

Chapter 6

Value at Risk

Value at risk (VaR) is probably the most basic and widely used measure of risk. It relies on estimating the amount that can potentially be lost on a given investment within a certain time range. This chapter starts with a review the concept of risk measure in general, including quantile risk measures, before providing a mathematical treatment of Value at Risk, together with experiments based on actual financial data sets.

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6.1 Risk Measures

Risk measures have two objectives:

- i) to provide a measure for risk, and
- ii) to determine an adequate level of capital reserves that matches the current level of risk.

In what follows, the potential losses associated to a given risk will be modeled by the values of a random variable X .

Definition 6.1. *A risk measure is a mapping that assigns a value V_X to a given loss random variable X .*

For insurance companies, which need to hold a capital in order to meet future liabilities, the capital C_X required to face the risk induced by a potential loss $X \geq 0$ can be defined as

$$C_X := V_X - L_X, \quad (6.1)$$

where

- a) V_X stands for an upper “reasonable” estimate of the potential loss associated to X .
 b) L_X represents the *liabilities* of the company.

In other words, managing risk means here determining a level V_X of provision or capital requirement that will not be “too much” exceeded by X . When $L_X < 0$ the amount $-L_X > 0$ corresponds to a debt owed by the company, while $L_X > 0$ corresponds to positive liabilities such as deferred revenue or to a debt owed to the company.

Some examples of risk measures (Hardy (2006))

- a) The *expected value premium principle* is the risk measure defined by

$$V_X := \mathbb{E}[X] + \alpha \mathbb{E}[X]$$

for some $\alpha \geq 0$. For $\alpha = 0$, $V_X := \mathbb{E}[X]$ it is called the *pure premium* risk measure.

- b) The *standard deviation premium principle* is the risk measure defined by

$$V_X := \mathbb{E}[X] + \alpha \sqrt{\text{Var}[X]}$$

for some $\alpha \geq 0$, where $\text{Var}[X]$ denotes the variance of X .

In order to proceed with more examples of risk measures, we will need to use conditional expectations, see *e.g.* Lemma A.15 for the following proposition. The what follows, we let $\mathbb{1}_A$ denote the *indicator function* of any event A subset of Ω , defined as

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

Lemma 6.2. *Let A be an event such that $\mathbb{P}(A) > 0$. The conditional expectation of $X : \Omega \rightarrow \mathbb{N}$ given the event A satisfies the relation*

$$\mathbb{E}[X | A] := \frac{1}{\mathbb{P}(A)} \mathbb{E}[X \mathbb{1}_A].$$

For example, consider the sample space $\Omega = \{1, 3, -1, -2, 5, 7\}$ with the non-uniform probability measure given by

$$\mathbb{P}(\{-1\}) = \mathbb{P}(\{-2\}) = \mathbb{P}(\{1\}) = \mathbb{P}(\{3\}) = \mathbb{P}(\{7\}) = \frac{1}{7}, \quad \mathbb{P}(\{5\}) = \frac{2}{7},$$

and the random variable

$$X : \Omega \longrightarrow \mathbb{Z}$$

given by

$$X(k) = k, \quad k = 1, 3, -1, -2, 5, 7.$$

Here, $\mathbb{E}[X \mid X > 1]$ denotes the expected value of X given the event

$$A := \{X > 1\} = \{3, 5, 7\} \subset \Omega,$$

i.e. the mean value of X given that X is strictly greater than one. This conditional expectation can be computed as

$$\begin{aligned} \mathbb{E}[X \mid X > 1] &= 3 \times \mathbb{P}(X = 3 \mid X > 1) + 5 \times \mathbb{P}(X = 5 \mid X > 1) + 7 \times \mathbb{P}(X = 7 \mid X > 1) \\ &= 3 \times \frac{1}{4} + 5 \times \frac{2}{4} + 7 \times \frac{1}{4} \\ &= \frac{3 + 2 \times 5 + 7}{4}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\frac{1}{\mathbb{P}(X > 1)} \mathbb{E}[X \mathbb{1}_{\{X > 1\}}] \\ &= \frac{1}{\mathbb{P}(X > 1)} (3 \times \mathbb{P}(X = 3) + 5 \times \mathbb{P}(X = 5) + 7 \times \mathbb{P}(X = 7)) \\ &= \frac{1}{4/7} \left(3 \times \frac{1}{7} + 5 \times \frac{2}{7} + 7 \times \frac{1}{7} \right) \\ &= \frac{3 + 5 \times 2 + 7}{4}, \end{aligned}$$

where $\mathbb{P}(X > 1) = 4/7$ and the truncated expectation $\mathbb{E}[X \mathbb{1}_{\{X > 1\}}]$ is given by

$$\mathbb{E}[X \mathbb{1}_{\{X > 1\}}] = \frac{3 + 2 \times 5 + 7}{7}.$$

c) The *Conditional Tail Expectation* (CTE) of X given that $X > 0$ is the risk measure defined as the conditional mean

$$V^X := \mathbb{E}[X \mid X > 0] = \frac{\mathbb{E}[X \mathbb{1}_{\{X > 0\}}]}{\mathbb{P}(X > 0)}. \quad (6.2)$$

Next, we consider the following market returns data.

```

1 library(quantmod)
2 getSymbols("^HSI",from="2013-06-01",to="2014-10-01",src="yahoo")
3 stock<-Ad(" HSI");returns <- as.vector((stock-lag(stock))/lag(stock));
4 times<=index(stock);m=mean(returns[returns<0]),na.rm=TRUE)
5 dev.new(width=16,height=7);par(oma=c(0,1,0,0))
6 plot(times,returns,pch=19,cex=0.4,col="blue",ylab="",xlab="",main="",las=1,cex.lab=1.8,
7      cex.axis=1.8,lwd=3)
8 segments(x0=times,x1=times,y0=0,y1=returns,col="blue")
9 abline(h=m,col="red",lwd=3);length(returns)

```

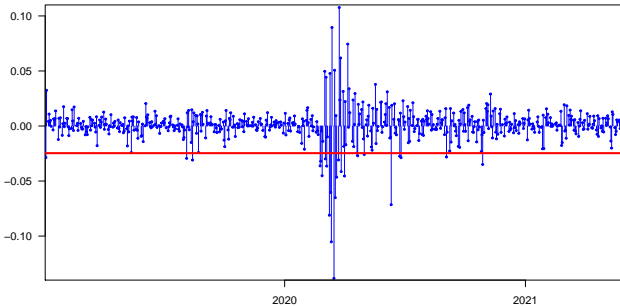


Fig. 6.1: Estimating liabilities by the conditional mean $\mathbb{E}[X | X < 0]$ over 346 market returns.

The conditional tail expectation (CTE) (6.2) estimated in Figure 6.1 can also be computed using the next `R` code, which also implements the statement of Lemma 6.2.

```

1 returns <- returns[!is.na(returns)]
2 condmean<-mean(returns[returns<0])
3 n <- length(returns); sum<-sum(returns[returns<0])
4 proportion<-length(returns[returns<0])/length(returns)
5 condmean; sum/proportion/n
6 condmean<-mean(returns[returns<(-0.025)])
7 n <- length(returns); sum<-sum(returns[returns<(-0.025)])
8 proportion<-length(returns[returns<(-0.025)])/length(returns)
9 condmean; sum/proportion/n

```

Coherent risk measures

Definition 6.3. A risk measure V is said to be coherent if it satisfies the following four properties, for any two random variables X, Y :

i) *Monotonicity:*

$$X \leq Y \implies V_X \leq V_Y,$$

ii) *(Positive) homogeneity:*

$$V_{\lambda X} = \lambda V_X, \quad \text{for constant } \lambda > 0,$$

iii) *Translation invariance:*

$$V_{\mu+X} = \mu + V_X, \quad \text{for constant } \mu > 0,$$

iv) *Subadditivity:*

$$V_{X+Y} \leq V_X + V_Y.$$

Subadditivity means that the combined risk of several portfolios is lower than the sum of risks of those portfolios, as happens usually through *portfolio diversification*. For example, one person traveling might insure the unlikely loss of her phone for $V_X = \$100$. However, two people traveling together might want to insure the phone loss event at a level V_{X+Y} lower than $V_X + V_Y = \$100 + \100 as the simultaneous loss of both phones during a same trip seems even more unlikely.

The concept of subadditivity is common in most pricing engines, as shown in the following example:

$$\text{Price}(\text{🍟} \text{ 🍔} \text{ 🥤}) \leq \text{Price}(\text{🍟}) + \text{Price}(\text{🍔}) + \text{Price}(\text{🥤}).$$

The *expectation* of random variables

$$V_X := \mathbb{E}[X],$$

or *pure premium* risk measure, is an example of a coherent (and additive) risk measure satisfying the above conditions (i)-(iv).

Definition 6.4. A distortion risk measure is a risk measure of the form

$$M_X = \mathbb{E}[X f_X(X)],$$

where f_X is a distortion function, i.e. a nonnegative, non-decreasing function such that

$$i) f_{\mu+X}(\mu+x) = f_X(x), \quad x \in \mathbb{R}, \mu \in \mathbb{R},$$

$$ii) f_{\lambda X}(\lambda x) = f_X(x), \quad x \in \mathbb{R}, \lambda > 0,$$

$$iii) \mathbb{E}[f_X(X)] = 1.$$

We note that distortion risk measures are positive homogeneous and translation invariant.

i) Positive homogeneity. For any $\lambda > 0$, we have

$$\begin{aligned} M_{\lambda X} &= \mathbb{E}[\lambda X f_{\lambda X}(\lambda X)] = \mathbb{E}[\lambda X f_X(X)] \\ &= \lambda \mathbb{E}[X f_X(X)] \\ &= \lambda M_X. \end{aligned}$$

ii) Translation invariance. For any $\mu \in \mathbb{R}$, we have

$$\begin{aligned} M_{\mu+X} &= \mathbb{E}[(\mu + X)f_{\mu+X}(\mu + X)] \\ &= \mathbb{E}[(\mu + X)f_X(X)] \\ &= \mathbb{E}[Xf_X(X)] + \mu\mathbb{E}[f_X(X)] \\ &= \mu + \mathbb{E}[Xf_X(X)] \\ &= \mu + M_X. \end{aligned}$$

See (7.2) and (7.7) below for examples of distortion risk measures.

6.2 Quantile Risk Measures

Definition 6.5. *The Cumulative Distribution Function (CDF) of a random variable X is the function*

$$F_X : \mathbb{R} \longrightarrow [0, 1]$$

defined by

$$F_X(x) := \mathbb{P}(X \leq x), \quad x \geq 0.$$

Any cumulative distribution function F_X satisfies the following properties:

- i) $x \mapsto F_X(x)$ is non-decreasing,
- ii) $x \mapsto F_X(x)$ is right-continuous,
- iii) $\lim_{x \rightarrow \infty} F_X(x) = 1$,
- iv) $\lim_{x \rightarrow -\infty} F_X(x) = 0$.

Cumulative distribution functions can be discontinuous functions, as illustrated in Figure 6.2 with

$$\mathbb{P}(X = 0) = \mathbb{P}(X \leq 0) - \mathbb{P}(X < 0) = 0.25 > 0.$$

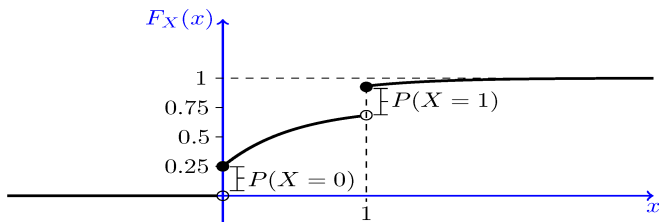


Fig. 6.2: Cumulative distribution function with discontinuities.*

Proposition 6.6 shows in particular that cumulative distribution functions admit left limits.

Proposition 6.6. *For any non-decreasing sequence $(x_n)_{n \geq 1}$ converging to $x \in \mathbb{R}$, we have*

$$\lim_{n \rightarrow \infty} F_X(x_n) = \lim_{n \rightarrow \infty} \mathbb{P}(X \leq x_n) = \mathbb{P}(X < x). \quad (6.3)$$

Proof. By (A.7), we have

$$\begin{aligned} \mathbb{P}(X < x) &= \mathbb{P}(X \in (-\infty, x)) \\ &= \mathbb{P}\left(X \in \bigcup_{n \geq 1} (-\infty, x_n]\right) \\ &= \lim_{n \rightarrow \infty} \mathbb{P}(X \in (-\infty, x_n]) \\ &= \lim_{n \rightarrow \infty} F_X(x_n). \end{aligned}$$

□

As a consequence of Proposition 6.6, the gap generated by a discontinuity of a CDF at a point $x \in \mathbb{R}$ is given by

$$F_X(x) - \lim_{y \nearrow x} F_X(y) = \mathbb{P}(X \leq x) - \mathbb{P}(X < x) = \mathbb{P}(X = x).$$

* Picture taken from <https://www.probabilitycourse.com/>.

```

1 x <- seq(-4, 4, length=1000)
  plot(x, pnorm(x, mean=0, sd=1), type="l", lwd=3, xlab = 'x', ylab = '', main = '', col='blue',
       ylim=c(-0.001,1.002), las=1, cex.lab=1, cex.axis=1, xaxs='t', yaxs='t'); grid(4, 10, lwd =
3 plot(x, pexp(x, 1), type="l", lwd=3, xlab = 'x', ylab = '', main = '', col='blue',
       ylim=c(-0.001,1.002), las=1, cex.lab=1, cex.axis=1, xaxs='t', yaxs='t'); grid(4, 10, lwd =
2)
  plot(x, ppois(x, 1), type="l", lwd=3, xlab = 'x', ylab = '', main = '', col='blue',
       ylim=c(-0.001,1.002), las=1, cex.lab=1, cex.axis=1, xaxs='t', yaxs='t'); grid(4, 10, lwd =
2)

```

Figure 6.3-(a) shows the continuous Cumulative Distribution Function

$$F_X(x) := \mathbb{P}(X \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy, \quad x \geq 0,$$

of a Gaussian random variable $X \simeq \mathcal{N}(0, 1)$.

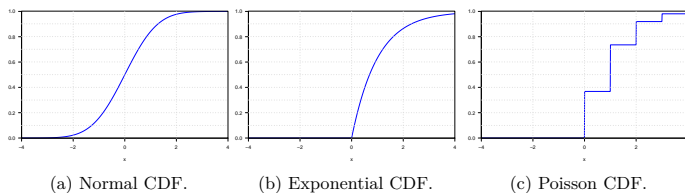


Fig. 6.3: Cumulative distribution functions.

On the other hand, if $F_X(x)$ is differentiable in $x \in \mathbb{R}$ then the distribution of the random variable X is said to admit a *probability density function* (PDF) $f_X(x)$ given as the derivative

$$f_X(x) = F'_X(x), \quad x \geq 0.$$

Definition 6.7. Given X a random variable with cumulative distribution function $F_X : \mathbb{R} \rightarrow [0, 1]$ and a level $p \in (0, 1)$, the p -quantile q_X^p of X is defined by

$$q_X^p := \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq p\}. \quad (6.4)$$

We note that by (6.4), the function $p \mapsto q_X^p$ is the *generalized inverse* $F_X^{-1}(x)$ of the *Cumulative Distribution Function*

$$x \mapsto F_X(x) := \mathbb{P}(X \leq x), \quad x \geq 0.$$

of X , see Definition 1 in [Embrechts and Hofert \(2013\)](#). As a consequence, we have the following.

Proposition 6.8.

- i) The function $p \mapsto q_X^p$ is a non-decreasing, left-continuous function of $p \in [0, 1]$, and it admits limits on the right.
 ii) For all $p \in [0, 1]$ and $x \in \mathbb{R}$, we have

$$p \leq F_X(x) \iff q_X^p \leq x.$$

Proof. (i) follows from Proposition 1-(2) in [Embrechts and Hofert \(2013\)](#), since $F_X(x)$ is non-decreasing in $x \in \mathbb{R}$, and (ii) follows from Proposition 1-(5) therein, since $F_X(x)$ is right-continuous in $x \in \mathbb{R}$. \square

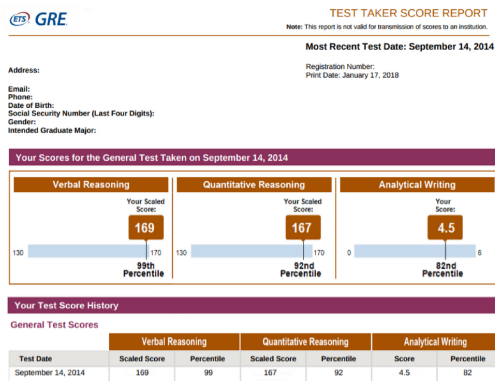


Fig. 6.4: Example of quantiles given as percentiles.

Quantiles of common distributions

The quantiles of various distributions can be obtained in R.

- *Gaussian distribution.* The command

```
1 qnorm(.95, mean=0, sd=1)
```

shows that the 95%-quantile of a $\mathcal{N}(0, 1)$ Gaussian random variable is 1.644854.

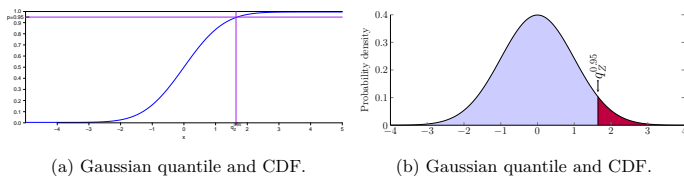


Fig. 6.5: Gaussian quantile $q_Z^p = 1.644854$ at $p = 0.95$.

- *Exponential distribution.* The command

```
1 qexp(.95, 1)
```

displays the 95%-quantile of an exponentially distributed random variable with CDF

$$\mathbb{P}(X \leq x) = 1 - e^{-\lambda x}, \quad x \geq 0.$$

By equating $\mathbb{P}(X \leq q_X^p) = p$, we find

$$\begin{aligned} q_X^p &= \inf \{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq p\} \\ &= -\frac{1}{\lambda} \log(1 - p) \\ &= \mathbb{E}[X] \log \frac{1}{1 - p}, \end{aligned}$$

and when $p = 95\%$ and $\lambda = 1$ this yields

$$q_X^p = 2.995732 \simeq 2.996\mathbb{E}[X].$$

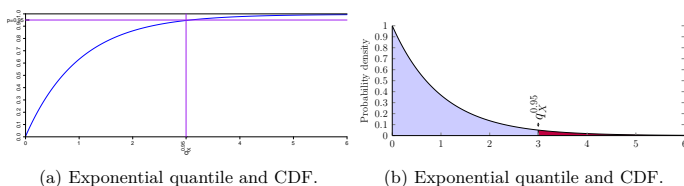


Fig. 6.6: Exponential quantile $q_X^p = 2.995732$ at $p = 0.95$.

- *Student distribution.* The command

```
1 qt(.90, df=5)
```

displays the 90%-quantile of a Student t -distributed random variable with 5 degrees of freedom, which is 1.475884.

- *Bernoulli distribution.* Consider the Bernoulli random variable $X \in \{0, 1\}$ with the distribution

$$\mathbb{P}(X = 1) = 2\%, \quad \mathbb{P}(X = 0) = 98\%.$$

In this case, we check from Figure 6.7 that $q_X^{0.99} = 1$.

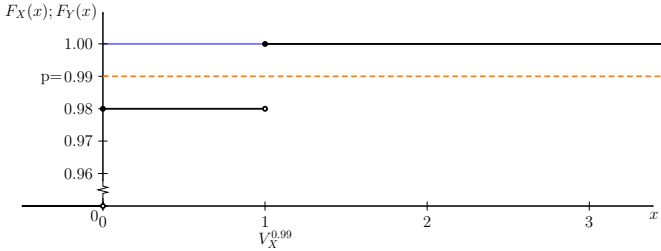


Fig. 6.7: Cumulative distribution function of X .

Empirical Cumulative Distribution Function

Definition 6.9. The empirical Cumulative Distribution Function (CDF) of an N -point data set $\{x_1, x_2, x_3, \dots, x_N\}$ is estimated as

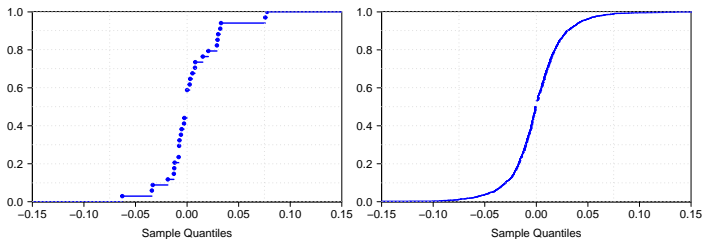
$$F_N(x) := \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\{x_i \leq x\}}, \quad x \geq 0.$$

```
1 getSymbols("~/STI",from="1990-01-03",to="2015-01-03",src="yahoo")
2 getSymbols("1800.HK",from=Sys.Date()-50,to=Sys.Date(),src="yahoo")
3 stock=Ad(`1800.HK`);stock.rtn=(stock-lag(stock))/lag(stock);
4 stock.rtn <- stock.rtn[!is.na(stock.rtn)]
5 stock.ecdf=ecdf(as.vector(stock.rtn))
6 plot(stock.ecdf, xlab = "Sample Quantiles", ylim=c(-0.001,1.002), xlim=c(-0.15,0.15), ylab = "",
7     lwd = 3, main = "", col='blue', las=1, cex.lab=1.5, cex.axis=1.5, xaxs='t', yaxs='t');
8 grid(4, 10, lwd = 2)
```

```

1  getSymbols("1800.HK",from=Sys.Date()-3650,to=Sys.Date(),src="yahoo")
2  stock=Ad("1800.HK");stock.rtn=(stock-lag(stock))/lag(stock);
   stock.ecdf=ecdf(as.vector(stock.rtn))
4  plot(stock.ecdf, xlab = 'Sample Quantiles', ylim=c(-0.001,1.002), xlim=c(-0.15,0.15), ylab = '',
   lwd = 2, main = '',col='blue', cex=1, las=1, cex.lab=1.5, cex.axis=1.5, xaxs='l', yaxs='l')
   grid(4, 10, lwd = 2)

```



(a) Empirical CDF on 50 samples.

(b) Empirical CDF on 2463 samples.

Fig. 6.8: Empirical cumulative distribution functions.

Note that the empirical distribution function in Figure 6.8-a) has a visible discontinuity (or gap) at $x = 0$, whose height 0.05483347 is given by

```

1  length(stock.rtn[stock.rtn==0])/length(stock.rtn)

```

6.3 Value at Risk (VaR)

Consider a random variable X used to model the potential losses associated to a given risk. The probability $\mathbb{P}(X > V)$ that X exceeds the level V is of a capital importance. Choosing the value of V such that for example

$$\mathbb{P}(X \leq V) \geq 0.95, \quad \text{i.e.} \quad \mathbb{P}(X > V) \leq 0.05,$$

means that insolvency will occur with probability less than 5%. In this setting, the 95%-quantile risk measure is the smallest value of $x \in \mathbb{R}$ such that

$$\mathbb{P}(X \leq x) \geq 0.95, \quad \text{i.e.} \quad \mathbb{P}(X > x) \leq 0.05.$$

More precisely, we have the following definition.

Definition 6.10. *The Value at Risk V_X^p of a random variable X at the level $p \in (0, 1)$ is the p -quantile of X defined by*

$$V_X^p := \inf\{x \in \mathbf{R} : \mathbb{P}(X \leq x) \geq p\}. \quad (6.5)$$

In other words, for some decreasing sequence $(x_n)_{n \geq 1}$ such that

$$\mathbb{P}(X \leq x_n) \geq p \quad \text{for all } n \geq 1,$$

we have

$$V_X^p := \lim_{n \rightarrow \infty} x_n. \quad (6.6)$$

Similarly to the above, the function $p \mapsto V_X^p$ is the *generalized inverse* $F_X^{-1}(x)$ of the *Cumulative Distribution Function* $\mapsto F_X$ of X , and from Proposition 6.8-(i) we have the following result.

Proposition 6.11. *The function $p \mapsto V_X^p$ is a non-decreasing, left-continuous function of $p \in [0, 1]$, and it admits limits on the right.*

In particular, if F_X is continuous and strictly increasing it admits an inverse F_X^{-1} , and in this case V_X^p is given by

$$V_X^p = F_X^{-1}(p), \quad p \in (0, 1).$$

Proposition 6.12. *The Value at Risk V_X^p of X at the level $p \in (0, 1)$ satisfies the properties*

$$\mathbb{P}(X < V_X^p) \leq p \leq \mathbb{P}(X \leq V_X^p), \quad (6.7)$$

and

$$\mathbb{P}(X > V_X^p) \leq 1 - p \leq \mathbb{P}(X \geq V_X^p). \quad (6.8)$$

In particular, if $\mathbb{P}(X = V_X^p) = 0$, then we have

$$p = \mathbb{P}(X < V_X^p) = \mathbb{P}(X \leq V_X^p). \quad (6.9)$$

Proof. Using the decreasing sequence $(x_n)_{n \geq 1}$ in (6.6) and the right continuity of the cumulative distribution function F_X , we have

$$\begin{aligned} \mathbb{P}(X \leq V_X^p) &= \mathbb{P}(X \leq \lim_{n \rightarrow \infty} x_n) \\ &= F_X(\lim_{n \rightarrow \infty} x_n) \\ &= \lim_{n \rightarrow \infty} F_X(x_n) \\ &= \lim_{n \rightarrow \infty} \mathbb{P}(X \leq x_n) \\ &\geq p. \end{aligned}$$

On the other hand, if $\mathbb{P}(X < V_X^p) > p$ then there is a strictly increasing sequence $(y_n)_{n \geq 1}$ such that

$$\lim_{n \rightarrow \infty} y_n = V_X^p$$

and by (6.3) we have

$$\mathbb{P}(X < V_X^p) = \lim_{n \rightarrow \infty} \mathbb{P}(X \leq y_n) > p,$$

in which case there would exist $n \geq 1$ such that $y_n < V_X^p$ and $\mathbb{P}(X \leq y_n) > p$, which contradicts (6.5). Relations (6.8)-(6.9) are direct consequences of (6.7). \square

When $\mathbb{P}(X = V_X^p) > 0$ we may have $\mathbb{P}(X > V_X^p) = 0$, for example in the case of a Bernoulli random variable $X \in \{0, 1\}$ with the distribution

$$\mathbb{P}(X = 1) = 2\%, \quad \mathbb{P}(X = 0) = 98\%,$$

see Figure 6.7. The next proposition also follows from the Definition 6.10 of V_X^p and Proposition 6.8-(ii).

Proposition 6.13. *For all $x \in \mathbb{R}$ we have*

$$V_X^p \leq x \iff \mathbb{P}(X \leq x) \geq p. \tag{6.10}$$

Proof. \Leftarrow) If $\mathbb{P}(X \leq x) \geq p$ then we have

$$V_X^p = \inf\{y \in \mathbb{R} : \mathbb{P}(X \leq y) \geq p\} \leq x.$$

\Rightarrow) On the other hand, choosing a strictly decreasing sequence $(x_n)_{n \geq 1}$ such that

$$\lim_{n \rightarrow \infty} x_n = V_X^p \quad \text{and} \quad \mathbb{P}(X \leq x_n) \geq p, \quad n \geq 1,$$

if $V_X^p \leq x$ we have

$$\mathbb{P}(X \leq x) \geq \mathbb{P}(X \leq V_X^p) = \lim_{n \rightarrow \infty} \mathbb{P}(X \leq x_n) \geq p$$

by the right continuity of the cumulative distribution function F_X of X . \square

On the other hand, the Value at Risk V_X^p does not reveal any information on *how large* losses can be beyond V_X^p , see Chapter 7 for details. The next proposition shows how to estimate Value at Risk when switching the sign of the data.

Proposition 6.14. *Assume that the cumulative distribution function F_X is continuous and strictly increasing. Then, we have*

$$V_{-X}^p = -V_X^{1-p}, \quad p \in (0, 1). \tag{6.11}$$

Proof. Since F_X is continuous, we have

$$F_{-X}(x) = \mathbb{P}(-X \leq x)$$

$$\begin{aligned}
&= \mathbb{P}(X \geq -x) \\
&= 1 - \mathbb{P}(X < -x) \\
&= 1 - \mathbb{P}(X \leq -x) \\
&= 1 - F_X(-x),
\end{aligned}$$

hence, taking

$$x := F_{-X}^{-1}(p),$$

we have

$$p = F_{-X}(F_{-X}^{-1}(p)) = 1 - F_X(-F_{-X}^{-1}(p)),$$

or

$$F_X(-F_{-X}^{-1}(p)) = 1 - p$$

i.e.

$$F_{-X}^{-1}(p) = -F_X^{-1}(1 - p),$$

which yields

$$V_{-X}^p = F_{-X}^{-1}(p) = -F_X^{-1}(1 - p) = -V_X^{1-p}, \quad p \in (0, 1).$$

□

In Figure 6.9 we choose a continuous CDF F_X with

$$F_{-X}(x) = 1 - F_X(-x), \quad x \in \mathbb{R}.$$

In this case, the continuity of F_X ensures the symmetry property

$$V_{-X}^p = -V_X^{1-p}$$

of Proposition 6.14.

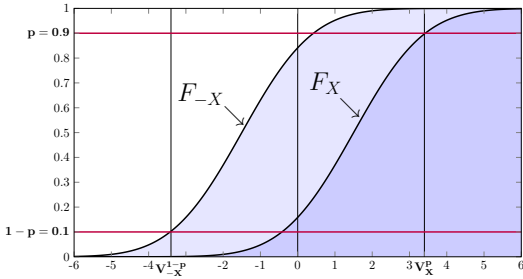


Fig. 6.9: Continuous CDF.

On the other hand, in Figure 6.10 we consider X with distribution

$$\mathbb{P}(X = 1) = 0.5, \quad \mathbb{P}(X = 2) = 0.3, \quad \mathbb{P}(X = 3.4) = 0.2,$$

hence

$$\mathbb{P}(-X = -3.4) = 0.2, \quad \mathbb{P}(-X = -2) = 0.3, \quad \mathbb{P}(-X = -1) = 0.5,$$

and we check that in this discontinuous case the relation $V_{-X}^q = -V_X^{1-q}$ fails for $p = 0.8$, although it still holds for $p' = 0.9$.

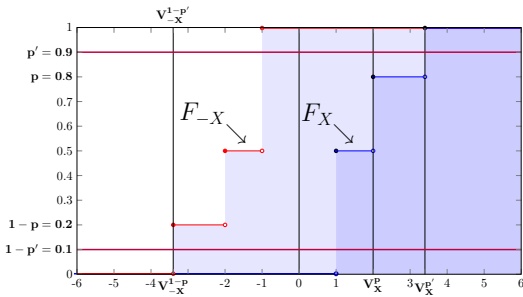


Fig. 6.10: Discontinuous CDF.

Next, we check the properties of Value at Risk. Although Value at Risk satisfies the monotonicity, positive homogeneity and translation invariance

properties, it is *not* subadditive in general. Namely, the Value at Risk V_{X+Y}^p of $X + Y$ may be larger than the sum $V_X^p + V_Y^p$.

Proposition 6.15. *Value at Risk V_X^p is a monotone, positive homogeneous and translation invariant risk measure. However, it is not subadditive, and therefore it is not a coherent risk measure.*

Proof.

a) *Monotonicity.*

Value at Risk is a monotone risk measure. If $X \leq Y$ then

$$\mathbb{P}(Y \leq x) = \mathbb{P}(X \leq Y \leq x) \leq \mathbb{P}(X \leq x), \quad x \geq 0,$$

hence

$$\mathbb{P}(Y \leq x) \geq p \implies \mathbb{P}(X \leq x) \geq p, \quad x \geq 0,$$

which shows that

$$V_X^p \leq V_Y^p$$

by (6.5).

b) *Positive homogeneity and translation invariance.*

Value at Risk satisfies the positive homogeneity and translation invariance properties. For any $\mu \in \mathbb{R}$ and $\lambda > 0$, we have

$$\begin{aligned} V_{\mu+\lambda X}^p &= \inf\{x \in \mathbb{R} : \mathbb{P}(\mu + \lambda X \leq x) \geq p\} \\ &= \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq (x - \mu)/\lambda) \geq p\} \\ &= \inf\{\mu + \lambda y \in \mathbb{R} : \mathbb{P}(X \leq y) \geq p\} \\ &= \mu + \lambda \inf\{y \in \mathbb{R} : \mathbb{P}(X \leq y) \geq p\} \\ &= \mu + \lambda V_X^p. \end{aligned}$$

c) *Subadditivity and coherence.*

We show that Value at Risk is *not subadditive* by considering two *independent* Bernoulli random variables $X, Y \in \{0, 1\}$ having the same distribution

$$\begin{cases} \mathbb{P}(X = 1) = \mathbb{P}(Y = 1) = 2\%, \\ \mathbb{P}(X = 0) = \mathbb{P}(Y = 0) = 98\%, \end{cases}$$

hence $V_X^{0.975} = V_Y^{0.975} = 0$.

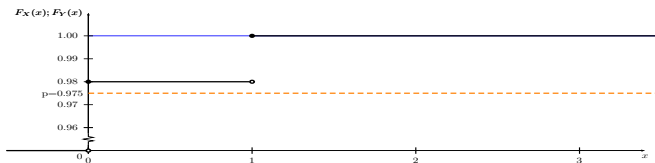


Fig. 6.11: Cumulative distribution function of X and Y .

On the other hand, we have

$$\begin{cases} \mathbb{P}(X + Y = 2) = \mathbb{P}(X = 1 \text{ and } Y = 1) = (0.02)^2 = 0.04\%, \\ \mathbb{P}(X + Y = 1) = 2 \times 0.02 \times 0.98 = 3.92\%, \\ \mathbb{P}(X + Y = 0) = \mathbb{P}(X = 0 \text{ and } Y = 0) = (0.98)^2 = 96.04\%, \end{cases}$$

hence

$$V_{X+Y}^{0.975} = 1 > V_X^{0.975} + V_Y^{0.975} = 0.$$

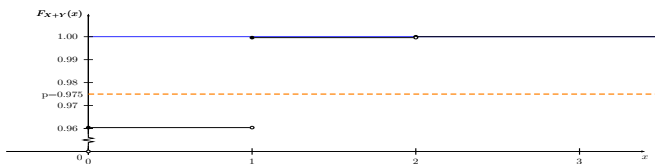


Fig. 6.12: Cumulative distribution function of $X + Y$.

□

In the next proposition, we use the standard Gaussian Cumulative Distribution Function (CDF)

$$\Phi(x) := \int_{-\infty}^x e^{-y^2/2} \frac{dy}{\sqrt{2\pi I}}, \quad x \in \mathbb{R},$$

of a standard normal random variable $Z \simeq \mathcal{N}(0, 1)$.

Proposition 6.16. Gaussian Value at Risk. *Given $X \simeq \mathcal{N}(\mu_X, \sigma_X^2)$, we have*

$$V_X^p = \mu_X + \sigma_X q_Z^p \tag{6.12}$$



where the normal quantile $q_Z^p = V_Z^p$ at the level p satisfies

$$\Phi(q_Z^p) = \mathbb{P}(Z \leq q_Z^p) = p \quad \text{for } Z \simeq \mathcal{N}(0, 1),$$

i.e.

$$q_Z^p = \Phi^{-1}(p) \quad \text{and} \quad V_X^p = \mu_X + \sigma_X \Phi^{-1}(p).$$

Proof. We represent the random variable $X \simeq \mathcal{N}(\mu_X, \sigma_X^2)$ as

$$X = \mu_X + \sigma_X Z,$$

where $Z \simeq \mathcal{N}(0, 1)$ is a standard normal random variable, and use the relation

$$\begin{aligned} p &= \mathbb{P}(X \leq V_X^p) \\ &= \mathbb{P}(\mu_X + \sigma_X Z \leq V_X^p) \\ &= \mathbb{P}(Z \leq (V_X^p - \mu_X)/\sigma_X) \\ &= \mathbb{P}(Z \leq q_Z^p), \end{aligned}$$

which holds provided that $V_X^p = \mu_X + \sigma_X q_Z^p$. □

We also note that if $X \simeq \mathcal{N}(\mu_X, \sigma_X^2)$ then $-X \simeq \mathcal{N}(-\mu_X, \sigma_X^2)$, hence

$$\begin{aligned} V_{-X}^p &= -\mu_X + \sigma_X q_Z^p \\ &= -\mu_X - \sigma_X q_Z^{1-p} \\ &= -V_X^{1-p}, \end{aligned}$$

which is consistent with (6.11).

The next remark shows that, although Value at Risk is *not sub-additive* in general, it is sub-additive (and therefore coherent) on (not necessarily independent) Gaussian random variables.

Remark 6.17. *If X and Y are two Gaussian random variables, we have*

$$V_{X+Y}^p \leq V_X^p + V_Y^p.$$

Proof. By (6.12), for any two random variables X and Y , we have

$$\begin{aligned} \sigma_{X+Y}^2 &= \text{Var}[X + Y] \\ &= \mathbb{E}[(X + Y)^2] - (\mathbb{E}[X + Y])^2 \\ &= \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[XY] - \mathbb{E}[X]^2 - \mathbb{E}[Y]^2 - 2\mathbb{E}[X]\mathbb{E}[Y] \\ &= \text{Var}[X] + \text{Var}[Y] + 2(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]) \end{aligned}$$

$$\begin{aligned}
&= \text{Var}[X] + \text{Var}[Y] + 2\mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \\
&= \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}(X, Y) \tag{6.13}
\end{aligned}$$

$$\begin{aligned}
&\leq \text{Var}[X] + \text{Var}[Y] + 2\sqrt{\mathbb{E}[(X - \mathbb{E}[X])^2]}\sqrt{\mathbb{E}[(Y - \mathbb{E}[Y])^2]} \\
&= \text{Var}[X] + \text{Var}[Y] + 2\sqrt{\text{Var}[X]}\sqrt{\text{Var}[Y]} \tag{6.14}
\end{aligned}$$


$$= (\sqrt{\text{Var}[X]} + \sqrt{\text{Var}[Y]})^2,$$

where, from (6.13) to (6.14) we applied the *Cauchy-Schwarz* inequality, hence $\sigma_{X+Y} \leq \sigma_X + \sigma_Y$. Assuming that X and Y are Gaussian, by (6.12) we find

$$\begin{aligned}
V_{X+Y}^p &= \mu_{X+Y} + \sigma_{X+Y}q_Z^p \\
&= \mu_X + \mu_Y + \sigma_{X+Y}q_Z^p \\
&\leq \mu_X + \mu_Y + (\sigma_X + \sigma_Y)q_Z^p \\
&= V_X^p + V_Y^p.
\end{aligned}$$

□

6.4 Numerical estimates

In this section we are using the PerformanceAnalytics  package, see also § 6.1.1 of [Mina and Xiao \(2001\)](#). In case we care about negative return values, Definition 6.10 is replaced with

$$\bar{V}_X^p := \text{Sup}\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \leq 1 - p\}. \tag{6.15}$$

In case the CDF of X is continuous, we note the relation

$$\begin{aligned}
\bar{V}_X^p &= \text{Sup}\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \leq 1 - p\} \\
&= -\inf\{-x \in \mathbb{R} : \mathbb{P}(X \leq x) \leq 1 - p\} \\
&= -\inf\{x \in \mathbb{R} : \mathbb{P}(X \leq -x) \leq 1 - p\} \\
&= -\inf\{x \in \mathbb{R} : \mathbb{P}(-X \geq x) \leq 1 - p\} \\
&= -\inf\{x \in \mathbb{R} : 1 - \mathbb{P}(-X \geq x) \geq p\} \\
&= -\inf\{x \in \mathbb{R} : \mathbb{P}(-X \leq x) \geq p\} \\
&= -V_{-X}^p,
\end{aligned}$$

hence the relation

$$\bar{V}_X^p = -V_{-X}^p = V_X^{1-p}$$

which is obtained from Proposition 6.14 when the cumulative distribution function F_X is continuous and strictly increasing.

```

1  install.packages("PerformanceAnalytics")
   library(PerformanceAnalytics)
3  getSymbols("0700.HK",from="2010-01-03",to="2018-02-01",src="yahoo")
   stock=Ad(`0700.HK`);chartSeries(stock,up.col="blue",theme="white")
   stock.rtn=(stock-lag(stock))/lag(stock)[-1];stock.rtn <- stock.rtn[!is.na(stock.rtn)]
5  dev.new(width=16,height=7); chart.CumReturns(stock.rtn, main="Cumulative Returns")
   var=VaR(stock.rtn, p=.95, method="historical");var
7  length(stock.rtn[stock.rtn<var[1]])/length(stock.rtn)
   times=index(stock);chartSeries(stock.rtn,up.col="blue",theme="white")
9  abline(h=var,col="red",lwd=3)

```

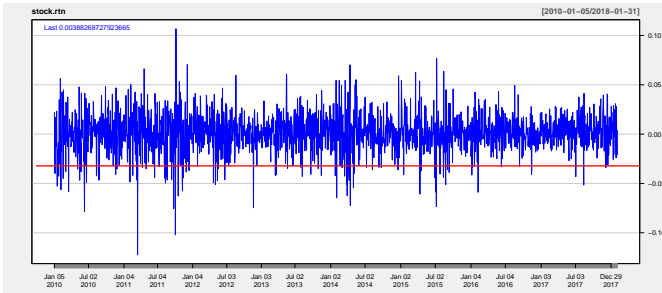


Fig. 6.13: Market returns *vs.* Value at Risk level in red.

The historical 95%-Value at Risk over N samples $(x_i)_{i=1,2,\dots,N}$ can be estimated by inverting the *empirical cumulative distribution function* $F_N(x)$, and is found to be $\bar{V}_X^{95\%} = -0.03165963$.

```

1  VaR(stock.rtn, p=.95, method="gaussian",invert="FALSE")
2  VaR(stock.rtn, p=.95, method="gaussian",invert="TRUE")

```

The Gaussian 95%-Value at Risk is estimated from (6.12) with $p = 0.95$ as

$$\bar{V}_X^p = V_X^{1-p} = \mu + \sigma q_Z^{1-p} = \mu - \sigma q_Z^p,$$

where $-\mu = \mathbb{E}[-X]$ and $\sigma^2 = \text{Var}[-X]$, and is found equal to

$$\bar{V}_X^{95\%} = -0.03115425.$$

It can be recovered up to approximation according to Proposition 6.16 from the following `R` code, which yields -0.0311592 .

```
1 m=mean(stock.rtn,na.rm=TRUE); s=sd(stock.rtn,na.rm=TRUE)
2 q=qnorm(.95, mean=0, sd=1); m-s*q
```

Note that here we are concerned about large negative returns, which explains the negative sign in $m - s * q$.

The next lemma is useful for random simulation purposes, and it will also be used in the proof of Propositions 7.6 and 7.12 below.

Lemma 6.18. *Any random variable X can be represented as*

$$X = V_X^U = F_X^{-1}(U),$$

where U a uniformly distributed random variable on $[0, 1]$.

Proof. It suffices to note that by (6.10) we have

$$\mathbb{P}(V_X^U \leq x) = \mathbb{P}(U \leq \mathbb{P}(X \leq x)) = \mathbb{P}(X \leq x) = F_X(x), \quad x \geq 0.$$

□

Exercises

Exercise 6.1 Consider a random variable X having the Pareto distribution with probability density function

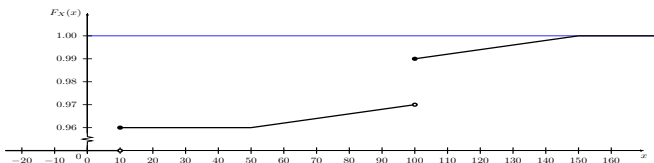
$$f_X(x) = \frac{\gamma\theta^\gamma}{(\theta+x)^{\gamma+1}}, \quad x \geq 0.$$

a) Compute the cumulative distribution function

$$F_X(x) := \int_0^x f_X(y)dy, \quad x \geq 0.$$

b) Compute the value at risk V_X^p at the level p for any θ and γ , and then for $p = 99\%$, $\theta = 40$ and $\gamma = 2$.

Exercise 6.2 Consider a random variable X with the following cumulative distribution function:

Fig. 6.14: Cumulative distribution function of X .

- Give the value of $\mathbb{P}(X = 100)$.
- Give the value of V_X^q for all q in the interval $[0.97, 0.99]$.
- Compute the value of V_X^q for all q in the interval $[0.99, 1]$.

Hint: We have

$$F_X(x) = \mathbb{P}(X \leq x) = 0.99 + 0.01 \times \frac{x - 100}{50}, \quad x \in [100, 150].$$

Exercise 6.3 Discrete distribution. Consider $X \in \{10, 100, 110\}$ with the distribution

$$\mathbb{P}(X = 10) = 90\%, \quad \mathbb{P}(X = 100) = 9.5\%, \quad \mathbb{P}(X = 110) = 0.5\%.$$

Compute the value at risk $V_X^{99\%}$.

Exercise 6.4 Exponential distribution. Assume that X has an exponential distribution with parameter $\lambda > 0$ and mean $1/\lambda$, *i.e.*

$$\mathbb{P}(X \leq x) = 1 - e^{-\lambda x}, \quad x \geq 0.$$

- Compute

$$V_X^p := \inf \{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq p\}$$

and $V_X^{95\%}$.

- Assuming that the liabilities of a company are estimated by $\mathbb{E}[X]$, compute the amount of required capital C_X from (6.1).

Exercise 6.5 Given X a random variable having the geometric distribution with

$$\mathbb{P}(X = k) = (1 - p)^k p, \quad k \geq 0,$$

compute the conditional expectation $\mathbb{E}[X \mid X \geq a]$ for $a > 0$.

Exercise 6.6 Estimating risk probabilities from moments.

a) Show that for every $r > 0$

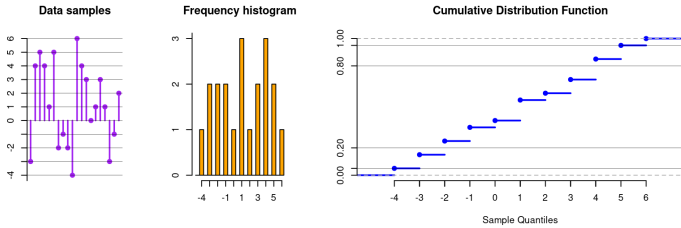
$$V_X^p \leq \left(\frac{\mathbb{E}[|X|^r]}{1-p} \right)^{1/r} = \frac{\|X\|_{L^r(\Omega)}}{(1-p)^{1/r}},$$

where $\|X\|_{L^r(\Omega)} := (\mathbb{E}[|X|^r])^{1/r}$.

Hint: Use the argument of the Markov inequality.

b) Give an upper bound for $V_X^{95\%}$ when $p = 95\%$ and $r = 1$.

Exercise 6.7 We consider a discrete random variable X having the following distribution.



a) Find the following quantities for the above data set, and mark their values on the graph.

i) Historical “Academic” Value at Risk at $p = 0.95$. $\text{VaR}_{Ac-H}^{95} = \underline{\hspace{2cm}}$

ii) Historical “Academic” Value at Risk at $p = 0.80$. $\text{VaR}_{Ac-H}^{80} = \underline{\hspace{2cm}}$

iii) Historical “Practitioner” Value at Risk at $p = 0.95$. $\overline{\text{VaR}}_{Pr-H}^{95} = \underline{\hspace{2cm}}$

iv) Historical “Practitioner” Value at Risk at $p = 0.80$. $\overline{\text{VaR}}_{Pr-H}^{80} = \underline{\hspace{2cm}}$

b) Knowing that mean=1.15, sd=3.048, qnorm(0.95)=1.645 and qnorm(0.80)=0.842, compute (from Proposition 6.16):

i) Gaussian “Academic” Value at Risk at $p = 0.95$. $\text{VaR}_{Ac-G}^{95} = \underline{\hspace{2cm}}$

ii) Gaussian “Academic” Value at Risk at $p = 0.80$. $\text{VaR}_{Ac-G}^{80} = \underline{\hspace{2cm}}$

iii) Gaussian “Practitioner” Value at Risk at $p = 0.95$. $\overline{\text{VaR}}_{Pr-G}^{95} = \underline{\hspace{2cm}}$

iv) Gaussian “Practitioner” Value at Risk at $p = 0.80$. $\overline{\text{VaR}}_{Pr-G}^{80} = \underline{\hspace{2cm}}$

