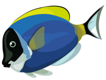
Chapter 3
Processes with Jumps

Modeling insurance risk requires to use continuous-time stochastic processes that allow for jumps in addition to a continuous component. This chapter presents the construction of Poisson and compound Poisson processes which are used for the modeling of insurance claim and reserve processes. Applications will be given to the closed form computation of ruin probabilities in the Cramér-Lundberg model.

3.1 The Poisson Process

The most elementary and useful jump process is the *standard Poisson process* $(N_t)_{t\in\mathbb{R}_+}$ which is a *counting process*, *i.e.* $(N_t)_{t\in\mathbb{R}_+}$ has jumps of size $+1$ only and its paths are constant in between two jumps, with $N_0 := 0$.



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The counting process $(N_t)_{t\in\mathbb{R}_+}$ that can be used to model discrete arrival times such as claim dates in insurance, or connection logs.

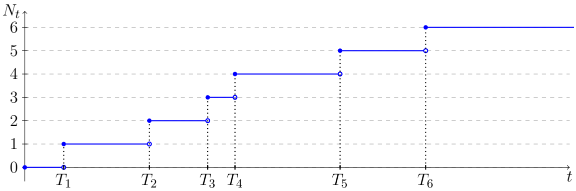

Fig. 3.1: Sample path of a counting process $(N_t)_{t\in\mathbb{R}_+}$.

Using the indicator functions

$$\mathbb{1}_{[T_k,\infty)}(t) = \begin{cases} 1 \text{ if } t \geqslant T_k, \\ 0 \text{ if } 0 \leqslant t < T_k, \quad k \geqslant 1, \end{cases}$$

the value of N_t at time t can be written as

$$N_t = \sum_{k\geqslant 1} \mathbb{1}_{[T_k,\infty)}(t), \qquad t \geqslant 0, \tag{3.1}$$

where and $(T_k)_{k\geqslant 1}$ is the increasing family of jump times of $(N_t)_{t\in\mathbb{R}_+}$ such that

$$\lim_{k\to\infty} T_k = +\infty.$$

The operation defined in (3.1) can be implemented in R using the following code.

```
T=10; Tn=c(1,3,4,7,9); dev.new(width=T, height=5)
plot(stepfun(Tn,c(0,1,2,3,4,5)),xlim =c(0,T),xlab="t",ylab=expression('N'[t]),pch=1, cex=0.8,
     col='blue', lwd=2, main="", cex.axis=1.2, cex.lab=1.4,xaxs='i'); grid()
```

In order for the counting process $(N_t)_{t\in\mathbb{R}_+}$ to be a Poisson process, it has to satisfy the following conditions:

1. Independence of increments: for all $0 \leqslant t_0 < t_1 < \cdots < t_n$ and $n \geqslant 1$ the increments
$$N_{t_1} - N_{t_0}, \ldots, \; N_{t_n} - N_{t_{n-1}},$$
are mutually independent random variables.

2. Stationarity of increments: $N_{t+h} - N_{s+h}$ has the same distribution as $N_t - N_s$ for all $h > 0$ and $0 \leqslant s \leqslant t$.

The meaning of the above stationarity condition is that for all fixed $k \geqslant 0$ we have

$$\mathbb{P}(N_{t+h} - N_{s+h} = k) = \mathbb{P}(N_t - N_s = k),$$

for all $h > 0$, *i.e.*, the value of the probability

$$\mathbb{P}(N_{t+h} - N_{s+h} = k)$$

does not depend on $h > 0$, for all fixed $0 \leqslant s \leqslant t$ and $k \geqslant 0$.

Based on the above assumption, given $T > 0$ a time value, a natural question arises:

what is the probability distribution of the random variable N_T?

We already know that N_t takes values in $\mathbb{N}$ and therefore it has a discrete distribution for all $t \in \mathbb{R}_+$.

It is a remarkable fact that the distribution of the increments of $(N_t)_{t\in\mathbb{R}_+}$, can be completely determined from the above conditions, as shown in the following theorem.

As seen in the next result, cf. Theorem 4.1 in Bosq and Nguyen (1996), the Poisson increment $N_t - N_s$ has the Poisson distribution with parameter $(t-s)\lambda$.

Theorem 3.1. *Assume that the counting process $(N_t)_{t\in\mathbb{R}_+}$ satisfies the above independence and stationarity Conditions 1 and 2 on page 76. Then, for all fixed $0 \leqslant s \leqslant t$ the increment $N_t - N_s$ follows the Poisson distribution with parameter $(t-s)\lambda$, i.e. we have*

$$\mathbb{P}(N_t - N_s = k) = \mathrm{e}^{-(t-s)\lambda} \frac{((t-s)\lambda)^k}{k!}, \qquad k \geqslant 0, \tag{3.2}$$

for some constant $\lambda > 0$.

The parameter $\lambda > 0$ is called the intensity of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ and it is given by

$$\lambda := \lim_{h\to 0} \frac{1}{h} \mathbb{P}(N_h = 1). \tag{3.3}$$

The proof of the above Theorem 3.1 is technical and not included here, cf. *e.g.* Bosq and Nguyen (1996) for details, and we could in fact take this distribution property (3.2) as one of the hypotheses that define the Poisson process.

Precisely, we could restate the definition of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$ as being a stochastic process defined by (3.1), which is assumed to have independent increments distributed according to the Poisson distribution, in the sense that for all $0 \leqslant t_0 \leqslant t_1 < \cdots < t_n$,

$$(N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}})$$

is a vector of independent Poisson random variables with respective parameters

$$((t_1 - t_0)\lambda, \ldots, (t_n - t_{n-1})\lambda).$$

In particular, N_t has the Poisson distribution with parameter λt, *i.e.*,

$$\mathbb{P}(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \qquad t > 0.$$

The *expected value* $\mathbb{E}[N_t]$ and the variance of N_t can be computed as

$$\mathbb{E}[N_t] = \mathrm{Var}[N_t] = \lambda t, \tag{3.4}$$

see Exercise A.1. As a consequence, the *dispersion index* of the Poisson process is

$$\frac{\mathrm{Var}[N_t]}{\mathbb{E}[N_t]} = 1, \qquad t \geqslant 0. \tag{3.5}$$

Short time behaviour

From (3.3) above we deduce the *short time asymptotics**

$$\begin{cases} \mathbb{P}(N_h = 0) = e^{-\lambda h} = 1 - \lambda h + o(h), \qquad h \to 0, \\ \mathbb{P}(N_h = 1) = \lambda h e^{-\lambda h} \simeq \lambda h, \qquad h \to 0. \end{cases}$$

By stationarity of the Poisson process we also find more generally that

$$\begin{cases} \mathbb{P}(N_{t+h} - N_t = 0) = e^{-\lambda h} = 1 - \lambda h + o(h), \qquad h \to 0, \\ \mathbb{P}(N_{t+h} - N_t = 1) = \lambda h e^{-\lambda h} \simeq \lambda h, \qquad h \to 0, \\ \mathbb{P}(N_{t+h} - N_t = 2) \simeq h^2 \frac{\lambda^2}{2} = o(h), \qquad h \to 0, \qquad t > 0, \end{cases} \tag{3.6}$$

for all $t > 0$. This means that within a "short" time interval $[t, t+h]$ of length h, the increment $N_{t+h} - N_t$ behaves like a Bernoulli random variable with parameter λh. This fact can be used for the random simulation of Poisson process paths.

* The notation $f(h) = o(h^k)$ means $\lim_{h \to 0} f(h)/h^k = 0$, and $f(h) \simeq h^k$ means $\lim_{h \to 0} f(h)/h^k = 1$.

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The next R code and Figure 3.2 present a simulation of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ according to its short time behavior (3.6).

```
lambda = 0.6;T=10;N=1000*lambda;h=T*1.0/N
t=0;s=c();for (k in 1:N) {if (runif(1)<lambda*h) {s=c(s,t)};t=t+h}
dev.new(width=T, height=5)
plot(stepfun(s,cumsum(c(0,rep(1,length(s))))),xlim
      =c(0,T),xlab="t",ylab=expression('N'[t]),pch=1, cex=0.8, col='blue', lwd=2, main="",
      cex.axis=1.2, cex.lab=1.4,xaxs='i'); grid()
```

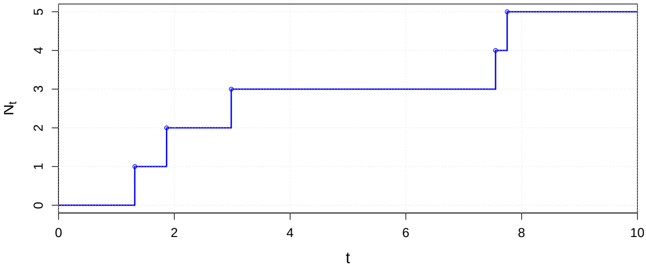

Fig. 3.2: Sample path of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

More generally, for $k \geqslant 1$ we have

$$\mathbb{P}(N_{t+h} - N_t = k) \simeq h^k \frac{\lambda^k}{k!}, \qquad h \to 0, \qquad t > 0.$$

Time-dependent intensity

The intensity of the Poisson process can in fact be made time-dependent (*e.g.* by a time change), in which case we have

$$\mathbb{P}(N_t - N_s = k) = \exp\left(-\int_s^t \lambda(u)du\right) \frac{\left(\int_s^t \lambda(u)du\right)^k}{k!}, \quad k = 0, 1, 2, \ldots .$$

Assuming that $\lambda(t)$ is a continuous function of time t we have in particular, as h tends to zero,

$$\mathbb{P}(N_{t+h} - N_t = k)$$

$$= \begin{cases} \exp\left(-\int_t^{t+h}\lambda(u)du\right) = 1-\lambda(t)h+o(h), & k=0, \\ \exp\left(-\int_t^{t+h}\lambda(u)du\right)\int_t^{t+h}\lambda(u)du = \lambda(t)h+o(h), & k=1, \\ o(h), & k\geqslant 2. \end{cases}$$

The intensity process $(\lambda(t))_{t\in\mathbb{R}_+}$ can also be made random, as in the case of Cox processes.

Poisson process jump times

In order to determine the distribution of the first jump time T_1 we note that we have the equivalence

$$\{T_1 > t\} \Longleftrightarrow \{N_t = 0\},$$

which implies

$$\mathbb{P}(T_1 > t) = \mathbb{P}(N_t = 0) = \mathrm{e}^{-\lambda t}, \qquad t \geqslant 0,$$

i.e., T_1 has an exponential distribution with parameter $\lambda > 0$.

In order to prove the next proposition we note that more generally, we have the equivalence

$$\{T_n > t\} \Longleftrightarrow \{N_t \leqslant n-1\},$$

for all $n \geqslant 1$. This allows us to compute the distribution of the random jump time T_n with its probability density function. It coincides with the *gamma* distribution with integer parameter $n \geqslant 1$, also known as the Erlang distribution in queueing theory.

Proposition 3.2. *For all $n \geqslant 1$, the probability distribution of T_n has the gamma probability density function*

$$t \longmapsto \lambda^n \mathrm{e}^{-\lambda t}\frac{t^{n-1}}{(n-1)!}$$

with shape parameter $n \geqslant 1$ and scaling parameter $\lambda > 0$ on $\mathbb{R}_+$, i.e., for all $t > 0$ the probability $\mathbb{P}(T_n \geqslant t)$ is given by

$$\mathbb{P}(T_n \geqslant t) = \lambda^n \int_t^\infty \mathrm{e}^{-\lambda s}\frac{s^{n-1}}{(n-1)!}ds.$$

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Proof. We have

$$\mathbb{P}(T_1 > t) = \mathbb{P}(N_t = 0) = \mathrm{e}^{-\lambda t}, \qquad t \geqslant 0,$$

and by induction, assuming that

$$\mathbb{P}(T_{n-1} > t) = \lambda \int_t^\infty \mathrm{e}^{-\lambda s} \frac{(\lambda s)^{n-2}}{(n-2)!} ds, \qquad n \geqslant 2,$$

we obtain

$$\begin{aligned}
\mathbb{P}(T_n > t) &= \mathbb{P}(T_n > t \geqslant T_{n-1}) + \mathbb{P}(T_{n-1} > t) \\
&= \mathbb{P}(N_t = n-1) + \mathbb{P}(T_{n-1} > t) \\
&= \mathrm{e}^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} + \lambda \int_t^\infty \mathrm{e}^{-\lambda s} \frac{(\lambda s)^{n-2}}{(n-2)!} ds \\
&= \lambda \int_t^\infty \mathrm{e}^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!} ds, \qquad t \geqslant 0,
\end{aligned}$$

where we applied an integration by parts to derive the last line. □

In particular, for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}_+$, we have

$$\mathbb{P}(N_t = n) = p_n(t) = \mathrm{e}^{-\lambda t} \frac{(\lambda t)^n}{n!},$$

i.e., $p_{n-1} : \mathbb{R}_+ \to \mathbb{R}_+$, $n \geqslant 1$, is the probability density function of the random jump time T_n.

In addition to Proposition 3.2 we could show the following proposition which relies on the *strong Markov property*, see *e.g.* Theorem 6.5.4 of Norris (1998).

Proposition 3.3. *The (random) interjump times*

$$\tau_k := T_{k+1} - T_k$$

spent at state $k \geqslant 0$, with $T_0 = 0$, form a sequence of independent identically distributed random variables having the exponential distribution with parameter $\lambda > 0$, i.e.,

$$\mathbb{P}(\tau_0 > t_0, \ldots, \tau_n > t_n) = \mathrm{e}^{-(t_0 + t_1 + \cdots + t_n)\lambda}, \qquad t_0, t_1, \ldots, t_n \geqslant 0.$$

As the expectation of the exponentially distributed random variable τ_k with parameter $\lambda > 0$ is given by

$$\mathbb{E}[\tau_k] = \lambda \int_0^\infty x e^{-\lambda x} dx = \frac{1}{\lambda},$$

we can check that the nth jump time $T_n = \tau_0 + \cdots + \tau_{n-1}$ has the mean

$$\mathbb{E}[T_n] = \frac{n}{\lambda}, \qquad n \geqslant 1.$$

Consequently, the higher the intensity $\lambda > 0$ is (*i.e.*, the higher the probability of having a jump within a small interval), the smaller the time spent in each state $k \geqslant 0$ is on average.

As a consequence of Proposition 3.2, random samples of Poisson process jump times can be generated from Poisson jump times using the following R code according to Proposition 3.3.

```
lambda = 0.6;T=10;Tn=c();n=0;
S=0; while (S<T) {S=S+rexp(1,rate=lambda); Tn=c(Tn,S); n=n+1}
Z<-cumsum(c(0,rep(1,n))); dev.new(width=T, height=5)
plot(stepfun(Tn,Z),xlim =c(0,T),ylim=c(0,8),xlab="t",ylab=expression('N'[t]),pch=1, cex=1,
      col="blue", lwd=2, main="", las = 1, cex.axis=1.2, cex.lab=1.4,xaxs='i',yaxs='i'); grid()
```

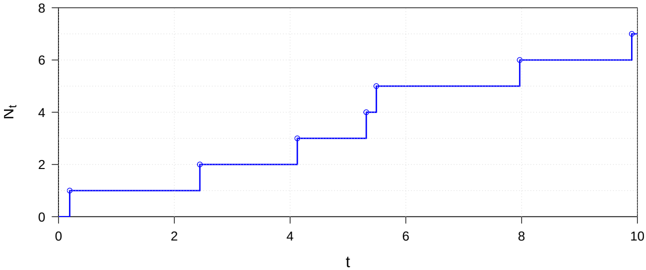

Fig. 3.3: Sample path of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

In addition, conditionally to $\{N_T = n\}$, the n jump times on $[0,T]$ of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ are independent uniformly distributed random variables on $[0,T]^n$, cf. *e.g.* § 11.1 in Privault (2018). This fact can also be useful for the random simulation of Poisson process paths.

```
lambda = 0.6;T=10;n = rpois(1,lambda*T);Tn <- sort(runif(n,0,T)); Z<-cumsum(c(0,rep(1,n)));
    dev.new(width=T, height=5)
plot(stepfun(Tn,Z),xlim =c(0,T),ylim=c(0,8),xlab="t",ylab=expression('N'[t]),pch=1, cex=1,
    col="blue", lwd=2, main="", las = 1, cex.axis=1.2, cex.lab=1.4,xaxs='i',tick.ratio = 0.5);
    grid()
```

Compensated Poisson martingale

From (3.4) above we deduce that

$$\mathbb{E}[N_t - \lambda t] = 0, \tag{3.7}$$

i.e., the compensated Poisson process $(N_t - \lambda t)_{t\in\mathbb{R}_+}$ has *centered increments*.

```
lambda = 0.6;T=10;Tn=c();S=0;n=0;
while (S<T) {S=S+rexp(1,rate=lambda); Tn=c(Tn,S); n=n+1}
Z<-cumsum(c(0,rep(1,n)));
N <- function(t) {return(stepfun(Tn,Z)(t))};t <- seq(0,10,0.01)
dev.new(width=T, height=5)
plot(t,N(t)-lambda*t,xlim = c(0,10),ylim =
    c(-2,2),xlab="t",ylab=expression(paste('N'[t],'-t')),type="l",lwd=2,col="blue",main="",
    xaxs = "i", yaxs = "i", xaxs = "i", yaxs = "i", las = 1, cex.axis=1.2, cex.lab=1.4)
abline(h = 0, col="black", lwd =2)
points(Tn,N(Tn)-lambda*Tn,pch=1,cex=0.8,col="blue",lwd=2)
```

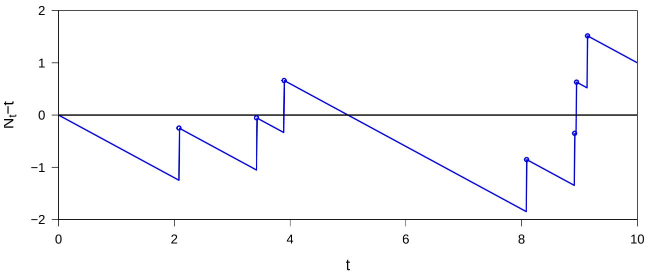

Fig. 3.4: Sample path of the compensated Poisson process $(N_t - \lambda t)_{t\in\mathbb{R}_+}$.

Since in addition $(N_t - \lambda t)_{t\in\mathbb{R}_+}$ also has independent increments, we get the following proposition. We let

$$\mathcal{F}_t := \sigma\big(N_s \ : \ s \in [0, t]\big), \qquad t \geqslant 0,$$

denote the *filtration* generated by the Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

Proposition 3.4. *The compensated Poisson process*

$$(N_t - \lambda t)_{t \in \mathbb{R}_+}$$

is a martingale *with respect* $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$.

The Poisson process belong to the family of *renewal processes*, which are counting processes of the form

$$N_t = \sum_{n \geqslant 1} \mathbb{1}_{[T_n, \infty)}(t), \qquad t \geqslant 0,$$

for which $\tau_k := T_{k+1} - T_k$, $k \geqslant 0$, is a sequence of independent identically distributed random variables.

3.2 Compound Poisson Process

The Poisson process itself appears to be too limited to develop realistic price models as its jumps are of constant size. Therefore there is some interest in considering jump processes that can have random jump sizes.

Let $(Z_k)_{k \geqslant 1}$ denote a sequence of independent, identically distributed (*i.i.d.*) square-integrable random variables, distributed as a common random variable Z with probability distribution $\nu(dy)$ on $\mathbb{R}$, independent of the Poisson process $(N_t)_{t \in \mathbb{R}_+}$. We have

$$\mathbb{P}(Z \in [a,b]) = \nu([a,b]) = \int_a^b \nu(dy), \quad -\infty < a \leqslant b < \infty, \quad k \geqslant 1,$$

and when the distribution $\nu(dy)$ admits a probability density $\varphi(y)$ on $\mathbb{R}$, we write $\nu(dy) = \varphi(y)dy$ and

$$\mathbb{P}(Z \in [a,b]) = \int_a^b \varphi(y)dy, \quad -\infty < a \leqslant b < \infty, \quad k \geqslant 1.$$

Figure 3.5 shows an example of Gaussian jump size distribution.

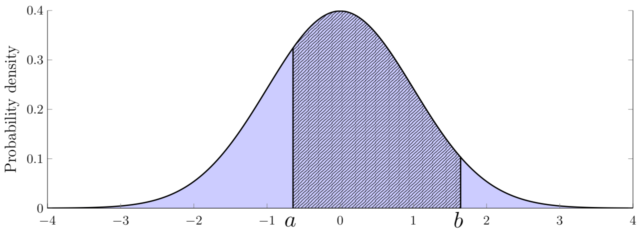

Fig. 3.5: Probability density function φ.

Definition 3.5. *The process* $(Y_t)_{t\in\mathbb{R}_+}$ *given by the random sum*

$$Y_t := Z_1 + Z_2 + \cdots + Z_{N_t} = \sum_{k=1}^{N_t} Z_k, \qquad t \geqslant 0, \tag{3.8}$$

*is called a compound Poisson process.**

Letting Y_{t^-} denote the left limit

$$Y_{t^-} := \lim_{s \nearrow t} Y_s, \qquad t > 0,$$

we note that the jump size

$$\Delta Y_t := Y_t - Y_{t^-}, \qquad t \geqslant 0,$$

of $(Y_t)_{t\in\mathbb{R}_+}$ at time t is given by the relation

$$\Delta Y_t = Z_{N_t} \Delta N_t, \qquad t \geqslant 0, \tag{3.9}$$

where

$$\Delta N_t := N_t - N_{t^-} \in \{0,1\}, \qquad t \geqslant 0,$$

denotes the jump size of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$, and N_{t^-} is the left limit

$$N_{t^-} := \lim_{s \nearrow t} N_s, \qquad t > 0,$$

The next Figure 3.6 represents a sample path of a compound Poisson process, with here $Z_1 = 0.9$, $Z_2 = -0.7$, $Z_3 = 1.4$, $Z_4 = 0.6$, $Z_5 = -2.5$, $Z_6 = 1.5$, $Z_7 = -0.5$, with the relation

$$Y_{T_k} = Y_{T_k^-} + Z_k, \qquad k \geqslant 1.$$

* We use the convention $\sum_{k=1}^{n} Z_k = 0$ if $n = 0$, so that $Y_0 = 0$.

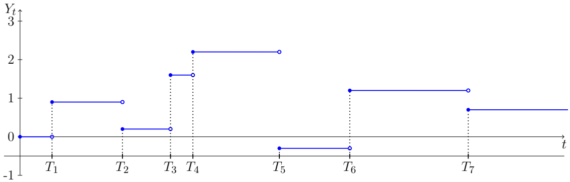

Fig. 3.6: Sample path of a compound Poisson process $(Y_t)_{t\in\mathbb{R}_+}$.

Example. Assume that the jump sizes Z are Gaussian distributed with mean δ and variance η^2, with

$$\nu(dy) = \frac{1}{\sqrt{2\pi\eta^2}} \mathrm{e}^{-(y-\delta)^2/(2\eta^2)} dy.$$

```
N<-50;Tk<-cumsum(rexp(N,rate=0.5)); Zk<-rexp(N,rate=0.5); Yk<-cumsum(c(0,Zk))
plot(stepfun(Tk,Yk),xlim = c(0,10),lwd=2,do.points = F,main="L=0.5",col="blue")
Zk<-rnorm(N,mean=0,sd=1); Yk<-cumsum(c(0,Zk))
plot(stepfun(Tk,Yk),xlim = c(0,10),lwd=2,do.points = F,main="L=0.5",col="blue")
```

Given that $\{N_T = n\}$, the n jump sizes of $(Y_t)_{t\in\mathbb{R}_+}$ on $[0, T]$ are independent random variables which are distributed on $\mathbb{R}$ according to $\nu(dx)$. Based on this fact, the next proposition allows us to compute the *Moment Generating Function* (MGF) of the increment $Y_T - Y_t$.

Proposition 3.6. *For any $t \in [0, T]$ and $\alpha \in \mathbb{R}$ we have*

$$\mathbb{E}\left[\mathrm{e}^{(Y_T - Y_t)\alpha}\right] = \exp\left((T-t)\lambda\left(\mathbb{E}\left[\mathrm{e}^{\alpha Z}\right] - 1\right)\right). \tag{3.10}$$

Proof. Since N_t has a Poisson distribution with parameter $t > 0$ and is independent of $(Z_k)_{k\geqslant 1}$, for all $\alpha \in \mathbb{R}$ we have, by conditioning on the value of $N_T - N_t = n$,

$$\begin{aligned}
\mathbb{E}\left[\mathrm{e}^{(Y_T - Y_t)\alpha}\right] &= \mathbb{E}\left[\exp\left(\alpha \sum_{k=N_t+1}^{N_T} Z_k\right)\right] = \mathbb{E}\left[\exp\left(\alpha \sum_{k=1}^{N_T - N_t} Z_{k+N_t}\right)\right] \\
&= \mathbb{E}\left[\exp\left(\alpha \sum_{k=1}^{N_T - N_t} Z_k\right)\right] \\
&= \sum_{n\geqslant 0} \mathbb{E}\left[\exp\left(\alpha \sum_{k=1}^{N_T - N_t} Z_k\right) \Bigg| N_T - N_t = n\right] \mathbb{P}(N_T - N_t = n)
\end{aligned}$$

$$= \sum_{n\geqslant 0} \mathbb{E}\left[\exp\left(\alpha\sum_{k=1}^n Z_k\right)\right] \mathbb{P}(N_T - N_t = n)$$
$$= \mathrm{e}^{-(T-t)\lambda} \sum_{n\geqslant 0} \frac{\lambda^n}{n!}(T-t)^n \mathbb{E}\left[\exp\left(\alpha\sum_{k=1}^n Z_k\right)\right]$$
$$= \mathrm{e}^{-(T-t)\lambda} \sum_{n\geqslant 0} \frac{\lambda^n}{n!}(T-t)^n \prod_{k=1}^n \mathbb{E}\big[\mathrm{e}^{\alpha Z_k}\big]$$
$$= \mathrm{e}^{-(T-t)\lambda} \sum_{n\geqslant 0} \frac{\lambda^n}{n!}(T-t)^n \big(\mathbb{E}\big[\mathrm{e}^{\alpha Z}\big]\big)^n$$
$$= \exp\Big((T-t)\lambda\big(\mathbb{E}\big[\mathrm{e}^{\alpha Z}\big]-1\big)\Big).$$

□

As a consequence of Proposition 3.6, we can derive the following version of the Lévy-Khintchine formula, after approximating $f : [0, T] \longrightarrow \mathbb{R}$ a bounded deterministic function of time by indicator functions:

$$\mathbb{E}\left[\exp\left(\int_0^T f(t)dY_t\right)\right] = \exp\left(\lambda\int_0^T\int_{-\infty}^{\infty}\big(\mathrm{e}^{yf(t)}-1\big)\nu(dy)dt\right). \quad (3.11)$$

We note that we can also write

$$\mathbb{E}\big[\mathrm{e}^{(Y_T-Y_t)\alpha}\big] = \exp\Big((T-t)\lambda\int_{-\infty}^{\infty}(\mathrm{e}^{\alpha y}-1)\nu(dy)\Big)$$
$$= \exp\Big((T-t)\lambda\int_{-\infty}^{\infty}\mathrm{e}^{\alpha y}\nu(dy) - (T-t)\lambda\int_{-\infty}^{\infty}\nu(dy)\Big),$$

since the probability distribution $\nu(dy)$ of Z satisfies

$$\mathbb{E}\big[\mathrm{e}^{\alpha Z}\big] = \int_{-\infty}^{\infty}\mathrm{e}^{\alpha y}\nu(dy) \quad \text{and} \quad \int_{-\infty}^{\infty}\nu(dy) = 1.$$

From the moment generating function (3.10) we can compute the expectation and variance of Y_t for fixed t. Note that the proofs of those identities require to exchange the differentiation and expectation operators, which is possible when the moment generating function (3.10) takes finite values for all α in a certain neighborhood $(-\varepsilon, \varepsilon)$ of 0.

Proposition 3.7. *i) The expectation of Y_t is given as the product of the mean number of jump times $\mathbb{E}[N_t] = \lambda t$ and the mean jump size $\mathbb{E}[Z]$, i.e.,*

$$\mathbb{E}[Y_t] = \mathbb{E}[N_t]\mathbb{E}[Z] = \lambda t \mathbb{E}[Z]. \tag{3.12}$$

ii) Regarding the variance, we have

$$\text{Var}\,[Y_t] = \mathbb{E}[N_t]\mathbb{E}\big[|Z|^2\big] = \lambda t \mathbb{E}\big[|Z|^2\big]. \tag{3.13}$$

Proof. (i) We use the relation

$$\mathbb{E}[Y_t] = \frac{\partial}{\partial \alpha}\mathbb{E}[\mathrm{e}^{\alpha Y_t}]_{|\alpha=0} = \lambda t \int_{-\infty}^{\infty} y\nu(dy) = \lambda t \mathbb{E}[Z].$$

(ii) By (3.10), we have

$$\begin{aligned}
\mathbb{E}\big[Y_t^2\big] &= \frac{\partial^2}{\partial \alpha^2}\mathbb{E}[\mathrm{e}^{\alpha Y_t}]_{|\alpha=0} \\
&= \frac{\partial^2}{\partial \alpha^2}\exp\big(\lambda t(\mathbb{E}\big[\mathrm{e}^{\alpha Z}\big]-1)\big)_{|\alpha=0} \\
&= \frac{\partial}{\partial \alpha}\Big(\lambda t \mathbb{E}\big[Z\mathrm{e}^{\alpha Z}\big]\exp\big(\lambda t(\mathbb{E}\big[\mathrm{e}^{\alpha Z}\big]-1)\big)\Big)_{|\alpha=0} \\
&= \lambda t \mathbb{E}\big[Z^2\big] + (\lambda t \mathbb{E}[Z])^2 \\
&= \lambda t \int_{-\infty}^{\infty} y^2\nu(dy) + (\lambda t)^2\left(\int_{-\infty}^{\infty} y\nu(dy)\right)^2 \\
&= \lambda t \mathbb{E}[Z^2] + (\lambda t \mathbb{E}[Z])^2.
\end{aligned}$$

□

Relation (3.12) can be directly recovered using series summations, as

$$\begin{aligned}
\mathbb{E}[Y_t] &= \mathbb{E}\left[\sum_{k=1}^{N_t} Z_k\right] \\
&= \sum_{n\geqslant 1}\mathbb{E}\left[\sum_{k=1}^{N_t} Z_k \,\middle|\, N_t = n\right]\mathbb{P}(N_t = n) \\
&= \mathrm{e}^{-\lambda t}\sum_{n\geqslant 1}\frac{\lambda^n t^n}{n!}\mathbb{E}\left[\sum_{k=1}^{n} Z_k \,\middle|\, N_t = n\right] \\
&= \mathrm{e}^{-\lambda t}\sum_{n\geqslant 1}\frac{\lambda^n t^n}{n!}\mathbb{E}\left[\sum_{k=1}^{n} Z_k\right]
\end{aligned}$$

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$$= \lambda t e^{-\lambda t} \mathbb{E}[Z] \sum_{n \geqslant 1} \frac{(\lambda t)^{n-1}}{(n-1)!}$$
$$= \lambda t \mathbb{E}[Z]$$
$$= \mathbb{E}[N_t]\mathbb{E}[Z].$$

As a consequence, the *dispersion index* of the compound Poisson process

$$\frac{\mathrm{Var}\,[Y_t]}{\mathbb{E}[Y_t]} = \frac{\mathbb{E}\left[|Z|^2\right]}{\mathbb{E}[Z]}, \qquad t \geqslant 0.$$

coincides with the dispersion index of the random jump size Z. By a multivariate version of Theorem A.19, Proposition 3.6 can be used to show the next result.

Proposition 3.8. (i) *The compound Poisson process*

$$Y_t = \sum_{k=1}^{N_t} Z_k, \qquad t \geqslant 0,$$

has independent increments, i.e. for any finite sequence of times $t_0 < t_1 < \cdots < t_n$*, the increments*

$$Y_{t_1} - Y_{t_0},\ Y_{t_2} - Y_{t_1}, \ldots,\ Y_{t_n} - Y_{t_{n-1}}$$

are mutually independent random variables.

(ii) *In addition, the increment* $Y_t - Y_s$ *is stationary,* $0 \leqslant s \leqslant t$*, i.e. the distribution of* $Y_{t+h} - Y_{s+h}$ *does not depend of* $h \geqslant 0$.

Proof. This result relies on the fact that the result of Proposition 3.6 can be extended to sequences $0 \leqslant t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, as

$$\begin{aligned}
\mathbb{E}\left[\prod_{k=1}^n e^{i\alpha_k(Y_{t_k} - Y_{t_{k-1}})}\right] &= \mathbb{E}\left[\exp\left(i\sum_{k=1}^n \alpha_k (Y_{t_k} - Y_{t_{k-1}})\right)\right] \\
&= \exp\left(\lambda \sum_{k=1}^n (t_k - t_{k-1}) \int_{-\infty}^{\infty} (e^{i\alpha_k y} - 1)\nu(dy)\right) \qquad (3.14) \\
&= \prod_{k=1}^n \exp\left((t_k - t_{k-1})\lambda \int_{-\infty}^{\infty} (e^{i\alpha_k y} - 1)\nu(dy)\right) \\
&= \prod_{k=1}^n \mathbb{E}\left[e^{i\alpha_k(Y_{t_k} - Y_{t_{k-1}})}\right],
\end{aligned}$$

which also shows the stationarity in distribution of $Y_{t+h} - Y_{s+h}$ in $h \geqslant 0$, for $0 \leqslant s \leqslant t$. □

Since the compensated compound Poisson process also has independent and centered increments by (3.7) we have the following counterpart of Proposition 3.4.

Proposition 3.9. *The compensated compound Poisson process*

$$M_t := Y_t - \lambda t \mathbb{E}[Z], \qquad t \geqslant 0,$$

is a martingale*.*

```r
lambda = 0.6;T=10;Tn=c();S=0;n=0;
while (S<T) {S=S+rexp(1,rate=lambda); Tn=c(Tn,S); n=n+1}
Z<-cumsum(c(0,rep(1,n))); Zn<-cumsum(c(0,rexp(n,rate=2)));
Y <- function(t) {return(stepfun(Tn,Zn)(t))};t <- seq(0,10,0.01)
par(oma=c(0,0.1,0,0))
plot(t,Y(t)-0.5*lambda*t,xlim = c(0,10),ylim =
c(-2,2),xlab="t",ylab=expression(paste('Y'[t],'-t')),type="l",lwd=2,col="blue",main="", xaxs =
       "i", yaxs = "i", xaxs = "i", yaxs = "i", las = 1, cex.axis=1.2, cex.lab=1.4)
abline(h = 0, col="black", lwd =2)
points(Tn,Y(Tn)-0.5*lambda*Tn,pch=1,cex=0.8,col="blue",lwd=2);grid()
```

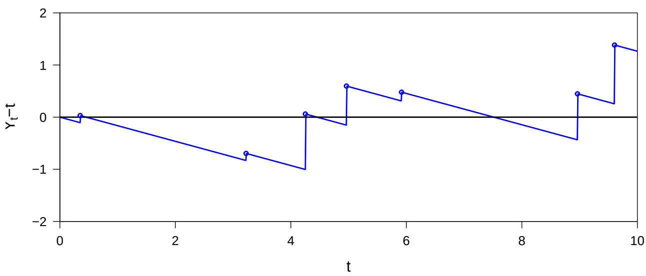

Fig. 3.7: Sample path of a compensated compound Poisson process $(Y_t - \lambda t \mathbb{E}[Z])_{t\in\mathbb{R}_+}$.

3.3 Claim and Reserve Processes

We consider

- a number N_t of claims made until $t \geqslant 0$, which is modeled by a Poisson process $(N_t)_{t\geqslant 0}$ with intensity $\lambda > 0$,
- a sequence $(Z_k)_{k\geqslant 1}$ of nonnegative independent, identically-distributed random variables, which represent the claim amounts.

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We assume that the claim amounts $(Z_k)_{k\geqslant 1}$ and the process of arrivals $(N_t)_{t\geqslant 0}$ are independent. In the next definition we use the convention $S(t) = 0$ if $N_t = 0$.

Definition 3.10. *The* aggregate claim amount *up to time t is defined as the compound Poisson process*

$$S(t) = \sum_{k=1}^{N_t} Z_k.$$

The aggregate claim amount $(S(t))_{t\in\mathbb{R}_+}$ can also be written as

$$S(t) = Y_{N_t}, \qquad t \in \mathbb{R}_+,$$

where $(Y_k)_{k\geqslant 1}$ is the sequence of random variables independent of $(N_t)_{t\in\mathbb{R}_+}$ given by

$$Y_k = \sum_{j=1}^{k} Z_j = Z_1 + \cdots + Z_k, \qquad k \geqslant 1,$$

with $Y_0 := 0$. In the next definition, $f : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function mapping $t > 0$ to the premium income $f(t)$ received between time 0 and time t, with $f(0) = 0$.

Definition 3.11. Standard compound Poisson risk model. *The surplus (or reserve) process $(R_x(t))_{t\geqslant 0}$ is defined as*

$$R_x(t) = x + f(t) - S(t), \qquad t \geqslant 0,$$

where $x \geqslant 0$ is the amount of initial reserves and $f(t)$ is the premium income received between time 0 and time $t > 0$.

In the next Figures 3.8 and 3.9 we take $f(t) := ct$ with $c = 0.5$.

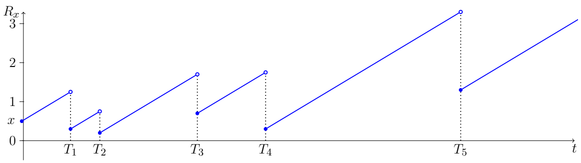

Fig. 3.8: Sample path (without ruin) of a reserve process $(R_x(t))_{t\in\mathbb{R}_+}$.

Unlike the above figure, the next Figure 3.9 contains a ruin event.

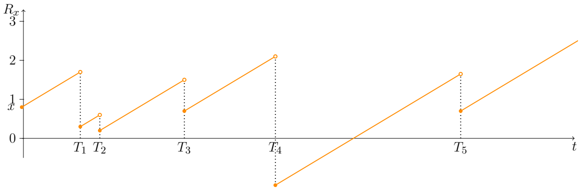

Fig. 3.9: Sample path (with ruin) of a reserve process $(R_x(t))_{t\in\mathbb{R}_+}$.

3.4 Ruin Probabilities

We will consider the infinite time ruin probability

$$\Psi(x) = \mathbb{P}\big(\exists\, t \geqslant 0 \ : \ R_x(t) < 0\big),$$

with $\Psi(x) = 1$ for $x < 0$, and the finite-time ruin probability defined as

$$\Psi_T(x) = \mathbb{P}\big(\exists\, t \in [0, T] \ : \ R_x(t) < 0\big),$$

given $T > 0$ a finite time horizon, with $\Psi_T(x) = 1$ for $x < 0$.

Denoting by m_0^T the infimum

$$m_0^T := \min_{0\leqslant t\leqslant T}(f(t) - S(t)),$$

the ruin probability $\Psi_T(x)$ can also be written as

$$\Psi_T(x) = \mathbb{P}\big(m_0^T < -x\big), \qquad x \geqslant 0.$$

Cramér-Lundberg Model

In Proposition 3.12 we compute the ruin probability in infinite time starting from an initial reserve $x \geqslant 0$.

Proposition 3.12. *Assume that the premium income function satisfies $f(t) = ct$ with premium rate $c > 0$.*

a) The ruin probability in infinite time starting from the initial reserve $x = 0$ is given by

$$\Psi(0) = \mathbb{P}\big(\exists\, t \geqslant 0 \ : \ R_0(t) < 0\big) = \frac{\lambda\mu}{c},$$

provided that $c \geqslant \lambda\mu$, where $\mu = \mathbb{E}[Z]$, and $\Psi(0) = 1$ if $c < \lambda\mu$.

b) Assume that the claim sizes $(Z_k)_{k\geqslant 1}$ form a sequence of independent, exponentially distributed random variables with mean $\mu > 0$, i.e. with parameter $1/\mu$. Then, the ruin probability in infinite time starting from the initial reserve $x \geqslant 0$ is given by

$$\Psi(x) = \frac{\lambda\mu}{c}\mathrm{e}^{(\lambda/c-1/\mu)x}, \qquad x \geqslant 0, \tag{3.15}$$

provided that $c \geqslant \lambda\mu$, with $\Psi(x) = 1$ if $c < \lambda\mu$.

Proof. Let

$$\Phi(x) := 1 - \Psi(x) = \mathbb{P}\left(R_x(t) \geqslant 0,\ \forall t \geqslant 0\right)$$

denote the probability of non-ruin starting from an initial reserve $x \geqslant 0$. Since $c \geqslant 0$, letting

$$F(z) := \mathbb{P}(Z_1 \leqslant z), \qquad z \geqslant 0,$$

denote the cumulative distribution function of the claim size Z_1, for all $y \geqslant 0$ we have

$$\begin{aligned}
\Phi(y) &= \mathbb{P}(R_y(t) \geqslant 0,\ \forall\, t \geqslant 0) \\
&= \mathbb{P}\left(y + ct - \sum_{k=1}^{N_t} Z_k \geqslant 0,\ \forall\, t \geqslant 0\right) \\
&= \mathbb{P}\left(y + ct - \sum_{k=1}^{N_t} Z_k \geqslant 0,\ \forall\, t \geqslant T_1\right) \\
&= \mathbb{E}\left[\mathbb{1}_{\left\{y+cT_1-Z_1+c(t-T_1)-\sum_{k=2}^{N_t} Z_k \geqslant 0,\ \forall\ t\geqslant T_1\right\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\left\{y+cT_1-Z_1+c(t-T_1)-\sum_{k=2}^{N_t} Z_k \geqslant 0,\ \forall\ t\geqslant T_1\right\}}\ \Big|\ T_1\right]\right] \\
&= \mathbb{E}\left[\mathbb{P}\left(y + cT_1 - Z_1 + c(t-T_1) - \sum_{k=2}^{N_t} Z_k \geqslant 0,\ \forall\, t \geqslant T_1\ \Big|\ T_1\right)\right] \\
&= \mathbb{E}\big[\Phi(y + cT_1 - Z_1)\big] \\
&= \lambda \int_0^\infty \mathrm{e}^{-\lambda s} \int_0^\infty \Phi(y + cs - z) dF(z) ds \\
&= \lambda \int_0^\infty \mathrm{e}^{-\lambda s} \int_0^{y+cs} \Phi(y + cs - z) dF(z) ds \\
&= \frac{\lambda}{c} \int_0^\infty \mathrm{e}^{-\lambda u/c} \int_0^{y+u} \Phi(y + u - z) dF(z) du
\end{aligned}$$

$$= \frac{\lambda}{c} e^{\lambda y/c} \int_y^\infty e^{-\lambda u/c} \int_0^u \Phi(u-z) dF(z) du. \tag{3.16}$$

By differentiating (3.16) with respect to y, we find

$$\Phi'(y) = \frac{\lambda}{c} \left(\Phi(y) - \int_0^y \Phi(y-z) dF(z) ds \right), \tag{3.17}$$

hence by integration by parts with respect to $z \in [0, y]$, we get

$$\begin{aligned} \Phi(y) &= \Phi(0) + \int_0^y \Phi'(u) du \\ &= \Phi(0) + \frac{\lambda}{c} \int_0^y \Phi(u) du - \frac{\lambda}{c} \int_0^y \int_0^u \Phi(u-z) dF(z) du \\ &= \Phi(0) + \frac{\lambda}{c} \int_0^y \Phi(y-z)(1-F(z)) dz, \end{aligned}$$

a) Case $x = 0$. Letting y tend to infinity in the above inequality, we deduce

$$\begin{aligned} \Phi(\infty) &= \Phi(0) + \frac{\lambda}{c} \int_0^\infty \Phi(\infty - z)(1-F(z)) dz \\ &= \Phi(0) + \frac{\lambda}{c} \Phi(\infty) \int_0^\infty (1-F(z)) dz \\ &= \Phi(0) + \Phi(\infty) \frac{\lambda}{c} \int_0^\infty \mathbb{P}(Z > z) dz \\ &= \Phi(0) + \Phi(\infty) \frac{\lambda}{c} \mathbb{E}[Z] \\ &= \Phi(0) + \Phi(\infty) \frac{\lambda \mu}{c}, \end{aligned} \tag{3.18}$$

since

$$\begin{aligned} \int_0^\infty \mathbb{P}(Z_1 > z) dz &= \int_0^\infty \mathbb{E}[\mathbb{1}_{\{Z_1 > z\}}] dz \\ &= \mathbb{E}\left[\int_0^\infty \mathbb{1}_{\{Z_1 > z\}} dz \right] \\ &= \mathbb{E}\left[\int_0^{Z_1} dz \right] \\ &= \mathbb{E}[Z_1] \\ &= \mu \end{aligned}$$

is the average claim size. From (3.18) we have

$$\Phi(\infty) = \Phi(0) + \Phi(\infty) \frac{\lambda \mu}{c}.$$

When $\lambda\mu > c$ we find $\Phi(0) = \Phi(\infty) = 0$, whereas when $\lambda\mu \leqslant c$ we have $\Phi(\infty) = 1$ and we obtain $\Phi(0) = 1 - \lambda\mu/c$. In particular, the infinite time ruin probability $\Psi(0)$ starting from the initial reserve $x = 0$ is given by

$$\begin{aligned}\Psi(0) &= 1 - \Phi(0) \\ &= \mathbb{P}\big(\exists\, t \geqslant 0 \ : \ R_x(t) < 0\big) \\ &= \frac{\lambda\mu}{c},\end{aligned} \tag{3.19}$$

provided that $\lambda\mu \leqslant c$.

b) Case $x > 0$. We refer to Exercise 3.2 for the computation of the ruin probability $\Psi(x)$ starting from any $x > 0$. when the claim sizes $(Z_k)_{k\geqslant 1}$ are exponentially distributed. □

Analytic expressions for finite time ruin probabilities have also been obtained when $(Y_k)_{k\geqslant 1}$ are independent, exponentially distributed random variables with parameter $\mu > 0$ and $f(t) = ct$ is linear, $c \geqslant 0$, Theorem 4.1 and Relation (4.6) of Dozzi and Vallois (1997) show that

$$\begin{aligned}\Psi_T(x) &= \mathbb{P}\big(m_0^T < -x\big) \\ &= \lambda \int_0^T \left(x \sum_{n\geqslant 0} \frac{(\lambda\mu t(x+ct))^n}{(n!)^2} + ct \sum_{n\geqslant 0} \frac{(\lambda\mu t(x+ct))^n}{n!(n+1)!} \right) \frac{\mathrm{e}^{-\mu(x+ct)-\lambda t}}{x+ct} dt,\end{aligned}$$

see also Theorem 3.1 of León and Villa (2009) for other related expressions.

R simulation*

The following R code provides an approximation of the infinite time ruin probability (3.15) of Proposition 3.12 by Monte Carlo simulation when T is sufficiently large, see also (3.21) in Exercise 3.2.

* See Kaas et al. (2009), Example 4.3.7.

```
T=20; # Use T>=500 to approximate infinite time dev.new(width=T, height=5)
nSim = 50; lambda = 0.1; x = 7.5; mu = 10; c = 3; N <- rep(Inf, nSim)
for (k in 1:nSim){tauk <- rexp(10*T*lambda,lambda);Ti <- cumsum(tauk)
n=length(Ti[Ti<T]);if (n>=1) {Zk <- rexp(n,1/mu);Si <- x + Ti*c
Ri <- Si - cumsum(Zk);RRi <- Si - c(0,cumsum(Zk)[1:n-1]);
Si<-Si[Ti<T];Ri<-Ri[Ti<T];RRi<-RRi[Ti<T]
Ri <- c(Ri,Ri[n]+c*(T-Ti[n]));RRi <- c(RRi,Ri[n+1]);Ti <- c(Ti[Ti<T],T);
ruin <- !all(Ri[1:n]>=0);}
else {ruin<-FALSE;Ti=c(T);RRi=x+c*T;Ri=x+c*T;};color="blue";
if (ruin) {N[k] <- min(which(Ri<0));color="orange"}
par(mgp=c(0.8,1,1));par(mar=c(2,2,2,2))
plot(c(0,rbind(Ti,Ti)),c(x,rbind(RRi,Ri)),xlab="Time
     t",xlim=c(0,T*0.99),ylim=c(-c*T/3,x+c*T),lwd=3,ylab="R(t)",type="l",col=color,
     main=paste(length(N[N<Inf]),"/",k,"=",format(length(N[N<Inf])/k,digits=4)),
     axes=FALSE, cex.lab=1.4)
axis(1, pos=0, las = 1, cex.axis=1.2);axis(2, pos=0, las = 1, cex.axis=1.2);Sys.sleep(0.2)}
N <- N[N<Inf];length(N);mean(N);sd(N);max(N)
cat('Theoretical value:',lambda*mu*exp(-x*(1/mu-lambda/c))/c,'\n')
cat('Simulation:',length(N)/nSim,'\n')
```

Figure 3.10 computes an estimate of the infinite time ruin probability $\Psi(x)$ by generating the sample paths of the reserve process $(R_x(t))_{t\in\mathbb{R}_+}$.

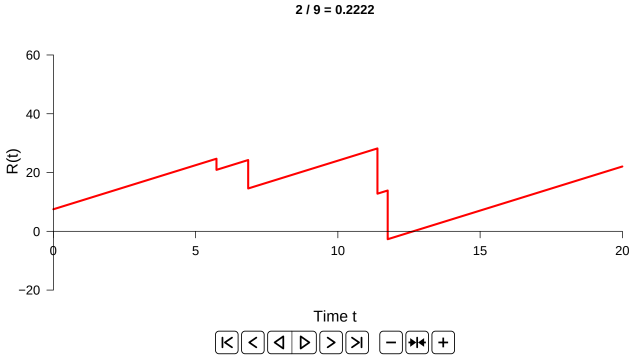

Fig. 3.10: Sample paths of a reserve process $(R_x(t))_{t\in\mathbb{R}_+}$.*

Probability density function

The probability density function of m_0^T at $-x<0$ can be computed as

$$-\frac{\partial\Psi_T}{\partial x}(x).$$

* The animation works in Acrobat Reader on the entire pdf file.

An important practical problem is to obtain numerical values of the sensitivity of the finite-time ruin probability with respect to the initial reserve

$$\frac{\partial \Psi_T}{\partial x}(x),$$

in particular due to new solvency regulations in Europe. The problem of computing the corresponding sensitivity for the finite-time ruin probability $\Psi_T(x)$ has been covered in Loisel and Privault (2009) based on multiple integration. Formulas for the finite-time ruin probability

$$\Psi_T(x) = \mathbb{P}\big(\exists\, t \in [0,T] \ :\ R_x(t) < 0\big)$$

have been proposed in Picard and Lefèvre (1997), see also De Vylder (1999) and Ignatova et al. (2001), Rullière and Loisel (2004). In Privault and Wei (2004; 2007), the Malliavin calculus has been used to provide a way to compute the sensitivity of the probability

$$\mathbb{P}\left(R_x(T) < 0\right)$$

that the terminal surplus is negative with respect to parameters such as the initial reserve or the interest rate of the model.

Non-constant rate of income

When the company income is an arbitrary function $f(t)$ of time such that $f(0) := 0$ we clearly we have $m_0^T \leqslant 0 = f(0)$, hence the distribution of m_0^T is supported on $(-\infty, 0]$. On the other hand, we have $m_0^T = 0$ if and only if $N_T = 0$ or $f(T_k) - Y_k > 0$ for all $k \geqslant 1$ such that $T_k \leqslant T$, hence the distribution of m_0^T has a Dirac mass at 0 with weight

$$\begin{aligned}
&\mathbb{P}\big(m_0^T = 0\big) = \mathbb{P}(N_T = 0) + \mathbb{P}\big(\{m_0^T \geqslant 0\} \cap \{N_T \geqslant 1\}\big) \\
&= \mathrm{e}^{-\lambda T} + \mathrm{e}^{-\lambda T}\mathbb{E}\left[\sum_{k\geqslant 1} \lambda^k \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \mathbb{1}_{\{f(t_1)>Y_1\}} \cdots \mathbb{1}_{\{f(t_k)>Y_k\}} dt_1 \cdots dt_k\right],
\end{aligned}$$

where we used the fact that Poisson jump times are independent uniformly distributed on the square $[0,T]^n$ given that $\{N_T = n\}$.

On the other hand, since f is increasing we have

$$m_0^T = \inf_{T_k \leqslant T,\ k\geqslant 0} (f(T_k) - Y_k) = \mathbb{1}_{\{N_T \geqslant 1\}} \inf_{T_k \leqslant T,\ k\geqslant 1} (f(T_k) - Y_k),$$

with $T_0 = 0$. Hence we have the integral expression

$$\begin{aligned}
&P\big(\{m_0^T \geqslant y\} \cap \{N_T \geqslant 1\}\big) \qquad (3.20)\\
&= \mathrm{e}^{-\lambda T}\mathbb{E}\left[\sum_{k\geqslant 1}\lambda^k \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \mathbb{1}_{\{y<\inf_{1\leqslant l\leqslant k}(f(t_l)-Y_l)\}} dt_1 \cdots dt_k\right]\\
&= \lambda \mathrm{e}^{-\lambda T}\mathbb{E}\left[\sum_{k\geqslant 0}\lambda^k \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} \mathbb{1}_{\{f(t_1)>Y_1+y\}} \cdots \mathbb{1}_{\{f(t_{k+1})>Y_{k+1}+y\}} dt_1 \cdots dt_{k+1}\right]
\end{aligned}$$

Random rate of income

Here, we consider the infimum

$$m_0^T = \inf_{0\leqslant t\leqslant T}(X_t - S(t))$$

where $(X_t)_{t\in\mathbb{R}_+}$ is a stochastic process with independent increments and $X_0 = 0$, independent of $(S(t))_{t\in\mathbb{R}_+}$, and such that

$$\inf_{t\in[a,b]} X_t, \qquad 0 \leqslant a < b,$$

has a probability density function denoted by $\phi_{a,b}(x)$. For example, if $(X_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion then $\phi_{a,b}(x)$ is given by

$$\begin{aligned}
&\int_x^\infty \phi_{a,b}(z)dz = \mathbb{P}\left(\inf_{t\in[a,b]} X_t \geqslant x\right)\\
&= \mathbb{E}\left[\mathbb{1}_{\{X_a<x\}}\mathbb{P}\left(\inf_{t\in[a,b]} X_t \geqslant x \;\middle|\; X_a\right)\right] + \mathbb{E}\left[\mathbb{1}_{\{X_a\geqslant x\}}\mathbb{P}\left(\inf_{t\in[a,b]} X_t \geqslant x \;\middle|\; X_a\right)\right]\\
&= \mathbb{E}\left[\mathbb{1}_{\{X_a<x\}}\mathbb{P}\left(\inf_{t\in[0,b-a]} B_t \geqslant x - X_a \;\middle|\; X_a\right)\right] + \mathbb{P}(X_a \geqslant x)\\
&= 2\mathbb{E}\left[\mathbb{1}_{\{X_a<x\}}\mathbb{P}\left(B_{b-a} \geqslant x - X_a \mid X_a\right)\right] + \mathbb{P}(X_a \geqslant x)\\
&= \frac{1}{\pi\sqrt{a(b-a)}}\int_0^\infty \mathrm{e}^{-(x-y)^2/(2a)}\int_y^\infty \mathrm{e}^{-z^2/(2(b-a))}dzdy + \frac{1}{\sqrt{2\pi a}}\int_x^\infty \mathrm{e}^{-z^2/(2a)}dz.
\end{aligned}$$

We have $m_0^T \leqslant X_0 = 0$ *a.s.*, hence the distribution of m_0^T is carried by $(-\infty, 0]$.

Guaranteed Maturity Benefits

Variable annuity benefits offered by insurance companies are usually protected via different mechanisms such as Guaranteed Minimum Maturity Benefits (GMMBs) or Guaranteed Minimum Death Benefits (GMDBs). The computation of the corresponding risk measures is an important issue for the practitioner in risk management.

Given a fund value process $(F_t)_{t\in\mathbb{R}_+}$, an insurer is continuously charging annualized mortality and expense fees at the rate m from the account of variable annuities, resulting into a margin offset income M_t given by

$$M_t := mF_t \qquad t \in \mathbb{R}_+.$$

Denoting by τ_x the future lifetime of a policyholder at the age x, the future payment made by the insurer at maturity T is

$$(G - F_T)^+ \mathbb{1}_{\{\tau_x > T\}}$$

where G is the guarantee level expressed as a percentage of the initial fund value F_0, δ is a roll-up rate according to which the guarantee increases up to the payment time. In this case, the random variable X is taken equal to

$$X := \mathrm{e}^{-rT}(G - F_T)^+ \mathbb{1}_{\{\tau_x > T\}} - \int_0^{\min(T,\tau_x)} \mathrm{e}^{-rs} M_s \mathrm{d}s.$$

Exercises

Exercise 3.1 Consider N a Poisson random variable with distribution

$$\mathbb{P}(N = n) = \mathrm{e}^{-\lambda}\frac{\lambda^n}{n!}, \qquad n = 0, 1, 2, \ldots$$

where $\lambda > 0$, and let $Y := \sum_{k=1}^{N} Z_k$, where $(Z_k)_{k\geqslant 1}$ is a sequence of independent centered $\mathcal{N}(0,\sigma^2)$ Gaussian random variables with variance σ^2 and cumulative distribution function

$$\mathbb{P}(Z_k \leqslant x) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{x} \mathrm{e}^{-y^2/(2\sigma^2)} dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x/\sigma} \mathrm{e}^{-y^2/2} dy = \Phi\left(\frac{x}{\sigma}\right),$$

$x \in \mathbb{R}_+$.

a) Compute $\mathbb{P}(Y \geqslant y)$ using the conditioning

$$\mathbb{P}(Y \geqslant y) = \sum_{n\geqslant 1} \mathbb{P}\left(\sum_{k=1}^{N} Z_k \geqslant y \;\middle|\; N = n\right) \mathbb{P}(N = n) = \cdots$$

b) Find $\mathbb{E}[Y]$.

Exercise 3.2 Show that when the claim size distribution is exponential with mean $\mu > 0$, *i.e.* when $F(z) = 1 - \mathrm{e}^{-z/\mu}$, $z \geqslant 0$, the ruin probability is given by

$$\Psi(x) = \mathbb{P}\big(\exists\, t \in \mathbb{R}_+ \ : \ R_x(t) < 0\big) = \frac{\lambda\mu}{c}\mathrm{e}^{(\lambda/c-1/\mu)x}, \quad x \geqslant 0, \qquad (3.21)$$

provided that $c \geqslant \lambda\mu$.

Exercise 3.3 An insurance company receives continuous-time premium income at the rate \$$\mu$ per year. Claim payments are filed by subscribers according to a Poisson process $(N_t)_{t\in\mathbb{R}_+}$ of intensity $\lambda > 0$ claims per year. All claims have same constant amount \$$C > 0$.

a) Compute the expected value $\mathbb{E}[R_T]$ and variance $\mathbb{E}[(R_T - \mathbb{E}[R_T])^2]$ of the company's reserve $R_T := R_0 + \mu T - CN_T$ at time $T > 0$, with constant initial reserve R_0.
b) Express the probability $\mathbb{P}(R_T < 0)$ of ruin at time T using the Poisson probability mass function $\mathbb{P}(N_T = k) = \mathrm{e}^{-\lambda T}(\lambda T)^k/k!$, $k \geqslant 0$.

Exercise 3.4 Consider

- a number N_t of claims made until $t \geqslant 0$, which is modeled by a Poisson process $(N_t)_{t\geqslant 0}$ with intensity $\lambda > 0$,
- a sequence $(Z_k)_{k\geqslant 1}$ of nonnegative independent, identically-distributed random variables, which represent the claim amounts.

We assume that the claim amounts $(Z_k)_{k\geqslant 1}$ and the process $(N_t)_{t\geqslant 0}$ of arrivals are independent. The *aggregate claim amount* made up to time t to an insurance company is defined as the compound Poisson process

$$S(t) := \sum_{k=1}^{N_t} Z_k = Z_1 + Z_2 + \cdots + Z_{N_t}, \qquad t \in [0, T].$$

The initial reserve of the company is denoted by $x \geqslant 0$ and the premium income received up to time $t \geqslant 0$ is denoted by $f(t)$.

a) Give the mean and variance of $S(T)$.
Hint: Use the mean $\mathbb{E}[N_T] = \lambda T$ and the moments $\mathbb{E}[Z_1]$ and $\mathbb{E}[Z_1^2]$.
b) Using the Chebyshev inequality (3.22), provide an upper bound for the ruin probability $\mathbb{P}(x + f(T) - S(T) < 0)$ at time $T > 0$, provided that $x + f(T) - \lambda T\mathbb{E}[Z_1] > 0$.

Hint: By the Chebyshev inequality inequality, for any random variable X with mean $\mu > 0$ and variance σ^2 we have

$$\mathbb{P}(X \leqslant 0) = \mathbb{P}(X - \mu \leqslant -\mu) \leqslant \mathbb{P}(|X - \mu| \geqslant \mu) \leqslant \frac{\sigma^2}{\mu^2}. \qquad (3.22)$$

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