Chapter 5
Superhedging Risk Measure

In this chapter, we measure risk using the prices of financial derivatives such as options, that protect their holders against various kinds of market events. For this, we review some basic knowledge of call or put options and related financial derivatives, together with their pricing in the Black-Scholes framework. As a result, we introduce the superhedging risk measure, which can be defined from the price of a portfolio that hedges a given financial derivative.

5.1 Historical Sketch

Early accounts of option contracts can also be found in *The Politics* Aristotle (350 BCE) by Aristotle (384-322 BCE). Option credit contracts appear to have been used as early as the 10*th* century by traders in the Mediterranean.

Referring to the philosopher Thales of Miletus (c. 624 - c. 546 BCE), Aristotle writes:

> "He (Thales) knew by his skill in the stars while it was yet winter that there would be a great harvest of olives in the coming year; so, having a little money, he gave *deposits* for the use of all the olive-presses in Chios and Miletus, which he hired at a low price because no one bid against him. When the harvest-time came, and many were wanted all at once and of a sudden, he let them out at any rate which he pleased, and made a quantity of money".

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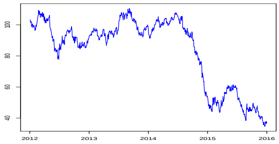
More recently, Robert Merton and Myron Scholes shared the 1997 Nobel Prize in economics: "In collaboration with Fisher Black, developed a pioneering formula for the valuation of stock options ... paved the way for economic valuations in many areas ... generated new types of financial instruments and facilitated more efficient risk management in society."* See Black and Scholes (1973) "The Pricing of Options and Corporate Liabilities". Journal of Political Economy 81 (3): 637-654.

The development of options pricing tools contributed greatly to the expansion of option markets and led to development several ventures such as the "Long Term Capital Management" (LTCM), founded in 1994. The fund yielded annualized returns of over 40% in its first years, but registered a loss of US$4.6 billion in less than four months in 1998, which resulted into its closure in early 2000.

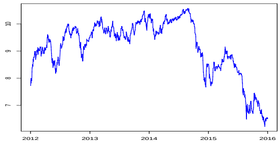
As of year 2015, the size of the financial derivatives market is estimated at over one quadrillion (or one million billions, or 10^{15}) USD, which is more than 10 times the size of the total Gross World Product (GWP).

5.2 Financial Derivatives

The following graphs exhibit a correlation between commodity (oil) prices and an oil-related asset price.



(a) WTI price graph.



(b) Graph of Keppel Corp. stock price

Fig. 5.1: Comparison of WTI *vs.* Keppel price graphs.

The study of financial derivatives aims at finding functional relationships between the price of an underlying asset (a company stock price, a commodity price, etc.) and the price of a related financial contract (an option, a financial derivative, etc.).
In the above quote by Aristotle, olive oil can be regarded as the underlying asset, while the oil press stands for the financial derivative.

* This has to be put in relation with the modern development of Risk Societies; "societies increasingly preoccupied with the future (and also with safety), which generates the notion of risk" (Wikipedia).

Next, we move to a description of (European) call and put options, which are at the basis of risk management.

European put option contracts

As previously mentioned, an important concern for the buyer of a stock at time t is whether its price S_T can decline at some future date T. The buyer of the stock may seek protection from a market crash by purchasing a contract that allows him to sell his asset at time T at a guaranteed price K fixed at time t. This contract is called a put option with strike price K and exercise date T.

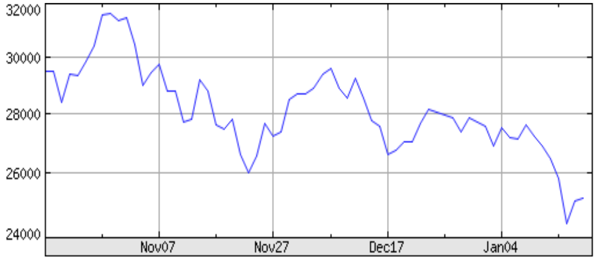


Fig. 5.2: Graph of the Hang Seng index - holding a put option might be useful here.

Definition 5.1. *A (European)* put *option is a contract that gives its holder the right (but not the obligation) to* sell *a quantity of assets at a predefined price* K *called the strike price (or exercise price) and at a predefined date* T *called the maturity.*

In case the price S_T falls down below the level K, exercising the contract will give the holder of the option a gain equal to $K - S_T$ in comparison to those who did not subscribe the option contract and have to sell the asset at the market price S_T. In turn, the issuer of the option contract will register a loss also equal to $K - S_T$ (in the absence of transaction costs and other fees).

If S_T is above K, then the holder of the option contract will not exercise the option as he may choose to sell at the price S_T. In this case, the profit derived from the option contract is 0.

Two possible scenarios (S_T finishing above K or below K) are illustrated in Figure 5.3.

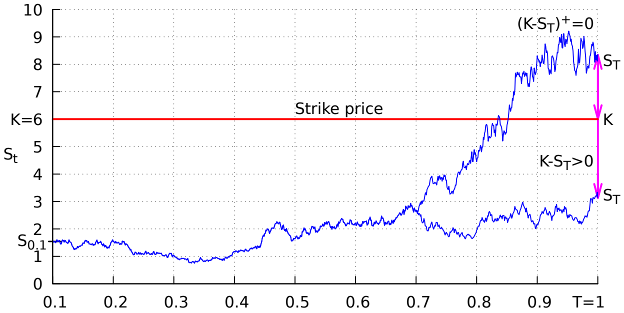


Fig. 5.3: Two put option scenarios.

In general, the payoff of a (so called *European*) put option contract can be written as

$$\phi(S_T) = (K - S_T)^+ := \begin{cases} K - S_T & \text{if } S_T \leqslant K, \\ 0, & \text{if } S_T \geqslant K. \end{cases}$$

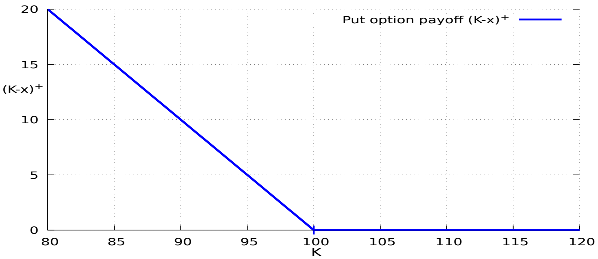


Fig. 5.4: Payoff function $x \mapsto (K - x)^+$ of a put option with strike price $K = 100$.

See *e.g.* https://optioncreator.com/stwwxvz.

Cash settlement *vs.* physical delivery

Physical delivery. In the case of physical delivery, the put option contract issuer will pay the strike price $\$K$ to the option contract holder in exchange for one unit of the risky asset priced S_T.

Cash settlement. In the case of a cash settlement, the put option issuer will satisfy the contract by transferring the amount $C = (K - S_T)^+$ to the option contract holder.

Examples of put options: The **buy back guarantee**[*] in currency exchange and the **price drop protection** in online ticket booking are common examples of European put options.

The derivatives market

As of year 2015, the size of the derivatives market was estimated at more that \$1.2 quadrillion,[†] or more than 10 times the Gross World Product (GWP). See **here** or **here** for up-to-date data on notional amounts outstanding and gross market value from the Bank for International Settlements (BIS).

European call option contracts

On the other hand, if the trader aims at buying some stock or commodity, his interest will be in prices not going up and he might want to purchase a call option, which is a contract allowing him to buy the considered asset at time T at a price not higher than a level K fixed at time t.

Definition 5.2. *A (European)* call *option is a contract that gives its holder the right (but not the obligation) to* purchase *a quantity of assets at a predefined price K called the strike price, and at a predefined date T called the maturity.*

Here, in the event that S_T goes above K, the buyer of the option contract will register a potential gain equal to $S_T - K$ in comparison to an agent who did not subscribe to the call option.

Two possible scenarios (S_T finishing above K or below K) are illustrated in Figure 5.5.

[*] Right-click to open or save the attachment.

[†] One thousand trillion, or one million billion, or 10^{15}.

Fig. 5.5: Two call option scenarios.

In general, the payoff of a (so called European) call option contract can be written as

$$\phi(S_T) = (S_T - K)^+ := \begin{cases} S_T - K & \text{if } S_T \geqslant K, \\ 0, & \text{if } S_T \leqslant K. \end{cases}$$

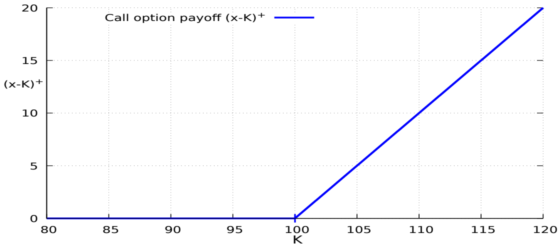


Fig. 5.6: Payoff function $x \mapsto (x - K)^+$ of a call option with strike price $K = 100$.

See *e.g.* https://optioncreator.com/stqhbgn.

Example of a call option: The **price lock guarantee*** is a common example of a European *call* option.

* Right-click to open or save the attachment.

According to market practice, options are often divided into a certain number n of *warrants*, the (possibly fractional) quantity n being called the *entitlement ratio*.

Cash settlement *vs.* physical delivery

Physical delivery. In the case of physical delivery, the call option contract issuer will transfer one unit of the risky asset priced S_T to the option contract holder in exchange for the strike price $\$K$. Physical delivery may include physical goods, commodities or assets such as coffee, airline fuel or live cattle.

Cash settlement. In the case of a cash settlement, the call option issuer will fulfill the contract by transferring the amount $C = (S_T - K)^+$ to the option contract holder.

Option pricing

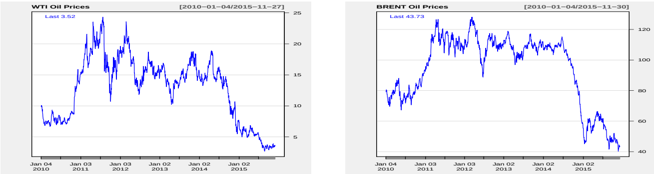
In order for an option contract to be fair, the buyer of the option contract should pay a fee (similar to an insurance fee) at the signature of the contract. The computation of this fee is an important issue, and is known as option *pricing*.

Option hedging

The second important issue is that of *hedging*, *i.e.* how to manage a given portfolio in such a way that it contains the required random payoff $(K - S_T)^+$ (for a put option) or $(S_T - K)^+$ (for a call option) at the maturity date T.

Example: Fuel hedging and the four-way zero-collar option

```
install.packages("Quandl"); library(Quandl); library(quantmod)
getSymbols("DCOILBRENTEU", src="FRED")
chartSeries(DCOILBRENTEU,up.col="blue",theme="white",name = "BRENT Oil Prices",lwd=5)
BRENT = Quandl("FRED/DCOILBRENTEU",start_date="2010-01-01", end_date="2015-11-30",type="xts")
chartSeries(BRENT,up.col="blue",theme="white",name = "BRENT Oil Prices",lwd=5)
getSymbols("WTI", from="2010-01-01", to="2015-11-30")
WTI <- Ad(`WTI`)
chartSeries(WTI,up.col="blue",theme="white",name = "WTI Oil Prices",lwd=5)
```



(a) WTI price graph.

(b) Brent price graph

Fig. 5.7: Brent and WTI price graphs.

(April 2011)

Fuel hedge promises Kenya Airways smooth ride in volatile oil market.*

(November 2015)

A close look at the role of fuel hedging in Kenya Airways $259 million loss.*
The four-way call collar call option requires its holder to purchase the underlying asset (here, airline fuel) at a price specified by the blue curve in Figure 5.8, when the underlying asset price is represented by the red line.

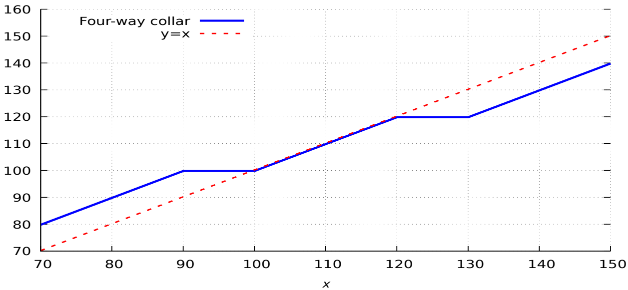


Fig. 5.8: Price map of a four-way collar option.

The four-way call collar option contract will result into a positive or negative payoff depending on current fuel prices, as illustrated in Figure 5.9.

* Right-click to open or save the attachment.

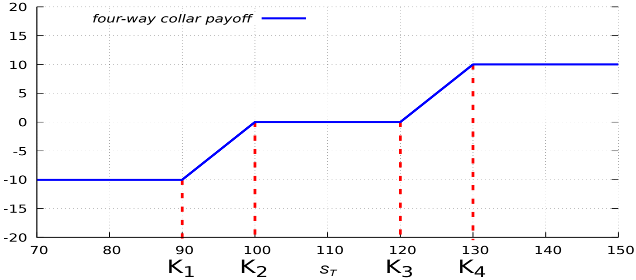


Fig. 5.9: Payoff function of a four-way call collar option.

The four-way call collar payoff can be written as a linear combination

$$\phi(S_T) = (K_1 - S_T)^+ - (K_2 - S_T)^+ + (S_T - K_3)^+ - (S_T - K_4)^+$$

of call and put option payoffs with respective strike prices

$$K_1 = 90, \quad K_2 = 100, \quad K_3 = 120, \quad K_4 = 130,$$

see *e.g.* https://optioncreator.com/st5rf51.

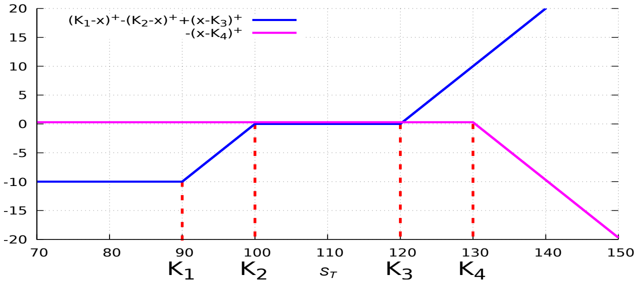


Fig. 5.10: Four-way call collar payoff as a combination of call and put options.*

Therefore, the four-way call collar option contract can be *synthesized* by:

* The animation works in Acrobat Reader on the entire pdf file.

1. purchasing a *put option* with strike price $K_1 = \$90$, and
2. selling (or issuing) a *put option* with strike price $K_2 = \$100$, and
3. purchasing a *call option* with strike price $K_3 = \$120$, and
4. selling (or issuing) a *call option* with strike price $K_4 = \$130$.

Moreover, the call collar option contract can be made *costless* by adjusting the boundaries K_1, K_2, K_3, K_4, in which case it becomes a *zero-collar* option.

Example - The one-step 4-5-2 model

We close this introduction with a simplified example of the pricing and hedging technique in a one-step binary model with two time instants $t = 0$ and $t = 1$. Consider:

i) A risky underlying stock valued $S_0 = \$4$ at time $t = 0$, and taking only two possible values

$$S_1 = \begin{cases} \$5 \\ \\ \$2 \end{cases}$$

at time $t = 1$.

ii) An option contract that promises a claim payoff C whose values are defined contingent to the market data of S_1 as:

$$C := \begin{cases} \$3 & \text{if } S_1 = \$5 \\ \\ \$0 & \text{if } S_1 = \$2. \end{cases}$$

Exercise: Does C represent the payoff of a put option contract? Of a call option contract? If yes, with which strike price K?

Quiz: Using this **online form**, input your own intuitive estimate for the price of the claim C.

At time $t = 0$ the option contract issuer (or writer) chooses to invest α units in the risky asset S, while keeping $\$\beta$ on our bank account, meaning that we invest a total amount

$$\alpha S_0 + \$\beta \qquad \text{at time } t = 0.$$

Here, the amount $\$\beta$ may be positive or negative, depending on whether it is corresponds to savings or to debt, and is interpreted as a *liability*.

The following issues can be addressed:

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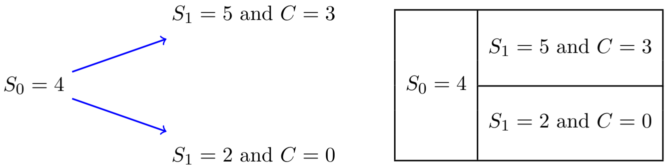
a) *Hedging*: How to choose the portfolio allocation $(\alpha, \$\beta)$ so that the value

$$\alpha S_1 + \$\beta$$

of the portfolio matches the future payoff C at time $t = 1$?

b) *Pricing*: How to determine the amount $\alpha S_0 + \$\beta$ to be invested by the option contract issuer in such a portfolio at time $t = 0$?



Hedging or *replicating* the contract means that at time $t = 1$ the portfolio value matches the future payoff C, *i.e.*

$$\alpha S_1 + \$\beta = C.$$

Hedge, then price. This condition can be rewritten as

$$C = \begin{cases} \$3 = \alpha \times \$5 + \$\beta & \text{if } S_1 = \$5, \\ \\ \$0 = \alpha \times \$2 + \$\beta & \text{if } S_1 = \$2, \end{cases}$$

i.e.

$$\begin{cases} 5\alpha + \beta = 3, \\ \\ 2\alpha + \beta = 0, \end{cases} \quad \text{which yields} \quad \begin{cases} \alpha = 1 \text{ stock}, \\ \\ \$\beta = -\$2. \end{cases}$$

In other words, the option contract issuer purchases 1 (one) unit of the stock S at the price $S_0 = \$4$, and borrows \$2 from the bank. The price of the option contract is then given by the portfolio value

$$\alpha S_0 + \$\beta = 1 \times \$4 - \$2 = \$2.$$

at time $t = 0$.

The above computation is implemented in the attached **IPython notebook***
that can be run **here** or **here**. This algorithm is scalable and can be extended to recombining binary trees over multiple time steps.

* Right-click to save as attachment (may not work on 🍎).

Definition 5.3. *The* arbitrage-free price *of the option contract is defined as the initial cost* $\alpha S_0 + \$\beta$ *of the portfolio hedging the claim payoff* C*.*

Conclusion: in order to deliver the random payoff $C = \begin{cases} \$3 & \text{if } S_1 = \$5 \\ \\ \$0 & \text{if } S_1 = \$2. \end{cases}$
to the option contract holder at time $t = 1$, the option contract issuer (or writer) will:

1. charge $\alpha S_0 + \$\beta = \$2$ (the option contract price) at time $t = 0$,
2. borrow $-\$\beta = \$2$ from the bank,
3. invest those $\$2 + \$2 = \$4$ into the purchase of $\alpha = 1$ unit of stock valued at $S_0 = \$4$ at time $t = 0$,
4. wait until time $t = 1$ to find that the portfolio value has evolved into

$$C = \begin{cases} \alpha \times \$5 + \$\beta = 1 \times \$5 - \$2 = \$3 & \text{if } S_1 = \$5, \\ \\ \alpha \times \$2 + \$\beta = 1 \times \$2 - \$2 = 0 & \text{if } S_1 = \$2, \end{cases}$$

so that the option contract and the equality $C = \alpha S_1 + \$\beta$ can be fulfilled, allowing the option issuer to break even whatever the evolution of the risky asset price S.

In a *cash settlement*, the stock is sold at the price $S_1 = \$5$ or $S_1 = \$2$, the payoff $C = (S_1 - K)^+ = \$3$ or $\$0$ is issued to the option contract holder, and the loan is refunded with the remaining $2.

In the case of *physical delivery* , $\alpha = 1$ share of stock is handed in to the option holder in exchange for the strike price $K = \$2$ which is used to refund the initial $2 loan subscribed by the issuer.

Here, the option contract price $\alpha S_0 + \$\beta = \$2$ is interpreted as the cost of hedging the option. We will see that this model is scalable and extends to discrete time.

We note that the initial option contract price of $2 can be turned to $C = \$3$ (%50 profit) ... or into $C = \$0$ (total ruin).

Thinking further

1) The expected claim payoff at time $t = 1$ is

$$\begin{aligned} \mathbb{E}[C] &= \$3 \times \mathbb{P}(C = \$3) + \$0 \times \mathbb{P}(C = \$0) \\ &= \$3 \times \mathbb{P}(S_1 = \$5). \end{aligned}$$

In absence of arbitrage opportunities ("fair market"), this expected payoff $\mathbb{E}[C]$ should equal the initial amount \$2 invested in the option. In that case we should have

$$\begin{cases} \mathbb{E}[C] = \$3 \times \mathbb{P}(S_1 = \$5) = \$2 \\ \\ \mathbb{P}(S_1 = \$5) + \mathbb{P}(S_1 = \$2) = 1. \end{cases}$$

from which we can *infer* the probabilities

$$\begin{cases} \mathbb{P}(S_1 = \$5) = \dfrac{2}{3} \\ \\ \mathbb{P}(S_1 = \$2) = \dfrac{1}{3}, \end{cases} \tag{5.1}$$

which are called *risk-neutral* probabilities. We see that under the risk-neutral probabilities, the stock S has twice more chances to go up than to go down in a "fair" market.

2) Based on the probabilities (5.1) we can also compute the expected value $\mathbb{E}[S_1]$ of the stock at time $t = 1$. We find

$$\begin{aligned} \mathbb{E}[S_1] &= \$5 \times \mathbb{P}(S_1 = \$5) + \$2 \times \mathbb{P}(S_1 = \$2) \\ &= \$5 \times \frac{2}{3} + \$2 \times \frac{1}{3} \\ &= \$4 \\ &= S_0. \end{aligned}$$

Here this means that, on average, no extra profit or loss can be made from an investment on the risky stock, hence the term "risk-neutral". In a more realistic model we can assume that the riskless bank account yields an interest rate equal to r, in which case the above analysis is modified by letting $\$\beta$ become $\$(1+r)\beta$ at time $t = 1$, nevertheless the main conclusions remain unchanged.

Market-implied probabilities

By matching the theoretical price $\mathbb{E}[C]$ to an actual market price data $\$M$ as

$$\$M = \mathbb{E}[C] = \$3 \times \mathbb{P}(C = \$3) + \$0 \times \mathbb{P}(C = \$0) = \$3 \times \mathbb{P}(S_1 = \$5)$$

we can infer the probabilities

$$\begin{cases} \mathbb{P}(S_1 = \$5) = \dfrac{\$M}{3} \\ \\ \mathbb{P}(S_1 = \$2) = \dfrac{3 - \$M}{3}, \end{cases} \tag{5.2}$$

which are *implied probabilities* estimated from market data, as illustrated in Figure 5.11. We note that the conditions

$$0 < \mathbb{P}(S_1 = \$5) < 1, \qquad 0 < \mathbb{P}(S_1 = \$2) < 1$$

are equivalent to $0 < \$M < 3$, which is consistent with financial intuition in a non-deterministic market. Figure 5.11 shows the time evolution of probabilities $p(t)$, $q(t)$ of two opposite outcomes.

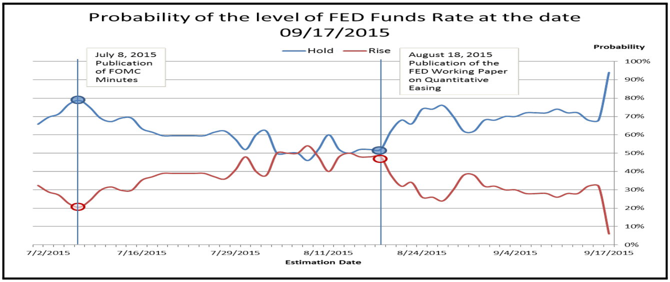


Fig. 5.11: Implied probabilities.

The *Practitioner* expects a good model to be:

- *Robust* with respect to missing, spurious or noisy data,
- *Fast* - prices have to be delivered daily in the morning,
- *Easy* to calibrate - parameter estimation,
- *Stable* with respect to re-calibration and the use of new data sets.

Typically, a medium size bank manages 5,000 options and 10,000 deals daily over 1,000 possible scenarios and dozens of time steps. This can mean a hundred million computations of $\mathbb{E}[C]$ daily, or close to a billion such computations for a large bank.

The *Mathematician* tends to focus on more theoretical features, such as:

- *Elegance*,
- *Sophistication*,

- *Existence* of analytical (closed-form) solutions / error bounds,
- *Significance* to mathematical finance.

This includes:

- *Creating* new payoff functions and structured products,
- *Defining* new models for underlying asset prices,
- *Finding* new ways to compute expectations $\mathbb{E}[C]$ and hedging strategies.

The methods involved include:

- Monte Carlo (60%),

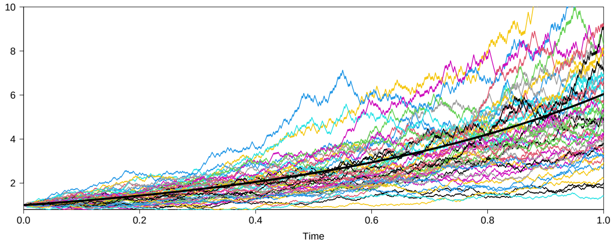


Fig. 5.12: One hundred sample price paths used for the Monte Carlo method.

- PDEs and finite differences (30%),
- Other analytic methods and approximations (10%),

\+ AI and Machine Learning techniques.

5.3 Black-Scholes Analysis

In this section we consider an asset price process $(S_t)_{t\in\mathbb{R}_+}$ modeled as the geometric Brownian motion (1.10) of Proposition 1.7. Recall that the evolution of the riskless bank account value $(A_t)_{t\in\mathbb{R}_+}$ is constructed from standard returns, defined from the differential equation

$$\frac{dA_t}{dt} = rA_t, \quad t \geqslant 0,$$

with solution

$$A_t = A_0 e^{rt}, \qquad t \geqslant 0,$$

where $r > 0$ is the risk-free interest rate. The risky asset price process $(S_t)_{t\in\mathbb{R}_+}$ is modeled from the equation

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$$dS_t = rS_t dt + \sigma S_t dB_t$$

with solution

$$S_t = S_0 \exp\left(\sigma B_t + \left(r - \frac{1}{2}\sigma^2\right) t\right), \qquad t \geqslant 0.$$

Let α_t and β_t be the numbers of units invested at time $t \geqslant 0$, respectively in the assets priced $(S_t)_{t\in\mathbb{R}_+}$ and $(A_t)_{t\in\mathbb{R}_+}$.

In the sequel, we will consider a portfolio whose value V_t at time $t \geqslant 0$ is given by

$$V_t = \beta_t A_t + \alpha_t S_t, \qquad t \geqslant 0.$$

Black-Scholes formula for European call options

In the case of European call options with payoff function $\phi(x) = (x - K)^+$ we have the following Black-Scholes formula.

Proposition 5.4. *The price at time $t \in [0, T]$ of the European call option with strike price K and maturity T is given by*

$$\begin{aligned} \mathrm{BS}_c(S_t, K, r, T-t, \sigma) &= \mathrm{e}^{-(T-t)r}\mathbb{E}[(S_T - K)^+ \mid \mathcal{F}_t] \qquad (5.3) \\ &= S_t \Phi\big(d_+(T-t)\big) - K\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big), \end{aligned}$$

$0 \leqslant t \leqslant T$, *with*

$$\begin{cases} d_+(T-t) := \dfrac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, & (5.4a) \\[2ex] d_-(T-t) := \dfrac{\log(S_t/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad 0 \leqslant t < T, & (5.4b) \end{cases}$$

where "log" denotes the natural logarithm *"ln" and*

$$\Phi(x) := \mathbb{P}(\mathcal{N} \leqslant x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-y^2/2} dy, \qquad x \in \mathbb{R},$$

is the standard Gaussian Cumulative Distribution Function (CDF).

We note the relation

$$d_+(T-t) = d_-(T-t) + |\sigma|\sqrt{T-t}, \qquad 0 \leqslant t < T. \qquad (5.5)$$

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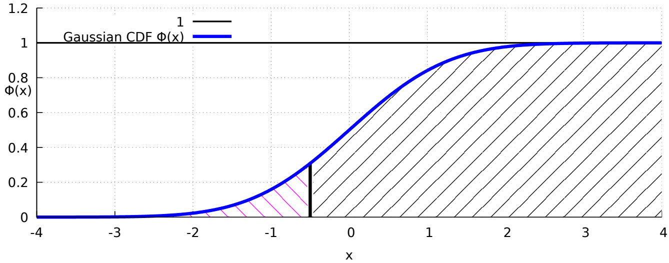


Fig. 5.13: Graph of the Gaussian Cumulative Distribution Function (CDF).

In other words, the European call option with strike price K and maturity T is priced at time $t \in [0, T]$ as

$$\mathrm{BS}_c(S_t, K, r, T-t, \sigma) = S_t \Phi\big(d_+(T-t)\big) - K\mathrm{e}^{-(T-t)r} \Phi\big(d_-(T-t)\big),$$

$0 \leqslant t \leqslant T$. The following R script implements the Black-Scholes formula for European call options in R.*

```
BSCall <- function(S, K, r, T, sigma)
{d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T));d2 <- d1 - sigma * sqrt(T)
BSCall = S*pnorm(d1) - K*exp(-r*T)*pnorm(d2)
BSCall}
```

In comparison with the discrete-time Cox-Ross-Rubinstein (CRR) model, the interest in the Black-Scholes formula is to provide an analytical solution that can be evaluated in a single step, which is computationally much more efficient.

* Download the corresponding **IPython notebook** that can be run **here** or **here**.

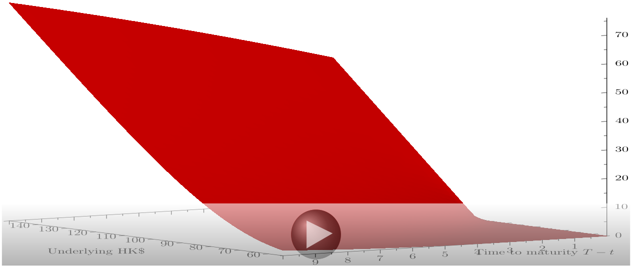


Fig. 5.14: Graph of the Black-Scholes call price map with strike price $K = 100$.*

Figure 5.14 presents an interactive graph of the Black-Scholes call price map, *i.e.* of the function

$$(t,x) \longmapsto \mathrm{BS}_c(x,K,r,T-t,\sigma) = x\Phi\big(d_+(T-t)\big) - K\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big).$$

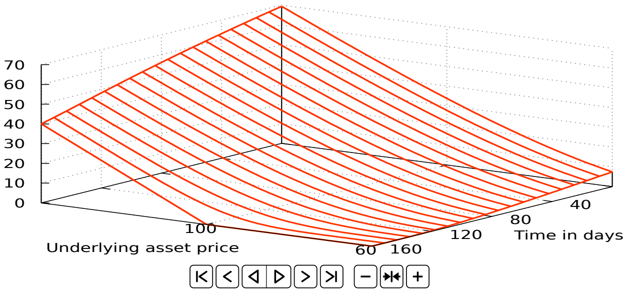


Fig. 5.15: Time-dependent solution of the Black-Scholes PDE (call option).†

The next proposition is proved by a direct differentiation of the Black-Scholes function.

Proposition 5.5. *The Black-Scholes* Delta *of the European call option is given by*

$$\alpha_t = \alpha_t(S_t) = \frac{\partial}{\partial x}\mathrm{BS}_c(x,K,r,T-t,\sigma)_{|x=S_t} = \Phi\big(d_+(T-t)\big) \in [0,1], \quad (5.6)$$

* Right-click on the figure for interaction and "Full Screen Multimedia" view.

† The animation works in Acrobat Reader on the entire pdf file.

where $d_+(T-t)$ *is defined in* (5.4a).

We note that the Black-Scholes call price splits into a risky component $S_t\Phi\big(d_+(T-t)\big)$ and a riskless component $-K\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big)$, as follows:

$$\mathrm{BS}_c(S_t,K,r,T-t,\sigma) = \underbrace{S_t\Phi\big(d_+(T-t)\big)}_{\text{Risky investment (held)}} - \underbrace{K\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big)}_{\text{Risk free investment (borrowed)}}, \tag{5.7}$$

$0 \leqslant t \leqslant T$, *i.e.* $\alpha_t = \Phi\big(d_+(T-t)\big)$ represents the quantity of assets invested in the risky asset priced at S_t. The following R script implements the Black-Scholes Delta for European call options.

```
DeltaCall <- function(S, K, r, T, sigma)
{d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T))
DeltaCall = pnorm(d1);DeltaCall}
```

In Figure 5.16 we plot the Delta of the European call option as a function of the underlying asset price and of the time remaining until maturity.

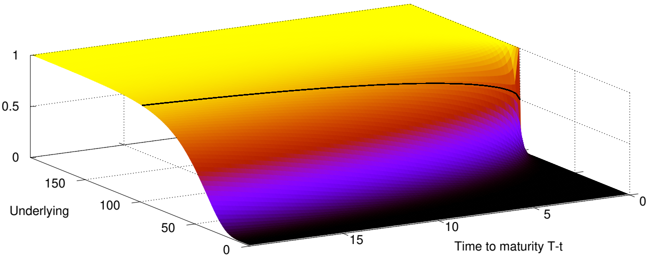


Fig. 5.16: Delta of a European call option with strike price $K = 100$, $r = 3\%$, $\sigma = 10\%$.

Black-Scholes formula for European Put Options

The European put option price is computed in the next proposition.

Proposition 5.6. *The price at time* $t \in [0, T]$ *of the European put option with strike price* K *and maturity* T *is given by*

$$\begin{aligned}\mathrm{BS}_p(S_t,K,r,T-t,\sigma) &= \mathrm{e}^{-(T-t)r}\mathbb{E}[(K-S_T)^+ \mid \mathcal{F}_t] \\ &= K\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - S_t\Phi\big(-d_+(T-t)\big),\end{aligned}$$

$0 \leqslant t \leqslant T$, *where* $d_+(T-t)$ *and* $d_-(T-t)$ *are defined in* (5.4a)-(5.4b).

The Black-Scholes formula for European Put Options is plotted in illustrated in Figure 5.17.

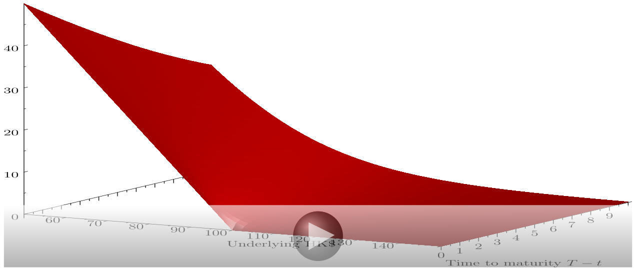


Fig. 5.17: Graph of the Black-Scholes put price function with strike price $K = 100$.*

In other words, the European put option with strike price K and maturity T is priced at time $t \in [0, T]$ as

$$Bl_p(S_t, K, r, T-t, \sigma) = K\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - S_t\Phi\big(-d_+(T-t)\big),$$

$0 \leqslant t \leqslant T$.

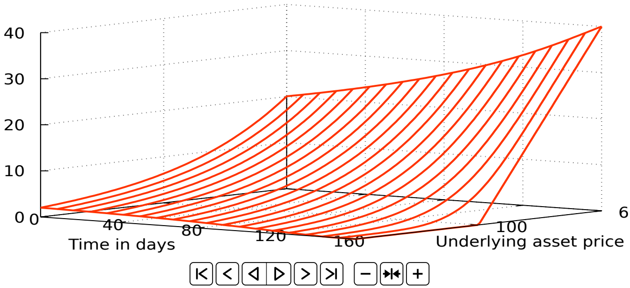


Fig. 5.18: Time-dependent solution of the Black-Scholes PDE (put option).†

The following R script implements the Black-Scholes formula for European put options in R.

* Right-click on the figure for interaction and "Full Screen Multimedia" view.

† The animation works in Acrobat Reader on the entire pdf file.

```
BSPut <- function(S, K, r, T, sigma)
{d1 = (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T));d2 = d1 - sigma * sqrt(T);
BSPut = K*exp(-r*T) * pnorm(-d2) - S*pnorm(-d1);BSPut}
```

The Black-Scholes *Delta* of the European put option is computed in the following proposition.

Proposition 5.7. *The Black-Scholes* Delta *of the European put option is given by*

$$\begin{aligned}\alpha_t = \alpha_t(S_t) &= \frac{\partial}{\partial x}\mathrm{BS}_p(x,K,r,T-t,\sigma)_{|x=S_t} \\ &= -(1-\Phi(d_+(T-t))) = -\Phi(-d_+(T-t)) \in [-1,0], \quad 0 \leqslant t \leqslant T,\end{aligned}$$

where $d_+(T-t)$ is defined in (5.4a).

We note that the Black-Scholes put price splits into a risky component $-S_t\Phi(-d_+(T-t))$ and a riskless component $K\mathrm{e}^{-(T-t)r}\Phi(-d_-(T-t))$, as follows:

$$\mathrm{BS}_p(S_t,K,r,T-t,\sigma) = \underbrace{K\mathrm{e}^{-(T-t)r}\Phi(-d_-(T-t))}_{\text{Risk−free investment (savings)}} - \underbrace{S_t\Phi(-d_+(T-t))}_{\text{Risky investment (short)}}, \tag{5.8}$$

$0 \leqslant t \leqslant T$, *i.e.* $-\Phi(-d_+(T-t))$ represents the quantity of assets invested in the risky asset priced at S_t.

```
DeltaPut <- function(S, K, r, T, sigma)
{d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T)); DeltaPut = -pnorm(-d1);DeltaPut}
```

In Figure 5.19 we plot the Delta of the European put option as a function of the underlying asset price and of the time remaining until maturity.

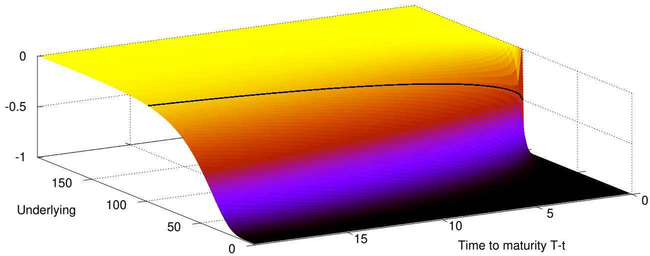


Fig. 5.19: Delta of a European put option with strike price $K = 100$, $r = 3\%$, $\sigma = 10\%$.

Exercises

Exercise 5.1 Consider a risky asset valued $S_0 = \$3$ at time $t = 0$ and taking only two possible values $S_1 \in \{\$1, \$5\}$ at time $t = 1$, and a financial claim given at time $t = 1$ by

$$C := \begin{cases} \$0 & \text{if } S_1 = \$5 \\ \\ \$2 & \text{if } S_1 = \$1. \end{cases}$$

Is C the payoff of a call option or of a put option? Give the strike price of the option.

Exercise 5.2 Consider a risky asset valued $S_0 = \$4$ at time $t = 0$, and taking only two possible values $S_1 \in \{\$2, \$5\}$ at time $t = 1$. Compute the initial value $V_0 = \alpha S_0 + \$\beta$ of the portfolio hedging the claim payoff

$$C = \begin{cases} \$0 & \text{if } S_1 = \$5 \\ \\ \$6 & \text{if } S_1 = \$2 \end{cases}$$

at time $t = 1$, and find the corresponding risk-neutral probability measure $\mathbb{P}^*$.

Exercise 5.3 Consider a risky asset valued $S_0 = \$4$ at time $t = 0$, and taking only two possible values $S_1 \in \{\$5, \$2\}$ at time $t = 1$, and the claim payoff

$$C = \begin{cases} \$3 & \text{if } S_1 = \$5 \\ \\ \$0 & \text{if } S_1 = \$2. \end{cases} \qquad \text{at time } t = 1.$$

We assume that the issuer charges $1 for the option contract at time $t = 0$.

a) Compute the portfolio allocation (α, β) made of α stocks and $\$\beta$ in cash, so that:

   i) the full $1 option price is invested into the portfolio at time $t = 0$,

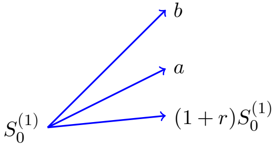
   and

   ii) the portfolio reaches the $C = \$3$ target if $S_1 = \$5$ at time $t = 1$.

b) Compute the loss incurred by the option issuer if $S_1 = \$2$ at time $t = 1$.

Exercise 5.4 Recall that an *arbitrage opportunity* consist of a portfolio allocation with zero or negative cost that can yield a nonnegative and possibly strictly positive payoff at maturity.

a) Consider the following market model:



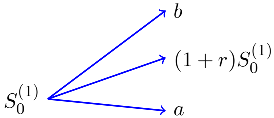
i) Does this model allow for arbitrage opportunities?

Yes | No |

ii) If this model allows for arbitrage opportunities, how can they be realized?

By shortselling | By borrowing on savings | N.A. |

b) Consider the following market model:



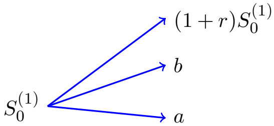
i) Does this model allow for arbitrage opportunities?

Yes | No |

ii) If this model allows for arbitrage opportunities, how can they be realized?

By shortselling | By borrowing on savings | N.A. |

c) Consider the following market model:



i) Does this model allow for arbitrage opportunities?

Yes |	No |

ii) If this model allows for arbitrage opportunities, how can they be realized?

By shortselling |	By borrowing on savings |	N.A. |

Exercise 5.5 In a market model with two time instants $t = 0$ and $t = 1$ and risk-free interest rate r, consider:

- a riskless asset valued $S_0^{(0)}$ at time $t = 0$, and value $S_1^{(0)} = (1 + r)S_0^{(0)}$ at time $t = 1$.

- a risky asset with price $S_0^{(1)}$ at time $t = 0$, and three possible values at time $t = 1$, with $a < b < c$, *i.e.*:

$$S_1^{(1)} = \begin{cases} S_0^{(1)}(1+a), \\ S_0^{(1)}(1+b), \\ S_0^{(1)}(1+c). \end{cases}$$

In general, is it possible to hedge (or replicate) a claim with three distinct claim payoff values C_a, C_b, C_c in this market?

Exercise 5.6 Superhedging risk measure. Consider a stock valued S_0 at time $t = 0$, and taking only two possible values $S_1 = \underline{S}_1$ or $S_1 = \overline{S}_1$ at time $t = 1$, with $\underline{S}_1 < \overline{S}_1$.

a) Compute the initial portfolio allocation (α, β) of a portfolio made of α units of stock and $\$\beta$ in cash, hedging the call option with strike price $K \in [\underline{S}_1, \overline{S}_1]$ and claim payoff

$$C = (S_1 - K)^+ = \begin{cases} \overline{S}_1 - K \text{ if } S_1 = \overline{S}_1 \\ \$0 \qquad\quad \text{if } S_1 = \underline{S}_1, \quad \text{at time } t = 1. \end{cases}$$

b) Show that the risky asset allocation α satisfies the condition $\alpha \in [0, 1]$.

c) Compute the Superhedging Risk Measure SRM_C* of the claim $C = (S_1 - K)^+$.

Exercise 5.7 Given two strike prices $K_1 < K_2$, we consider a long box spread option, realized as the combination of four legs with same maturity date:

- One *long* call with strike price K_1 and payoff function $(x - K_1)^+$,
- One *short* put with strike price K_1 and payoff function $-(K_1 - x)^+$,
- One *short* call with strike price K_2 and payoff function $-(x - K_2)^+$,
- One *long* put with strike price K_2 and payoff function $(K_2 - x)^+$.

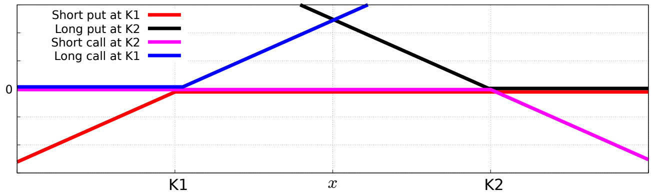


Fig. 5.20: Graphs of call/put payoff functions.

a) Find the payoff of the long box spread option in terms of K_1 and K_2.
b) From Table 5.1, find a choice of strike prices $K_1 < K_2$ that can be used to build a long box spread option on the Hang Seng Index (HSI).
c) Using Table 5.1, price the option built in part (b) in index points, and then in HK$.

   Hints.

   i) The closing prices in Table 5.1 are warrant prices quoted in index points.
   ii) Warrant prices are converted to option prices by multiplication by the number given in the "Entitlement Ratio" column.
   iii) The conversion rate from index points to HK$ is HK$50 per index point.

d) Would you buy the option priced in part (c) ?

* "*The smallest amount necessary to be paid for a portfolio at time $t = 0$ so that the value of this portfolio at time $t = 1$ is at least as great as C*".

DERIVATIVE WARRANT SEARCH

Link to Relevant Exchange Traded Options

Updated: 19 December 2022

Basic Data										Market Data								
DW Code	Issuer	UL	Call/ Put	DW Type	Listing (D-M-Y)	Maturity (D-M-Y)	Strike Currency	Strike	Entitle-ment Ratio^	Total Issue Size	O/S (%)	Delta (%)	IV. (%)	Trading Currency	Day High	Day Low	Closing Price	T/O ('000)
18606	SG	HSI	Put	Standard	22-11-2022	28-03-2023	HKD	25088	10000	300,000,000	8.01	(0.002)	30.968	HKD	0.054	0.042	0.053	459
19399	HT	HSI	Put	Standard	01-12-2021	28-03-2023	HKD	25200	10000	400,000,000	0.06	(0.002)	32.190	HKD	0.000	0.000	0.061	0
19485	BI	HSI	Put	Standard	02-12-2021	28-03-2023	HKD	25200	10000	150,000,000	21.41	(0.002)	28.154	HKD	0.044	0.037	0.044	59
22857	VT	HSI	Put	Standard	26-02-2021	28-03-2023	HKD	25000	8000	80,000,000	22.45	(0.002)	30.905	HKD	0.065	0.043	0.064	1,165
26601	BI	HSI	Call	Standard	27-12-2021	28-03-2023	HKD	25200	11000	150,000,000	0.00	0.018	25.347	HKD	0.390	0.360	0.370	84
27489	BP	HSI	Call	Standard	17-09-2021	28-03-2023	HKD	25000	7500	80,000,000	2.95	0.009	28.392	HKD	0.590	0.540	0.540	6
28231	HS	HSI	Call	Standard	29-09-2021	28-03-2023	HKD	25118	7500	200,000,000	0.00	0.012	24.897	HKD	0.000	0.000	0.570	0

^ The entitlement ratio in general represents the number of derivative warrants required to be exercised into one share or one unit of the underlying asset (subject to any adjustments as may be necessary to reflect any capitalization, rights issue, distribution or the like).

Delayed data on Delta and Implied Volatility of Derivative Warrants are provided by Reuters.
Users should not use such data provided by Reuters for commercial purposes without its prior written consent.

For underlying stock price, please refer to Securities Prices of Market Data.

Table 5.1: Call and put options on the Hang Seng Index (HSI).