 \leq \nu \leq \overline{\sigma}^2} \operatorname{tr}\left[A\begin{pmatrix}\nu & 1\\ 1 & \nu^{-1}\end{pmatrix}\right], \qquad A \in \mathbb{S}_2.$$

Let $(B_t)_{t\geq 0}$ and $\big(\widetilde{B}_t\big)_{t\geq 0}$ be the canonical process in the extended space, with $\langle B, \widetilde{B}\rangle_t = t$.

Proof of Proposition 2.10. Letting $\widehat{Y}_s = Y_s^{t,y} - Y_s^{t,x}$, $\widehat{Z}_s = Z_s^{t,y} - Z_s^{t,x}$, $\widehat{K}_s = K_s^{t,y} - K_s^{t,x}$ and $\widehat{Y}_T = \varphi\big(X_T^{t,y}\big) - \varphi\big(X_T^{t,x}\big)$ for $y \geq x$, $s \in [t,T]$, we have

$$\begin{aligned}\widehat{Y}_s \;=\; & \widehat{Y}_T + \int_s^T \big(f(u,X_u^{t,y},Y_u^{t,y},Z_u^{t,y}) - f(u,X_u^{t,x},Y_u^{t,x},Z_u^{t,x})\big)du - \int_s^T \widehat{Z}_u dB_u \\ & + \int_s^T \big(g(u,X_u^{t,y},Y_u^{t,y},Z_u^{t,y}) - g(u,X_u^{t,x},Y_u^{t,x},Z_u^{t,x})\big)d\langle B\rangle_u - \int_s^T d\widehat{K}_u.\end{aligned}$$

Defining the processes

$$\begin{aligned}p_u &:= f(u,X_u^{t,y},Y_u^{t,y},Z_u^{t,y}) - f(u,X_u^{t,x},Y_u^{t,y},Z_u^{t,y})\\ a_u &:= \frac{f(u,X_u^{t,x},Y_u^{t,y},Z_u^{t,y}) - f(u,X_u^{t,x},Y_u^{t,x},Z_u^{t,y})}{Y_u^{t,y} - Y_u^{t,x}} \mathbf{1}_{\{Y_u^{t,y} \neq Y_u^{t,x}\}},\\ b_u &:= \frac{f(u,X_u^{t,x},Y_u^{t,x},Z_u^{t,y}) - f(u,X_u^{t,x},Y_u^{t,x},Z_u^{t,x})}{Z_u^{t,y} - Z_u^{t,x}} \mathbf{1}_{\{Z_u^{t,x} \neq Z_u^{t,x}\}}, \qquad u \in [0,T],\end{aligned}$$

and

$$
\begin{aligned}
q_u &:= g(u, X_u^{t,y}, Y_u^{t,y}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,y}, Z_u^{t,y}) \\
c_u &:= \frac{g(u, X_u^{t,x}, Y_u^{t,y}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y})}{Y_u^{t,y} - Y_u^{t,x}} \mathbf{1}_{\{Y_u^{t,y} \neq Y_u^{t,x}\}}, \\
d_u &:= \frac{g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x})}{Z_u^{t,y} - Z_u^{t,x}} \mathbf{1}_{\{Z_u^{t,x} \neq Z_u^{t,x}\}}, \qquad u \in [0,T],
\end{aligned}
$$

which are $(\mathcal{F}_t)_{t\in[0,T]}$-adapted in $\mathcal{M}_G^2(0,T)$, and using the decompositions

$$
\begin{aligned}
& f(u, X_u^{t,y}, Y_u^{t,y}, Z_u^{t,y}) - f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x}) \\
&= f(u, X_u^{t,y}, Y_u^{t,y}, Z_u^{t,y}) - f(u, X_u^{t,x}, Y_u^{t,y}, Z_u^{t,y}) + f(u, X_u^{t,x}, Y_u^{t,y}, Z_u^{t,y}) - f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y}) \\
&\quad + f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y}) - f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x}),
\end{aligned}
$$

and

$$
\begin{aligned}
& g(u, X_u^{t,y}, Y_u^{t,y}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x}) \\
&= g(u, X_u^{t,y}, Y_u^{t,y}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,y}, Z_u^{t,y}) + g(u, X_u^{t,x}, Y_u^{t,y}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y}) \\
&\quad + g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,y}) - g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x}),
\end{aligned}
$$

we have

$$
\widehat{Y}_s = \widehat{Y}_T + \int_s^T \big(a_u \widehat{Y}_u + b_u \widehat{Z}_u + p_u\big) du - \int_s^T \widehat{Z}_u dB_u + \int_s^T \big(c_u \widehat{Y}_u + d_u \widehat{Z}_u + q_u\big) d\langle B\rangle_u - \int_s^T d\widehat{K}_u,
$$

$s \in [0,T]$. Hence, by Theorem 3.2 in Hu et al. (2014b) we get

$$
\widehat{Y}_s = \frac{1}{\Gamma_t} \widetilde{\mathbb{E}}_G \left[\Gamma_T \widehat{Y}_T + \int_t^T p_u \Gamma_u du + \int_t^T q_u \Gamma_u d\langle B\rangle_u \,\middle|\, \mathcal{F}_t \right], \tag{B.1}
$$

where $\Gamma_s = \exp\left(\int_t^s (a_u - b_u d_u) du + \int_t^u c_u d\langle B\rangle_u \right) \mathcal{E}_s^B \mathcal{E}_s^{\widetilde{B}}$, and

$$
\mathcal{E}_s^B := \exp\left(\int_t^s d_u dB_u - \frac{1}{2} \int_t^s d_u^2 d\langle B\rangle_u \right), \quad \mathcal{E}_s^{\widetilde{B}} := \exp\left(\int_t^s b_u d\widetilde{B}_u - \frac{1}{2} \int_t^s b_u^2 d\langle \widetilde{B}\rangle_u \right),
$$

$0 \le t \le s \le T$. By Lemma B.1, the solution $(X_s^{t,x})_{s\in[t,T]}$ of the forward SDE (2.9) satisfies $\widehat{X}_s = X_s^{t,y} - X_s^{t,x} \ge 0$ for all $s \in [t,T]$ if $x \le y$, and since $f(s,x,y,z)$ and $g(s,x,y,z)$ are non-decreasing in x we have $p_s \ge 0$ and $q_s \ge 0$ q.s., $0 \le t \le s \le T$. Since $\varphi(x)$ is non-decreasing we have $\widehat{Y}_T = \varphi(X_T^{t,x}) - \varphi(X_T^{t,y}) \ge 0$ q.s., hence by (B.1) we have $\widehat{Y}_s = Y_s^{t,x} - Y_s^{t,y} \ge 0$, $s \in [t,T]$, if $x \le y$, which implies monotonicity of $\big(Y_s^{t,x}\big)_{s\in[t,T]}$, hence we also get $u(t,x) \le u(t,y)$, $x \le y$, $t \in [0,T]$, since $u(t,x) = Y_t^{t,x}$. □

C Continuous dependence of G-FBSDEs solutions

Theorems C.1 and C.2 are stated using convergence in L_G^p, $p \geq 1$, and we note that if $\lim_{n\to\infty} \mathbb{E}_G[|Y_{n,t} - Y_t|^2] = 0$ then by Proposition 6.1.21 in Peng (2019) there exists a subsequence $(Y_{n_k,t})_{k\geq 1}$ converging to Y_t quasi-surely, $t \in [0,T]$. The following result deals with continuous dependence of G-BSDEs solutions on coefficients, see also Theorem 4.1 of Zhang and Chen (2011) for a related result.

Theorem C.1 *Consider the family of G-forward-backward stochastic differential equations*

$$\begin{cases} X_{n,t} = x_{n,0} + \int_0^t b_n(s, X_{n,s})ds + \int_0^t h_n(s, X_{n,s})d\langle B\rangle_s + \int_0^t \sigma_n(s, X_{n,s})dB_s, \\ Y_{n,t} = \varphi_n(X_{n,T}) + \int_t^T f_n(s, X_{n,s}, Y_{n,s}, Z_{n,s})ds + \int_t^T g_n(s, X_{n,s}, Y_{n,s}, Z_{n,s})d\langle B\rangle_s \\ \qquad - \int_t^T Z_{n,s}dB_s - \int_t^T dK_{n,s}, \quad t \in [0,T], \end{cases}$$

where, for $n \geq 1$, the coefficients b_n, h_n, σ_n, f_n, g_n, φ_n and b, h, σ, φ, f, g satisfy (A_1)-(A_3) for a same $C > 0$. Assume that $\lim_{n\to\infty} x_{n,0} = x_0$,

$$\lim_{n\to\infty} \left(\mathbb{E}_G[|b_n(t,X_t) - b(t,X_t)|^4] + \mathbb{E}_G[|h_n(t,X_t) - h(t,X_t)|^4] + \mathbb{E}_G[|\sigma_n(t,X_t) - \sigma(t,X_t)|^4]\right) = 0,$$

and

$$\lim_{n\to\infty} \left(\mathbb{E}_G[|f_n(t,X_t,Y_t,Z_t) - f(t,X_t,Y,Z_t)|^2] + \mathbb{E}_G[|g_n(t,X_t,Y_t,Z_t) - g(t,X_t,Y_t,Z_t)|^2]\right) = 0,$$

for all $t \in [0,T]$, and that $\lim_{n\to\infty} \mathbb{E}_G[|\varphi_n(X_T) - \varphi(X_T)|^2] = 0$. Then for all $t \in [0,T]$ we have $\lim_{n\to\infty} \mathbb{E}_G\left[\left|Y_{n,t} - Y_t\right|^2\right] = 0$, where $(Y_t)_{t\in\mathbb{R}_+}$ solves the G-BSDE

$$\begin{cases} X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t h(s, X_s)d\langle B\rangle_s + \int_0^t \sigma(s, X_s)dB_s, \\ Y_t = \varphi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds + \int_t^T g(s, X_s, Y_s, Z_s)d\langle B\rangle_s - \int_t^T Z_s dB_s - \int_t^T dK_s. \end{cases}$$

Proof. For $n \geq 1$, let $\widehat{Y}_{n,t} := Y_{n,t} - Y_t$, $\widehat{X}_{n,t} := X_{n,t} - X_t$, $\widehat{Z}_{n,t} := Z_{n,t} - Z_t$, $\widehat{\varphi}_n(x) := \varphi_n(x) - \varphi(x)$, $(t,x) \in [0,T] \times \mathbb{R}$. Applying the G-Itô formula to $\left|\widehat{Y}_{n,t}\right|^2$, taking G-expectation on both sides and noting that $\int_t^T 2\widehat{Y}_{n,s}dK_s$ is a martingale under $\mathbb{E}_G$, this yields

$$\mathbb{E}_G\left[\left|\widehat{Y}_{n,t}\right|^2\right] \leq \mathbb{E}_G\left[(\varphi_n(X_{n,T}) - \varphi(X_T))^2\right] + \mathbb{E}_G\left[\int_t^T 2\widehat{Y}_{n,s}\big(f_n(s, X_{n,s}, Y_{n,s}, Z_{n,s}) - f(s, X_s, Y_s, Z_s)\big)ds\right.$$

$$+\int_t^T 2\widehat{Y}_{n,s}\big(g_n(s,X_{n,s},Y_{n,s},Z_{n,s})-g(s,X_s,Y_s,Z_s)\big)d\langle B\rangle_s-\int_t^T\big|\widehat{Z}_{n,s}\big|^2 d\langle B\rangle_s\bigg].$$

By the inequality $2ab\le(12C^2)a^2+b^2/(12C^2)$, we have

$$\begin{aligned}&2\widehat{Y}_{n,s}\big(g_n(s,X_{n,s},Y_{n,s},Z_{n,s})-g(s,X_s,Y_s,Z_s)\big)\\&\quad\le\ 12C^2\big|\widehat{Y}_{n,s}\big|^2+\frac{1}{12C^2}\big(g_n(s,X_{n,s},Y_{n,s},Z_{n,s})-g(s,X_s,Y_s,Z_s)\big)^2.\end{aligned}$$

Letting $\hat{g}_n(t,x,y,z):=g_n(t,x,y,z)-g(t,x,y,z)$, $t\in[0,T]$, $x,y,z\in\mathbb{R}$, we have

$$\begin{aligned}&\big(g_n(s,X_{n,s},Y_{n,s},Z_{n,s})-g(s,X_s,Y_s,Z_s)\big)^2\\&\quad\le\ 2\big(g_n(s,X_{n,s},Y_{n,s},Z_{n,s})-g_n(s,X_s,Y_s,Z_s)\big)^2+2(\hat{g}_n(s,X_s,Y_s,Z_s))^2\\&\quad\le\ 6C^2\big((1+X_{n,s}^m+X_s^m)^2|X_{n,s}-X_s|^2+|Y_{n,s}-Y_s|^2+|Z_{n,s}-Z_s|^2\big)+2(\hat{g}_n(s,X_s,Y_s,Z_s))^2.\end{aligned}$$

Similarly, by the inequality $2ab\le(12C^2/\underline{\sigma}^2)a^2+b^2/(12C^2/\underline{\sigma}^2)$, we have

$$\begin{aligned}&2\widehat{Y}_{n,s}\big(f_n(s,X_{n,s},Y_{n,s},Z_{n,s})-f(s,X_s,Y_s,Z_s)\big)\\&\quad\le\ \frac{12C^2}{\underline{\sigma}^2}\big|\widehat{Y}_{n,s}\big|^2+\frac{\underline{\sigma}^2}{12C^2}\big(f_n(s,X_{n,s},Y_{n,s},Z_{n,s})-f(s,X_s,Y_s,Z_s)\big)^2.\end{aligned}$$

Letting $\hat{f}_n(t,x,y,z):=f_n(t,x,y,z)-f(t,x,y,z)$, $t\in[0,T]$, $x,y,z\in\mathbb{R}$, we then have

$$\begin{aligned}&\big(f_n(s,X_{n,s},Y_{n,s},Z_{n,s})-f(s,X_s,Y_s,Z_s)\big)^2\\&\quad\le\ 6C^2\big((1+X_{n,s}^m+X_s^m)^2|X_{n,s}-X_s|^2+|Y_{n,s}-Y_s|^2+|Z_{n,s}-Z_s|^2\big)+2\big(\hat{f}_n(s,X_s,Y_s,Z_s)\big)^2,\end{aligned}$$

In addition, by the locally Lipschitz property of φ_n, we have

$$\begin{aligned}&(\varphi_n(X_{n,T})-\varphi(X_T))^2=(\varphi_n(X_{n,T})-\varphi_n(X_T)+\widehat{\varphi}_n(X_T))^2\\&\le 2(\varphi_n(X_{n,T})-\varphi_n(X_T))^2+2\big(\widehat{\varphi}_n(X_T)\big)^2\le 2C^2(1+X_{n,T}^m+X_T^m)^2|X_{n,T}-X_T|^2+2\big(\widehat{\varphi}_n(X_T)\big)^2.\end{aligned}$$

Combining the above estimates and noting that $\underline{\sigma}^2\int_t^T\big|\widehat{Z}_{n,s}\big|^2ds\le\int_t^T\big|\widehat{Z}_{n,s}\big|^2d\langle B\rangle_s\le\overline{\sigma}^2\int_t^T\big|\widehat{Z}_{n,s}\big|^2ds$, by point of view of Corollary 3.5.5 in Peng (2019) we find

$$\begin{aligned}&\mathbb{E}_G\big[\big|\widehat{Y}_{n,t}\big|^2\big]\le 2C^2\mathbb{E}_G\big[(1+X_{n,T}^m+X_T^m)^2\big|\widehat{X}_{n,T}\big|^2\big]+2\mathbb{E}_G\big[\big(\widehat{\varphi}_n(X_T)\big)^2\big]\\&+\left(\frac{12C^2}{\underline{\sigma}^2}+\frac{\underline{\sigma}^2}{2}+12C^2\overline{\sigma}^2+\frac{\overline{\sigma}^2}{2}\right)\int_t^T\mathbb{E}_G\big[\big|\widehat{Y}_{n,s}\big|^2\big]ds+\frac{\underline{\sigma}^2+\overline{\sigma}^2}{2}\int_t^T\mathbb{E}_G\big[\big(1+X_{n,s}^m+X_s^m\big)^2\big|\widehat{X}_{n,s}\big|^2\big]ds\\&+\frac{\underline{\sigma}^2}{6C^2}\int_t^T\mathbb{E}_G\big[\big(\hat{f}_n(s,X_s,Y_s,Z_s)\big)^2\big]ds+\frac{\overline{\sigma}^2}{6C^2}\int_t^T\mathbb{E}_G\big[\hat{g}_n(s,X_s,Y_s,Z_s)\big]^2ds,\end{aligned}$$

By the Hölder inequality, and then by the Gronwall inequality, we get

$$\mathbb{E}_G\big[\big|\widehat{Y}_{n,t}\big|^2\big]\le C'\Big((1+x_{n,0}^{2m}+x_0^{2m})\big(\mathbb{E}_G\big[\big|\widehat{X}_{n,T}\big|^4\big]\big)^{1/2}+\mathbb{E}_G\big[\widehat{\varphi}_n^2(X_T)\big]$$

$$+\left(1+x_{n,0}^{2m}+x_0^{2m}\right)\int_t^T\!\left(\mathbb{E}_G\left[\left|\widehat{X}_{n,s}\right|^4\right]\right)^{1/2}ds+\int_t^T\left(\mathbb{E}_G\left[\hat{f}_n^2(s,X_s,Y_s,Z_s)\right]+\mathbb{E}_G\left[\hat{g}_n^2(s,X_s,Y_s,Z_s)\right]\right)ds\Big),$$

where we used $\mathbb{E}_G[|X_s|^p] \le C^*(1+|x_0|^p)$ for any $s\in[0,T]$ and $p\ge 2$, where $C^*>0$ is a constant depending on p, C, T, G, see e.g. Proposition 4.1 in Hu et al. (2014b). Therefore, we have

$$\begin{aligned}
&\mathbb{E}_G\left[\left|\widehat{Y}_{n,t}\right|^2\right]\\
&\le\ C''\left(\left(1+x_{n,0}^{2m}+x_0^{2m}\right)\left(\mathbb{E}_G\left[\left|\widehat{X}_{n,T}\right|^4\right]\right)^{1/2}+(T-t)\left(1+x_{n,0}^{2m}+x_0^{2m}\right)\left(\sup_{s\in[t,T]}\mathbb{E}_G\left[\left|\widehat{X}_{n,s}\right|^4\right]\right)^{1/2}\right)\\
&+C''\mathbb{E}_G\left[\widehat{\varphi}_n^2(X_T)\right]+C''\left(\int_t^T\mathbb{E}_G\left[\hat{f}_n^2(s,X_s,Y_s,Z_s)\right]ds+\int_t^T\mathbb{E}_G\left[\hat{g}_n^2(s,X_s,Y_s,Z_s)\right]ds\right).
\end{aligned}$$

By Theorem C.2 below we have $\lim_{n\to\infty}\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|\widehat{X}_{n,t}\right|^4\right]=0$, hence, we conclude from the bounds $|\hat{f}_n(t,x,y,z)|\le 2C(|x|^{m+1}+|y|+|z|)$, $|\hat{g}_n(t,x,y,z)|\le 2C(|x|^{m+1}+|y|+|z|)$ and $\left|\widehat{\varphi}_n(x)\right|\le 2C|x|^{m+1}$ and dominated convergence on the interval $[0,T]$. □

The following result extends Theorem 9.7 in Mishura and Shevchenko (2017) from SDEs under linear expectation framework to G-SDEs under G-expectation framework, see also Theorem 3.1 of Zhang and Chen (2011) for a related result.

Theorem C.2 *Let $p\ge 1$. Assume that the coefficients b, h, σ, b_n, h_n, σ_n, $n\ge 0$, satisfy (A_1)-(A_3) for a same $C>0$. Assume that $\lim_{n\to\infty}x_{n,0}=x_0$ and*

$$\lim_{n\to\infty}\left(\mathbb{E}_G\left[|b_n(t,X_t)-b(t,X_t)|^{2p}\right]+\mathbb{E}_G\left[|h_n(t,X_t)-h(t,X_t)|^{2p}\right]+\mathbb{E}_G\left[|\sigma_n(t,X_t)-\sigma(t,X_t)|^{2p}\right]\right)=0,$$

for all $t\in[0,T]$. Then we have $\lim_{n\to\infty}\mathbb{E}_G\left[\sup_{t\in[0,T]}\left|X_{n,t}-X_t\right|^{2p}\right]=0$.

Proof. Letting $\Delta_n(t):=\mathbb{E}_G\left[\sup_{s\in[0,t]}\left|X_{n,t}-X_t\right|^{2p}\right]$, we estimate

$$\begin{aligned}
\Delta_n(t)\le 2^{2p-1}\Bigg(&\left|x_{n,0}-x_0\right|^{2p}+\mathbb{E}_G\left[\sup_{s\in[0,t]}\left|\int_0^s(b_n(u,X_{n,u})-b(u,X_u))du\right|^{2p}\right]\\
&+\mathbb{E}_G\left[\sup_{s\in[0,t]}\left|\int_0^s(h_n(u,X_{n,u})-h(u,X_u))d\langle B\rangle_u\right|^{2p}\right]\\
&+\mathbb{E}_G\left[\sup_{s\in[0,t]}\left|\int_0^s(\sigma_n(u,X_{n,u})-\sigma(u,X_u))dB_u\right|^{2p}\right].
\end{aligned}$$

By the Burkholder-Davis-Gundy (B-D-G) inequality for G-Brownian motion, see Gao (2009), we find

$$\mathbb{E}_G\left[\sup_{s\in[0,t]}\left|\int_0^s(\sigma_n(u,X_{n,u})-\sigma(u,X_u))dB_u\right|^{2p}\right]\le C_1t^{p-1}\int_0^t\mathbb{E}_G\left[|\sigma_n(u,X_{n,u})-\sigma(u,X_u)|^{2p}\right]du$$

$$\le C_1 2^{2p-1} t^{p-1} \left(\int_0^t \mathbb{E}_G\big[|\sigma_n(u, X_{n,u}) - \sigma_n(u, X_u)|^{2p}\big] du + \int_0^t \mathbb{E}_G\big[|\sigma_n(u, X_u) - \sigma(u, X_u)|^{2p}\big] du \right)$$

$$\le C_1 2^{2p-1} t^{p-1} \left(C^{2p} \int_0^t \Delta_n(u) du + \int_0^t \mathbb{E}_G\big[|\sigma_n(u, X_u) - \sigma(u, X_u)|^{2p}\big] du \right),$$

and

$$\mathbb{E}_G \left[\sup_{s\in[0,t]} \left| \int_0^s (h_n(u, X_{n,u}) - h(u, X_u)) d\langle B\rangle_u \right|^{2p} \right] \le C_2 t^{2p-1} \int_0^t \mathbb{E}_G\big[\big|h_n(u, X_{n,u}) - h(u, X_u)\ \big|^{2p}\big] du$$

$$\le C_2 2^{2p-1} t^{2p-1} \left(\int_0^t \mathbb{E}_G\big[|h_n(u, X_{n,u}) - h_n(u, X_u)|^{2p}\big] du + \int_0^t \mathbb{E}_G\big[|\sigma_n(u, X_u) - h(u, X_u)|^{2p}\big] du \right)$$

$$\le C_2 2^{2p-1} t^{2p-1} \left(C^{2p} \int_0^t \Delta_n(u) du + \int_0^t \mathbb{E}_G\big[|h_n(u, X_u) - h(u, X_u)|^{2p}\big] du \right),$$

for some $C_1, C_2 > 0$. In the same fashion, by the Hölder inequality, we get

$$\mathbb{E}_G \left[\sup_{s\in[0,t]} \left| \int_0^s (b_n(u, X_{n,u}) - b(u, X_u)) du \right|^{2p} \right]$$
$$\le \quad 2^{2p-1} t^{2p-1} \left(C^{2p} \int_0^t \Delta_n(u) du + \int_0^t \mathbb{E}_G\big[|b_n(u, X_u) - b(u, X_u)|^{2p}\big] du \right),$$

Summing up, we obtain

$$\Delta_n(T) \le C' \bigg(\big|x_{n,0} - x_0\big|^{2p} + \int_0^T \Delta_n(u) du + \int_0^T \mathbb{E}_G\big[|b_n(u, X_u) - b(u, X_u)|^{2p}\big] du$$
$$+ \int_0^T \mathbb{E}_G\big[|h_n(u, X_u) - h(u, X_u)|^{2p}\big] du + \int_0^T \mathbb{E}_G\big[\sigma_n(u, X_u) - \sigma(u, X_u)|^{2p}\big] du \bigg),$$

and hence by the Gronwall inequality, we have

$$\Delta_n(T) \le C'' \bigg(\big|x_{n,0} - x_0\big|^{2p} + \int_0^T \mathbb{E}_G\big[|b_n(u, X_u) - b(u, X_u)|^{2p}\big] du$$
$$+ \int_0^T \mathbb{E}_G\big[|h_n(u, X_u) - h(u, X_u)|^{2p}\big] du + \int_0^T \mathbb{E}_G\big[|\sigma_n(u, X_u) - \sigma(u, X_u)|^{2p}\big] du \bigg)$$

Finally, by Proposition 4.1 in Hu et al. (2014b) we have

$$\mathbb{E}_G\big[|b_n(u, X_u) - b(u, X_u)|^{2p}\big] \le 2^{2p-1} \mathbb{E}_G\big[|b_n(u, X_u)|^{2p} + |b(u, X_u)|^{2p}\big]$$
$$\le \quad 2^{2p} C^{2p} \mathbb{E}_G\big[1 + \big|X_u\big|^{2p}\big] \le 2^{2p} C^{2p} C''' \big(1 + |x_0|^{2p}\big) < \infty$$

for some $C''' > 0$, and similarly for the sequences $(h_n)_{n\ge1}$, $(\sigma_n)_{n\ge1}$, hence we conclude by dominated convergence on the interval $[0, T]$. □

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