Pricing CIR yield options by conditional moment matching

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Abstract

We propose an approximation scheme for the pricing of yield options in the CIR model using conditional moment matching based on the gamma and lognormal distributions. This method is fast and simple to implement, and it shows a high degree of accuracy without being subject to the numerical instabilities that can be encountered with more sophisticated approaches.

Keywords: CIR model; Asian options; Asian caps, conditional moment matching; stratified approximation.
Mathematics Subject Classification (2010): 91B28, 60J60, 33C10, 97M30.

1 Introduction

Bonds and bond options on the Cox-Ingersoll-Ross [4] (CIR) stochastic differential equation

$$dS_t = (a - bS_t)dt + \sigma\sqrt{S_t}dW_t \tag{1.1}$$

have been priced by closed form expressions in [11] and [13] using the Girsanov theorem and change of numeraire. As for average rate options on the time integral

$$\Lambda_T = \int_0^T S_t dt$$

of the mean reverting process $(S_t)_{t\in\mathbb{R}_+}$, no closed form solutions are available and such options have been priced in [8] and in [3] by numerical Laplace and Fourier transform inversion. Closed form series expansions for the joint probability density function of (S_T, Λ_T) and for the marginal density of Λ_T have been derived in [6] and [5], respectively using special functions and complex recursion arguments. These expressions have been used for pricing via truncated series when $a = 0$ in [6], and when $a \geq 0$ in [5].

In this paper we develop a conditional moment matching technique for the fast approximation of option prices on average yields, using single and double integrals. For this, we fit the parameters of the lognormal and gamma densities based on the conditional moments of Λ_T given $S_T = y$. An option with payoff $\phi(S_T, \Lambda_T)$ will be priced via the approximation

$$\begin{aligned}\mathbb{E}[\phi(\Lambda_T, S_T)] &= \int_0^\infty \int_0^\infty \phi(x,y) f_{(\Lambda_T,S_T)}(x,y)dxdy \\ &\approx \int_0^\infty \int_0^\infty \phi(x,y) \tilde{f}_{\Lambda_T|S_T=x}(y)dx f_{S_T}(y)dy,\end{aligned}$$

where $f_{(\Lambda_T,S_T)}$ is the joint probability density of (Λ_T, S_T), $f_{S_T}(y)$ is the probability density of S_T, and $\tilde{f}_{\Lambda_T|S_T=x}(y)$ is a lognormal or gamma approximation of the conditional density of Λ_T given $S_T = y$. This approach can be regarded as an extension of existing moment matching techniques for the pricing of Asian options, see e.g. [10] for the approximation of $\int_0^T S_u du$ by a lognormal random variable, and [12] for the conditional case.

While retaining a high degree of accuracy, our method provides a simpler alternative in comparison with the existing literature that involves either numerical Fourier-Laplace inversion [8], [3], or complex recursions and series expansions [5], [6]. It is also more stable numerically, see e.g. the rows 3-5 of Table 1 and Figure 3, and it performs significantly faster, cf. Table 2. We note that in the CIR model, the gamma approximation is consistently better than the lognormal approximation, whereas in the geometric Brownian motion model the lognormal approximation performs better, cf. [12].

Other approaches to the approximation of average and Asian option prices based on (unconditional) moments and cumulants include [14] which relies on Gram-Charlier expansions, [2] which applies moment matching in exponential Lévy models, and [7] which deals with stochastic volatility.

We proceed as follows. After recalling the basics of the CIR model in Section 2, we present the conditional moment matching technique in Section 3. This technique is used for the approximations of option prices presented in Section 4, which contains numerical simulations for comparison with existing algorithms.

2 CIR model

The probability density function of S_T is known to be the non-central chi-square probability density function

$$f_{S_T}(y) = \frac{2b}{\sigma^2(1-e^{-bT})} \exp\left(-\frac{2b(S_0+ye^{bT})}{\sigma^2(e^{bT}-1)}\right)\left(\frac{ye^{bT}}{S_0}\right)^{a/\sigma^2-1/2} I_{2a/\sigma^2-1}\left(\frac{2b\sqrt{yS_0}}{\sigma^2\sinh(bT/2)}\right), \tag{2.1}$$

$y > 0$, where

$$I_\lambda(z) := \left(\frac{z}{2}\right)^\lambda \sum_{k=0}^\infty \frac{(z^2/4)^k}{k!\Gamma(\lambda+k+1)}, \qquad z, \lambda \in \mathbb{R},$$

is the modified Bessel function of the first kind, cf. Corollary 24 in [1], and

$$\Gamma(\lambda) := \int_0^\infty x^{\lambda-1} e^x dx$$

denotes the gamma function. Under the Feller condition $2a/\sigma^2 < 1$ the value of $f_{S_T}(x)$ is not defined at $x = 0$ and the probability distribution of S_T admits a point mass at 0. In large time T with $a, b > 0$, due to the asymptotics

$$I_\lambda(z) \approx \frac{1}{\Gamma(\lambda+1)}\left(\frac{z}{2}\right)^\lambda, \qquad [z \to 0],$$

the density (2.1) becomes the gamma density

$$f(y) = \frac{1}{\Gamma(2a/\sigma^2)}\left(\frac{2b}{\sigma^2}\right)^{2a/\sigma^2} y^{-1+2a/\sigma^2} e^{-2by/\sigma^2}, \qquad y > 0, \tag{2.2}$$

with shape parameter $2a/\sigma^2$ and scale parameter $\sigma^2/(2b)$, which is also the invariant distribution of $(S_t)_{t\in\mathbb{R}_+}$.

The joint density function $f_{(\Lambda_T,S_T)}(x,y)$ of (Λ_T,S_T) has been computed in Theorem 3 and Relation (30) in [6] using Hermite polynomials and parabolic cylinder functions by inversion of the joint moment generating function

$$M(\lambda,\eta) := \mathbb{E}\left[e^{\lambda S_T+\eta\Lambda_T}\right] = e^{-S_0\psi(\lambda,\eta)-a\phi(\lambda,\eta)}, \tag{2.3}$$

where

$$\psi(\lambda,\eta) := \frac{\lambda((\bar{b}-b)+e^{-\bar{b}T}(\bar{b}+b))+2\eta(1-e^{-\bar{b}T})}{\sigma^2\lambda(1-e^{-\bar{b}T})-\bar{b}-b-e^{-\bar{b}T}(\bar{b}-b)}$$

and

$$\phi(\lambda,\eta) := \frac{1}{\sigma^2}(\bar{b}-b)T+\frac{2}{\sigma^2}\log\frac{\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b)-\sigma^2\lambda(1-e^{-\bar{b}T})}{2\bar{b}},$$

with $\bar{b} := \sqrt{b^2-2\eta\sigma^2}$. The next proposition will be used for the derivation of conditional moments.

Proposition 2.1 *The conditional moment generating function of Λ_T given $S_T=y$ can be expressed as*

$$\begin{aligned}&\mathbb{E}\left[e^{\eta\Lambda_T}\;\middle|\;S_T=y\right] \\ &= \frac{\bar{b}\sinh(bT/2)}{b\sinh(\bar{b}T/2)}\exp\left(\frac{(S_0+y)}{\sigma^2}\left(b\coth\frac{bT}{2}-\bar{b}\coth\frac{\bar{b}T}{2}\right)\right)\frac{I_{2a/\sigma^2-1}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh(\bar{b}T/2)}\right)}{I_{2a/\sigma^2-1}\left(\frac{2b\sqrt{yS_0}}{\sigma^2\sinh(bT/2)}\right)},\end{aligned} \tag{2.4}$$

$y>0$, $\eta\le b^2/(2\sigma^2)$.

Proof. We use (2.1) and the relation

$$\begin{aligned}&\mathbb{E}\left[e^{\eta\Lambda_T}\;\middle|\;S_T=y\right] \\ &= \frac{2\bar{b}}{\sigma^2(1-e^{-\bar{b}T})f_{S_T}(y)}\exp\left(-\frac{\bar{b}T}{2}+\frac{1}{\sigma^2}\left(abT+b(S_0-y)-\bar{b}(S_0+y)\frac{e^{\bar{b}T}+1}{e^{\bar{b}T}-1}\right)\right) \\ &\quad\times\left(\frac{y}{S_0}\right)^{a/\sigma^2-1/2} I_{2a/\sigma^2-1}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh(\bar{b}T/2)}\right), \qquad y>0,\end{aligned}$$

cf. Relation (27) in [9] and § 3.3 of [1]. □

We note that

$$\mathbb{E}\left[e^{\eta\Lambda_T} \mid S_T = y\right] = \frac{g_b(\eta)}{g_b(0)}, \qquad y > 0,$$

where

$$g_b(\eta) := \frac{\bar{b}}{1-e^{-\bar{b}T}} \exp\left(-\frac{\bar{b}}{\sigma^2}\left(\frac{\sigma^2 T}{2} + (S_0+y)\frac{e^{\bar{b}T}+1}{e^{\bar{b}T}-1}\right)\right) I_{2a/\sigma^2-1}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh(\bar{b}T/2)}\right) \tag{2.5}$$

with the relation

$$f_{S_T}(y) = \frac{2}{\sigma^2} g_b(0)\left(\frac{y}{S_0}\right)^{a/\sigma^2-1/2} \exp\left(\frac{b}{\sigma^2}\left(aT+S_0-y\right)\right), \qquad y > 0.$$

3 Conditional moment matching

We have

$$\mathbb{E}[S_T] = S_0 e^{-bT} + \frac{a}{b}\left(1-e^{-bT}\right), \tag{3.1}$$

from which it follows

$$\mathbb{E}[\Lambda_T] = S_0\frac{1-e^{-bT}}{b} + a\frac{e^{-bT}+bT-1}{b^2}. \tag{3.2}$$

We also have

$$\mathrm{Var}[S_T] = S_0\frac{\sigma^2}{b}\left(e^{-bT}-e^{-2bT}\right) + \frac{a\sigma^2}{2b^2}\left(1-e^{-bT}\right)^2,$$

and from (2.3) we can derive the second moment as

$$\mathrm{Var}[\Lambda_T] = \sigma^2 S_0\frac{1-2bTe^{-bT}-e^{-2bT}}{b^3} + \sigma^2 a\frac{e^{-2bT}+2bT+4(bT+1)e^{-bT}-5}{2b^4}. \tag{3.3}$$

We use the conditional moment generating function of Proposition 2.1 to compute the conditional moments of Λ_T as in the next proposition.

Proposition 3.1 *We have*

$$\mathbb{E}[\Lambda_T \mid S_T = y] = -\frac{\sigma^2}{b^2} + \frac{1}{b(e^{bT}-1)^2}\left(\frac{\sigma^2 T}{2}(e^{2bT}-1) + (S_0+y)(e^{2bT}-2bTe^{bT}-1)\right.$$
$$\left. + \sqrt{yS_0e^{bT}}\left(e^{bT}(bT-2)+bT+2\right)\frac{I_{2a/\sigma^2}\left(\frac{2b\sqrt{yS_0}}{\sigma^2\sinh(bT/2)}\right) + I_{2a/\sigma^2-2}\left(\frac{2b\sqrt{yS_0}}{\sigma^2\sinh(bT/2)}\right)}{I_{2a/\sigma^2-1}\left(\frac{2b\sqrt{yS_0}}{\sigma^2\sinh(bT/2)}\right)}\right)$$
$$= -\frac{\sigma^2}{b^2} + \frac{1}{b(e^{bT}-1)^2}\left(\frac{\sigma^2 T}{2}(e^{2bT}-1) + (S_0+y)(e^{2bT}-2bTe^{bT}-1)\right.$$

$$+\sqrt{yS_0e^{bT}}\left(e^{bT}(bT-2)+bT+2\right)\left(\frac{I_{2a/\sigma^2-2}\left(\frac{2b\sqrt{yS_0}}{\sigma^2\sinh(bT/2)}\right)}{I_{2a/\sigma^2-1}\left(\frac{2b\sqrt{yS_0}}{\sigma^2\sinh(bT/2)}\right)}+(\sigma^2-2a)\frac{\sinh(bT/2)}{b\sqrt{yS_0}}\right)\Bigg)\Bigg). \tag{3.4}$$

Proof. We need to differentiate $\eta \mapsto g_b(\eta)$ with $\bar{b} := \sqrt{b^2 - 2\eta\sigma^2}$ and

$$\bar{b}' = \frac{-\sigma^2}{\sqrt{b^2-2\eta\sigma^2}} = -\frac{\sigma^2}{\bar{b}}.$$

From (2.5) we have

$$\frac{g_b'(\eta)}{g_b(\eta)} = -\frac{\sigma^2}{\bar{b}^2} + \frac{1}{\bar{b}}\left(\frac{\sigma^2 T}{2}\frac{(e^{\bar{b}T}+1)}{(e^{\bar{b}T}-1)} + (S_0+y)\frac{e^{2\bar{b}T}-2\bar{b}Te^{\bar{b}T}-1}{(e^{\bar{b}T}-1)^2}\right)$$
$$+\frac{\sqrt{yS_0e^{\bar{b}T}}\left(e^{\bar{b}T}(\bar{b}T-2)+\bar{b}T+2\right)\left(I_{2a/\sigma^2}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh\left(\bar{b}T/2\right)}\right)+I_{2a/\sigma^2-2}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh\left(\bar{b}T/2\right)}\right)\right)}{\bar{b}\left(e^{\bar{b}T}-1\right)^2 I_{2a/\sigma^2-1}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh\left(\bar{b}T/2\right)}\right)},$$

which yields (3.4) by taking $\eta = 0$ and using the relation

$$\frac{I_\nu(x)+I_{\nu-2}(x)}{I_{\nu-1}(x)} = \frac{I_{\nu-2}(x)}{I_{\nu-1}(x)} + 2\frac{1-\nu}{x}.$$

□

In the next proposition we compute the conditional variance of Λ_T.

Proposition 3.2 *We have*

$$\mathrm{Var}[\Lambda_T \mid S_T = y] = -2\frac{\sigma^4}{b^4} + \sigma^2\frac{(e^{bT}(2bT+1)-1)}{b^2(e^{bT}-1)}\left(\frac{\sigma^2}{b^2} + \mathbb{E}[\Lambda_T \mid S_T = y]\right)$$
$$+\frac{1}{b(e^{bT}-1)^2}\Bigg(-\sigma^4T^2\frac{e^{2bT}}{b} - 2\frac{\sigma^2T}{b}(S_0+y)\left(e^{2bT}-e^{bT}(bT+1)\right)$$
$$-\frac{\sigma^2T}{2b}\sqrt{yS_0e^{bT}}\left(e^{bT}(3bT-4)+bT+4\right)\frac{I_{2a/\sigma^2}\left(\frac{4b\sqrt{yS_0e^{-bT}}}{\sigma^2(1-e^{-bT})}\right)+I_{2a/\sigma^2-2}\left(\frac{4b\sqrt{yS_0e^{-bT}}}{\sigma^2(1-e^{-bT})}\right)}{I_{2a/\sigma^2-1}\left(\frac{2b\sqrt{yS_0}}{\sigma^2\sinh(bT/2)}\right)}$$
$$+\frac{yS_0e^{bT}}{b(1-e^{-bT})^2}\left(e^{bT}(bT-2)+bT+2\right)^2\left(2+\frac{I_{2a/\sigma^2-3}+I_{2a/\sigma^2+1}}{I_{2a/\sigma^2-1}}-\frac{(I_{2a/\sigma^2-2}+I_{2a/\sigma^2})^2}{(I_{2a/\sigma^2-1})^2}\right)\Bigg),$$

where we let $I_z := I_z\left(\frac{2b\sqrt{yS_0}}{\sigma^2\sinh(bT/2)}\right)$, $z \in \mathbb{R}$.

Proof. Letting

$$h_b(\eta) := \frac{g_b'(\eta)}{g_b(\eta)},$$

we have

$$g_b''(\eta) = (h_b(\eta)g_b(\eta))' = h_b'(\eta)g_b(\eta) + (h_b(\eta))^2 g_b(\eta),$$

hence by (2.5) we find

$$\begin{aligned}
\mathbb{E}[\Lambda_T^2 \mid S_T = y] &= \frac{g_b''(0)}{g_b(0)} \\
&= h_b'(0) + (h_b(0))^2 \\
&= h_b'(0) + (\mathbb{E}[\Lambda_T \mid S_T = y])^2,
\end{aligned}$$

and $\mathrm{Var}[\Lambda_T \mid S_T = y] = h_b'(0)$. Next, we have

$$\begin{aligned}
h_b'(\eta) = &-\frac{2\sigma^4}{\bar{b}^4} - \sigma^2 \frac{(1 - e^{\bar{b}T}(2\bar{b}T + 1))}{\bar{b}^3(e^{\bar{b}T} - 1)^3} \Bigg(\sigma^2 T(e^{2\bar{b}T} - 1)/2 + (S_0 + y)(e^{2\bar{b}T} - 2\bar{b}Te^{\bar{b}T} - 1) \\
&+ \sqrt{yS_0e^{\bar{b}T}}\big(e^{\bar{b}T}(\bar{b}T - 2) + \bar{b}T + 2\big) \frac{I_{2a/\sigma^2}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh(\bar{b}T/2)}\right) + I_{2a/\sigma^2-2}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh(\bar{b}T/2)}\right)}{I_{2a/\sigma^2-1}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh(\bar{b}T/2)}\right)} \Bigg) \\
&+ \frac{1}{\bar{b}(e^{\bar{b}T} - 1)^2} \Bigg(-\sigma^4 T^2 \frac{e^{2\bar{b}T}}{\bar{b}} - 2\frac{\sigma^2 T}{\bar{b}}(S_0 + y)\left(e^{2\bar{b}T} - e^{\bar{b}T}(1 + \bar{b}T)\right) \\
&- \frac{\sigma^2 T}{2\bar{b}}\sqrt{yS_0e^{\bar{b}T}}\big(e^{\bar{b}T}(3\bar{b}T - 4) + \bar{b}T + 4\big) \frac{I_{2a/\sigma^2}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh(\bar{b}T/2)}\right) + I_{2a/\sigma^2-2}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh(\bar{b}T/2)}\right)}{I_{2a/\sigma^2-1}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh(\bar{b}T/2)}\right)} \\
&- 4yS_0e^{\bar{b}T/2}\left(\frac{e^{-\bar{b}T/2}(1 - \bar{b}T/2)}{1 - e^{-\bar{b}T}} - \frac{e^{-3\bar{b}T/2}\bar{b}T}{(1 - e^{-\bar{b}T})^2}\right)\left(e^{\bar{b}T}(\bar{b}T - 2) + \bar{b}T + 2\right) \\
&\times \left(\frac{I_{2a/\sigma^2-1}(I_{2a/\sigma^2-3} + 2I_{2a/\sigma^2-1} + I_{2a/\sigma^2+1})}{2I_{2a/\sigma^2-1}^2} - \frac{(I_{2a/\sigma^2-2} + I_{2a/\sigma^2})^2}{2I_{2a/\sigma^2-1}^2} \right) \Bigg) \\
= &-\frac{2\sigma^4}{\bar{b}^4} - \sigma^2 \frac{(1 - e^{\bar{b}T}(2\bar{b}T + 1))}{\bar{b}^2(e^{\bar{b}T} - 1)} \left(\frac{\sigma^2}{\bar{b}^2} + \mathbb{E}[\Lambda_T \mid S_T = y] \right) \\
&+ \frac{1}{\bar{b}(e^{\bar{b}T} - 1)^2} \left(-\sigma^4 T^2 \frac{e^{2\bar{b}T}}{\bar{b}} - 2\frac{\sigma^2 T}{\bar{b}}(S_0 + y)\big(e^{2\bar{b}T} - e^{\bar{b}T}(\bar{b}T + 1)\big) \right) \\
&- \frac{\sigma^2 T}{2\bar{b}}\sqrt{yS_0e^{\bar{b}T}}\big(e^{\bar{b}T}(3\bar{b}T - 4) + \bar{b}T + 4\big) \frac{I_{2a/\sigma^2}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh(\bar{b}T/2)}\right) + I_{2a/\sigma^2-2}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh(\bar{b}T/2)}\right)}{I_{2a/\sigma^2-1}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh(\bar{b}T/2)}\right)}
\end{aligned}$$

$$
-\frac{4yS_0}{\bar{b}}\left(\frac{1-\bar{b}T/2}{1-e^{-\bar{b}T}}-\frac{e^{-\bar{b}T}\bar{b}T}{(1-e^{-\bar{b}T})^2}\right)\left(e^{\bar{b}T}(\bar{b}T-2)+\bar{b}T+2\right)
$$

$$
\times\left(\frac{I_{2a/\sigma^2-3}+2I_{2a/\sigma^2-1}+I_{2a/\sigma^2+1}}{2I_{2a/\sigma^2-1}}-\frac{(I_{2a/\sigma^2-2}+I_{2a/\sigma^2})^2}{2I^2_{2a/\sigma^2-1}}\right)\Bigg)
$$

$$
=-\frac{2\sigma^4}{\bar{b}^4}-\sigma^2\frac{(1-e^{\bar{b}T}(2\bar{b}T+1))}{\bar{b}^2(e^{\bar{b}T}-1)}\left(\frac{\sigma^2}{\bar{b}^2}+\mathbb{E}[\Lambda_T\mid S_T=y]\right)
$$

$$
+\frac{1}{\bar{b}(e^{\bar{b}T}-1)^2}\Bigg(-\sigma^4T^2\frac{e^{2\bar{b}T}}{\bar{b}}-2\frac{\sigma^2T}{\bar{b}}(S_0+y)\big(e^{2\bar{b}T}-e^{\bar{b}T}(\bar{b}T+1)\big)
$$

$$
-\frac{\sigma^2T}{2\bar{b}}\sqrt{yS_0e^{\bar{b}T}}\left(e^{\bar{b}T}\left(3\bar{b}T-4\right)+\bar{b}T+4\right)\frac{I_{2a/\sigma^2}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh\left(\bar{b}T/2\right)}\right)+I_{2a/\sigma^2-2}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh\left(\bar{b}T/2\right)}\right)}{I_{2a/\sigma^2-1}\left(\frac{2\bar{b}\sqrt{yS_0}}{\sigma^2\sinh\left(\bar{b}T/2\right)}\right)}
$$

$$
+\frac{yS_0e^{bT}}{\bar{b}(1-e^{-\bar{b}T})^2}\left(e^{\bar{b}T}(\bar{b}T-2)+\bar{b}T+2\right)^2\left(2+\frac{I_{2a/\sigma^2-3}+I_{2a/\sigma^2+1}}{I_{2a/\sigma^2-1}}-\frac{(I_{2a/\sigma^2-2}+I_{2a/\sigma^2})^2}{(I_{2a/\sigma^2-1})^2}\right)\Bigg).
$$

□

As an alternative to Proposition 3.2 one can also compute directly $\mathbb{E}[\Lambda_T^2 \mid S_T=y]$ and take advantage of algebraic cancellations for better numerical stability, before subtraction of $(\mathbb{E}[\Lambda_T \mid S_T=y])^2$.

Based on the first and second moments of Λ_T, we will fit its marginal density to the gamma and lognormal densities.

Conditional gamma approximation

Under the conditional gamma approximation we have

$$
f_{\Lambda_T|S_T=y}(x)\approx\frac{e^{-x/\theta_T(y)}}{\theta_T(y)}\frac{(x/\theta_T(y))^{-1+\nu_T(y)}}{\Gamma\left(\nu_T(y)\right)},\quad x,y>0, \tag{3.5}
$$

where $\theta_T(y)$, $\nu_T(y)$ are parameters estimated from Propositions 3.1 and 3.2, by matching the first and second conditional moments of Λ_T to those of the gamma distribution, as

$$
\theta_T(y)=\frac{\mathrm{Var}[\Lambda_T\mid S_T=y]}{\mathbb{E}[\Lambda_T\mid S_T=y]}\quad\text{and}\quad\nu_T(y)=\frac{\mathbb{E}[\Lambda_T\mid S_T=y]}{\theta_T(y)}=\frac{(\mathbb{E}[\Lambda_T\mid S_T=y])^2}{\mathrm{Var}[\Lambda_T\mid S_T=y]}. \tag{3.6}
$$

Conditional lognormal approximation

Under the conditional lognormal approximation we have

$$f_{\Lambda_T|S_T=y}(x) \approx \frac{1}{x\sigma_T(y)\sqrt{2\pi T}} e^{-(-\mu_T(y)+\log x)^2/(2T\sigma_T^2(y))}, \quad x, y > 0, \tag{3.7}$$

where $\mu_T(y)$, $\sigma_T(y)$ are parameters estimated from Propositions 3.1 and 3.2, by matching the first and second conditional moments

$$\mathbb{E}[\Lambda_T \mid S_T = y] = e^{\mu_T(y)+\sigma_T^2(y)T/2} \quad \text{and} \quad \mathbb{E}[\Lambda_T^2 \mid S_T = y] = e^{2(\mu_T(y)+\sigma_T^2(y)T)}$$

to those of the lognormal distribution with mean $\mu_T(y)$ and variance $\sigma_T^2(y)T$, i.e.

$$\mu_T(y) = -\frac{\sigma_T^2(y)T}{2} + \log \mathbb{E}[\Lambda_T \mid S_T = y] \text{ and } \sigma_T^2(y)T = \log\left(1 + \frac{\mathrm{Var}[\Lambda_T \mid S_T = y]}{(\mathbb{E}[\Lambda_T \mid S_T = y])^2}\right), \tag{3.8}$$

where $\mathbb{E}[\Lambda_T \mid S_T = y]$ and $\mathrm{Var}[\Lambda_T \mid S_T = y]$ are computed from Propositions 3.1 and 3.2. Figure 1 presents an example of fitting of the marginal density

$$f_{\Lambda_T}(x) = \int_0^\infty f_{(\Lambda_T,S_T)}(x,y)dy, \qquad x > 0,$$

where $f_{(\Lambda_T,S_T)}(x,y)$ is given by Theorem 3 of [6] with $a = 0$, using the gamma and lognormal densities similarly to (3.6) and (3.8).

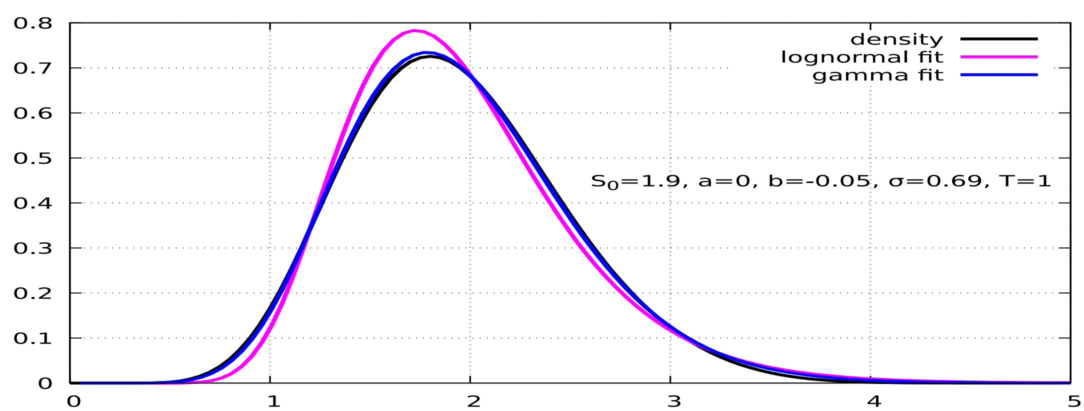

Figure 1: Marginal density $f_{\Lambda_T}(x)$ fitted to the gamma and lognormal densities.

Figure 2 presents an example of the fitting of the conditional density function

$$f_{\Lambda_T|S_T=x}(x,y) = \frac{f_{(\Lambda_T,S_T)}(x,y)}{f_{S_T}(y)},$$

where $f_{S_T}(y)$ is given in (2.4).

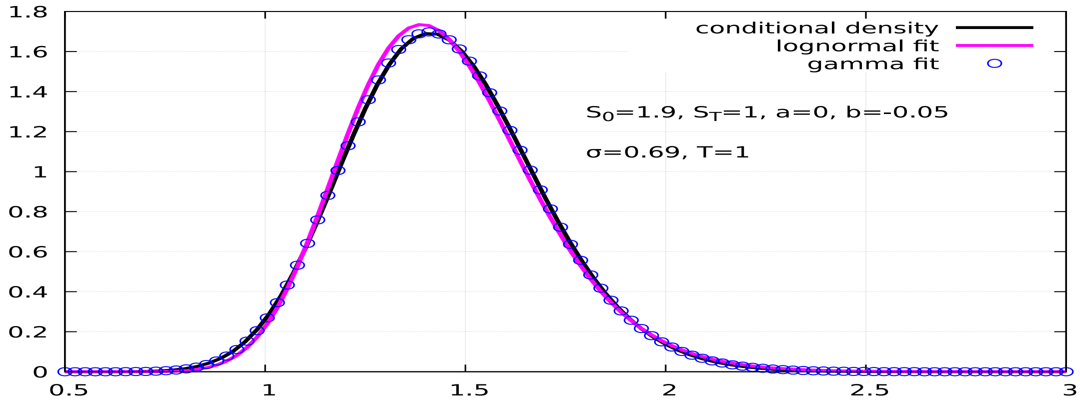

Figure 2: Conditional density $f_{\Lambda_T|S_T}$ fitted to the gamma and lognormal densities.

4 Option pricing

4.1 Equity Asian options

In this section we consider the Equity Asian call prices given by

$$\mathrm{EA}(K,T) := e^{bT}\,\mathbb{E}\left[\left(\frac{\Lambda_T}{T}-K\right)^+\right].$$

Gamma approximation

Under the unconditional gamma approximation we have

$$f_{\Lambda_T}(x) \approx \frac{1}{\Gamma(\nu_T)}\frac{x^{-1+\nu_T}}{(\theta_T)^{\nu_T}}, \tag{4.1}$$

where

$$\theta_T = \frac{\mathrm{Var}[\Lambda_T]}{\mathbb{E}[\Lambda_T]} \quad \text{and} \quad \nu_T = \frac{\mathbb{E}[\Lambda_T]}{\theta_T} = \frac{(\mathbb{E}[\Lambda_T])^2}{\mathrm{Var}[\Lambda_T]}, \tag{4.2}$$

and $\mathbb{E}[\Lambda_T]$, $\mathrm{Var}[\Lambda_T]$ are given by (3.2) and (3.3). Hence the Asian option price can be approximated as

$$\begin{aligned}\mathbb{E}\left[(\Lambda_T-KT)^+\right] &= \int_{KT}^{\infty}(x-KT)^+ f_{\Lambda_T}(x)dx \\ &\approx \frac{1}{\Gamma(\nu_T)}\int_{KT}^{\infty}(x-KT)e^{-x/\theta_T}\frac{x^{-1+\nu_T}}{(\theta_T)^{\nu_T}}dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\nu_T)}\int_{KT}^{\infty} e^{-x/\theta_T}(x/\theta_T)^{\nu_T}dx - \frac{KT}{\Gamma(\nu_T)}\int_{KT}^{\infty} e^{-x/\theta_T}\frac{x^{-1+\nu_T}}{(\theta_T)^{\nu_T}}dx \\
&= \frac{\theta_T}{\Gamma(\nu_T)}\int_{KT/\theta_T}^{\infty} e^{-x}x^{\nu_T}dx - \frac{KT}{\Gamma(\nu_T)}\int_{KT/\theta_T}^{\infty} e^{-x}x^{-1+\nu_T}dx \\
&= \theta_T\nu_T Q\left(1+\nu_T, \frac{KT}{\theta_T}\right) - KTQ\left(\nu_T, \frac{KT}{\theta_T}\right),
\end{aligned}
$$

where

$$
Q(\lambda, z) := \frac{1}{\Gamma(\lambda)}\int_z^{\infty} t^{\lambda-1}e^{-t}dt, \qquad z > 0,
$$

is the normalized upper incomplete gamma function, which yields

$$
\begin{aligned}
\mathrm{EA}(K,T) &= e^{bT}\,\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] \\
&\approx \frac{e^{bT}}{T}\left(\nu_T\theta_T Q\left(1+\nu_T, \frac{KT}{\theta_T}\right) - KTQ\left(\nu_T, \frac{KT}{\theta_T}\right)\right). \qquad (4.3)
\end{aligned}
$$

Stratified gamma approximation

Under the conditional gamma approximation (3.5), the Asian equity call option price will be approximated from (4.3) by

$$
\begin{aligned}
&\mathrm{EA}(K,T) = e^{bT}\,\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] \\
&= \frac{e^{bT}}{T}\int_{KT}^{\infty}\mathbb{E}\left[(\Lambda_T - KT)^+ \,\middle|\, S_T = y\right] f_{S_T}(y)dy \\
&\approx \frac{e^{bT}}{T}\int_0^{\infty}\left(\nu_T(y)\theta_T(y)Q\left(1+\nu_T(y), \frac{KT}{\theta_T(y)}\right) - KTQ\left(\nu_T(y), \frac{KT}{\theta_T(y)}\right)\right) f_{S_T}(y)dy,
\end{aligned}
\tag{4.4}
$$

with $f_{S_T}(y)$ given by (2.1), where $\nu_T(y)$ and $\theta_T(y)$ are given by (3.6).

Lognormal approximation

Under the unconditional lognormal approximation we have

$$
f_{\Lambda_T}(x) \approx \frac{1}{x\sigma_T\sqrt{2\pi T}}e^{-(\mu_T - \log x)^2/(2T\sigma_T^2)}, \quad x > 0, \tag{4.5}
$$

where

$$
\sigma_T^2 T = \log\left(1 + \mathrm{Var}[\Lambda_T]/(\mathbb{E}[\Lambda_T])^2\right) \quad \text{and} \quad \mu_T = -\frac{\sigma_T^2 T}{2} + \log\mathbb{E}[\Lambda_T],
$$

hence the lognormal approximation

$$\begin{aligned}
\mathrm{EA}(K,T) &= \frac{e^{bT}}{T}\,\mathbb{E}\left[(\Lambda_T - KT)^+\right] \qquad (4.6)\\
&= \frac{e^{bT}}{T}\int_{KT}^{\infty}(x-KT)^+ f_{\Lambda_T}(x)dx\\
&\approx \frac{e^{bT}}{T}\left(e^{\mu_T+\sigma_T^2T/2}\Phi(d_1) - KT\Phi(d_2)\right), \qquad (4.7)
\end{aligned}$$

where Φ denotes the standard Gaussian cumulative distribution function and

$$d_1 := \frac{\log(\mathbb{E}[\Lambda_T]/(KT))}{\sigma_T\sqrt{T}} + \sigma_T\frac{\sqrt{T}}{2}, \qquad d_2 := d_1 - \sigma_T\sqrt{T}.$$

Stratified lognormal approximation

Under the conditional lognormal approximation (3.7) we have

$$\begin{aligned}
\mathrm{EA}(K,T) &= \frac{e^{bT}}{T}\int_0^{\infty}\mathbb{E}\left[(\Lambda_T-KT)^+\,\Big|\,S_T=z\right]f_{S_T}(y)dy \qquad (4.8)\\
&\approx \frac{e^{bT}}{T}\int_0^{\infty}\left(e^{\mu_T(y)+\sigma_T^2(y)T/2}\Phi(d_1(y)) - KT\Phi(d_2(y))\right)f_{S_T}(y)dy,
\end{aligned}$$

where

$$d_1(y) = \frac{\log(\mathbb{E}[\Lambda_T \mid S_T=y]/(KT))}{\sigma_T(y)\sqrt{T}} + \sigma_T(y)\frac{\sqrt{T}}{2},$$

and $d_2(y) = d_1(y) - \sigma_T(y)\sqrt{T}$, where $\mu_T(y)$ and $\sigma_T(y)$ are given by (3.8).

In Table 1 we compare our conditional (or stratified) and unconditional moment matching estimates with the analytical prices computed in [6] with the parameters of [3] and $a = 0$. In this table and in Figure 3, the Monte Carlo method is implemented with 100,000 sample paths via a numerical solution of the CIR equation (1.1) by the Euler scheme, and the integrals are discretized over 100 time steps.

In rows 3-5 of Table 1 we note some differences with the values computed in [6], possibly connected to the observation that the joint density method may fail for small values of σ^2T, while our approximations remain consistent with the Monte Carlo estimates.

b	σ	T	K	S_0	DN*	JD†	StrG ‡	StrL §	Gam ¶	Log‖	MC**
-0.05	0.69	1	2	1.9	0.1902	0.1908	0.1908	0.1908	0.1904	0.1878	0.1903
-0.05	0.72	1	2	2.1	0.3098	0.3087	0.3087	0.3084	0.3075	0.3014	0.3099
-0.02	0.14	1	2	2	0.0197	inf	0.0560	0.0560	0.0555	0.0554	0.0554
-0.18	0.42	1	2	2	0.2189	inf	0.2179	0.2178	0.2174	0.2152	0.2190
-0.01	0.35	2	2	2	0.1725	inf	0.1689	0.1688	0.1686	0.1670	0.1681
-0.05	0.71	2	2	2	0.3339	0.3531	0.3531	0.3527	0.3508	0.3391	0.3567

Table 1: Equity Asian (EA) call prices.

Figure 3 presents a summary of Equity Asian call option prices compared to the double numerical integration of the joint density function given in [6].

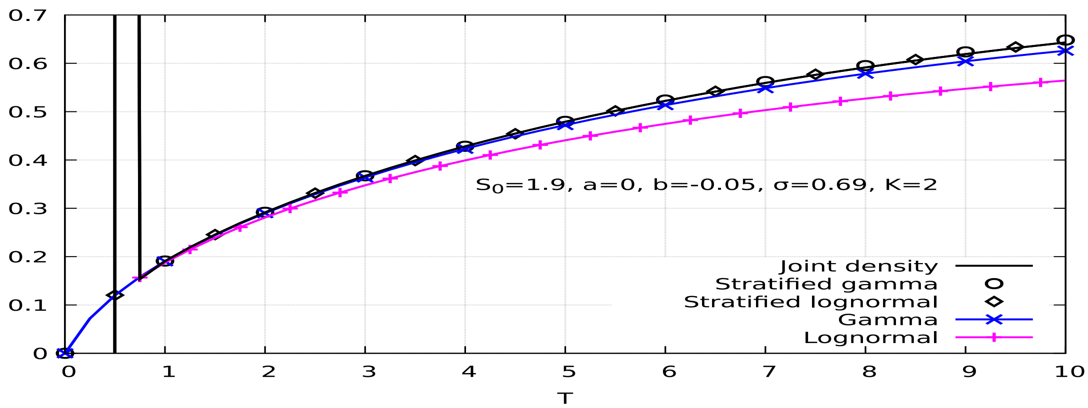

Figure 3: Equity Asian (EA) call prices for $T \in [0, 10]$.

Table 2 presents a sample of computation times for comparison of the different methods.

Parameters						Time					
S_0	a	b	σ	T	K	StrG	StrL	Gam	Log	MC	JD
2.1	0.0	-0.05	0.72	1.0	2.0	1.32e-02	1.23e-02	2.60e-5	1.80e-5	144.46	8.62

Table 2: Equity Asian (EA) call option - computation times in seconds.*

*Prices of [6].
†Double numerical integration of the joint density function $f_{(\Lambda_T, S_T)}(x, y)$ of [6].
‡Conditional or stratified gamma approximation (4.4).
§Conditional or stratified lognormal approximation (4.8).
¶Gamma approximation (4.3).
‖Lognormal approximation (4.6).
**Monte Carlo with 100,000 sample paths.

The Equity Asian put price can be computed from the parity relation

$$\begin{aligned}
\mathrm{EA}^p(K,T) &:= e^{bT}\,\mathbb{E}\left[\left(K-\frac{\Lambda_T}{T}\right)^+\right] \\
&= Ke^{bT}-\frac{e^{bT}}{T}\,\mathbb{E}\left[\Lambda_T\right]+\mathrm{EA}(K,T) \\
&= Ke^{bT}-\frac{S_0b(e^{bT}-1)+ae^{bT}(bT-1)+a}{b^2T}+\mathrm{EA}(K,T).
\end{aligned}$$

4.2 Cash Binary Asian caps

We consider the Cash Binary Asian cap price

$$\mathrm{CBA}(K,T):=\mathbb{E}\left[e^{-\Lambda_T}\mathbb{1}_{\{\Lambda_T>KT\}}\right].$$

Under the unconditional gamma approximation (4.1) we have

$$\begin{aligned}
\mathbb{E}\left[e^{-\Lambda_T}\mathbb{1}_{\{\Lambda_T>KT\}}\right] &= \int_{KT}^{\infty} e^{-x} f_{\Lambda_T}(x)dx \\
&\approx \frac{1}{\Gamma(\nu_T)}\int_{KT}^{\infty} e^{-x(1+/\theta_T)}\frac{x^{-1+\nu_T}}{(\theta_T)^{\nu_T}}dx \\
&= \frac{1}{(1+\theta_T)^{\nu_T}}\frac{1}{\Gamma(\nu_T)}\int_{KT(1+1/\theta_T)}^{\infty} x^{-1+\nu_T}e^{-x}dx \\
&= \frac{1}{(1+\theta_T)^{\nu_T}}Q\left(\nu_T, KT+\frac{KT}{\theta_T}\right),
\end{aligned}$$

where ν_T and θ_T are given by (4.2). By the conditional gamma approximation (3.5) we find

$$\begin{aligned}
\mathrm{CBA}(K,T) &= \int_0^{\infty}\mathbb{E}\left[e^{-\Lambda_T}\mathbb{1}_{\{\Lambda_T>KT\}}\;\middle|\; S_T=y\right] f_{S_T}(y)dy \\
&\approx \int_0^{\infty}\frac{1}{(1+\theta_T(y))^{\nu_T(y)}}Q\left(\nu_T(y), KT+\frac{KT}{\theta_T(y)}\right) f_{S_T}(y)dy,
\end{aligned}$$

with $f_{S_T}(y)$ given by (2.1). Such formulas are not available for the lognormal approximations (3.7) and (4.5), as the lognormal moment generating function does not admit a closed form expression. In this case we will rely on discretizations of the single integral

$$\mathrm{CBA}(K,T) = \int_{KT}^{\infty} e^{-x} f_{\Lambda_T}(x)dx$$

$$\approx \quad \frac{1}{\sigma_T\sqrt{2\pi T}} \int_{KT}^{\infty} e^{-x-(-\mu_T+\log x)^2/(2T\sigma_T^2)} \frac{dx}{x},$$

and double integral

$$\begin{aligned}
\mathrm{CBA}(K,T) &= \int_0^\infty \mathbb{E}\left[e^{-\Lambda_T}\mathbb{1}_{\{\Lambda_T>KT\}} \mid S_T=y\right] f_{S_T}(y)dy \\
&= \int_0^\infty \int_0^\infty e^{-x}\mathbb{1}_{\{x>KT\}} f_{\Lambda_T|S_T=y}(x)dx f_{S_T}(y)dy \\
&\approx \frac{1}{\sigma_T(y)\sqrt{2\pi T}} \int_0^\infty \int_{KT}^\infty e^{-x-(-\mu_T(y)+\log x)^2/(2T\sigma_T^2(y))} \frac{dx}{x} f_{S_T}(y)dy,
\end{aligned}$$

similarly to the Asian caps considered in Sections 4.3 and 4.4. Table 3 shows that our stratified gamma approximation generally matches the results of [5].

		Maturity					Maturity				
				T = 0.1					T = 0.5		
Strike	Type	DN*	StrG†	StrL‡	Gam§	Log¶	DN	StrG	StrL	Gam	Log
0.08	CBA	0.9631	0.9632	0.9678	0.9636	0.9732	0.8018	0.8020	0.8161	0.8019	0.8273
0.12	CBA	0.0387	0.0386	0.0451	0.0390	0.0489	0.1459	0.1458	0.1579	0.1458	0.1571
				T = 1					T = 2		
Strike	Type	DN	StrG	StrL	Gam	Log	DN	StrG	StrL	Gam	Log
0.08	CBA	0.7301	0.7306	0.7476	0.7302	0.7558	0.6644	0.6643	0.6832	0.6630	0.6867
0.12	CBA	0.1565	0.1563	0.1679	0.1575	0.1667	0.1317	0.1322	0.1421	0.1337	0.1421
				T = 5					T = 10		
Strike	Type	DN	StrG	StrL	Gam	Log	DN	StrG	StrL	Gam	Log
0.08	CBA	0.5354	0.5317	0.5465	0.5313	0.5474	0.3504	0.3480	0.3545	0.3479	0.3547
0.12	CBA	0.0631	0.0632	0.0704	0.0633	0.0704	0.0185	0.0179	0.0214	0.0179	0.0214

Table 3: Cash Binary Asian (CBA) cap prices, $S_0 = 0.1$, $a = 0.15$, $b = 1.5$, $\sigma = 0.2$.

Figure 4 uses the same parameters as in [3], [5], and Table 3. It shows that the gamma approximation performs better than the lognormal approximation, which is consistent with the fact that the limiting and invariant distribution of $(S_t)_{t\in\mathbb{R}_+}$ is gamma according to (2.2).

*Prices of [5].
†Stratified gamma approximation.
‡Stratified lognormal approximation.
§Non-stratified gamma approximation.
¶Non-stratified lognormal approximation.

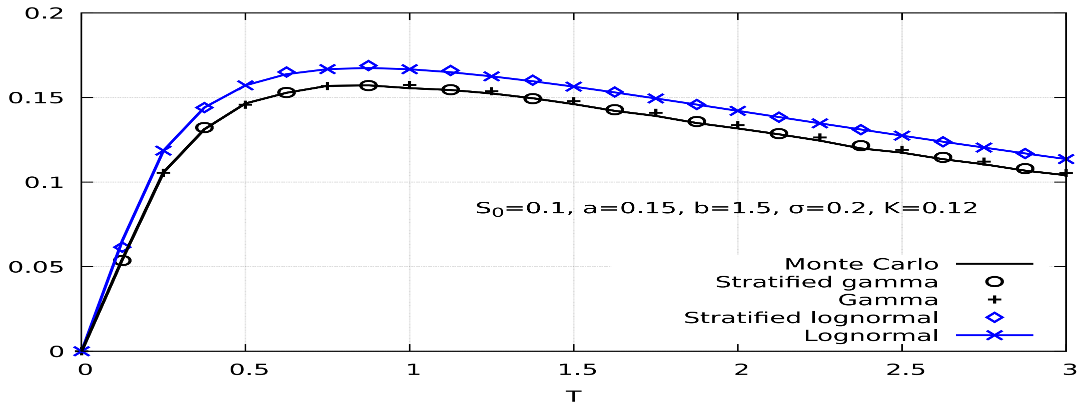

Figure 4: Cash Binary Asian (CBA) cap prices for $T \in [0, 3]$.

The Cash Binary floor price can be similarly estimated under the conditional gamma approximation as

$$\begin{aligned} \mathrm{CBA}^f(K,T) &:= \int_0^\infty \mathbb{E}\left[e^{-\Lambda_T}\mathbb{1}_{\{\Lambda_T<KT\}} \;\middle|\; S_T=y\right] f_{S_T}(y)dy \\ &\approx \int_0^\infty \frac{1}{(1+\theta_T(y))^{\nu_T(y)}} P\left(\nu_T(y), KT+\frac{KT}{\theta_T(y)}\right) f_{S_T}(y)dy, \end{aligned}$$

where

$$P(\lambda,z) := \frac{1}{\Gamma(\lambda)}\int_0^z t^{\lambda-1}e^{-t}dt, \qquad z>0,$$

is the normalized lower incomplete gamma function.

4.3 Rate Binary Asian caps

Next, we consider the Rate Binary Asian caps

$$\mathrm{RBA}(K,T) := \mathbb{E}\left[\Lambda_T e^{-\Lambda_T}\mathbb{1}_{\{\Lambda_T>KT\}}\right].$$

Under the unconditional gamma approximation (4.1) we have

$$\begin{aligned} \mathbb{E}\left[\Lambda_T e^{-\Lambda_T}\mathbb{1}_{\{\Lambda_T>KT\}}\right] &= \int_{KT}^\infty x e^{-x} f_{\Lambda_T}(x)dx \\ &= \frac{1}{\Gamma(\nu_T)}\int_{KT}^\infty e^{-x(1+1/\theta_T)}(x/\theta_T)^{\nu_T}dx \\ &= \frac{1}{\Gamma(\nu_T)}\frac{\theta_T}{(1+\theta_T)^{\nu_T+1}}\int_{KT(1+1/\theta_T)}^\infty e^{-x}x^{\nu_T}dx \\ &= \frac{\theta_T\nu_T}{(1+\theta_T)^{\nu_T+1}} Q\left(1+\nu_T, KT+\frac{KT}{\theta_T}\right). \end{aligned}$$

Using (3.5) we obtain the stratified gamma approximation

$$\begin{aligned}\mathrm{RBA}(K,T) &= \int_0^\infty \mathbb{E}\left[\Lambda_T e^{-\Lambda_T}\mathbb{1}_{\{\Lambda_T>KT\}} S_T = y\right] f_{S_T}(y)dy \\ &\approx \int_0^\infty \frac{\theta_T(y)\nu_T(y)}{(1+\theta_T(y))^{\nu_T(y)+1}} Q\left(1+\nu_T(y), KT+\frac{KT}{\theta_T(y)}\right) f_{S_T}(y)dy,\end{aligned}$$

where $f_{S_T}(y)$ is given by (2.1). Such formulas are not available for the lognormal approximations (3.7) and (4.5), and in this case we rely on the discretization of single and double integrals as in Section 4.2.

In Table 4 we compare our results with those obtained in [5] using the parameters of [3]. Again, the gamma approximation performs significantly better than the lognormal approximation, while it tends to slightly underestimate prices for large maturities.

				Maturity T = 0.1					Maturity T = 0.5		
Strike	Type	DN	StrG	StrL	Gam	Log	DN	StrG	StrL	Gam	Log
0.08	RBA	0.0097	0.0097	0.0097	0.0097	0.0098	0.0421	0.0421	0.0427	0.0421	0.0430
0.12	RBA	0.0005	0.0005	0.0006	0.0005	0.0006	0.0097	0.0097	0.0104	0.0096	0.0104
				T = 1					T = 2		
Strike	Type	DN	StrG	StrL	Gam	Log	DN	StrG	StrL	Gam	Log
0.08	RBA	0.0777	0.0777	0.0790	0.0778	0.0796	0.1402	0.1404	0.1432	0.1403	0.1437
0.12	RBA	0.0211	0.0211	0.0225	0.0212	0.0224	0.0354	0.0355	0.0379	0.0358	0.0380
				T = 5					T = 10		
Strike	Type	DN	StrG	StrL	Gam	Log	DN	StrG	StrL	Gam	Log
0.08	RBA	0.2731	0.2719	0.2775	0.2718	0.2778	0.3495	0.3477	0.3528	0.3477	0.3530
0.12	RBA	0.0410	0.0409	0.0454	0.0409	0.0454	0.0234	0.0226	0.0269	0.0226	0.0269

Table 4: Rate Binary Asian (RBA) cap prices, $S_0 = 0.1$, $a = 0.15$, $b = 1.5$, $\sigma = 0.2$.

Figure 5 uses increased values of the parameters S_0, a, b and K for a clearer comparison between the proposed methods.

The Rate Binary Asian floor price can be similarly estimated under the conditional gamma approximation as

$$\begin{aligned}\mathrm{RBA}^f(K,T)(K,T) &:= \int_0^\infty \mathbb{E}\left[\Lambda_T e^{-\Lambda_T}\mathbb{1}_{\{\Lambda_T<KT\}} S_T = y\right] f_{S_T}(y)dy \\ &\approx \int_0^\infty \frac{\theta_T(y)\nu_T(y)}{(1+\theta_T(y))^{\nu_T(y)+1}} P\left(1+\nu_T(y), KT+\frac{KT}{\theta_T(y)}\right) f_{S_T}(y)dy.\end{aligned}$$

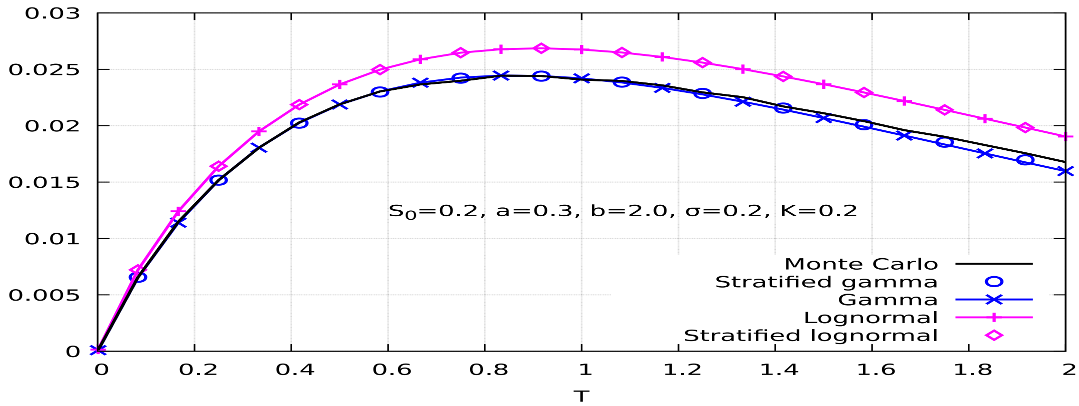

Figure 5: Rate Binary Asian (RBA) cap prices for $T \in [0, 2]$.

4.4 Regular Asian caps

Finally we consider the regular Asian cap with price

$$\mathrm{AO}^c(K,T) := \mathbb{E}\left[e^{-\Lambda_T}\left(\frac{\Lambda_T}{T} - K\right)^+\right]$$

which can be estimated as

$$\begin{aligned}
&\mathrm{AO}^c(K,T) := \mathbb{E}\left[e^{-\Lambda_T}\left(\frac{\Lambda_T}{T} - K\right)^+\right] \\
&= \frac{1}{T}\mathrm{RBA}(K,T) - K \times \mathrm{CBA}(K,T) \\
&\approx \frac{\theta_T}{(1+\theta_T)^{\nu_T+1}}\left(\nu_T Q\left(1+\nu_T, KT + \frac{KT}{\theta_T}\right) - \left(KT + \frac{KT}{\theta_T}\right) Q\left(\nu_T, KT + \frac{KT}{\theta_T}\right)\right)
\end{aligned}$$

under the unconditional gamma approximation (4.1). Using the conditional gamma approximation (3.5), we find

$$\begin{aligned}
\mathrm{AO}^c(K,T) &= \mathbb{E}\left[e^{-\Lambda_T}\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] \\
&\approx \frac{1}{T}\int_0^\infty \frac{\theta_T(y)}{(1+\theta_T(y))^{\nu_T(y)+1}}\left(\nu_T(y) Q\left(1+\nu_T(y), KT + \frac{KT}{\theta_T(y)}\right)\right. \\
&\qquad \left. - \left(KT + \frac{KT}{\theta_T(y)}\right) Q\left(\nu_T(y), KT + \frac{KT}{\theta_T(y)}\right)\right) f_{S_T}(y) dy,
\end{aligned}$$

with $f_{S_T}(y)$ given by (2.1). In Table 5 we compare our results to those of [5], with the parameters of [3].

Strike	Type	Maturity T = 0.1 DN	StrG	StrL	Gam	Log	Maturity T = 0.5 DN	StrG	StrL	Gam	Log
0.08	AO^c	0.0199	0.0199	0.0199	0.0199	0.0199	0.0201	0.0210	0.0200	0.0201	0.0199
0.12	AO^c	0.0002	0.0002	0.0002	0.0002	0.0002	0.0018	0.0018	0.0018	0.0018	0.0019
		T = 1					T = 2				
Strike	Type	DN	StrG	StrL	Gam	Log	DN	StrG	StrL	Gam	Log
0.08	AO^c	0.0193	0.0193	0.0192	0.0194	0.0191	0.0170	0.0170	0.0169	0.0171	0.0169
0.12	AO^c	0.0023	0.0023	0.0023	0.0023	0.0024	0.0019	0.0019	0.0019	0.0018	0.0019
		T = 5					T = 10				
Strike	Type	DN	StrG	StrL	Gam	Log	DN	StrG	StrL	Gam	Log
0.08	AO^c	0.0118	0.0118	0.0118	0.0118	0.0118	0.0069	0.0069	0.0069	0.0069	0.0069
0.12	AO^c	0.0006	0.0006	0.0006	0.0006	0.0006	0.0001	0.0001	0.0001	0.0001	0.0001

Table 5: Cash Regular Asian (AO) cap prices, $S_0 = 0.1$, $a = 0.15$, $b = 1.5$, $\sigma = 0.2$.

Figure 6 presents the evolution of prices according to maturity times. As can be seen in Table 5, all five methods show consistent numerical results in this case.

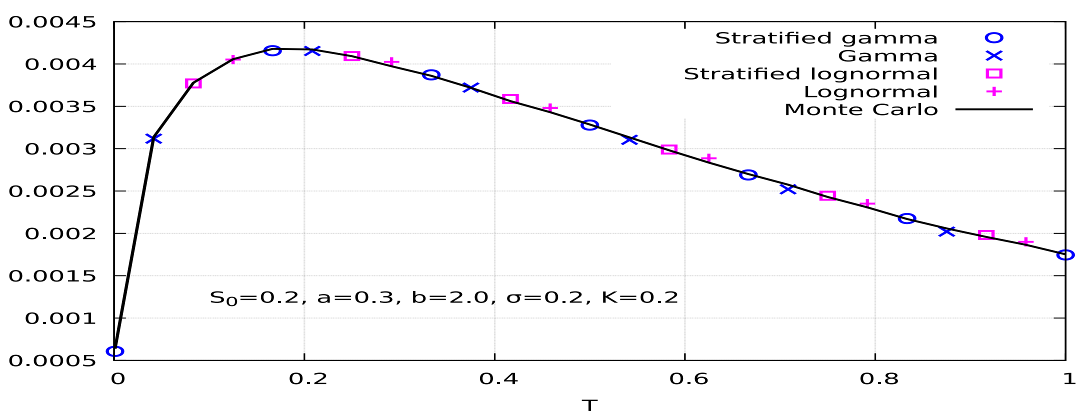

Figure 6: Regular Asian (AO) cap prices for $T \in [0, 1]$.

The regular Asian floor price can be similarly estimated from

$$\mathrm{AO}^f(K,T) := \mathbb{E}\left[e^{-\Lambda_T}\left(K - \frac{\Lambda_T}{T}\right)^+\right] = K \times \mathrm{CBA}^f(K,T) - \frac{1}{T}\mathrm{RBA}^f(K,T).(K,T).$$

Regular Asian put-call parity

From the relation

$$E\left[e^{\eta\Lambda_T}\right] = e^{-S_0\psi(\eta) - a\phi(\eta)},$$

where $\bar{b} := \sqrt{b^2 - 2\eta\sigma^2}$ and

$$\psi(\eta) := \frac{2\eta(e^{-\bar{b}T} - 1)}{\bar{b} + b + e^{-\bar{b}T}(\bar{b} - b)} \qquad \phi(\eta) := \frac{1}{\sigma^2}(\bar{b} - b)T + \frac{2}{\sigma^2}\log\frac{\bar{b} + b + e^{-\bar{b}T}(\bar{b} - b)}{2\bar{b}},$$

which satisfy

$$\begin{aligned}
\psi'(\eta) &= \frac{2(e^{-\bar{b}T}-1)}{\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b)} + \frac{2\eta T\sigma^2 e^{-\bar{b}T}}{\bar{b}(\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b))} \\
&\quad - \frac{2\eta(e^{-\bar{b}T}-1)}{(\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b))^2}(-\sigma^2/\bar{b}+(\sigma^2 T/\bar{b})e^{-\bar{b}T}(\bar{b}-b)-e^{-\bar{b}T}(\sigma^2/\bar{b})) \\
&= \frac{2\bar{b}(e^{-\bar{b}T}-1)+2\eta T\sigma^2 e^{-\bar{b}T}}{\bar{b}(\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b))} + \frac{2\eta\sigma^2(e^{-\bar{b}T}-1)(1-Te^{-\bar{b}T}(\bar{b}-b)+e^{-\bar{b}T})}{\bar{b}(\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b))^2}
\end{aligned}$$

and

$$\begin{aligned}
\phi'(\eta) &= -\frac{T}{\bar{b}} + \frac{2(-\sigma^2/\bar{b}+(\sigma^2 T/\bar{b})e^{-\bar{b}T}(\bar{b}-b)+e^{-\bar{b}T}(-\sigma^2/\bar{b}))}{\sigma^2(\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b))} + \frac{2(2\sigma^2/\bar{b})}{\sigma^2(2\bar{b})} \\
&= -\frac{T}{\bar{b}} - 2\frac{1-e^{-\bar{b}T}(\bar{b}-b)T+e^{-\bar{b}T}}{\bar{b}(\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b))} + \frac{2}{\bar{b}^2},
\end{aligned}$$

with $\bar{b}' = -\sigma^2/\bar{b}$ and $\eta = -1$, we find

$$\begin{aligned}
&\mathbb{E}\left[\Lambda_T e^{-\Lambda_T}\right] = -(S_0\psi'(-1)+a\phi'(-1))e^{-S_0\psi(-1)-a\phi(-1)} \\
&= -S_0\left(\frac{2\bar{b}(e^{-\bar{b}T}-1)-2\sigma^2 Te^{-\bar{b}T}}{\bar{b}(\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b))} - \frac{2\sigma^2(e^{-\bar{b}T}-1)(1-e^{-\bar{b}T}(\bar{b}-b)T+e^{-\bar{b}T})}{\bar{b}(\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b))^2}\right) E\left[e^{-\Lambda_T}\right] \\
&\quad + \frac{a}{\bar{b}}\left(T + 2\frac{1-e^{-\bar{b}T}(\bar{b}-b)T+e^{-\bar{b}T}}{\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b)} - \frac{2}{\bar{b}}\right) E\left[e^{-\Lambda_T}\right],
\end{aligned}$$

which allows us to estimate the regular Asian floor price from

$$\begin{aligned}
\mathrm{AO}^f(K,T) &= \mathbb{E}\left[e^{-\Lambda_T}\left(K-\frac{\Lambda_T}{T}\right)^+\right] \\
&= \mathbb{E}\left[e^{-\Lambda_T}\left(\frac{\Lambda_T}{T}-K\right)^+\right] - \left(\frac{1}{T}\,\mathbb{E}\left[\Lambda_T e^{-\Lambda_T}\right] - K\,\mathbb{E}\left[e^{-\Lambda_T}\right]\right) \\
&= \mathrm{AO}^c(K,T) + K\,\mathbb{E}\left[e^{-\Lambda_T}\right] - \frac{1}{T}\,\mathbb{E}\left[\Lambda_T e^{-\Lambda_T}\right].
\end{aligned}$$

5 Conclusion

The conditional moment matching method provides a fast way to price options in the CIR model, and it avoids technicalities linked to more sophisticated approaches involving numerical Fourier-Laplace inversion or complex closed form recursions. Our implementation of conditional moment matching shows that for yield options in the CIR model, the stratified gamma approximation performs consistently better than the lognormal approximations.

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