Exercise Solutions

Chapter 1

Exercise 1.1 According to Definition 1.3, we need to check the following five properties of Brownian motion:

(i) starts at 0 at time 0,

(ii) independence of increments,

(iii) almost sure continuity of trajectories,

(iv) stationarity of the increments,

(v) Gaussianity of increments.

Checking conditions (i) to (iv) does not pose any particular problem since the time changes $t \mapsto c + t$ and $t \mapsto t/c^2$ are deterministic and continuous.

a) Let $X_t := B_{c+t} - B_t$, $t \in \mathbb{R}_+$. For any finite sequence of times $t_0 < t_1 < \cdots < t_n$, the sequence

$$(X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}) \\ = (B_{c+t_1} - B_{c+t_0}, B_{c+t_2} - B_{c+t_1}, \ldots, B_{c+t_n} - B_{c+t_{n-1}})$$

is made of independent random variables. Concerning (v), $X_t - X_s = B_{c+t} - B_{c+s}$ is normally distributed with mean zero and variance $t + c - (c + s) = t - s$ for any $0 \leqslant s < t$.

b) Let $X_t := cB_{t/c^2}-$, $t \in \mathbb{R}_+$. For any finite sequence of times $t_0 < t_1 < \cdots < t_n$, the sequence

$$(X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}) \\ = (cB_{t_1/c^2} - cB_{t_0/c^2}, cB_{t_2/c^2} - cB_{t_1/c^2}, \ldots, cB_{t_n/c^2} - cB_{t_{n-1/c^2}})$$

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is made of independent random variables. Concerning (v), $X_t - X_s = cB_{t/c^2} - cB_{s/c^2}$ is normally distributed with mean zero and variance $c^2(t/c^2 - s/c^2) = t - s$ for any $0 \leqslant s < t$, since

$$\mathrm{Var}\left[cB_{t/c^2}\right] = c^2\mathrm{Var}\left[B_{t/c^2}\right] = c^2 t/c^2 = t, \qquad t \geqslant 0.$$

Exercise 1.2 The solution to (1.16) is given by

$$S_t = S_0 \mathrm{e}^{\sigma B_t + (\mu - \sigma^2/2)t}, \qquad t \geqslant 0,$$

see the proof of Proposition 1.7 for details. The next R code can be used to generate Figure 1.22.

```
N=1000; t <- 0:N; dt <- 1.0/N; nsim <- 10; sigma=0.6; mu=0.001
Z <- c(rnorm(n = N, sd = sqrt(dt)));
plot(t*dt, exp(mu*t), xlab = "time", ylab = "Geometric Brownian motion", type = "l", ylim =
      c(0, 4), col = 1,lwd=3)
lines(t*dt, exp(sigma*c(0,cumsum(Z))+mu*t-sigma*sigma*t*dt/2),xlab = "time",type = "l",ylim
      = c(0, 4), col = 4)
```

Exercise 1.3

a) Those quantities can be computed from the expression of S_t^n as a function of the $\mathcal{N}(0,t)$ random variable B_t for $n \geqslant 1$. Namely, we have

$$\begin{aligned}
\mathbb{E}[S_t^n] &= \mathbb{E}\left[S_0^n \mathrm{e}^{n\sigma B_t - n\sigma^2 t/2 + nrt}\right] \\
&= S_0 \mathrm{e}^{-n\sigma^2 t/2 + nrt}\mathbb{E}\left[\mathrm{e}^{n\sigma B_t}\right] \\
&= S_0 \mathrm{e}^{-n\sigma^2 t/2 + nrt + n^2\sigma^2 t/2} \\
&= S_0^n \mathrm{e}^{nrt + (n-1)n\sigma^2 t/2},
\end{aligned}$$

where we used the Gaussian moment generating function (MGF) formula (A.41), *i.e.*

$$\mathbb{E}\left[\mathrm{e}^{n\sigma B_t}\right] = \mathrm{e}^{n^2\sigma^2 t/2}$$

for the normal random variable $B_t \simeq \mathcal{N}(0,t)$, $t > 0$.

b) By the result of Question (a)), we have $\mathbb{E}[S_t] = S_0 \mathrm{e}^{rt}$ and

$$\begin{aligned}
\mathbb{E}[S_t^2] &= \mathbb{E}\left[S_0^2 \mathrm{e}^{2\sigma B_t - \sigma^2 t + 2rt}\right] \\
&= S_0^2 \mathrm{e}^{-\sigma^2 t + 2rt}\mathbb{E}\left[\mathrm{e}^{2\sigma B_t}\right] \\
&= S_0^2 \mathrm{e}^{\sigma^2 t + 2rt}, \qquad t \geqslant 0.
\end{aligned}$$

Exercise 1.4 From the solution of Exercise 1.3, we have

$$\mathbb{E}\big[S_t^{(i)}\big] = S_0^{(i)}\mathrm{e}^{\mu t}, \qquad t\in[0,T], \quad i=1,2,$$

and

$$\begin{aligned}\mathrm{Var}\left[S_t^{(i)}\right] &= \mathbb{E}\big[\big(S_t^{(i)}\big)^2\big] - \big(\mathbb{E}\big[S_t^{(i)}\big]\big)\\ &= \big(S_0^{(i)}\big)^2\mathrm{e}^{2\mu t+\sigma_i^2 t} - \big(S_0^{(i)}\big)^2\mathrm{e}^{2\mu t}\\ &= \big(S_0^{(i)}\big)^2\mathrm{e}^{2\mu t}\big(\mathrm{e}^{\sigma_i^2 t}-1\big), \quad t\in[0,T], \quad i=1,2.\end{aligned}$$

Hence, we have

$$\mathrm{Var}\left[S_t^{(2)}-S_t^{(1)}\right] = \mathrm{Var}\left[S_t^{(1)}\right] + \mathrm{Var}\left[S_t^{(2)}\right] - 2\,\mathrm{Cov}\left(S_t^{(1)},S_t^{(2)}\right)$$

with

$$\begin{aligned}\mathbb{E}\big[S_t^{(1)}S_t^{(2)}\big] &= \mathbb{E}\big[S_0^{(1)}S_0^{(2)}\mathrm{e}^{2\mu t+\sigma_1 W_t^{(1)}-\sigma_1^2 t/2+\sigma_2 W_t^{(2)}-\sigma_2^2 t/2}\big]\\ &= S_0^{(1)}S_0^{(2)}\mathrm{e}^{2\mu t-\sigma_1^2 t/2-\sigma_2^2 t/2}\mathbb{E}\big[\mathrm{e}^{\sigma_1 W_t^{(1)}+\sigma_2 W_t^{(2)}}\big]\\ &= S_0^{(1)}S_0^{(2)}\mathrm{e}^{2\mu t-\sigma_1^2 t/2-\sigma_2^2 t/2}\exp\left(\frac{1}{2}\mathbb{E}\big[\big(\sigma_1 W_t^{(1)}+\sigma_2 W_t^{(2)}\big)^2\big]\right),\end{aligned}$$

with

$$\begin{aligned}\mathbb{E}\big[\big(\sigma_1 W_t^{(1)}+\sigma_2 W_t^{(2)}\big)^2\big] &= \mathbb{E}\big[\big(\sigma_1 W_t^{(1)}\big)^2\big] + 2\mathbb{E}\big[\sigma_1 W_t^{(1)}\sigma_2 W_t^{(2)}\big] + \mathbb{E}\big[\big(\sigma_2 W_t^{(2)}\big)^2\big]\\ &= \sigma_1^2 t + 2\rho\sigma_1\sigma_2 t + \sigma_2^2 t,\end{aligned}$$

hence

$$\mathbb{E}\big[S_t^{(1)}S_t^{(2)}\big] = S_0^{(1)}S_0^{(2)}\mathrm{e}^{2\mu t+\rho\sigma_1\sigma_2 t},$$

and

$$\mathrm{Cov}\left(S_t^{(1)},S_t^{(2)}\right) = \mathbb{E}\big[S_t^{(1)}S_t^{(2)}\big] - \mathbb{E}\big[S_t^{(1)}\big]\mathbb{E}\big[S_t^{(2)}\big] = S_0^{(1)}S_0^{(2)}\mathrm{e}^{2\mu t}(\mathrm{e}^{\rho\sigma_1\sigma_2 t}-1),$$

and therefore

$$\begin{aligned}&\mathrm{Var}\left[S_t^{(2)}-S_t^{(1)}\right]\\ &= \big(S_0^{(1)}\big)^2\mathrm{e}^{2\mu t}(\mathrm{e}^{\sigma_1^2 t}-1) + \big(S_0^{(2)}\big)^2\mathrm{e}^{2\mu t}(\mathrm{e}^{\sigma_2^2 t}-1) - 2S_0^{(1)}S_0^{(2)}\mathrm{e}^{2\mu t}(\mathrm{e}^{\rho\sigma_1\sigma_2 t}-1)\\ &= \mathrm{e}^{2\mu t}\big(\big(S_0^{(1)}\big)^2\mathrm{e}^{\sigma_1^2 t} + \big(S_0^{(2)}\big)^2\mathrm{e}^{\sigma_2^2 t} - 2S_0^{(1)}S_0^{(2)}\mathrm{e}^{\rho\sigma_1\sigma_2 t} - \big(S_0^{(2)}-S_0^{(1)}\big)^2\big).\end{aligned}$$

Exercise 1.5 We have

$$\mathbb{E}\big[S_0\mathrm{e}^{\sigma B_t+\mu t-\sigma^2 t/2}\big] = S_0\mathrm{e}^{\mu t-\sigma^2 t/2}\mathbb{E}\big[\mathrm{e}^{\sigma B_t}\big] = S_0\mathrm{e}^{\mu t-\sigma^2 t/2}\mathrm{e}^{\sigma^2 t/2} = S_0\mathrm{e}^{\mu t}$$

and

$$\mathbb{E}[\log S_t] = \mathbb{E}\left[\log S_0 + \sigma B_t + \mu t - \frac{\sigma^2 t}{2}\right] = (\log S_0) + \mu t - \frac{\sigma^2 t}{2},$$

hence

$$\text{Theil}_t = \log \mathbb{E}[S_t] - \mathbb{E}[\log S_t] = \log S_0 + \mu t - \left((\log S_0) + \mu t - \frac{\sigma^2 t}{2}\right) = \frac{\sigma^2 t}{2}.$$

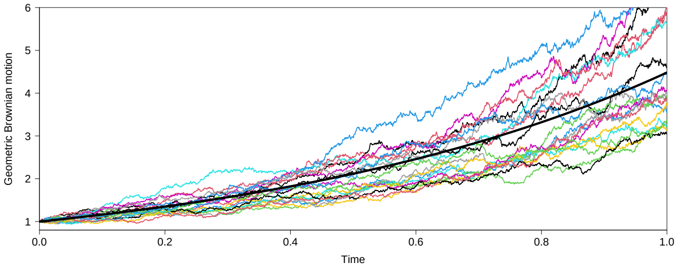

Fig. S.1: Twenty sample paths of geometric Brownian motion $(S_t)_{t\in\mathbb{R}_+}$.

Chapter 2

Exercise 2.1

a) i) By calculation (expected answer). We have

$$\begin{aligned}\rho(0) &= \text{Cov}(X_n, X_n)\\ &= \text{Var}[X_n]\\ &= \mathbb{E}[X_n^2]\\ &= \mathbb{E}[(Z_n - aZ_{n-1})^2]\\ &= \mathbb{E}[Z_n^2 - 2aZ_{n-1}Z_n + a^2 Z_{n-1}^2]\\ &= \mathbb{E}[Z_n^2] - 2a\mathbb{E}[Z_{n-1}Z_n] + a^2\mathbb{E}[Z_{n-1}^2]\\ &= 1 - 2a\mathbb{E}[Z_{n-1}]\mathbb{E}[Z_n] + a^2\\ &= 1 + a^2,\end{aligned}$$

and

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$$\begin{aligned}
&\rho(1) = \mathrm{Cov}(X_n, X_{n+1}) \\
&= \mathrm{Cov}(Z_n + aZ_{n-1}, Z_{n+1} + aZ_n) \\
&= \mathrm{Cov}(Z_n, Z_{n+1}) + a\,\mathrm{Cov}(Z_{n-1}, Z_{n+1}) + a\,\mathrm{Cov}(Z_{n-1}, Z_n) + a^2\,\mathrm{Cov}(Z_{n-1}, Z_n) \\
&= \mathrm{Cov}(Z_n, Z_{n+1}) + a\,\mathrm{Cov}(Z_{n-1}, Z_{n+1}) + a\,\mathrm{Cov}(Z_n, Z_n) + a^2\,\mathrm{Cov}(Z_{n-1}, Z_n) \\
&= a\,\mathrm{Var}[Z_n] \\
&= a,
\end{aligned}$$

and for $k \geqslant 2$,

$$\begin{aligned}
\rho(k) &= \mathrm{Cov}(X_n, X_{n+k}) \\
&= \mathrm{Cov}(Z_n + aZ_{n-1}, Z_{n+k} + aZ_{n+k-1}) \\
&= \mathrm{Cov}(Z_n, Z_{n+k}) + a\,\mathrm{Cov}(Z_{n-1}, Z_{n+k}) \\
&\quad + a\,\mathrm{Cov}(Z_n, Z_{n+k-1}) + a^2\,\mathrm{Cov}(Z_{n-1}, Z_{n+k-1}) \\
&= 0,
\end{aligned}$$

since $n < n + k - 1$. By a similar argument, we obtain $\rho(k) = \rho(-k)$ for $k \leqslant 0$.

ii) Confirmation by simulation with an MA(1) time series constructed by hand:

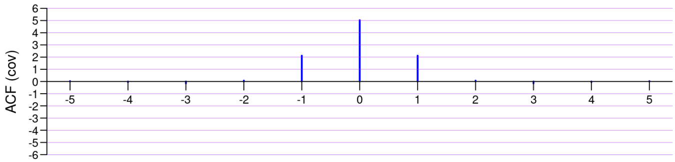
```
library(zoo)
N=10000;Zn<-zoo(rnorm(N,0,1))
Xn<-Zn+2*lag(Zn,-1, na.pad = TRUE);Xn<-Xn[-1]
k=0;cov(Xn[1:(length(Xn)-k)],lag(Xn,k))
```

or with an MA(1) time series constructed using arima.sim:

```
n=2000;a=2;
Xn<-arima.sim(model=list(ma=c(a)),n.start=100,n)
x=seq(100,100+n-1)
plot(x,Xn,pch=19, ylab="X", xlab="n", main = 'MA(1) Samples',col='blue')
lines(x,Xn,col='blue')
Xn<-zoo(Xn)
k=1;cov(Xn[1:(length(Xn)-k)],lag(Xn,k))
```

Using the command ccf to compute the autocovariance empirically, we find:

```
ccf(Xn,Xn,lag = 5, type="covariance", plot=T, lwd=2, col='blue', axes=FALSE,
      ylim=c(-1-a**2,1+a**2), main="")
axis(side = 1, at = seq(-5,5,1))
axis(side = 2, at = c(-1-a**2,-a,0,a,1+a**2), labels=c(expression(paste("-1-|a|"^"2")),
      "a",0,"a", expression(paste("1+|a|"^"2"))))
```



b) We note that $\mathbb{E}[X_n] = 0$, $n \in \mathbb{Z}$, and in addition the autocovariance $\text{Cov}(X_n, X_{n+k})$ depends only of $|k|$ and not on $n \in \mathbb{Z}$. Therefore, the time series $(X_n)_{n\in\mathbb{Z}}$ is *weakly stationary* by Definition 2.11. In addition, by Theorem 2.12 this time series is strictly stationary only if $|a| \neq 1$.

Exercise 2.2

a) We rewrite the equation defining $(X_n)_{n\geqslant 1}$ as

$$X_n = Z_n + LX_n = Z_n + \phi(L)X_n, \qquad n \geqslant 1.$$

where L is the lag operator $LX_n = X_{n-1}$ and $\phi(L) = L$. Taking $\phi(z) := z$, by Theorem 2.12 we need to check whether the solutions of the equation $\phi(z) = 1$ lie on the complete unit circle. As $\phi(z) = 1$ admits the unique solution $z = 1$ which lies on the complete unit circle, we conclude that the AR(1) time series $(X_n)_{n\geqslant 1}$ is not weakly stationary.

b) As in part (a)), we rewrite the AR(2) equation for $(Y_n)_{n\geqslant 1}$ as

$$Y_n = Z_n + 0.75 \times LY_n - 0.125 \times L^2 Y_n = Z_n + \phi(L)Y_n, \qquad n \geqslant 2.$$

with $\phi(L) = 0.75 \times L - 0.125 \times L^2$. The equation $\phi(z) = 1$ with $\phi(z) = 0.75z - 0.125z^2$ reads $z^2 - 6z + 8 = (z-2)(z-4) = 0$. This equation has two solutions $z = 2, 4$ which lie outside the complex unit circle, hence by Theorem 2.12 the AR(2) time series $(Y_n)_{n\geqslant 2}$ is weakly stationary.

Exercise 2.3

a) We have

$$\begin{aligned}\mathbb{E}[X_{n+1}] &= \mathbb{E}\big[Z_{n+1} + \alpha X_n\big] \\ &= \mathbb{E}[Z_{n+1}] + \alpha \mathbb{E}\big[X_n\big] \\ &= \alpha \mathbb{E}\big[X_n\big], \qquad n \geqslant 0,\end{aligned}$$

hence by induction we have $\mathbb{E}[X_n] = \alpha^n \mathbb{E}[X_0] = 0$ for all $n \geqslant 0$.

b) We have

$$\begin{aligned}
\mathrm{Cov}(X_{n+k+1}, X_n) &= \mathbb{E}[X_{n+k+1}X_n] - \mathbb{E}[X_{n+k+1}]\mathbb{E}[X_n] \\
&= \mathbb{E}[X_{n+k+1}X_n] \\
&= \mathbb{E}[(Z_{n+k+1} + \alpha X_{n+k})X_n] \\
&= \mathbb{E}[Z_{n+k+1}X_n + \alpha X_{n+k}X_n] \\
&= \mathbb{E}[Z_{n+k+1}X_n] + \mathbb{E}[\alpha X_{n+k}X_n] \\
&= \mathbb{E}[Z_{n+k+1}]\mathbb{E}[X_n] + \alpha\mathbb{E}[X_{n+k}X_n] \\
&= \alpha\mathbb{E}[X_{n+k}X_n] \\
&= \alpha(\mathbb{E}[X_{n+k}X_n] - \mathbb{E}[X_{n+k}]\mathbb{E}[X_n]) \\
&= \alpha\,\mathrm{Cov}(X_{n+k}, X_n), \qquad n \geqslant 0,
\end{aligned}$$

hence, since $\mathbb{E}[X_n] = 0$, $n \geqslant 0$, we find

$$\mathrm{Cov}(X_{n+k}, X_n) = \alpha^k\,\mathrm{Cov}(X_n, X_n) = \alpha^k\,\mathrm{Var}[X_n], \qquad k, n \geqslant 0.$$

c) We have

$$\begin{aligned}
\mathrm{Var}[X_{n+1}] &= \mathbb{E}[X_{n+1}^2] \\
&= \mathbb{E}[(Z_{n+1} + \alpha X_n)^2] \\
&= \mathbb{E}[Z_{n+1}^2 + 2\alpha Z_{n+1}X_n + \alpha^2 X_n^2] \\
&= \mathbb{E}[Z_{n+1}^2] + 2\alpha\mathbb{E}[Z_{n+1}X_n] + \alpha^2\mathbb{E}[X_n^2] \\
&= 1 + 2\alpha\mathbb{E}[Z_{n+1}]\mathbb{E}[X_n] + \alpha^2\mathbb{E}[X_n^2] \\
&= 1 + \alpha^2\mathbb{E}[X_n^2] \\
&= 1 + \alpha^2\,\mathrm{Var}[X_n].
\end{aligned}$$

By applying the above relation recursively and using the geometric series identity (13.51), we obtain

$$\begin{aligned}
\mathrm{Var}[X_n] &= 1 + \alpha^2\,\mathrm{Var}[X_{n-1}] \\
&= 1 + \alpha^2(1 + \alpha^2\,\mathrm{Var}[X_{n-2}]) \\
&= 1 + \alpha^2(1 + \alpha^2(1 + \alpha^2\,\mathrm{Var}[X_{n-2}])) \\
&= 1 + \alpha^2 + \cdots + \alpha^{2n} \\
&= \sum_{k=0}^{n} \alpha^{2k} \\
&= \begin{cases} \dfrac{1 - \alpha^{2n+2}}{1 - \alpha^2}, & \alpha \neq \pm 1, \\[2ex] n + 1, & \alpha = \pm 1, \qquad n \geqslant 0. \end{cases}
\end{aligned}$$

d) We check that the solution of $\phi(z) := \alpha z$ is $z = 1/\alpha$, hence by Theorem 2.12 there exists an AR(1) solution of (2.28) which is weakly stationary when $\alpha \neq \pm 1$. However, the present time series $(X_n)_{n\geqslant 0}$ started at $X_0 = 0$ is *not* weakly stationary because $\mathrm{Cov}(X_n, X_n) = \mathbb{E}[X_n^2]$ is not constant in $n \geqslant 0$.

Exercise 2.4

a) We have

$$\begin{aligned}
\mathrm{Var}[X_n] &= \mathrm{Var}[Z_{n-1} - Z_{n-2} + \alpha Z_{n-3}] \\
&= \mathrm{Var}[Z_{n-1}] + \mathrm{Var}[-Z_{n-2}] + \mathrm{Var}[\alpha Z_{n-3}] \\
&= \mathrm{Var}[Z_{n-1}] + \mathrm{Var}[Z_{n-2}] + \alpha^2 \mathrm{Var}[Z_{n-3}] \\
&= 2 + \alpha^2.
\end{aligned}$$

Next, since using the linearity relation

$$\mathrm{Cov}(X + Y, Z) = \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z)$$

and the fact that $\mathrm{Cov}(X, Z) = 0$ when X and Z are independent random variables, we have

$$\begin{aligned}
\mathrm{Cov}(X_{n+1}, X_n) &= \mathrm{Cov}(Z_n - Z_{n-1} + \alpha Z_{n-2}, Z_{n-1} - Z_{n-2} + \alpha Z_{n-3}) \\
&= \mathrm{Cov}(Z_n - Z_{n-1} + \alpha Z_{n-2}, Z_{n-1}) \\
&\quad + \mathrm{Cov}(Z_n - Z_{n-1} + \alpha Z_{n-2}, -Z_{n-2}) \\
&\quad + \mathrm{Cov}(Z_n - Z_{n-1} + \alpha Z_{n-2}, \alpha Z_{n-3}) \\
&= \mathrm{Cov}(-Z_{n-1}, Z_{n-1}) + \mathrm{Cov}(\alpha Z_{n-2}, -Z_{n-2}) \\
&= -\mathrm{Cov}(Z_{n-1}, Z_{n-1}) - \alpha\, \mathrm{Cov}(Z_{n-2}, Z_{n-2}) \\
&= -\alpha - 1,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Cov}(X_{n+2}, X_n) &= \mathrm{Cov}(Z_{n+2} - Z_n + \alpha Z_{n-1}, Z_{n-1} - Z_{n-2} + \alpha Z_{n-3}) \\
&= \mathrm{Cov}(Z_{n+2} - Z_n + \alpha Z_{n-1}, Z_{n-1}) \\
&\quad + \mathrm{Cov}(Z_{n+2} - Z_n + \alpha Z_{n-1}, -Z_{n-2}) \\
&\quad + \mathrm{Cov}(Z_{n+2} - Z_n + \alpha Z_{n-1}, \alpha Z_{n-3}) \\
&= \mathrm{Cov}(\alpha Z_{n-1}, Z_{n-1}) \\
&= \alpha\, \mathrm{Cov}(Z_{n-1}, Z_{n-1}) \\
&= \alpha,
\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cov}(X_{n+k}, X_n) &= \mathrm{Cov}(Z_{n+k-1} - Z_{n+k-2} + \alpha Z_{n+k-3}, Z_{n-1} - Z_{n-2} + \alpha Z_{n-3}) \\ &= \mathrm{Cov}(Z_{n+k-1} - Z_{n+k-2} + \alpha Z_{n+k-3}, Z_{n-1}) \\ &\quad + \mathrm{Cov}(Z_{n+k-1} - Z_{n+k-2} + \alpha Z_{n+k-3}, -Z_{n-2}) \\ &\quad + \mathrm{Cov}(Z_{n+k-1} - Z_{n+k-2} + \alpha Z_{n+k-3}, \alpha Z_{n-3}) \\ &= 0\end{aligned}$$

for $k \geqslant 3$.

b) Since the white noise sequence $(Z_n)_{n\in\mathbb{Z}}$ is made of independent identically distributed random variables, we have the identity in distribution

$$X_n = Z_{n-1} - Z_{n-2} + \alpha Z_{n-3} \stackrel{d}{\simeq} Z_n - Z_{n-1} + \alpha Z_{n-2}, \qquad n \geqslant 2,$$

which shows that $(X_n)_{n\geqslant 3}$ has the same distribution as an MA(2) time series the form

$$Y_n = Z_n + \beta_1 Z_{n-1} + \beta_2 Z_{n-2},$$

with $\beta_1 = -1$ and $\beta_1 = \alpha$.

Exercise 2.5

a) We have

$$\begin{aligned}\nabla X_n &= X_n - X_{n-1} \\ &= Z_n + \alpha_1 X_{n-1} - Z_{n-1} - \alpha_1 X_{n-2} \\ &= Z_n - Z_{n-1} + \alpha_1 \nabla X_{n-1}, \qquad n \geqslant 2,\end{aligned}$$

hence $(\nabla X_n)_{n\geqslant 2}$ forms an ARMA$(1,1)$ time series.

b) We have

$$\begin{aligned}\nabla^2 X_n &= \nabla X_n - \nabla X_{n-1} \\ &= X_n - X_{n-1} - (X_{n-1} - X_{n-2}) \\ &= X_n - 2X_{n-1} + X_{n-2} \\ &= Z_n + \alpha_1 X_{n-1} - 2Z_{n-1} - 2\alpha_1 X_{n-2} + Z_{n-2} + \alpha_1 X_{n-3} \\ &= Z_n - 2Z_{n-1} + Z_{n-2} + \alpha_1 \nabla^2 X_{n-1}, \qquad n \geqslant 3,\end{aligned}$$

which forms an ARMA$(1,2)$ time series.

Exercise 2.6

a) We have

$$\frac{\partial}{\partial a}\sum_{k=1}^{n}\big(r_k^{(2)}-a-br_k^{(1)}\big)^2 = -2\left(\sum_{k=1}^{n}\big(r_k^{(2)}-a-br_k^{(1)}\big)\right)$$
$$= 2an - 2\sum_{k=1}^{n} r_k^{(2)} + 2b\sum_{k=1}^{n} r_k^{(1)},$$

and

$$\frac{\partial}{\partial b}\sum_{k=1}^{n}\big(r_k^{(2)}-a-br_k^{(1)}\big)^2 = 2\sum_{k=1}^{n} r_k^{(1)}\big(-a+r_k^{(2)}-br_k^{(1)}\big)$$
$$= 2\sum_{k=1}^{n} r_k^{(1)}\left(r_k^{(2)}-br_k^{(1)}-\frac{1}{n}\sum_{l=1}^{n}\big(r_l^{(2)}-br_l^{(1)}\big)\right)$$
$$= 2\sum_{k=1}^{n} r_k^{(1)}r_k^{(2)} - \frac{2}{n}\sum_{k,l=1}^{n} r_k^{(1)}r_l^{(2)} - 2b\left(\sum_{k=1}^{n}(r_k^{(1)})^2 - \frac{1}{n}\sum_{k,l=1}^{n} r_k^{(1)}r_l^{(1)}\right).$$

b) In order to minimize the residual (2.29) over a and b we equate the above derivatives to zero, which yields the equations

$$\frac{\partial}{\partial a}\sum_{k=1}^{n}\big(r_k^{(2)}-a-br_k^{(1)}\big)^2_{|a=\hat{a},\ b=\hat{b}} = 2\widehat{a}n - 2\sum_{k=1}^{n} r_k^{(2)} + 2\widehat{b}\sum_{k=1}^{n} r_k^{(1)} = 0$$

and

$$\frac{\partial}{\partial b}\sum_{k=1}^{n}\big(r_k^{(2)}-a-br_k^{(1)}\big)^2_{|a=\hat{a},\ b=\hat{b}}$$
$$= 2\sum_{k=1}^{n} r_k^{(1)}r_k^{(2)} - \frac{2}{n}\sum_{k,l=1}^{n} r_k^{(1)}r_l^{(2)} - 2\widehat{b}\left(\sum_{k=1}^{n}(r_k^{(1)})^2 - \frac{1}{n}\sum_{k,l=1}^{n} r_k^{(1)}r_l^{(1)}\right)$$
$$= 0.$$

This leads to estimators $\widehat{a}$, $\widehat{b}$ of the parameters a and b respectively as the empirical mean and covariance of $(r_k^{(1)})_{k=1,2,\ldots,n}$, *i.e.*

$$\begin{cases} \widehat{a} = \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} \left(r_k^{(2)} - \widehat{b} r_k^{(1)} \right), \\ \\ \text{and} \\ \\ \widehat{b} = \dfrac{\displaystyle\sum_{k=1}^{n} r_k^{(1)} r_k^{(2)} - \frac{1}{n} \sum_{k,l=0}^{n} r_k^{(1)} r_l^{(2)}}{\displaystyle\sum_{k=1}^{n} (r_k^{(1)})^2 - \frac{1}{n} \sum_{k,l=0}^{n} r_k^{(1)} r_l^{(1)}} = \dfrac{\displaystyle\sum_{k=1}^{n} \left(r_k^{(1)} - \frac{1}{n} \sum_{l=0}^{n} r_l^{(1)} \right) \left(r_k^{(2)} - \frac{1}{n} \sum_{l=0}^{n} r_l^{(2)} \right)}{\displaystyle\sum_{k=1}^{n} \left(r_k^{(1)} - \frac{1}{n} \sum_{k=1}^{n} r_k^{(1)} \right)^2}. \end{cases}$$

Exercise 2.7 Since the p-value $= 0.02377$ is lower than the 5% confidence level, we can reject the nonstationarity (null) hypothesis H_0 at that level.

Exercise 2.8

a) We consider the equation

$$\varphi(z) = \alpha_1 z + \alpha_2 z^2 = 1,$$

i.e.

$$\alpha_2 z^2 + \alpha_1 z - 1 = 0,$$

with solutions

$$z_\pm = \frac{-\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_2}}{2\alpha_2} = \frac{-a \pm \sqrt{a^2 + 8a^2}}{4a^2} = \frac{-a \pm 3a}{4a^2} = \begin{cases} \dfrac{1}{2a} \\ -\dfrac{1}{a}, \end{cases}$$

hence by Theorem 2.12 the time series $(X_n)_{n \geqslant 1}$ is stationary for $a \notin \{-1, -1/2, 1/2, 1\}$.

b) We have

$$\begin{aligned} \mathbb{E}[X_n] &= \mathbb{E}[Z_n + \alpha_1 X_{n-1} + \alpha_2 X_{n-2}] \\ &= \mathbb{E}[Z_n] + \alpha_1 \mathbb{E}[X_{n-1}] + \alpha_2 \mathbb{E}[X_{n-2}] \\ &= \alpha_1 \mathbb{E}[X_{n-1}] + \alpha_2 \mathbb{E}[X_{n-2}] \\ &= \alpha_1 \mathbb{E}[X_n] + \alpha_2 \mathbb{E}[X_n], \end{aligned}$$

hence

$$(1 - \alpha_1 - \alpha_2) \mathbb{E}[X_n] = 0,$$

which implies $\mathbb{E}[X_n] = 0$, $n \in \mathbb{Z}$, since $1 - \alpha_1 - \alpha_2 \neq 0$.

c) We have

$$\begin{aligned}\operatorname{Cov}(X_n, Z_n) &= \operatorname{Cov}(Z_n + \alpha_1 X_{n-1} + \alpha_2 X_{n-2}, Z_n) \\ &= \operatorname{Cov}(Z_n, Z_n) + \alpha_1 \operatorname{Cov}(Z_n, X_{n-1}) + \alpha_2 \operatorname{Cov}(Z_n, X_{n-2}) \\ &= \operatorname{Cov}(Z_n, Z_n) \\ &= \sigma^2.\end{aligned}$$

d) We have

$$\begin{aligned}\operatorname{Cov}(X_{n+1}, X_n) &= \operatorname{Cov}(Z_{n+1} + \alpha_1 X_n + \alpha_2 X_{n-1}, X_n) \\ &= \operatorname{Cov}(Z_{n+1}, X_n) + \alpha_1 \operatorname{Cov}(X_n, X_n) + \alpha_2 \operatorname{Cov}(X_{n-1}, X_n), \\ &= 16\alpha_1 + \alpha_2 \operatorname{Cov}(X_{n-1}, X_n) \\ &= 4 + \frac{1}{2} \operatorname{Cov}(X_{n-1}, X_n),\end{aligned}$$

hence

$$\operatorname{Cov}(X_{n+1}, X_n) = 8, \qquad n \in \mathbb{Z}.$$

Chapter 3

Exercise 3.1

a) Since $Z_1 + Z_2 + \cdots + Z_n$ has the centered Gaussian $\mathcal{N}(0, n\sigma^2)$ distribution with variance $n\sigma^2$, we have

$$\begin{aligned}\mathbb{P}(Y \geqslant y) &= \sum_{n \geqslant 1} \mathbb{P}\left(\sum_{k=1}^{N} Z_k \geqslant y \;\middle|\; N = n\right) \mathbb{P}(N = n) \\ &= \sum_{n \geqslant 1} \left(1 - \mathbb{P}\left(\sum_{k=1}^{n} Z_k < y \;\middle|\; N = n\right)\right) \mathbb{P}(N = n) \\ &= \sum_{n \geqslant 1} \left(1 - \mathbb{P}\left(\sum_{k=1}^{n} Z_k \leqslant y \;\middle|\; N = n\right)\right) \mathbb{P}(N = n) \\ &= \sum_{n \geqslant 1} \left(1 - \Phi\left(\frac{y}{\sqrt{n\sigma^2}}\right)\right) \mathbb{P}(N = n) \\ &= \mathrm{e}^{-\lambda} \sum_{n \geqslant 1} \frac{\lambda^n}{n!} \Phi\left(-\frac{y}{\sqrt{n\sigma^2}}\right), \qquad y > 0.\end{aligned}$$

b) Since $\mathbb{E}[Z_k] = 0$ for all $k \geqslant 1$, we have

$$\begin{aligned}\mathbb{E}[Y] &= \sum_{n\geqslant 1} \mathbb{E}\left[\sum_{k=1}^{N} Z_k \;\Big|\; N=n\right] \mathbb{P}(N=n) \\ &= \sum_{n\geqslant 1} \mathbb{E}\left[\sum_{k=1}^{n} Z_k \;\Big|\; N=n\right] \mathbb{P}(N=n) \\ &= \sum_{n\geqslant 1} \mathbb{P}(N=n) \sum_{k=1}^{n} \mathbb{E}[Z_k \mid N=n] \\ &= \sum_{n\geqslant 1} \mathbb{P}(N=n) \sum_{k=1}^{n} \mathbb{E}[Z_k] \\ &= 0,\end{aligned}$$

as in (3.12).

Exercise 3.2 By (3.17), we have

$$\begin{aligned}\Phi'(y) &= \frac{\lambda}{c}\Phi(y) - \frac{\lambda}{c}\int_0^y \Phi(y-z)dF(z) \\ &= \frac{\lambda}{c}\Phi(y) - \frac{\lambda}{\mu c}\int_0^y \Phi(y-z)\mathrm{e}^{-z/\mu}dz \\ &= \frac{\lambda}{c}\Phi(y) - \frac{\lambda}{\mu c}\int_0^y \Phi(z)\mathrm{e}^{-(y-z)/\mu}dz,\end{aligned}$$

hence the differential equation

$$\begin{aligned}\Phi''(y) &= \frac{\lambda}{c}\Phi'(y) - \frac{\lambda}{\mu c}\Phi(y) + \frac{\lambda}{\mu^2 c}\int_0^y \Phi(z)\mathrm{e}^{-(y-z)/\mu}dz \\ &= \frac{\lambda}{c}\Phi'(y) - \frac{\lambda}{\mu c}\Phi(y) + \frac{1}{\mu}\left(\frac{\lambda}{c}\Phi(y) - \Phi'(y)\right) \\ &= \left(\frac{\lambda}{c} - \frac{1}{\mu}\right)\Phi'(y),\end{aligned}$$

which can be solved as

$$\Phi(y) = 1 - \frac{\lambda\mu}{c}\mathrm{e}^{(\lambda/c-1/\mu)y},$$

given the boundary conditions $\Phi(\infty) = 1$ and $\Phi(0) = 1 - \lambda\mu/c$, cf. (3.19). We conclude that

$$\Psi(y) = \frac{\lambda\mu}{c}\mathrm{e}^{(\lambda/c-1/\mu)y}, \qquad y \geqslant 0,$$

provided that $c < \lambda\mu$.

Exercise 3.3

a) We have

$$\begin{aligned}\mathbb{E}[R_T] &= \mathbb{E}[R_0 + \mu T - CN_T] \\ &= \mathbb{E}[R_0] + \mathbb{E}[\mu T] - \mathbb{E}[CN_T] \\ &= R_0 + \mu T - C\mathbb{E}[N_T] \\ &= R_0 + \mu T - C\lambda T \\ &= R_0 + (\mu - \lambda C)T,\end{aligned}$$

and similarly

$$\mathrm{Var}[R_T] = \mathrm{Var}[\mu T - CN_T] = \mathrm{Var}[-CN_T] = C^2\,\mathrm{Var}[N_T] = \lambda C^2 T.$$

b) We find

$$\begin{aligned}\mathbb{P}(R_T < 0) &= \mathbb{P}(R_0 + \mu T - CN_T < 0) \\ &= \mathbb{P}(N_T > (R_0 + \mu T)/C) \\ &= \sum_{k > (R_0+\mu T)/C} \mathbb{P}(N_T = k) \\ &= \mathrm{e}^{-\lambda T} \sum_{k > (R_0+\mu T)/C} \frac{(\lambda T)^k}{k!}.\end{aligned}$$

Exercise 3.4

a) We have $\mathbb{E}[S(T)] = \lambda T\mathbb{E}[Z]$ and $\mathrm{Var}[S(T)] = \lambda T\mathbb{E}[Z^2]$.

b) We have

$$\begin{aligned}\mathbb{P}(x + f(T) - S(T) < 0) &\leqslant \frac{\mathrm{Var}[x + f(T) - S(T)]}{(\mathbb{E}[x + f(T) - S(T)])^2} \\ &= \frac{\mathrm{Var}[S(T)]}{(x + f(T) - \mathbb{E}[S(T)])^2} \\ &= \frac{\lambda T\mathbb{E}[Z_1^2]}{(x + f(T) - \lambda T\mathbb{E}[Z_1])^2}.\end{aligned}$$

Chapter 4

Exercise 4.1

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a) Taking $(U,V) = (U,U)$, we have

$$\begin{aligned}
\mathbb{P}(U \leqslant u \text{ and } V \leqslant v) &= \mathbb{P}(U \leqslant u \text{ and } U \leqslant v) \\
&= \mathbb{P}(U \leqslant \min(u,v)) \\
&= \min(u,v) \\
&= C_M(u,v), \qquad u,v \in [0,1].
\end{aligned}$$

b) Taking $(U,V) = (U, 1-U)$, we have

$$\begin{aligned}
\mathbb{P}(U \leqslant u \text{ and } V \leqslant v) &= \mathbb{P}(U \leqslant u \text{ and } 1-U \leqslant v) \\
&= \mathbb{P}(U \leqslant u \text{ and } U \geqslant 1-v) \\
&= \mathbb{P}(1-v \leqslant U \leqslant u) \\
&= \mathbb{1}_{\{0 \leqslant 1-v \leqslant u \leqslant 1\}} \mathbb{P}(1-v \leqslant U \leqslant u) \\
&= \mathbb{1}_{\{0 \leqslant u+v-1 \leqslant 1\}} (u-(1-v)) \\
&= (u+v-1)^+,
\end{aligned}$$

$u,v \in [0,1]$.

c) We have

$$C(u,v) = \mathbb{P}(U \leqslant u \text{ and } V \leqslant v) \leqslant \mathbb{P}(U \leqslant u \text{ and } V \geqslant 1) \leqslant \mathbb{P}(U \leqslant u) = u,$$

$u,v \in [0,1]$, and similarly we find $C(u,v) \leqslant \mathbb{P}(U \leqslant v) = v$ for all $u,v \in [0,1]$, which yields (4.8).

d) For fixed $v \in [0,1]$ we have

$$\begin{aligned}
\frac{\partial C}{\partial u}(u,v) &= \lim_{\varepsilon \to 0} \frac{C(u+\varepsilon, v) - C(u,v)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\mathbb{P}(U \leqslant u+\varepsilon \text{ and } V \leqslant v) - \mathbb{P}(U \leqslant u \text{ and } V \leqslant v)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\mathbb{P}(u \leqslant U \leqslant u+\varepsilon \text{ and } V \leqslant v)}{P(u \leqslant U \leqslant u+\varepsilon)} \\
&= \lim_{\varepsilon \to 0} \mathbb{P}(V \leqslant v \mid u \leqslant U \leqslant u+\varepsilon) \\
&= \mathbb{P}(V \leqslant v \mid U = u) \\
&\leqslant 1,
\end{aligned}$$

$u,v \in [0,1]$, hence

$$h'(u) = \frac{\partial C}{\partial u}(u,v) - 1 = \mathbb{P}(V \leqslant v \mid U = u) - 1 \leqslant 0,$$

$u, v \in [0,1]$, and since $h(1) = C(1,v) - v = \mathbb{P}(V \leqslant v) - v = 0$, $v \in [0,1]$ we conclude that $h(u) \geqslant 0$, $u \in [0,1]$, which shows (4.9).

Exercise 4.2

a) When $\rho = 1$, we have

$$\begin{cases} \mathbb{P}(X = 1 \text{ and } Y = 1) = p_X p_Y + \sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \\ \mathbb{P}(X = 0 \text{ and } Y = 1) = (1-p_X)p_Y - \sqrt{p_X p_Y (1-p_X)(1-p_Y)} \geqslant 0, \\ \mathbb{P}(X = 1 \text{ and } Y = 0) = p_X(1-p_Y) - \sqrt{p_X p_Y (1-p_X)(1-p_Y)} \geqslant 0, \\ \mathbb{P}(X = 0 \text{ and } Y = 0) = (1-p_X)(1-p_Y) + \sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \end{cases}$$

hence

$$\begin{cases} (1-p_X)p_Y \geqslant \sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \\ p_X(1-p_Y) \geqslant \sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \end{cases}$$

hence

$$(1-p_X)p_Y \geqslant p_X(1-p_Y) \quad \text{and} \quad p_X(1-p_Y) \geqslant p_Y(1-p_X),$$

showing that $(1-p_X)p_Y = p_X(1-p_Y)$, which implies $p_X = p_Y$, and

$$\begin{cases} \mathbb{P}(X = 1 \text{ and } Y = 1) = p_X^2 + p_X(1-p_X) = p_X = p_Y, \\ \mathbb{P}(X = 0 \text{ and } Y = 1) = 0, \\ \mathbb{P}(X = 1 \text{ and } Y = 0) = 0, \\ \mathbb{P}(X = 0 \text{ and } Y = 0) = 1 - p_X = 1 - p_Y. \end{cases}$$

b) When $\rho = -1$, we have

$$\begin{cases} \mathbb{P}(X = 1 \text{ and } Y = 1) = p_X p_Y - \sqrt{p_X p_Y (1-p_X)(1-p_Y)} \geqslant 0, \\ \mathbb{P}(X = 0 \text{ and } Y = 1) = (1-p_X)p_Y + \sqrt{p_X p_Y (1-p_X)(1-p_Y)} \geqslant 0, \\ \mathbb{P}(X = 1 \text{ and } Y = 0) = p_X(1-p_Y) + \sqrt{p_X p_Y (1-p_X)(1-p_Y)} \geqslant 0, \\ \mathbb{P}(X = 0 \text{ and } Y = 0) = (1-p_X)(1-p_Y) - \sqrt{p_X p_Y (1-p_X)(1-p_Y)} \geqslant 0, \end{cases}$$

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hence

$$\begin{cases} p_X p_Y \geqslant \sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \\ p_X p_Y \geqslant \sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \end{cases}$$

hence

$$p_X p_Y \geqslant (1-p_X)(1-p_Y) \quad \text{and} \quad p_X p_Y \geqslant (1-p_X)(1-p_Y),$$

showing that $p_X p_Y = (1-p_X)(1-p_Y)$, which implies $p_X = 1-p_Y$, and

$$\begin{cases} \mathbb{P}(X=1 \text{ and } Y=1)=0, \\ \mathbb{P}(X=0 \text{ and } Y=1)=1, \\ \mathbb{P}(X=1 \text{ and } Y=0)=1, \\ \mathbb{P}(X=0 \text{ and } Y=0)=0. \end{cases}$$

Exercise 4.3

a) We have

$$\mathbb{P}(X \geqslant x) = \mathbb{P}(X \geqslant x \text{ and } Y \geqslant 0) = \mathrm{e}^{-(\lambda+\nu)x},$$

and

$$\mathbb{P}(Y \geqslant y) = \mathbb{P}(X \geqslant 0 \text{ and } Y \geqslant y) := \mathrm{e}^{-(\mu+\nu)y},$$

$x, y \geqslant 0$, *i.e.* X and Y are exponentially distributed with respective parameters $\lambda+\nu$ and $\mu+\nu$.

b) We have

$$\begin{aligned} &\mathbb{P}(X \leqslant x \text{ and } Y \leqslant 0) \\ &= \mathbb{P}(X \geqslant x \text{ and } Y \geqslant 0) - (\mathbb{P}(X \geqslant x) - \mathbb{P}(X \geqslant x \text{ and } Y \geqslant 0)) \\ &\quad -(\mathbb{P}(Y \geqslant x) - \mathbb{P}(X \geqslant x \text{ and } Y \geqslant 0)) \\ &= \mathbb{P}(X \geqslant x \text{ and } Y \geqslant 0) - \mathbb{P}(X \geqslant x) - \mathbb{P}(Y \geqslant x) + \mathbb{P}(X \geqslant x \text{ and } Y \geqslant 0)), \end{aligned}$$

$x, y \geqslant 0$, *i.e.* X and Y are exponentially distributed with respective parameters $\lambda+\nu$ and $\mu+\nu$.

c) Since $\mathrm{e}^{-(\lambda+\nu)X}$ and $\mathrm{e}^{-(\mu+\nu)Y}$ are uniformly distributed on $[0,1]$, a copula function $C(u,v)$ can be defined by

$$\begin{aligned} C(u,v) &:= \mathbb{P}(\mathrm{e}^{-(\lambda+\nu)X} \leqslant u \text{ and } \mathrm{e}^{-(\mu+\nu)Y} \leqslant v) \\ &= \mathbb{P}(X \leqslant -(\lambda+\nu)^{-1} \log u \text{ and } Y \leqslant -(\lambda+\nu)^{-1} \log v) \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{e}^{\lambda(\lambda+\nu)^{-1}\log u + \mu(\lambda+\nu)^{-1}\log v)y - \nu\,\mathrm{Max}(-(\lambda+\nu)^{-1}\log u, -(\lambda+\nu)^{-1}\log v))} \\
&= u^{\lambda/(\lambda+\nu)} v^{\mu/(\lambda+\nu)} \mathrm{e}^{-\nu\,\mathrm{Max}(-(\lambda+\nu)^{-1}\log u, -(\lambda+\nu)^{-1}\log v))} \\
&= u^{\lambda/(\lambda+\nu)} v^{\mu/(\lambda+\nu)} \mathrm{e}^{\nu \min(\log u^{(\lambda+\nu)^{-1}}, \log v^{(\lambda+\nu)^{-1}}))} \\
&= u^{\lambda/(\lambda+\nu)} v^{\mu/(\lambda+\nu)} \mathrm{e}^{\log \min(u^{\nu/(\lambda+\nu)}, v^{\nu/(\lambda+\nu)}))} \\
&= u^{\lambda/(\lambda+\nu)} v^{\mu/(\lambda+\nu)} \min(u^{\nu/(\lambda+\nu)}, v^{\nu/(\lambda+\nu)})) \\
&= u^{\lambda/(\lambda+\nu)} v^{\mu/(\lambda+\nu)} (\min(u,v))^{\nu/(\lambda+\nu)}, \qquad x, y \geqslant 0.
\end{aligned}
$$

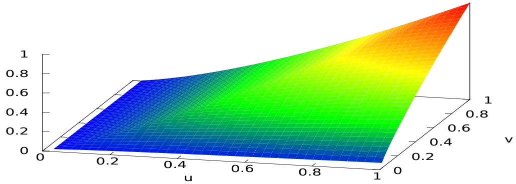

Fig. S.2: Exponential copula function $u, v \mapsto C(u,b)$ with $\lambda = 1$, $\mu = 2$, $\nu = 4$.

Exercise 4.4

a) We have

$$
F_X(x) = \mathbb{P}(X \leqslant x) = \mathbb{P}(X \leqslant x \text{ and } Y \leqslant \infty) = \frac{1}{1+\mathrm{e}^{-x}}
$$

and

$$
F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}(X \leqslant \infty \text{ and } Y \leqslant y) = \frac{1}{1+\mathrm{e}^{-y}}, \qquad x, y \in \mathbb{R}.
$$

The probability densities are given by

$$
f_X(x) = f_Y(x) = F_X'(x) = F_Y'(x) = \frac{\mathrm{e}^{-x}}{(1+\mathrm{e}^{-x})^2}, \qquad x \in \mathbb{R}.
$$

b) We have

$$
F_X^{-1}(u) = F_Y^{-1}(u) = -\log\frac{1-u}{u}, \qquad u \in (0,1),
$$

and the corresponding copula is given by

$$
\begin{aligned}
C(u,v) &= F_{(X,Y)}(F_X^{-1}(u), F_Y^{-1}(v)) \\
&= F_{(X,Y)}\left(-\log\frac{1-u}{u}, -\log\frac{1-v}{v}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{1 + (1-u)/u + (1-v)/v} \\
&= \frac{1}{1 + (1-u)/u + (1-v)/v} \\
&= \frac{uv}{u + v - uv}, \qquad u, v \in [0,1],
\end{aligned}
$$

which is a particular case of the Ali-Mikhail-Haq copula.

Exercise 4.5

a) We show that (X, Y) have Gaussian marginals $\mathcal{N}(0, \sigma^2)$ and $\mathcal{N}(0, \eta^2)$, according to the following computation:

$$
\begin{aligned}
\int_{-\infty}^{\infty} \widetilde{f}(x,y)dy &= \frac{1}{\pi\sigma\eta} \int_{-\infty}^{\infty} \mathbb{1}_{\mathbb{R}_-^2 \cup \mathbb{R}_+^2}(x,y) \mathrm{e}^{-x^2/(2\sigma^2) - y^2/(2\eta^2)} dy \\
&= \frac{1}{\pi\sigma\eta} \mathrm{e}^{-x^2/(2\sigma^2)} \mathbb{1}_{\mathbb{R}_+}(x) \int_0^{\infty} \mathrm{e}^{-y^2/(2\eta^2)} dy + \\
&\quad \frac{1}{\pi\sigma\eta} \mathrm{e}^{-x^2/(2\sigma^2)} \mathbb{1}_{\mathbb{R}_-}(x) \int_{-\infty}^{0} \mathrm{e}^{-y^2/(2\eta^2)} dy \\
&= \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-x^2/(2\sigma^2)} \mathbb{1}_{\mathbb{R}_+}(x) + \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-x^2/(2\sigma^2)} \mathbb{1}_{\mathbb{R}_-}(x) \\
&= \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-x^2/(2\sigma^2)}, \qquad x \in \mathbb{R}.
\end{aligned}
$$

b) The couple (X, Y) does *not* have a joint Gaussian distribution, and its joint probability density function does *not* coincide with $f_\Sigma(x, y)$.

c) When $\sigma = \eta = 1$, the random variable $X + Y$ has the probability density function

$$
\begin{aligned}
&\frac{\partial}{\partial a} \mathbb{P}(X + Y \leqslant a) = \frac{\partial}{\partial a} \int_{-\infty}^{\infty} \int_{-\infty}^{a-x} \widetilde{f}(x,y) dy dx \\
&= \frac{1}{\pi} \frac{\partial}{\partial a} \int_0^a \int_0^{a-x} \mathrm{e}^{-x^2/2 - y^2/2} dy dx \\
&= \frac{1}{\pi} \frac{\partial}{\partial a} \int_0^a \mathrm{e}^{-(a-z)^2/2} \int_0^z \mathrm{e}^{-y^2/2} dy dz \\
&= \frac{1}{\pi} \int_0^a \mathrm{e}^{-y^2/2} dy - \frac{1}{\pi} \int_0^a (a-z) \mathrm{e}^{-(a-z)^2/2} \int_0^z \mathrm{e}^{-y^2/2} dy dz \\
&= \frac{1}{\pi} \int_0^a \mathrm{e}^{-y^2/2} dy - \frac{1}{\pi} \int_0^a (a-z) \mathrm{e}^{-(a-z)^2/2} dz \int_0^a \mathrm{e}^{-y^2/2} dy \\
&\quad + \frac{1}{\pi} \int_0^a \mathrm{e}^{-y^2/2} \int_0^y \mathrm{e}^{-(a-z)^2/2} dz dy
\end{aligned}
$$

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$$
\begin{aligned}
&= \frac{1}{\pi}\mathrm{e}^{-a^2/2}\int_0^a \mathrm{e}^{-y^2/2}dy + \frac{1}{\pi}\int_0^a \mathrm{e}^{-y^2/2}(\mathrm{e}^{-(a-y)^2/2} - \mathrm{e}^{-a^2/2})dy \\
&= \frac{1}{\pi}\mathrm{e}^{-a^2/2}\int_0^a \mathrm{e}^{-y^2/2-(a-y)^2/2}dy \\
&= \frac{1}{\pi}\int_0^a \mathrm{e}^{-((\sqrt{2}y-a/\sqrt{2})^2-a^2/2)/2}dy \\
&= \frac{\mathrm{e}^{-a^2/4}}{\pi\sqrt{2}}\int_0^{a\sqrt{2}} \mathrm{e}^{-((y-a/\sqrt{2})^2)/2}dy \\
&= \frac{\mathrm{e}^{-a^2/4}}{\pi\sqrt{2}}\int_{-a/\sqrt{2}}^{a(\sqrt{2}-1/\sqrt{2})} \mathrm{e}^{-y^2/2}dy \\
&= \frac{\mathrm{e}^{-a^2/4}}{\sqrt{\pi}\sqrt{2\pi}}\int_{-a/\sqrt{2}}^{a/\sqrt{2}} \mathrm{e}^{-y^2/2}dy \\
&= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-a^2/4}\frac{1}{\sqrt{\pi}}\big(2\Phi\big(a/\sqrt{2}\big)-1\big), \quad a \geqslant 0,
\end{aligned}
$$

which vanishes at $a = 0$.

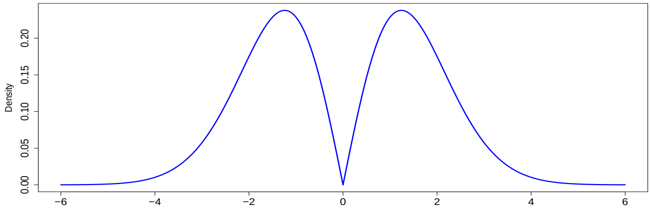

Fig. S.3: Density of $X + Y$.

d) The random variables X and Y are positively correlated, as

$$
\begin{aligned}
\int_{-\infty}^{\infty} y f_\Sigma(x,y)dy &= \frac{1}{\pi\sigma\eta}\int_{-\infty}^{\infty} \mathbb{1}_{\mathbb{R}_-^2\cup\mathbb{R}_+^2}(x,y)y\mathrm{e}^{-x^2/(2\sigma^2)-y^2/(2\eta^2)}dy \\
&= \frac{1}{\pi\sigma\eta}\mathrm{e}^{-x^2/(2\sigma^2)}\mathbb{1}_{\mathbb{R}_+}(x)\int_0^\infty y\mathrm{e}^{-y^2/(2\eta^2)}dy \\
&\quad + \frac{1}{\pi\sigma\eta}\mathrm{e}^{-x^2/(2\sigma^2)}\mathbb{1}_{\mathbb{R}_-}(x)\int_{-\infty}^0 y\mathrm{e}^{-y^2/(2\eta^2)}dy \\
&= \frac{\eta}{\pi\sigma}\mathrm{e}^{-x^2/(2\sigma^2)}\mathbb{1}_{\mathbb{R}_+}(x) - \frac{\eta}{\pi\sigma}\mathrm{e}^{-x^2/(2\sigma^2)}\mathbb{1}_{\mathbb{R}_-}(x),
\end{aligned}
$$

hence

$$
\mathbb{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_\Sigma(x,y)dydx
$$

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$$
\begin{aligned}
&= \frac{\eta}{\pi\sigma}\int_0^\infty x\mathrm{e}^{-x^2/(2\sigma^2)}dx - \frac{\eta}{\pi\sigma}\int_{-\infty}^0 x\mathrm{e}^{-x^2/(2\sigma^2)}dx \\
&= \frac{2\sigma\eta}{\pi},
\end{aligned}
$$

and

$$
\rho = \frac{\mathbb{E}[XY]}{\sigma\eta} = \frac{2}{\pi}.
$$

Under a rotation

$$
R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},
$$

of angle $\theta \in [0, 2\pi]$ we would find

$$
\begin{aligned}
&\mathbb{E}[(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta)] \\
&= \sin\theta\cos\theta\mathbb{E}[X^2] + (\cos^2\theta - \sin^2\theta)\mathbb{E}[XY] - \sin\theta\cos\theta\mathbb{E}[Y^2] \\
&= \sigma^2\sin\theta\cos\theta + (\cos^2\theta - \sin^2\theta)\frac{2\sigma\eta}{\pi} - \eta^2\sin\theta\cos\theta \\
&= \frac{\sigma^2}{2}\sin(2\theta) + \cos(2\theta)\frac{2\sigma\eta}{\pi} - \frac{\eta^2}{2}\sin(2\theta),
\end{aligned}
$$

and

$$
\rho = \frac{\sigma}{2\eta}\sin(2\theta) + \cos(2\theta)\frac{2}{\pi} - \frac{\eta}{2\sigma}\sin(2\theta),
$$

i.e. $\theta = \pi/4$ and $\sigma = \eta$ would lead to uncorrelated random variables.

Exercise 4.6

a) We have

$$
\begin{aligned}
\mathbb{P}(\tau_i \wedge \tau \geqslant s) &= \mathbb{P}(\tau_i \geqslant s \text{ and } \tau \geqslant s) \\
&= \mathbb{P}(\tau_i \geqslant s)\mathbb{P}(\tau \geqslant s) \\
&= \mathrm{e}^{-\lambda_i s}\mathrm{e}^{-\lambda s} \\
&= \mathrm{e}^{-(\lambda_i+\lambda)s}, \qquad s \geqslant 0,
\end{aligned}
$$

hence $\tau_i \wedge \tau$ is an exponentially distributed random variable with parameter $\lambda_i + \lambda$, $i = 1, 2$.

b) Next, we have

$$
\begin{aligned}
\mathbb{P}(\tau_1 \wedge \tau > s \text{ and } \tau_2 \wedge \tau > t) &= \mathbb{P}(\tau_1 > s \text{ and } \tau > s \text{ and } \tau_2 > t \text{ and } \tau > t) \\
&= \mathbb{P}(\tau_1 > s \text{ and } \tau_2 > t \text{ and } \tau > \mathrm{Max}(s,t)) \\
&= \mathbb{P}(\tau_1 > s)\mathbb{P}(\tau_2 > t)\mathbb{P}(\tau > \mathrm{Max}(s,t)) \\
&= \mathrm{e}^{-\lambda_1 s}\mathrm{e}^{-\lambda_2 t}\mathrm{e}^{-\lambda\,\mathrm{Max}(s,t)}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{e}^{-\lambda_1 s - \lambda_2 t - \lambda \operatorname{Max}(s,t)} \\
&= \mathrm{e}^{-(\lambda_1+\lambda)s - (\lambda_2+\lambda)t + \lambda \min(s,t)} \\
&= (1 - F_X(s))(1 - F_Y(t)) \min(\mathrm{e}^{\lambda s}, \mathrm{e}^{\lambda t}),
\end{aligned}
$$

$s, t \geqslant 0$.

c) We have

$$
\begin{aligned}
F_{X,Y}(s,t) &= \mathbb{P}(\tau_1 \wedge \tau \leqslant s \text{ and } \tau_2 \wedge \tau \leqslant t) \\
&= \mathbb{P}(\tau_1 \wedge \tau \leqslant s) - \mathbb{P}(\tau_1 \wedge \tau \leqslant s \text{ and } \tau_2 \wedge \tau > t) \\
&= \mathbb{P}(\tau_1 \wedge \tau \leqslant s) - \big(\mathbb{P}(\tau_2 \wedge \tau > t) - \mathbb{P}(\tau_1 \wedge \tau > s \text{ and } \tau_2 \wedge \tau > t)\big) \\
&= \mathbb{P}(\tau_1 \wedge \tau \leqslant s) + \mathbb{P}(\tau_2 \wedge \tau \leqslant t) + \mathbb{P}(\tau_1 \wedge \tau > s \text{ and } \tau_2 \wedge \tau > t) - 1 \\
&= F_X(s) + F_Y(t) + (1 - F_X(s))(1 - F_Y(t)) \min(\mathrm{e}^{\lambda s}, \mathrm{e}^{\lambda t}) - 1.
\end{aligned}
$$

d) We find

$$
\begin{aligned}
C(u,v) &= F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)) \\
&= F_X(F_X^{-1}(u)) + F_Y(F_X^{-1}(v)) \\
&\quad + (1 - F_X(F_X^{-1}(u)))(1 - F_Y(F_X^{-1}(v))) \min\left(\mathrm{e}^{\lambda F_X^{-1}(u)}, \mathrm{e}^{\lambda F_X^{-1}(v)}\right) - 1 \\
&= u + v - 1 + (1-u)(1-v) \min\left(\mathrm{e}^{\lambda F_X^{-1}(u)}, \mathrm{e}^{\lambda F_Y^{-1}(v)}\right) \\
&= u + v - 1 + (1-u)(1-v) \min\left(\mathrm{e}^{-\lambda \log(1-u)/(\lambda_1+\lambda)}, \mathrm{e}^{-\lambda \log(1-v)/(\lambda_2+\lambda)}\right) \\
&= u + v - 1 + \min\left((1-v)(1-u)^{1-\lambda/(\lambda_1+\lambda)}, (1-u)(1-v)^{1-\lambda/(\lambda_2+\lambda)}\right) \\
&= u + v - 1 + \min\left((1-v)(1-u)^{1-\theta_1}, (1-u)(1-v)^{1-\theta_2}\right), \quad u, v \in [0,1],
\end{aligned}
$$

with

$$
\theta_1 = \frac{\lambda}{\lambda_1 + \lambda} \quad \text{and} \quad \theta_2 = \frac{\lambda}{\lambda_2 + \lambda}.
$$

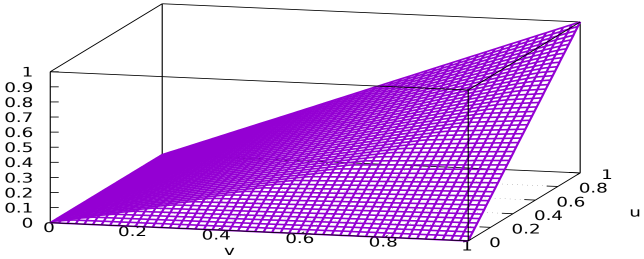

Fig. S.4: Survival copula graph with $\theta_1 = 0.3$ and $\theta_2 = 0.7$.

e) We have

$$C(u,v) = u + v - 1 + (1-u)(1-v)^{1-\theta_2} \mathbb{1}_{\{(1-u)^{\theta_1} < (1-v)^{\theta_2}\}} \\ +(1-v)(1-u)^{1-\theta_1} \mathbb{1}_{\{(1-u)^{\theta_1} > (1-v)^{\theta_2}\}}, \quad u, v \in [0,1],$$

hence

$$\frac{\partial C}{\partial u}(u,v) = -(1-v)^{1-\theta_2} \mathbb{1}_{\{(1-u)^{\theta_1} < (1-v)^{\theta_2}\}} \\ -(1-\theta_1)(1-v)(1-u)^{-\theta_1} \mathbb{1}_{\{(1-u)^{\theta_1} > (1-v)^{\theta_2}\}}$$

and the survival copula density is given by

$$\frac{\partial^2 C}{\partial u \partial v}(u,v) = (1-\theta_2)(1-v)^{-\theta_2} \mathbb{1}_{\{(1-u)^{\theta_1} < (1-v)^{\theta_2}\}} \\ +(1-\theta_1)(1-u)^{-\theta_1} \mathbb{1}_{\{(1-u)^{\theta_1} > (1-v)^{\theta_2}\}}, \quad u, v \in [0,1],$$

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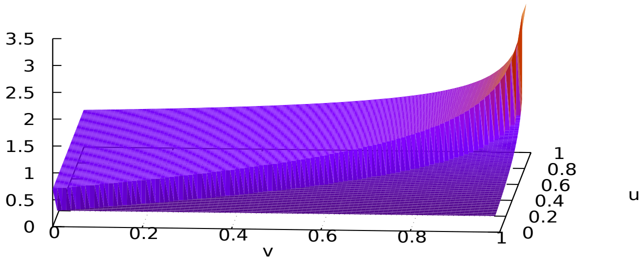

Fig. S.5: Survival copula density graph with $\theta_1 = 0.3$ and $\theta_2 = 0.7$.

Remark: When $\lambda = 0$ we have $\theta_1 = \theta_2 = 0$ and $\tau = +\infty$ *a.s.*, therefore we have

$$\min(\tau_1, \tau) = \tau_1 \quad \text{and} \quad \min(\tau_2, \tau) = \tau_2,$$

hence the copula $C(u, v)$ is given by

$$C(u, v) = u + v - 1 + (1 - v)(1 - u) = uv, \qquad u, v \in [0, 1],$$

which coincides with the copula of independence.

Chapter 5

Exercise 5.1 The payoff C is that of a *put* option with strike price $K = \$3$.

Exercise 5.2 Each of the two possible scenarios yields one equation:

$$\begin{cases} 5\xi + \eta = 0 \\ 2\xi + \eta = 6, \end{cases} \qquad \text{with solution} \qquad \begin{cases} \xi = -2 \\ \eta = +10. \end{cases}$$

The hedging strategy at $t = 0$ is to **shortsell** $-\xi = +2$ units of the asset S priced $S_0 = 4$, and to put $\eta = \$10$ on the savings account. The price $V_0 = \xi S_0 + \eta$ of the initial portfolio at time $t = 0$ is

$$V_0 = \xi S_0 + \eta = -2 \times 4 + 10 = \$2,$$

which yields the price of the claim at time $t = 0$. In order to hedge then option, one should:

i) At time $t = 0$,

a. Charge the \$2 option price.
b. Shortsell $-\xi = +2$ units of the stock priced $S_0 = 4$, which yields \$8.
c. Put $\eta = \$8 + \$2 = \$10$ on the savings account.

ii) At time $t = 1$,

a. If $S_1 = \$5$, spend \$10 from savings to buy back $-\xi = +2$ stocks.
b. If $S_1 = \$2$, spend \$4 from savings to buy back $-\xi = +2$ stocks, and deliver a \$10 - \$4 = \$6 payoff.

Pricing the option by the expected value $\mathbb{E}^*[C]$ yields the equality

$$\begin{aligned}\$2 &= \mathbb{E}^*[C] \\ &= 0 \times \mathbb{P}^*(C = 0) + 6 \times \mathbb{P}^*(C = 6) \\ &= 0 \times \mathbb{P}^*(S_1 = 2) + 6 \times \mathbb{P}^*(S_1 = 5) \\ &= 6 \times q^*,\end{aligned}$$

hence the risk-neutral probability measure $\mathbb{P}^*$ is given by

$$p^* = \mathbb{P}^*(S_1 = 5) = \frac{2}{3} \quad \text{and} \quad q^* = \mathbb{P}^*(S_1 = 2) = \frac{1}{3}.$$

Exercise 5.3

a) Each of the stated conditions yields one equation, *i.e.*

$$\begin{cases} 4\xi + \eta = 1 \\ 5\xi + \eta = 3, \end{cases} \qquad \text{with solution} \qquad \begin{cases} \xi = 2 \\ \eta = -7. \end{cases}$$

Therefore, the portfolio allocation at $t = 0$ consists to purchase $\xi = 2$ unit of the asset S priced $S_0 = 4$, and to borrow $-\eta = \$7$ in cash.

We can check that the price $V_0 = \xi S_0 + \eta$ of the initial portfolio at time $t = 0$ is

$$V_0 = \xi S_0 + \eta = 2 \times 4 - 7 = \$1.$$

b) This loss is expressed as

$$\xi \times \$2 + \eta = 2 \times 2 - 7 = -\$3.$$

Note that the \$1 received when selling the option is not counted here because it has already been fully invested into the portfolio.

Exercise 5.4

a) i) Does this model allow for arbitrage? | Yes | ✓ | No | |

ii) If this model allows for arbitrage opportunities, how can they be realized? | By shortselling | | By borrowing on savings | ✓ | N.A. | |

b) i) Does this model allow for arbitrage? | Yes | | No | ✓ |

ii) If this model allows for arbitrage opportunities, how can they be realized? | By shortselling | | By borrowing on savings | | N.A. | ✓ |

c) i) Does this model allow for arbitrage? | Yes | ✓ | No | |

ii) If this model allows for arbitrage opportunities, how can they be realized? | By shortselling | ✓ | By borrowing on savings | | N.A. | |

Exercise 5.5 Hedging a claim with possible payoff values C_a, C_b, C_c would require to solve

$$\begin{cases} (1+a)\xi S_0^{(1)} + (1+r)\eta S_0^{(0)} = C_a \\ \\ (1+b)\xi S_0^{(1)} + (1+r)\eta S_0^{(0)} = C_b \\ \\ (1+c)\xi S_0^{(1)} + (1+r)\eta S_0^{(0)} = C_c, \end{cases}$$

for ξ and η, which is not possible in general due to the existence of three conditions with only two unknowns.

Exercise 5.6

a) Each of two possible scenarios yields one equation:

$$\begin{cases} \alpha \overline{S}_1 + \beta = \overline{S}_1 - K \\ \\ \alpha \underline{S}_1 + \beta = 0, \end{cases} \quad \text{with solution} \quad \begin{cases} \alpha = \dfrac{\overline{S}_1 - K}{\overline{S}_1 - \underline{S}_1} \\ \\ \beta = -\underline{S}_1 \dfrac{\overline{S}_1 - K}{\overline{S}_1 - \underline{S}_1}. \end{cases}$$

b) We have

$$0 \leqslant \alpha = \frac{\overline{S}_1 - K}{\overline{S}_1 - \underline{S}_1} \leqslant 1$$

since $K \in [\underline{S}_1, \overline{S}_1]$.

c) We find

$$\begin{aligned}\mathrm{SRM}_{\mathrm{C}} &= \alpha S_0 + \beta \\ &= \alpha(S_0 - \underline{S}_1) \\ &= (S_0 - \underline{S}_1)\frac{\overline{S}_1 - K}{\overline{S}_1 - \underline{S}_1}.\end{aligned}$$

We note that when $S_0 < \underline{S}_1$ the value of $\mathrm{SRM}_{\mathrm{C}}$ is negative because in this case, investing in the zero-cost portfolio $(\alpha, -\alpha S_0)$ that would yield a payoff at least equal to $\alpha(\underline{S}_1 - S_0) = -\alpha(\underline{S}_0 - S_1) > 0$, which represents an *arbitrage opportunity*.

Exercise 5.7

a) The payoff of the long box spread option is given in terms of K_1 and K_2 as

$$\begin{aligned}(x-K_1)^+ - (K_1-x)^+ - (x-K_2)^+ + (K_2-x)^+ &= x - K_1 - (x - K_2) \\ &= K_2 - K_1.\end{aligned}$$

b) From Table 5.1 we check that the strike prices suitable for a long box spread option on the Hang Seng Index (HSI) are $K_1 = 25,000$ and $K_2 = 25,200$.

c) Based on the data provided, we note that the long box spread can be realized in two ways.

   i) Using the put option issued by BI (BOCI Asia Ltd.) at 0.044.

   In this case, the box spread option represents a short position priced

$$\underbrace{0.540}_{\text{Long call}} \times 7,500 \underbrace{-0.064}_{\text{Short put}} \times 8,000 \underbrace{-0.370}_{\text{Short call}} \times 11,000 \underbrace{+0.044}_{\text{Long put}} \times 10,000 = -92$$

   index points, or $-92 \times \$50 = -\$4,600$.

   Note that option prices are quoted in index points (to be multiplied by the relevant option/warrant entitlement ratio), and every index point is worth $50.

   ii) Using the put option issued by HT (Haitong Securities) at 0.061.

   In this case, the box spread option represents a long position priced

$$\underbrace{0.540}_{\text{Long call}} \times 7,500 \underbrace{-0.044}_{\text{Short put}} \times 8,000 \underbrace{-0.370}_{\text{Short call}} \times 11,000 \underbrace{+0.061}_{\text{Long put}} \times 10,000 = +78$$

   index points, or $78 \times \$50 = \$3,900$.

d) As the option built in i)) represents a short position paying \$4,600 today with an additional $\$50 \times (K_2 - K_1) = 200 = \$10,000$ payoff at maturity on March 28, I would definitely enter this position.

   As for the option built in ii)), it is less profitable because it costs \$3,900, however it is still profitable taking into account the \$10,000 payoff at maturity on March 28.

Chapter 6

Exercise 6.1

a) We have

$$\begin{aligned} F_X(x) &= \mathbb{P}(X \leqslant x) \\ &= \int_0^x f_X(y)dy \\ &= \gamma\theta^\gamma \int_0^x \frac{1}{(\theta+y)^{\gamma+1}}dy \\ &= \left[-\left(\frac{\theta}{\theta+y}\right)^\gamma\right]_0^x \\ &= 1-\left(\frac{\theta}{\theta+x}\right)^\gamma, \qquad x \in \mathbb{R}_+. \end{aligned}$$

b) Since the distribution of X admits a probability density function, the cumulative distribution function $x \longmapsto F_X(x)$ is continuous in x and we have $\mathbb{P}(X = x) = 0$ for all $x > 0$. Hence the Value at Risk V_X^p at the level p is given by the relation $F_X(V_X^p) = p$, *i.e.*

$$\left(\frac{\theta}{\theta+V_X^p}\right)^\gamma = 1-p,$$

   which gives

$$V_X^p = \theta\left(\frac{1}{(1-p)^{1/\gamma}} - 1\right).$$

   In particular, with $p = 99\%$, $\theta = 40$ and $\gamma = 2$, we find

$$V_X^p = ((1-p)^{-1/\gamma} - 1)\theta = 40\big(\sqrt{100} - 1\big) = \$360.$$

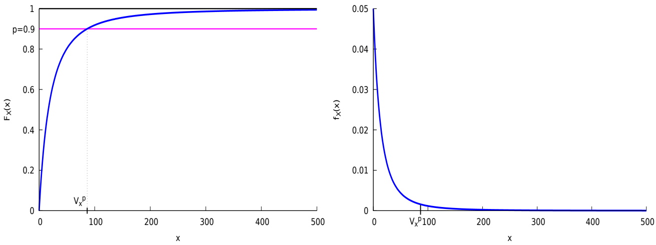

Fig. S.6: Pareto CDF $x \mapsto F_X(x)$ and PDF $x \mapsto f_X(x)$ with $V_X^{99\%} = \$86.49$.

Exercise 6.2

a) We have $\mathbb{P}(X = 100) = 0.02$.
b) We have $V_X^q = 100$ for all $q \in [0.97, 0.99]$.
c) The value at risk V_X^q at the level $q \in [0.99, 1]$ satisfies

$$F_X(V_X^q) = \mathbb{P}(X \leqslant V_X^q) = 0.99 + 0.01 \times (V_X^q - 100)/50 = q,$$

hence

$$V_X^q = 100 + 50(100q - 99) = 5000q - 4850, \qquad q \in [0.99, 1].$$

Exercise 6.3 We find $V_X^{99\%} = 100$ according to the following cumulative distribution function:

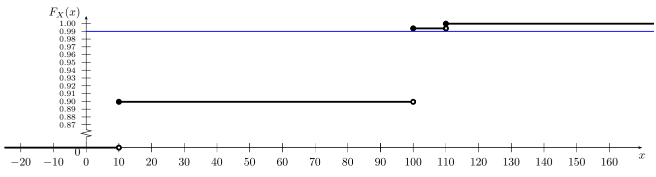

Fig. S.7: Cumulative distribution function of X and Y.

Exercise 6.4

a) We have

$$V_X^p := \inf \left\{ x \in \mathbb{R} \; : \; \mathbb{P}(X \leqslant x) \geqslant p \right\} = -\frac{1}{\lambda} \log(1-p) = \mathbb{E}[X] \log \frac{1}{1-p}.$$

When $p = 95\%$ this yields

$$V_X^p \simeq 2.996\mathbb{E}[X].$$

b) We find that the required capital C_X satisfies

$$C_X = V_X^p - \mathbb{E}[X] = \mathbb{E}[X] \log \frac{1}{1-p} - \mathbb{E}[X],$$

i.e.

$$C_X = V_X^{95\%} - \mathbb{E}[X] \simeq 1.996\mathbb{E}[X],$$

which means doubling the estimated amount of liabilities.

Exercise 6.5 By Proposition 6.2 and the geometric series identity (13.53), we have

$$\begin{aligned}
\mathbb{E}[X \mid X \geqslant a] &= \frac{1}{\mathbb{P}(X \geqslant a)} \mathbb{E}\big[X \mathbb{1}_{\{X \geqslant a\}}\big] \\
&= \frac{1}{\mathbb{P}(X \geqslant a)} \sum_{k \geqslant a} k \mathbb{P}(X = k) \\
&= \frac{1}{\displaystyle\sum_{k \geqslant a} (1-p)^k} \sum_{k \geqslant a} k(1-p)^k \\
&= \frac{(1-p)^a}{(1-p)^a \displaystyle\sum_{k \geqslant 0} (1-p)^k} \sum_{k \geqslant 0} (k+a)(1-p)^k \\
&= a + \frac{1}{\displaystyle\sum_{k \geqslant 0} (1-p)^k} \sum_{k \geqslant 0} k(1-p)^k \\
&= a + p \sum_{k \geqslant 0} k(1-p)^k \\
&= a + \frac{1}{p} \\
&= a + \mathbb{E}[X].
\end{aligned}$$

This can be recovered numerically for example with $a = 11$ using the R code below.

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```
geo_samples <- rgeom(100000, prob = 1/4)
mean(geo_samples)
mean(geo_samples[geo_samples>=10])
```

Exercise 6.6

a) As in the proof of the Markov inequality, for every $x > 0$ and $r > 0$ we have

$$\begin{aligned} x^r \mathbb{P}(X \geqslant x) &= x^r \mathbb{E}\left[\mathbb{1}_{\{X \geqslant x\}}\right] \\ &\leqslant \mathbb{E}\left[X^r \mathbb{1}_{\{X \geqslant x\}}\right] \\ &\leqslant \mathbb{E}\left[|X|^r\right], \end{aligned}$$

hence

$$\mathbb{P}(X \leqslant x) \geqslant 1 - \frac{1}{x^r}\mathbb{E}[|X|^r], \qquad x > 0. \tag{A.1}$$

From the inequality (A.1), it follows that

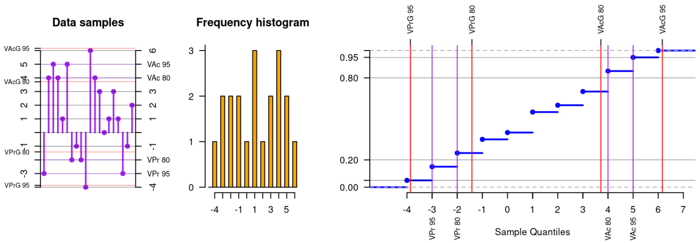
$$\begin{aligned} V_X^p &= \inf\left\{x \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant x) \geqslant p\right\} \\ &\leqslant \inf\left\{x \in \mathbb{R} \ : \ 1 - \frac{1}{x^r}\mathbb{E}[|X|^r] \geqslant p\right\} \\ &= \inf\left\{x \in \mathbb{R} \ : \ x^r \geqslant \frac{1}{1-p}\mathbb{E}[|X|^r]\right\} \\ &= \left(\frac{\mathbb{E}[|X|^r]}{1-p}\right)^{1/r} \\ &= \frac{\|X\|_{L^r(\Omega)}}{(1-p)^{1/r}}. \end{aligned}$$

b) Taking $p = 95\%$ and $r = 1$ we get

$$V_X^{95\%} \leqslant \frac{1}{1-p}\mathbb{E}[|X|] = 20\mathbb{E}[|X|].$$

To summarize, a smaller L^r-norm of X tends to make the value at risk V_X smaller.

Exercise 6.7



See the attached **code*** for a solution using R.

a) i) $\mathrm{VaR}^{95}_{\mathrm{A}c\text{-H}} = 5.$

ii) $\mathrm{VaR}^{80}_{\mathrm{A}c\text{-H}} = 4.$

iii) $\mathrm{VaR}^{95}_{\mathrm{P}r\text{-H}} = -3.$

iv) $\mathrm{VaR}^{80}_{\mathrm{P}r\text{-H}} = -2.$

b) By Proposition 6.16, we have:

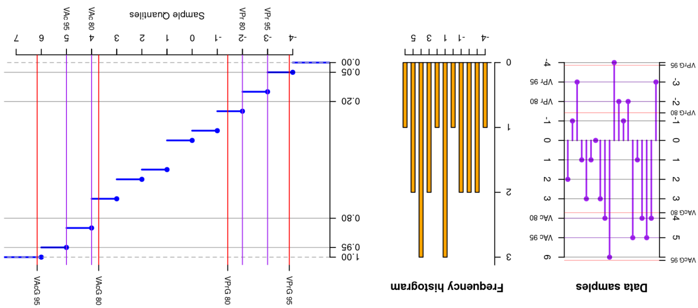
i) $\mathrm{VaR}^{95}_{\mathrm{A}c\text{-G}} =$ 1.15+3.048× qnorm(0.95)=6.164,

ii) $\mathrm{VaR}^{80}_{\mathrm{A}c\text{-G}} =$ 1.15+3.048× qnorm(0.80)=3.71551,

iii) $\mathrm{VaR}^{95}_{\mathrm{P}r\text{-G}} =$ 1.15-3.048× qnorm(0.95)=-3.864.

iv) $\mathrm{VaR}^{80}_{\mathrm{P}r\text{-G}} =$ 1.15-3.048× qnorm(0.80)=-1.41551,

Remark. The "Practitioner" Values at Risk can be better visualized after applying top-down and left-right symmetries (or a 180^o rotation) to the original CDF, as in the next figure.



* Right-click to save as attachment (may not work on 🍎).

Chapter 7

Exercise 7.1

a) Noting that $p = 1 - \mathrm{e}^{-\lambda \mathrm{VaR}_X^p}$ and using integration by parts on $\left[\mathrm{VaR}_X^p, \infty\right)$ with $u(x) = x$ and $v'(x) = \mathrm{e}^{-\lambda x}$, we have

$$\begin{aligned}
\mathbb{E}[X \mid X > \mathrm{VaR}_X^p] &= \frac{1}{\mathbb{P}(X > \mathrm{VaR}_X^p)} \int_{\mathrm{VaR}_X^p}^{\infty} x f_X(x) dx \\
&= \frac{\lambda}{1-p} \int_{\mathrm{VaR}_X^p}^{\infty} x \mathrm{e}^{-\lambda x} dx \\
&= \frac{\lambda}{1-p} \int_{\mathrm{VaR}_X^p}^{\infty} u(x) v'(x) dx \\
&= \frac{\lambda}{1-p} \left(\left[u(x)v(x)\right]_{\mathrm{VaR}_X^p}^{\infty} - \int_{\mathrm{VaR}_X^p}^{\infty} u'(x) v(x) dx \right) \\
&= \frac{\lambda}{1-p} \left(\left[-\frac{x}{\lambda} \mathrm{e}^{-\lambda x}\right]_{\mathrm{VaR}_X^p}^{\infty} + \frac{1}{\lambda} \int_{\mathrm{VaR}_X^p}^{\infty} \mathrm{e}^{-\lambda x} dx \right) \\
&= \frac{\lambda}{1-p} \left(\frac{\mathrm{VaR}_X^p}{\lambda} \mathrm{e}^{-\lambda \mathrm{VaR}_X^p} + \frac{1}{\lambda^2} \mathrm{e}^{-\lambda \mathrm{VaR}_X^p} \right) \\
&= \frac{\lambda}{1-p} \left(\frac{\mathrm{VaR}_X^p}{\lambda} (1-p) + \frac{1-p}{\lambda^2} \right) \\
&= \mathrm{VaR}_X^p + \frac{1}{\lambda} \\
&= \frac{1}{\lambda} - \frac{\log(1-p)}{\lambda}.
\end{aligned}$$

b) We have

$$\begin{aligned}
\mathrm{TV}_X^p &= \frac{1}{1-p} \int_p^1 V_X^q dq \\
&= -\frac{1}{\lambda(1-p)} \int_p^1 \log(1-q) dq \\
&= -\frac{1}{\lambda(1-p)} \int_0^{1-p} (\log q) dq \\
&= \frac{1 - p + (1-p) \log \frac{1}{1-p}}{\lambda(1-p)} \\
&= \frac{1}{\lambda} + \frac{1}{\lambda} \log \frac{1}{1-p} \\
&= \mathbb{E}[X] \left(1 + \log \frac{1}{1-p} \right)
\end{aligned}$$

$$= \mathbb{E}[X] + V_X^p.$$

Exercise 7.2

a) If $\mathbb{P}(X > z) > 0$ we have $\mathbb{E}\big[(X - z)\mathbb{1}_{\{X>z\}}\big] > 0$, hence

$$\mathbb{E}\big[X\mathbb{1}_{\{X>z\}}\big] > \mathbb{E}\big[z\mathbb{1}_{\{X>z\}}\big] = z\mathbb{P}(X > z),$$

and

$$\mathbb{E}[X \mid X > z] = \frac{\mathbb{E}\big[X\mathbb{1}_{\{X>z\}}\big]}{\mathbb{P}(X > z)} > z. \tag{A.2}$$

Recall that $\mathbb{E}[X \mid X > z]$ is not defined if $\mathbb{P}(X > z) = 0$.

b) We have

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}\big[X\mathbb{1}_{\{X\leqslant z\}}\big] + \mathbb{E}\big[X\mathbb{1}_{\{X>z\}}\big] \\
&\leqslant z\mathbb{E}\big[\mathbb{1}_{\{X\leqslant z\}}\big] + \mathbb{E}\big[X\mathbb{1}_{\{X>z\}}\big] \\
&= z\mathbb{P}(X \leqslant z) + \mathbb{E}\big[X\mathbb{1}_{\{X>z\}}\big] \\
&\leqslant \mathbb{E}[X \mid X > z]\mathbb{P}(X \leqslant z) + \mathbb{E}[X \mid X > z]\mathbb{P}(X > z) \\
&= \mathbb{E}[X \mid X > z].
\end{aligned}$$

Note that $\mathbb{E}[X] = \mathbb{E}[X \mid X > z]$ when $\mathbb{P}(X \leqslant z) = 0$, *i.e.* $\mathbb{P}(X > z) = 1$.

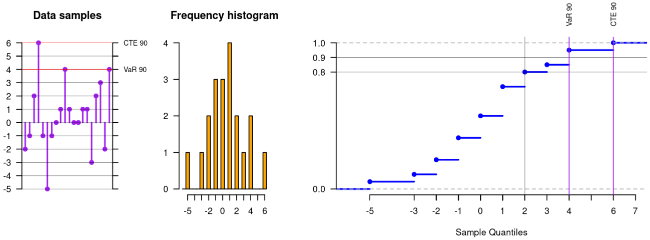
c) When $\mathbb{P}(X \leqslant z) > 0$, from (A.2) we find

$$\begin{aligned}
\mathbb{E}[X] &\leqslant z\mathbb{P}(X \leqslant z) + \mathbb{E}\big[X\mathbb{1}_{\{X>z\}}\big] \\
&< \mathbb{E}[X \mid X > z]\mathbb{P}(X \leqslant z) + \mathbb{E}[X \mid X > z]\mathbb{P}(X > z) \\
&= \mathbb{E}[X \mid X > z].
\end{aligned}$$

d) By (6.7) we have $\mathbb{P}\big(X \leqslant V_X^p\big) \geqslant p > 0$, hence by (c)) above we find $\mathrm{CTE}_X^p = \mathbb{E}\big[X \,\big|\, X > V_X^p\big] > \mathbb{E}[X]$.

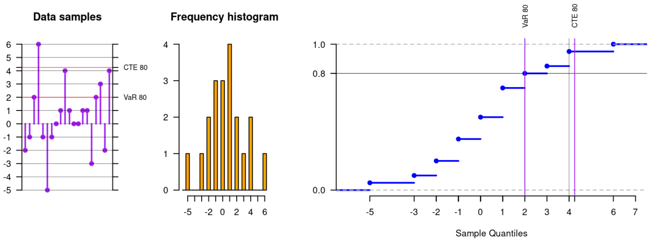
Exercise 7.3

a) We have $\mathrm{VaR}_X^{0.9} = 4$ and $\mathrm{CTE}_X^{0.9} = 6$.



b) We have $\mathrm{VaR}_X^{0.8} = 2$ and

$$\mathrm{CTE}_X^{0.8} = \frac{3 + 2 \times 4 + 6}{4} = \frac{17}{4} = 4.25.$$



Equivalently, we have

$$\begin{aligned}\mathrm{CTE}_X^{0.8} &= \frac{0.05 \times 3 + 0.1 \times 4 + 0.05 \times 6}{0.05 + 0.1 + 0.05} \\ &= \frac{0.05 \times 3 + 0.1 \times 4 + 0.05 \times 6}{0.2} \\ &= \frac{0.85}{0.2} = 4.25.\end{aligned}$$

Exercise 7.4

a) $\mathrm{VaR}_X^{90\%} = 4$.

b) $\mathbb{E}\big[X \mathbb{1}_{\{X > V_X^{90\%}\}}\big] = \dfrac{5+6}{23} = \dfrac{11}{23}$.

c) $\mathbb{P}\big(X > V_X^{90\%}\big) = \dfrac{2}{23}$.

d) $\mathrm{CTE}_X^{90\%} = \mathbb{E}\big[X \mid X > V_X^{90\%}\big] = \dfrac{\mathbb{E}\big[X \mathbb{1}_{\{X > V_X^{90\%}\}}\big]}{\mathbb{P}\big(X > V_X^{90\%}\big)} = \dfrac{5+6}{2} = \dfrac{11}{2} = 5.50.$

e) $\mathbb{E}\big[X \mathbb{1}_{\{X \geqslant V_X^{90\%}\}}\big] = \dfrac{4+5+6}{23} = \dfrac{15}{23}.$

f) $\mathbb{P}\big(X \geqslant V_X^{90\%}\big) = \dfrac{3}{23}.$

g) $\mathrm{ES}_X^{90\%} = \dfrac{1}{1-p}\Big(\mathbb{E}\big[X \mathbb{1}_{\{X \geqslant V_X^{90\%}\}}\big] + V_X^{90\%}\big(1-p-\mathbb{P}\big(X \geqslant V_X^{90\%}\big)\big)\Big) = 10 \times \dfrac{4+5+6}{23} + 10 \times 4\left(0.1 - \dfrac{3}{23}\right) = \dfrac{150}{23} + 40 \times \dfrac{2.3-3}{23} = \dfrac{150 - 40 \times 0.7}{23} = \dfrac{122}{23} = 5.304.$

h)
$$\begin{aligned}
\mathrm{TV}_X^{90\%} &= \frac{1}{1-p}\int_p^1 V_X^q dq = \frac{1}{1-p}\left(\int_p^{21/23} V_X^q dq + \int_{21/23}^{22/23} V_X^q dq + \int_{22/23}^1 V_X^q dq\right) \\
&= \frac{1}{1-p}\left(\int_p^{21/23} 4dq + \int_{21/23}^{22/23} 5dq + \int_{22/23}^1 6dq\right) \\
&= \frac{1}{1-p}\left(4\left(\frac{21}{23} - p\right) + \frac{5}{23} + \frac{6}{23}\right) = \frac{84 - 92p + 5 + 6}{23(1-p)} = \frac{122}{23} = 5.304.
\end{aligned}$$

We note that $\mathrm{ES}_X^{90\%} = \mathrm{TV}_X^{90\%}$ according to Proposition 7.12. The attached **R code** computes the above risk measures, as illustrated in Figure S.8.

```
> source("var-cte_quiz.R")
VaR90= 4, Threshold= 0.9130435
CTE90= 5.5
ES90= 5.304348
```

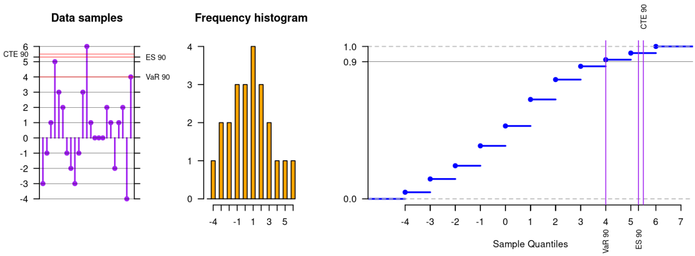


Fig. S.8: Value at Risk and Expected Shortfall for small data.

Exercise 7.5

a) The value at risk is $V_X^{98\%} = 100$.

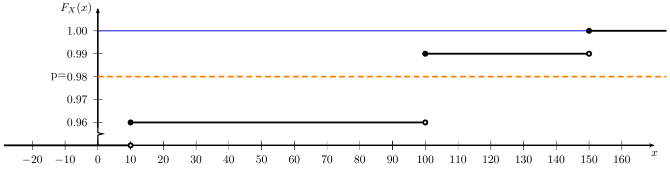


Fig. S.9: Cumulative distribution function of X.

b) Taking $p = 0.98$, we have

$$\begin{aligned}\mathrm{TV}_X^{98\%} &= \frac{1}{1-p}\int_p^1 V_X^q dq \\ &= \frac{1}{0.02}((0.99-0.98)\times 100 + (1-0.99)\times 150) = 125.\end{aligned}$$

c) We have

$$\begin{aligned}\mathrm{CTE}_X^{98\%} &= \frac{1}{\mathbb{P}(X > V_X^p)}\mathbb{E}\left[X\mathbb{1}_{\{X>V_X^p\}}\right] \\ &= \frac{1}{0.01}\times 150 \times 0.01 = 150.\end{aligned}$$

d) We have

$$\begin{aligned}\mathrm{ES}_X^{98\%} &= \frac{1}{1-p}\mathbb{E}\left[X\mathbb{1}_{\{X\geqslant V_X^p\}}\right] + \frac{V_X^p}{1-p}(1-p-\mathbb{P}(X\geqslant V_X)) \\ &= \frac{1}{0.02}(100\times 0.03 + 150\times 0.01) + \frac{100}{0.02}(0.02-(0.03+0.01)) \\ &= \frac{4.5}{0.02} + \frac{100}{0.02}(0.02-(0.03+0.01)) = 125.\end{aligned}$$

Note that we also have

$$\begin{aligned}\mathrm{ES}_X^{98\%} &= \frac{1}{1-p}\mathbb{E}\left[X\mathbb{1}_{\{X>V_X^p\}}\right] + \frac{V_X^p}{1-p}(1-p-\mathbb{P}(X>V_X)) \\ &= \frac{1}{0.02}(150\times 0.01) + \frac{100}{0.02}(0.02-0.01) \\ &= 125,\end{aligned}$$

hence the Expected Shortfall $\mathrm{ES}_X^{98\%}$ does coincide with the tail value at risk $TV_X^{98\%}$.

Exercise 7.6

a) The cumulative distribution function of X is given by the following graph:

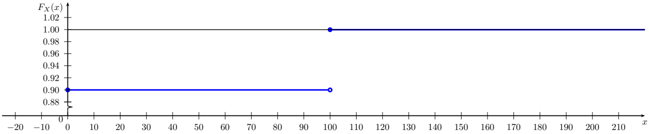


Fig. S.10: Cumulative distribution function of X.

b) The distribution of $X+Y$ is given by

$$\mathbb{P}(X+Y=0)=81\%,\quad \mathbb{P}(X+Y=100)=18\%,\quad \mathbb{P}(X+Y=200)=1\%.$$

The cumulative distribution function of $X+Y$ is given by the following graph:

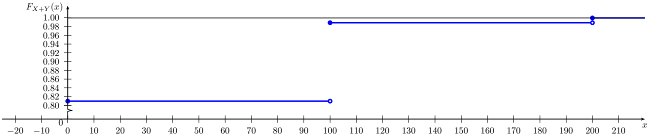


Fig. S.11: Cumulative distribution function of $X+Y$.

c) We have $V_{X+Y}^{99\%}=V_{X+Y}^{95\%}=V_{X+Y}^{90\%}=100$.

Note that we have $V_{X+Y}^{99\%}=100$ because

$$V_X^{99\%}=\inf\{x\in\mathbb{R} \ : \ \mathbb{P}(X\leqslant x)\geqslant 0.99\}=100.$$

d) We have

$$\mathrm{TV}_X^{90\%}=\frac{1}{1-0.9}\int_{0.9}^1 V_X^q dq=100\times\frac{1-0.9}{1-0.9}=100.$$

e) We have

$$\begin{aligned}
\mathrm{TV}^{99\%}_{X+Y} &= \frac{1}{1-0.9}\int_{0.9}^{1} V^q_{X+Y}dq \\
&= \frac{1}{0.1}\left(\int_{0.9}^{0.99} 100dq + \int_{0.99}^{1} 200dq\right) \\
&= \frac{1}{0.1}\,(100\times 0.09 + 200\times 0.01) \\
&= 110,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{TV}^{80\%}_{X+Y} &= \frac{1}{1-0.8}\int_{0.9}^{1} V^q_{X+Y}dq \\
&= \frac{1}{0.2}\left(\int_{0.8}^{0.81} 0dq + \int_{0.81}^{0.99} 100dq + \int_{0.99}^{1} 200dq\right) \\
&= \frac{1}{0.2}\,(100\times 0.18 + 200\times 0.01) = 100.
\end{aligned}$$

Exercise 7.7 (Exercise 6.2 continued).

a) For all $p \in [0.99, 1]$ we have

$$\begin{aligned}
\mathrm{TV}^{p}_{X} &= \frac{1}{1-p}\int_{p}^{1} V^q_{X}dq \\
&= \frac{1}{1-p}\int_{p}^{1} (5000q - 4850)dq \\
&= \frac{1}{1-p}\left(5000\frac{(1-p^2)}{2} - (1-p)4850\right) \\
&= 2500p - 2350.
\end{aligned}$$

In particular,

$$\mathrm{TV}^{99\%}_{X} = 2500\times 0.99 - 2350 = 125 \geqslant V^{99\%}_X = 100.$$

b) We have $V^{98\%}_X = 100$ and

$$\begin{aligned}
\mathrm{CTE}^{98\%}_{X} &= \mathbb{E}\big[X \mid X > V^{98\%}_X\big] \\
&= \frac{1}{\mathbb{P}\big(X > V^{98\%}_X\big)}\mathbb{E}\left[X\mathbb{1}_{\{X > V^{98\%}_X\}}\right] \\
&= \frac{1}{0.01}\int_{100}^{\infty} x f_X(x)dx \\
&= \frac{1}{0.01}\int_{100}^{\infty} x\frac{dF_X(x)}{dx}dx
\end{aligned}$$

$$= \frac{1}{0.01} \frac{0.01}{50} \int_{100}^{150} x dx$$
$$= \frac{150^2 - 100^2}{2 \times 50}$$
$$= 125.$$

Note that

$$\begin{aligned} \mathrm{TV}_X^{98\%} &= \frac{0.01}{0.02} \times 100 + \frac{1}{0.02} \left(5000 \frac{(1 - 0.99^2)}{2} - 0.01 \times 4850 \right) \\ &= 112.50 \\ &\geqslant V_X^{98\%} = 100, \end{aligned}$$

which differs from $\mathrm{CTE}_X^{98\%} = 125$ since

$$\mathbb{P}\big(X = V_X^{98\%}\big) = \mathbb{P}(X = 100) = 0.02 > 0.$$

Exercise 7.8

a) We have

$$V_X^p := \inf \big\{ x \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant x) \geqslant p \big\} = \log \frac{p}{1-p}.$$

b) We have

$$\begin{aligned} &\mathbb{E}[X \mid X > \mathrm{VaR}_X^p] = \frac{1}{\mathbb{P}(X > \mathrm{VaR}_X^p)} \int_{\mathrm{VaR}_X^p}^\infty x f_X(x) dx \\ &= \frac{1}{1-p} \int_{\mathrm{VaR}_X^p}^\infty x \mathrm{e}^{-\lambda x} dx \\ &= \frac{1}{1-p} \int_{\mathrm{VaR}_X^p}^\infty \frac{x \mathrm{e}^{-x}}{(1+\mathrm{e}^{-x})^2} dx \\ &= \frac{1}{1-p} \log(1 + \mathrm{e}^{\mathrm{VaR}_X^p}) - \frac{\mathrm{VaR}_X^p \mathrm{e}^{\mathrm{VaR}_X^p}}{1 + \mathrm{e}^{\mathrm{VaR}_X^p}} \\ &= \frac{1}{1-p} \log \left(1 + \frac{p}{1-p} \right) - \frac{1}{1-p} \frac{p}{1-p} \frac{1}{1 + \frac{p}{1-p}} \log \frac{p}{1-p} \\ &= \frac{1}{1-p} \log \frac{1}{1-p} - \frac{p}{1-p} \log \frac{p}{1-p} \\ &= -\frac{p}{1-p} \log p - \log(1-p). \end{aligned}$$

c) We have

$$
\begin{aligned}
\mathrm{TV}_X^p &= \frac{1}{1-p}\int_p^1 V_X^q dq \\
&= \frac{1}{1-p}\int_p^1 \log\frac{q}{1-q} dq \\
&= \frac{1}{1-p}\int_p^1 \log q dq - \frac{1}{1-p}\int_p^1 \log(1-q) dq \\
&= \frac{1}{1-p}\int_p^1 \log q dq - \frac{1}{1-p}\int_0^{1-p} \log q dq \\
&= \frac{1}{1-p}\int_p^1 \log q dq - \frac{1}{1-p}\left(\int_0^1 \log q dq - \int_{1-p}^1 \log q dq\right) \\
&= \frac{p-1-p\log p}{1-p} - \frac{-1+p+(1-p)\log(1-p)}{1-p} \\
&= -\frac{p}{1-p}\log p - \log(1-p).
\end{aligned}
$$

Exercise 7.9

a) We have

$$
q\mathbb{P}(Z \geqslant q) = \mathbb{E}\big[q\mathbb{1}_{\{Z\geqslant q\}}\big] \leqslant \mathbb{E}\big[Z\mathbb{1}_{\{Z\geqslant q\}}\big] = \int_q^\infty x f_Z(x) dx, \qquad q \geqslant 0.
$$

b) We have

$$
\begin{aligned}
\int_q^\infty x f_Z(x) dx &= \int_q^\infty x\phi(x) dx \\
&= \frac{1}{\sqrt{2\pi}}\int_q^\infty x\mathrm{e}^{-x^2/2} dx \\
&= -\frac{1}{\sqrt{2\pi}}\left[\mathrm{e}^{-x^2/2}\right]_q^\infty \\
&= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-q^2/2} \\
&= \phi(q), \qquad q \geqslant 0,
\end{aligned}
$$

and $1-p = \mathbb{P}(Z \geqslant q)$, hence

$$
(1-p)q \leqslant \int_q^\infty x f_Z(x) dx = \phi(q), \qquad q \geqslant 0.
$$

c) Taking $q := q_Z^p$ with $1-p = \mathbb{P}(Z \geqslant q_Z^p)$, we recover

$$
V_X^p = \mu_X + \sigma_X q_Z^p \leqslant \mu_X + \frac{\sigma_X}{1-p}\phi(q_Z^p) = \mathrm{CTE}_X^p,
$$

see Proposition 7.5.

Chapter 8

Exercise 8.1

a) We have

$$\mathbb{E}[X \mid G] = \lambda_G \int_0^\infty x \mathrm{e}^{-\lambda_G x} dx = \frac{1}{\lambda_G}$$

and

$$\mathbb{E}[X \mid B] = \lambda_B \int_0^\infty x \mathrm{e}^{-\lambda_B x} dx = \frac{1}{\lambda_B}.$$

b) We find

$$\begin{aligned}\mathbb{P}(B \mid X = x) &= \frac{f_X(x \mid B)\mathbb{P}(B)}{f_X(x \mid G)\mathbb{P}(G) + f_X(x \mid B)\mathbb{P}(B)} \\ &= \frac{\lambda_B \mathrm{e}^{-\lambda_B x}\mathbb{P}(B)}{\lambda_G \mathrm{e}^{-\lambda_G x}\mathbb{P}(G) + \lambda_B \mathrm{e}^{-\lambda_B x}\mathbb{P}(B)} \\ &= \frac{1}{1 + \dfrac{\lambda_G \mathbb{P}(G)}{\lambda_B \mathbb{P}(B)} \mathrm{e}^{(\lambda_B - \lambda_G)x}} \\ &= \frac{1}{1 + \lambda(x)\dfrac{\mathbb{P}(G)}{\mathbb{P}(B)}},\end{aligned}$$

where $\lambda(x)$ is the *likelihood ratio*

$$\lambda(x) = \frac{f_X(x \mid G)}{f_X(x \mid B)} = \frac{\lambda_G}{\lambda_B} \mathrm{e}^{(\lambda_B - \lambda_G)x}, \qquad x > 0.$$

c) The condition

$$D\mathbb{P}(B \mid X = x) \leqslant L\mathbb{P}(G \mid X = x)$$

rewrites as

$$D\mathbb{P}(B \mid X = x) \leqslant L(1 - \mathbb{P}(B \mid X = x)),$$

i.e.

$$(L + D)\mathbb{P}(B \mid X = x) \leqslant L$$

or

$$\frac{L + D}{1 + \lambda(x)\dfrac{\mathbb{P}(G)}{\mathbb{P}(B)}} \leqslant L,$$

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or

$$\lambda(x) = \frac{\lambda_G}{\lambda_B} e^{(\lambda_B - \lambda_G)x} \geqslant \frac{D}{L} \frac{\mathbb{P}(B)}{\mathbb{P}(G)}.$$

This condition holds if and only if

$$x \geqslant \frac{1}{\lambda_B - \lambda_G} \log\left(\frac{D}{L} \frac{\lambda_B \mathbb{P}(B)}{\lambda_G \mathbb{P}(G)}\right),$$

provided that $\lambda_B > \lambda_G$. Therefore we have

$$\begin{aligned} \mathscr{A} &= \left\{ x \in \mathbb{R} \ : \ \lambda(x) \geqslant \frac{D}{L} \frac{\mathbb{P}(B)}{\mathbb{P}(G)} \right\} \\ &= \left[\frac{1}{\lambda_B - \lambda_G} \log\left(\frac{D}{L} \frac{\lambda_B \mathbb{P}(B)}{\lambda_G \mathbb{P}(G)}\right), \infty \right), \end{aligned}$$

under the condition

$$\mathbb{E}[X \mid B] = \frac{1}{\lambda_B} < \frac{1}{\lambda_G} = \mathbb{E}[X \mid G].$$

Exercise 8.2

a) We find

$$\begin{aligned} \mathbb{P}(B \mid X = x) &= \frac{f_X(x \mid B)\mathbb{P}(B)}{f_X(x \mid G)\mathbb{P}(G) + f_X(x \mid B)\mathbb{P}(B)} \\ &= \mathbb{1}_{[0,\lambda_B]}(x) \frac{\mathbb{P}(B)/\lambda_B}{\mathbb{P}(G)/\lambda_G + \mathbb{P}(B)/\lambda_B} \\ &= \mathbb{1}_{[0,\lambda_B]}(x) \frac{1}{1 + \dfrac{\lambda_B \mathbb{P}(G)}{\lambda_G \mathbb{P}(B)}}. \end{aligned}$$

b) We have

$$\mathbb{E}[X \mid G] = \int_{-\infty}^{\infty} y f_X(y \mid G) dy = \frac{1}{\lambda_G} \int_0^{\lambda_G} y dy = \frac{\lambda_G}{2},$$

and similarly

$$\mathbb{E}[X \mid B] = \int_{-\infty}^{\infty} y f_X(y \mid B) dy = \frac{1}{\lambda_B} \int_0^{\lambda_B} y dy = \frac{\lambda_B}{2}.$$

c) The condition

$$D\mathbb{P}(B \mid X = x) \leqslant L\mathbb{P}(G \mid X = x)$$

rewrites as

$$D\mathbb{P}(B \mid X = x) \leqslant L(1 - \mathbb{P}(B \mid X = x)),$$

i.e.

$$(L + D)\mathbb{P}(B \mid X = x) \leqslant L$$

or

$$D\mathbb{1}_{[0,\lambda_B]}(x) \leqslant L\mathbb{1}_{(\lambda_B,\infty)}(x) + L\frac{\lambda_B \mathbb{P}(G)}{\lambda_G \mathbb{P}(B)}.$$

This condition holds if and only if

$$\frac{\lambda_B}{\lambda_G} \geqslant \frac{D}{L}\frac{\mathbb{P}(B)}{\mathbb{P}(G)}.$$

when $x \in [0, \lambda_B]$, and is always satisfied when $x \in (\lambda_B, \infty)$. Therefore, we have $\mathscr{A} = \mathbb{R}$ if

$$\frac{\lambda_B}{\lambda_G} \geqslant \frac{D}{L}\frac{\mathbb{P}(B)}{\mathbb{P}(G)},$$

and $\mathscr{A} = (\lambda_B, \infty)$ if

$$\frac{\lambda_B}{\lambda_G} < \frac{D}{L}\frac{\mathbb{P}(B)}{\mathbb{P}(G)},$$

under the condition

$$\mathbb{E}[X \mid B] = \lambda_B < \lambda_G = \mathbb{E}[X \mid G].$$

Exercise 8.3

a) We have

$$\overline{F}_G(x) = \mathrm{e}^{-\lambda_G x} \quad \text{and} \quad \overline{F}_B(x) = \mathrm{e}^{-\lambda_B x}, \quad x \geqslant 0,$$

hence

$$\overline{F}_B^{-1}(y) = -\frac{\log y}{\lambda_B}, \quad y \in (0, 1],$$

and

$$\begin{aligned}\overline{F}_G(\overline{F}_B^{-1}(y)) &= \overline{F}_G\left(-\frac{\log y}{\lambda_B}\right) \\ &= \mathrm{e}^{\lambda_G (\log y)/\lambda_B} \\ &= y^{\lambda_G/\lambda_B}, \qquad y \in [0, 1].\end{aligned}$$

We check that, according to Proposition 8.7,

$$\begin{aligned}\frac{d}{dy}\overline{F}_G\big(\overline{F}_B^{-1}(y)\big) &= \frac{d}{dy}y^{\lambda_G/\lambda_B}\\ &= \frac{d}{dy}\mathrm{e}^{\lambda_G(\log y)/\lambda_B}\\ &= \frac{\lambda_G}{y\lambda_B}\mathrm{e}^{\lambda_G(\log y)/\lambda_B}\\ &= \frac{\lambda_G}{\lambda_B}\mathrm{e}^{(\lambda_G-\lambda_B)(\log y)/\lambda_B}\\ &= \frac{\lambda_G}{\lambda_B}\mathrm{e}^{(\lambda_B-\lambda_G)\overline{F}_B^{-1}(y)}\\ &= \lambda\big(\overline{F}_B^{-1}(y)\big), \qquad x\in[0,1].\end{aligned}$$

Figure S.12 presents three samples of exponential ROC curves, with successively $(\lambda_B,\lambda_G)=(10,1)$, $(\lambda_B,\lambda_G)=(2,1)$, and $(\lambda_B,\lambda_G)=(1,1)$.

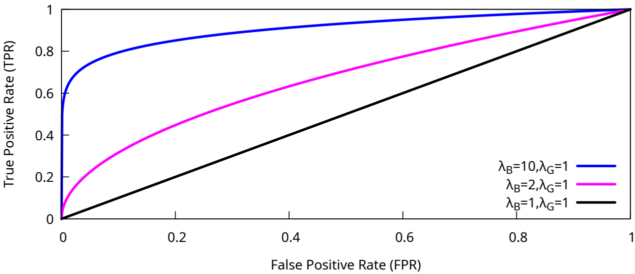

Fig. S.12: Exponential ROC curves.

b) We have

$$\overline{F}_G(x) := 1-\frac{x}{\lambda_G}, \qquad x\in[0,\lambda_G],$$

and

$$\overline{F}_B(x) := 1-\frac{x}{\lambda_B}, \qquad x\in[0,\lambda_B],$$

hence

$$\overline{F}_B^{-1}(y) := \lambda_B(1-y), \qquad y\in[0,1],$$

hence

$$\overline{F}_G\big(\overline{F}_B^{-1}(x)\big) = 1-\frac{\lambda_B}{\lambda_G}(1-y) = \frac{\lambda_G-\lambda_B}{\lambda_G}+\frac{\lambda_B}{\lambda_G}y, \quad y\in[0,1].$$

Figure S.13 presents three samples of uniform ROC curves, with successively $(\lambda_B, \lambda_G) = (1, 8)$, $(\lambda_B, \lambda_G) = (1, 2)$, and $(\lambda_B, \lambda_G) = (1, 1)$.

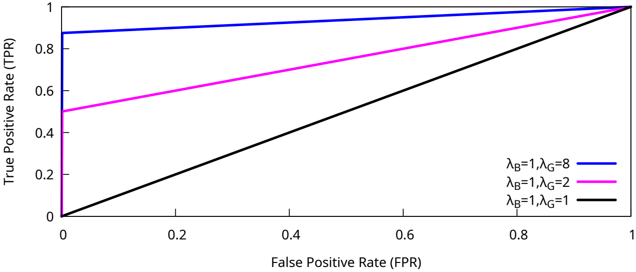

Fig. S.13: Uniform ROC curves.

Exercise 8.4

a) We have

$$\begin{aligned}\mathbb{P}(B \mid X = x) &= \frac{\mathbb{P}(B) f_X(x \mid B)}{\mathbb{P}(G) f_X(x \mid G) + \mathbb{P}(B) f_X(x \mid B)} \\ &= \frac{\mathbb{P}(B)\mathrm{e}^{-(x-\mu_B)^2/(2\sigma^2)}}{\mathbb{P}(G)\mathrm{e}^{-(x-\mu_G)^2/(2\sigma^2)} + \mathbb{P}(B)\mathrm{e}^{-(x-\mu_B)^2/(2\sigma^2)}} \\ &= \frac{1}{1+\mathrm{e}^{\alpha+\beta x}}, \qquad x \in \mathbb{R},\end{aligned}$$

with

$$\beta := \frac{\mu_G - \mu_B}{\sigma^2} > 0$$

and

$$\alpha := -\beta \frac{\mu_G + \mu_B}{2} + \log\left(\frac{\mathbb{P}(G)}{\mathbb{P}(B)}\right).$$

b) We have

$$\begin{aligned}\lambda(x) &= \frac{f_X(x \mid G)}{f_X(x \mid B)} \\ &= \mathrm{e}^{-(x-\mu_G)^2/(2\sigma^2)+(x-\mu_B)^2/(2\sigma^2)} \\ &= \mathrm{e}^{-(\mu_G^2-\mu_B^2-2x(\mu_G-\mu_B))/(2\sigma^2)} \\ &= \mathrm{e}^{\beta x-(\mu_G^2-\mu_B^2)/(2\sigma^2)}, \qquad x \in \mathbb{R}.\end{aligned}$$

c) The condition

$$\lambda(x) = \mathrm{e}^{\beta x - (\mu_G^2 - \mu_B^2)/(2\sigma^2)} \geqslant \frac{D(x)}{L(x)} \frac{\mathbb{P}(B)}{\mathbb{P}(G)}$$

is equivalent to

$$\begin{aligned}\beta x &\geqslant \frac{\mu_G^2 - \mu_B^2}{2\sigma^2} + \log\left(\frac{D(x)}{L(x)}\frac{\mathbb{P}(B)}{\mathbb{P}(G)}\right)\\ &\geqslant \frac{\mu_G^2 - \mu_B^2}{2\sigma^2} + \log\frac{\mathbb{P}(B)}{\mathbb{P}(G)} + \log\frac{D(x)}{L(x)}\\ &\geqslant \frac{\mu_G^2 - \mu_B^2}{2\sigma^2} + \log\frac{\mathbb{P}(B)}{\mathbb{P}(G)} + x(a+b),\end{aligned}$$

hence

$$x \geqslant \frac{\mu_G^2 - \mu_B^2}{2\sigma^2(\beta - a - b)} + \frac{1}{\beta - a - b}\log\frac{\mathbb{P}(B)}{\mathbb{P}(G)},$$

provided that

$$\beta := \frac{\mu_G - \mu_B}{\sigma^2} > a + b.$$

In this case, we have

$$\mathscr{A}^* = [x^*, \infty) = \left[\frac{\mu_G^2 - \mu_B^2}{2\sigma^2(\beta - a - b)} + \frac{1}{\beta - a - b}\log\frac{\mathbb{P}(B)}{\mathbb{P}(G)}, \infty\right),$$

where

$$x^* := \frac{\mu_G^2 - \mu_B^2}{2\sigma^2(\beta - a - b)} + \frac{1}{\beta - a - b}\log\frac{\mathbb{P}(B)}{\mathbb{P}(G)}.$$

Chapter 9

Exercise 9.1 By differentiation of (9.2), *i.e.*

$$\begin{aligned}\mathbb{P}(\tau < T \mid \mathcal{F}_t) &:= \mathbb{P}(S_T < K \mid \mathcal{F}_t)\\ &= \Phi\left(-\frac{(\mu - \sigma^2/2)(T-t) + \log(S_t/K))}{\sigma\sqrt{T-t}}\right), \qquad T \geqslant t,\end{aligned}$$

with respect to T, we find

$$d\mathbb{P}(\tau \leqslant T \mid \mathcal{F}_t) = \frac{dT}{2\sigma\sqrt{2\pi(T-t)}}\left(\frac{\sigma^2}{2} - \mu + \frac{\log(S_t/K)}{T-t}\right)$$

$$\times \exp\left(-\frac{\left((\mu-\sigma^2/2))(T-t)+\log(S_t/K)\right)\right)^2}{2(T-t)\sigma^2}\right),$$

provided that $\mu < \sigma^2/2$.

Exercise 9.2 Consider the first hitting time

$$\tau_K := \inf\{u \geqslant t \ : \ S_u \leqslant K\}$$

of the level $K > 0$ starting from $S_t > K$. By Lemma 15.1 in Privault (2022), we have

$$\mathbb{E}^*\left[e^{-(\tau_K-t)r} \mid \mathcal{F}_t\right] = \left(\frac{K}{S_t}\right)^{2r/\sigma^2},$$

provided that $S_t \geqslant K$.

Exercise 9.3

a) We have

$$\begin{aligned}
\mathbb{E}[X_k X_l] &= \mathbb{E}\left[(a_k M + \sqrt{1-a_k^2} Z_k)(a_l M + \sqrt{1-a_l^2} Z_l)\right] \\
&= \mathbb{E}\left[a_k a_l M^2 + a_k M\sqrt{1-a_l^2} Z_l + a_l M\sqrt{1-a_k^2} Z_k + \sqrt{1-a_k^2}\sqrt{1-a_l^2} Z_k Z_l\right] \\
&= a_k a_l \mathbb{E}\left[M^2\right] + a_k\sqrt{1-a_l^2}\mathbb{E}[Z_l M] + a_l\sqrt{1-a_k^2}\mathbb{E}[Z_k M] \\
&\quad + \sqrt{1-a_k^2}\sqrt{1-a_l^2}\mathbb{E}[Z_k Z_l] \\
&= a_k a_l \mathbb{E}\left[M^2\right] + a_k\sqrt{1-a_l^2}\mathbb{E}[Z_l]\mathbb{E}[M] + a_l\sqrt{1-a_k^2}\mathbb{E}[Z_k]\mathbb{E}[M] \\
&\quad + \sqrt{1-a_k^2}\sqrt{1-a_l^2}\mathbb{1}_{\{k=l\}} \\
&= a_k a_l + (1-a_k^2)\mathbb{1}_{\{k=l\}} \\
&= \mathbb{1}_{\{k=l\}} + a_k a_l \mathbb{1}_{\{k\neq l\}}, \qquad k, l = 1, 2, \ldots, n,
\end{aligned}$$

b) We check that the vector $(X_1, \ldots, X_n)$, with covariance matrix (9.12) has the probability density function

$$\begin{aligned}
&\varphi(x_1, \ldots, x_n) \\
&= \frac{1}{(2\pi)^{n/2}} \prod_{k=1}^n (1-a_k^2)^{-1/2} \int_{-\infty}^{\infty} e^{-\frac{(x_1-a_1 m)^2}{2(1-a_1^2)}} \cdots e^{-\frac{(x_n-a_n m)^2}{2(1-a_n^2)}} \frac{e^{-m^2/2}}{\sqrt{2\pi}} dm
\end{aligned}$$

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which is jointly Gaussian, with marginals given by

$$\begin{aligned}
x_k \longmapsto & \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(x_1, \ldots, x_n) dx_1 \cdots dx_{k-1} dx_{k+1} \cdots dx_n \\
&= \frac{1}{\sqrt{2\pi(1-a_k^2)}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(x_k - a_k m)^2}{2(1-a_k^2)}} \frac{\mathrm{e}^{-m^2/2}}{\sqrt{2\pi}} dm \\
&= \frac{1}{2\pi\sqrt{1-a_k^2}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(x_k - a_k m)^2}{2(1-a_k^2)} - m^2/2} dm \\
&= \frac{1}{2\pi\sqrt{1-a_k^2}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{x_k^2 - 2a_k x_k m + m^2}{2(1-a_k^2)}} dm \\
&= \frac{\mathrm{e}^{-x_k^2/2}}{2\pi\sqrt{1-a_k^2}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(m - a_k x_k)^2}{2(1-a_k^2)}} dm \\
&= \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x_k^2/2}, \qquad x_k \in \mathbb{R}.
\end{aligned}$$

c) We have

$$\begin{aligned}
\varphi(x_1, \ldots, x_n) &= \frac{1}{(2\pi)^{n/2}} \prod_{k=1}^{n} (1-a_k^2)^{-1/2} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(x_1 - a_1 m)^2}{2(1-a_1^2)}} \cdots \mathrm{e}^{-\frac{(x_n - a_n m)^2}{2(1-a_n^2)}} \frac{\mathrm{e}^{-m^2/2}}{\sqrt{2\pi}} dm \\
&= \frac{1}{(2\pi)^{n/2}} \prod_{k=1}^{n} (1-a_k^2)^{-1/2} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}\left(\frac{x_1^2 + a_1^2 m^2 - 2x_1 a_1 m}{1-a_1^2} + \cdots + \frac{x_n^2 + a_n^2 m^2 - 2x_n a_n m}{1-a_n^2} + m^2\right)} \frac{dm}{\sqrt{2\pi}} \\
&= \frac{1}{(2\pi)^{n/2}\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\left(\frac{x_1^2}{1-a_1^2} + \cdots + \frac{x_n^2}{1-a_n^2}\right)} \prod_{k=1}^{n} (1-a_k^2)^{-1/2} \\
&\quad \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{m^2}{2}\left(1 + \frac{a_1^2}{1-a_1^2} + \cdots + \frac{a_n^2}{1-a_n^2}\right) + 2m\left(\frac{x_1 a_1}{2(1-a_1^2)} + \cdots + \frac{x_n a_n}{2(1-a_n^2)}\right)} dm \\
&= \frac{\mathrm{e}^{-\frac{1}{2}\left(\frac{x_1^2}{1-a_1^2} + \cdots + \frac{x_n^2}{1-a_n^2}\right)}}{\sqrt{(2\pi)^n (1-a_1^2) \cdots (1-a_n^2)}} \exp\left(\frac{\frac{1}{2}\left(\frac{x_1 a_1}{1-a_1^2} + \cdots + \frac{x_n a_n}{1-a_n^2}\right)^2}{1 + \frac{a_1^2}{1-a_1^2} + \cdots + \frac{a_n^2}{1-a_n^2}}\right) \\
&\quad \times \left(1 + \frac{a_1^2}{1-a_1^2} + \cdots + \frac{a_n^2}{1-a_n^2}\right)^{-1/2} \\
&= \frac{\mathrm{e}^{-\frac{1}{2}\left(\frac{x_1^2}{1-a_1^2} + \cdots + \frac{x_n^2}{1-a_n^2}\right)}}{\sqrt{(2\pi)^n \alpha^2 (1-a_1^2) \cdots (1-a_n^2)}} \exp\left(\frac{1}{2\alpha^2}\left(\frac{x_1 a_1}{1-a_1^2} + \cdots + \frac{x_n a_n}{1-a_n^2}\right)^2\right)
\end{aligned}$$

$$= \frac{e^{-\frac{1}{2}\left(\frac{x_1^2}{1-a_1^2}+\cdots+\frac{x_n^2}{1-a_n^2}\right)}}{\sqrt{(2\pi)^n\alpha^2(1-a_1^2)\cdots(1-a_n^2)}} \exp\left(\frac{1}{2\alpha^2}\left(\frac{x_1a_1}{1-a_1^2}+\cdots+\frac{x_na_n}{1-a_n^2}\right)^2\right)$$
$$= \frac{e^{-\frac{1}{2}\left(\frac{x_1^2}{1-a_1^2}\left(1-\frac{a_1^2}{\alpha^2(1-a_1^2)}\right)+\cdots+\frac{x_n^2}{1-a_n^2}\left(1-\frac{a_n^2}{\alpha^2(1-a_n^2)}\right)\right)}}{\sqrt{(2\pi)^n\alpha^2(1-a_1^2)\cdots(1-a_n^2)}} \exp\left(\frac{1}{2\alpha^2}\sum_{1\leqslant p\neq l\leqslant n}\frac{x_px_la_pa_l}{(1-a_p^2)(1-a_l^2)}\right)$$
$$= \frac{1}{\sqrt{(2\pi)^n\det\Sigma}}e^{-\frac{1}{2}\langle x,\Sigma^{-1}x\rangle},$$

where

$$\alpha^2 := 1+\frac{a_1^2}{1-a_1^2}+\cdots+\frac{a_n^2}{1-a_n^2},$$

and

$$\Sigma^{-1} = \frac{1}{\alpha^2}\begin{bmatrix} \frac{\alpha^2(1-a_1^2)-a_1^2}{(1-a_1^2)^2} & \frac{-a_1a_2}{(1-a_1^2)(1-a_2^2)} & \cdots & \frac{-a_1a_n}{(1-a_1^2)(1-a_n^2)} \\ \frac{-a_2a_1}{(1-a_2^2)(1-a_1^2)} & \frac{\alpha^2(1-a_2^2)-a_2^2}{(1-a_2^2)^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \frac{\alpha^2(1-a_{n-1}^2)-a_{n-1}^2}{(1-a_{n-1}^4)} & \frac{-a_{n-1}a_n}{(1-a_{n-1}^2)(1-a_n^2)} \\ \frac{-a_na_1}{(1-a_n^2)(1-a_1^2)} & \ddots & \frac{-a_na_{n-1}}{(1-a_n^2)(1-a_{n-1}^2)} & \frac{\alpha^2(1-a_n^2)-a_n^2}{(1-a_n^2)^2} \end{bmatrix}.$$

Exercise 9.4 We have

$$\Sigma = \begin{bmatrix} 1 & a_1a_2 \\ a_2a_1 & 1 \end{bmatrix},$$

and letting

$$\begin{aligned}\alpha^2 &:= 1+\frac{a_1^2}{1-a_1^2}+\frac{a_2^2}{1-a_2^2} \\ &= \frac{(1-a_1^2)(1-a_2^2)+a_1^2(1-a_2^2)+a_2^2(1-a_1^2)}{(1-a_1^2)(1-a_2^2)} \\ &= \frac{1-a_2^2a_1^2}{(1-a_1^2)(1-a_2^2)},\end{aligned}$$

we find

$$\Sigma^{-1} = \frac{1}{\alpha^2}\begin{bmatrix} \frac{\alpha^2(1-a_1^2)-a_1^2}{(1-a_1^2)^2} & -\frac{a_1a_2}{(1-a_1^2)(1-a_2^2)} \\ -\frac{a_2a_1}{(1-a_2^2)(1-a_1^2)} & \frac{\alpha^2(1-a_2^2)-a_2^2}{(1-a_2^2)^2} \end{bmatrix}$$
$$= \frac{1}{\alpha^2}\begin{bmatrix} \frac{\alpha^2}{1-a_1^2}\left(1-\frac{(1-a_2^2)a_1^2}{1-a_2^2a_1^2}\right) & -\frac{a_1a_2}{(1-a_1^2)(1-a_2^2)} \\ -\frac{a_2a_1}{(1-a_2^2)(1-a_1^2)} & \frac{\alpha^2}{1-a_2^2}\left(1-\frac{(1-a_1^2)a_2^2}{1-a_2^2a_1^2}\right) \end{bmatrix}$$
$$= \frac{1}{\alpha^2}\begin{bmatrix} \frac{\alpha^2}{1-a_2^2a_1^2} & -\frac{a_1a_2}{(1-a_1^2)(1-a_2^2)} \\ -\frac{a_2a_1}{(1-a_2^2)(1-a_1^2)} & \frac{\alpha^2}{1-a_2^2a_1^2} \end{bmatrix}$$
$$= \frac{(1-a_1^2)(1-a_2^2)}{1-a_2^2a_1^2}\begin{bmatrix} \frac{\alpha^2}{1-a_2^2a_1^2} & -\frac{a_1a_2}{(1-a_1^2)(1-a_2^2)} \\ -\frac{a_2a_1}{(1-a_2^2)(1-a_1^2)} & \frac{\alpha^2}{1-a_2^2a_1^2} \end{bmatrix}$$
$$= \frac{1}{\alpha^2}\begin{bmatrix} \frac{\alpha^2}{1-a_2^2a_1^2} & -\frac{a_1a_2}{(1-a_1^2)(1-a_2^2)} \\ -\frac{a_2a_1}{(1-a_2^2)(1-a_1^2)} & \frac{\alpha^2}{1-a_2^2a_1^2} \end{bmatrix}$$
$$= \frac{1}{1-a_2^2a_1^2}\begin{bmatrix} 1 & -a_1a_2 \\ -a_1a_2 & 1 \end{bmatrix}.$$

In particular, the case $n = 2$ is able to recover all two-dimensional copulas by setting the correlation coefficient $\rho = a_1a_2$. In the general case, Σ is parametrized by n numbers, which offers less degrees of freedom compared with the joint Gaussian copula correlation method which relies on $n(n-1)/2$ coefficients, see also Exercise 9.3.

Chapter 10

Exercise 10.1 By absence of arbitrage we have $(1-\alpha)\mathrm{e}^{r_dT} = \mathrm{e}^{rT}$, hence $\alpha = 1 - \mathrm{e}^{(r-r_d)T}$.

Exercise 10.2

a) The bond payoff $\mathbb{1}_{\{\tau > T-t\}}$ is discounted according to the risk-free rate, before taking expectation.
b) We have $\mathbb{E}\left[\mathbb{1}_{\{\tau > T-t\}}\right] = \mathrm{e}^{-\lambda(T-t)}$, hence $P_d(t,T) = \mathrm{e}^{-(\lambda+r)(T-t)}$.
c) We have $P_M(t,T) = \mathrm{e}^{-(\lambda+r)(T-t)}$, hence $\lambda = -r - \frac{1}{T-t}\log P_M(t,T)$.

Exercise 10.3

a) We have

$$r_t = -a\int_0^t r_s ds + \sigma B_t^{(1)}, \quad t \geqslant 0,$$

hence

$$\int_0^t r_s ds = \frac{1}{a}\big(\sigma B_t^{(1)} - r_t\big)$$
$$= \frac{\sigma}{a}\left(B_t^{(1)} - \int_0^t \mathrm{e}^{-(t-s)a} dB_s^{(1)}\right)$$
$$= \frac{\sigma}{a}\int_0^t (1-\mathrm{e}^{-(t-s)a}) dB_s^{(1)},$$

and

$$\int_t^T r_s ds = \int_0^T r_s ds - \int_0^t r_s ds$$
$$= \frac{\sigma}{a}\int_0^T (1-\mathrm{e}^{-(T-s)a}) dB_s^{(1)} - \frac{\sigma}{a}\int_0^t (1-\mathrm{e}^{-(t-s)a}) dB_s^{(1)}$$
$$= -\frac{\sigma}{a}\left(\int_0^t (\mathrm{e}^{-(T-s)a} - \mathrm{e}^{-(t-s)a}) dB_s^{(1)} + \int_t^T (\mathrm{e}^{-(T-s)a} - 1) dB_s^{(1)}\right)$$
$$= -\frac{\sigma}{a}(\mathrm{e}^{-(T-t)a} - 1)\int_0^t \mathrm{e}^{-(t-s)a} dB_s^{(1)} - \frac{\sigma}{a}\int_t^T (\mathrm{e}^{-(T-s)a} - 1) dB_s^{(1)}$$
$$= -\frac{1}{a}(\mathrm{e}^{-(T-t)a} - 1) r_t - \frac{\sigma}{a}\int_t^T (\mathrm{e}^{-(T-s)a} - 1) dB_s^{(1)}.$$

The answer for λ_t is similar.

b) As a consequence of the answer to the previous question, we have

$$\mathbb{E}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \;\Big|\; \mathcal{F}_t\right] = C(a,t,T) r_t + C(b,t,T)\lambda_t,$$

and

$$\mathrm{Var}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \;\Big|\; \mathcal{F}_t\right]$$
$$= \mathrm{Var}\left[\int_t^T r_s ds \;\Big|\; \mathcal{F}_t\right] + \mathrm{Var}\left[\int_t^T \lambda_s ds \;\Big|\; \mathcal{F}_t\right]$$
$$+2\,\mathrm{Cov}\left(\int_t^T X_s ds, \int_t^T Y_s ds \;\Big|\; \mathcal{F}_t\right)$$
$$= \frac{\sigma^2}{a^2}\int_t^T (\mathrm{e}^{-(T-s)a} - 1)^2 ds$$
$$+2\rho\frac{\sigma\eta}{ab}\int_t^T (\mathrm{e}^{-(T-s)a} - 1)(\mathrm{e}^{-(T-s)b} - 1) ds$$
$$+\frac{\eta^2}{b^2}\int_t^T (\mathrm{e}^{-(T-s)b} - 1)^2 ds$$
$$= \sigma^2 \int_t^T C^2(a,s,T) ds + 2\rho\sigma\eta\int_t^T C(a,s,T) C(b,s,T) ds$$

$$+\eta^2 \int_t^T C^2(b, sT)ds,$$

from the Itô isometry.

Exercise 10.4 (Exercise 10.3 continued).

a) We use the fact that $(r_t, \lambda_t)_{t\in[0,T]}$ is a Markov process.
b) We use the tower property (A.33) of the conditional expectation given $\mathcal{F}_t$.

c) Writing $F(t, r_t, \lambda_t) = P(t, T)$, we have

$$\begin{aligned}
& d\left(\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} P(t,T)\right) \\
& = -(r_t+\lambda_t)\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} P(t,T)dt + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} dP(t,T) \\
& = -(r_t+\lambda_t)\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} P(t,T)dt + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} dF(t,r_t,\lambda_t) \\
& = -(r_t+\lambda_t)\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} P(t,T)dt + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \frac{\partial F}{\partial x}(t,r_t,\lambda_t)dr_t \\
& + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \frac{\partial F}{\partial y}(t,r_t,\lambda_t)d\lambda_t + \frac{1}{2}\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \frac{\partial^2 F}{\partial x^2}(t,r_t,\lambda_t)\sigma_1^2(t,r_t)dt \\
& + \frac{1}{2}\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \frac{\partial^2 F}{\partial y^2}(t,r_t,\lambda_t)\sigma_2^2(t,\lambda_t)dt \\
& + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \rho \frac{\partial^2 F}{\partial x \partial y}(t,r_t,\lambda_t)\sigma_1(t,r_t)\sigma_2(t,\lambda_t)dt + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \frac{\partial F}{\partial t}(t,r_t,\lambda_t)dt \\
& = \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \frac{\partial F}{\partial x}(t,r_t,\lambda_t)\sigma_1(t,r_t)dB_t^{(1)} + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \frac{\partial F}{\partial y}(t,r_t,\lambda_t)\sigma_2(t,\lambda_t)dB_t^{(2)} \\
& + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \left(-(r_t+\lambda_t)P(t,T) + \frac{\partial F}{\partial x}(t,r_t,\lambda_t)\mu_1(t,r_t)\right. \\
& + \frac{\partial F}{\partial y}(t,r_t,\lambda_t)\mu_2(t,\lambda_t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,r_t,\lambda_t)\sigma_1^2(t,r_t) + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(t,r_t,\lambda_t)\sigma_2^2(t,\lambda_t) \\
& \left. + \rho\frac{\partial^2 F}{\partial x \partial y}(t,r_t,\lambda_t)\sigma_1(t,r_t)\sigma_2(t,\lambda_t) + \frac{\partial F}{\partial t}(t,r_t,\lambda_t)\right) dt,
\end{aligned}$$

hence the bond pricing PDE is

$$\begin{aligned}
& -(x+y)F(t,x,y) + \mu_1(t,x)\frac{\partial F}{\partial x}(t,x,y) \\
& + \mu_2(t,y)\frac{\partial F}{\partial y}(t,x,y) + \frac{1}{2}\sigma_1^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x,y) \\
& + \frac{1}{2}\sigma_2^2(t,y)\frac{\partial^2 F}{\partial y^2}(t,x,y) + \rho\sigma_1(t,x)\sigma_2(t,y)\frac{\partial^2 F}{\partial x \partial y}(t,x,y) + \frac{\partial F}{\partial t}(t,r_t,\lambda_t) = 0.
\end{aligned}$$

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d) We have

$$
\begin{aligned}
P(t,T) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^T r_s ds - \int_t^T \lambda_s ds\right) \ \Big| \ \mathcal{F}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}} \exp\left(-\mathbb{E}\left[\int_t^T r_s ds \ \Big| \ \mathcal{F}_t\right] - \mathbb{E}\left[\int_t^T \lambda_s ds \ \Big| \ \mathcal{F}_t\right]\right) \\
&\quad \times \exp\left(\frac{1}{2}\operatorname{Var}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \ \Big| \ \mathcal{F}_t\right]\right) \\
&= \mathbb{1}_{\{\tau>t\}} \exp\left(-C(a,t,T)r_t - C(b,t,T)\lambda_t\right) \\
&\quad \times \exp\left(\frac{\sigma^2}{2}\int_t^T C^2(a,s,T)ds + \frac{\eta^2}{2}\int_t^T C^2(b,s,T)\mathrm{e}^{-(T-s)b}ds\right) \\
&\quad \times \exp\left(\rho\sigma\eta\int_t^T C(a,s,T)C(b,s,T)ds\right).
\end{aligned}
$$

e) This is a direct consequence of the answers to Questions (c)) and (d)).

f) The above analysis shows that

$$
\begin{aligned}
\mathbb{P}(\tau > T \mid \mathcal{G}_t) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^T \lambda_s ds\right) \ \Big| \ \mathcal{F}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}} \exp\left(-C(b,t,T)\lambda_t + \frac{\eta^2}{2}\int_t^T C^2(b,s,T)ds\right),
\end{aligned}
$$

for $a = 0$ and

$$
\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) \ \Big| \ \mathcal{F}_t\right] = \exp\left(-C(a,t,T)r_t + \frac{\sigma^2}{2}\int_t^T C^2(a,s,T)ds\right),
$$

for $b = 0$, and this implies

$$
\begin{aligned}
U_\rho(t,T) &= \exp\left(\rho\sigma\eta\int_t^T C(a,s,T)C(b,s,T)ds\right) \\
&= \exp\left(\rho\frac{\sigma\eta}{ab}\left(T-t-C(a,t,T)-C(b,t,T)+C(a+b,t,T)\right)\right).
\end{aligned}
$$

g) We have

$$
\begin{aligned}
& f(t,T) = -\mathbb{1}_{\{\tau>t\}}\frac{\partial}{\partial T}\log P(t,T) \\
& = \mathbb{1}_{\{\tau>t\}}\left(r_t\mathrm{e}^{-(T-t)a} - \frac{\sigma^2}{2}C^2(a,t,T) + \lambda_t\mathrm{e}^{-(T-t)b} - \frac{\eta^2}{2}C^2(b,t,T)\right) \\
& \quad -\mathbb{1}_{\{\tau>t\}}\rho\sigma\eta C(a,t,T)C(b,t,T).
\end{aligned}
$$

h) We use the relation

$$\begin{aligned}\mathbb{P}(\tau > T \mid \mathcal{G}_t) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^T \lambda_s ds\right) \;\Big|\; \mathcal{F}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}} \exp\left(-C(b,t,T)\lambda_t + \frac{\eta^2}{2}\int_t^T C^2(b,s,T)ds\right) \\ &= \mathbb{1}_{\{\tau>t\}}\mathrm{e}^{-\int_t^T f_2(t,u)du},\end{aligned}$$

where $f_2(t,T)$ is the Vasicek forward rate corresponding to λ_t, *i.e.*

$$f_2(t,u) = \lambda_t \mathrm{e}^{-(u-t)b} - \frac{\eta^2}{2}C^2(b,t,u).$$

i) In this case we have $\rho = 0$ and

$$P(t,T) = \mathbb{P}(\tau > T \mid \mathcal{G}_t)\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) \;\Big|\; \mathcal{F}_t\right],$$

since $U_\rho(t,T) = 0$.

Chapter 11

Exercise 11.1 It suffices to check that as λ tends to ∞, the ratio

$$\frac{S_{T_i}^{i,j}\sum_{k=i}^{j-1}\delta_k \exp\left(-\sum_{p=i}^{k}\delta_p r_p - \lambda\sum_{p=i}^{k}\delta_p\right)}{(1-\xi)\sum_{k=i}^{j-1}\left(1-\mathrm{e}^{-\lambda\delta_k}\right)\exp\left(-\sum_{p=i}^{k}\delta_p r_p - \lambda\sum_{p=i}^{k}\delta_p\right)}$$

converges to 0, while it tends to $+\infty$ as λ goes to 0. Therefore, the equation (11.4) admits a numerical solution.

Exercise 11.2 Equation (11.4) reads

$$\begin{aligned}&S_{T_i}^{i,j}\sum_{k=i}^{j-1}\delta_k \exp\left(-\sum_{p=i}^{k}\delta_p r_p - \lambda\sum_{p=i}^{k}\delta_p\right) \\ &\quad = (1-\xi)\sum_{k=i}^{j-1}\left(\mathrm{e}^{\lambda\delta_k}-1\right)\exp\left(-\sum_{p=i}^{k}\delta_p r_p - \lambda\sum_{p=i}^{k}\delta_p\right),\end{aligned}$$

or

$$S_{T_i}^{i,j}\sum_{k=i}^{j-1}\delta_k P(0,T_{k+1})\exp\left(-\lambda\sum_{p=i}^{k}\delta_p\right)$$
$$=(1-\xi)\sum_{k=i}^{j-1}\big(\mathrm{e}^{\lambda\delta_k}-1\big)P(0,T_{k+1})\exp\left(-\lambda\sum_{p=i}^{k}\delta_p\right),$$

since

$$P(0,T_{k+1})=\exp\left(-\sum_{p=i}^{k}\delta_p r_p\right),\quad k=0,1,2.$$

From the terminal data of Figure 11.6 we infer with

$$\begin{cases} i=0,\\ j=3,\\ T_0=03/20/2015,\\ T_1=06/22/2015,\\ t=04/12/2015,\\ T_2=09/21/2015,\\ T_3=12/21/2015,\\ \delta_1=\delta_2=\delta_3=0.25,\\ \xi=0.4,\\ S_{T_1}^{1,3}=0.1079.\end{cases}$$

Hence, from the data of Figure S.14, Equation (11.4) rewrites as

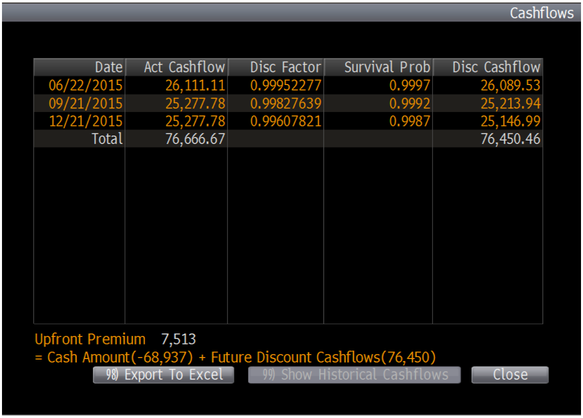
$$\begin{aligned}&0.1079\times 0.25\\ &\times\big(0.99952277\times\mathrm{e}^{-\lambda\times 0.25}+0.99827639\times\mathrm{e}^{-\lambda\times 0.5}+0.99607821\times\mathrm{e}^{-\lambda\times 0.75}\big)\\ &=(1-0.4)\times\big(\mathrm{e}^{\lambda\times 0.25}-1\big)\\ &\times\big(0.99952277\times\mathrm{e}^{-\lambda\times 0.25}+0.99827639\times\mathrm{e}^{-\lambda\times 0.5}+0.99607821\times\mathrm{e}^{-\lambda\times 0.75}\big),\end{aligned}$$

with solution $\lambda=0.0017987468$. The default probability is given by $p=1-\mathrm{e}^{-\lambda\times 0.75}=0.001348151$.

Next, from the discount factors of Figure S.14 we solve the Equation (11.4) numerically in Table S.1 below to find the default rate $\lambda_1=0.0017987468$ and default probability 0.0012460256, which is consistent with the value of 0.0013 in Figure 11.6, see also Castellacci (2008).

Date	Delta	Discount Factor	Premium Leg	Protection Leg
Jun 22, 2015	0.2611111	0.99952277	0.0002814722	0.0002814708
Sep 21, 2015	0.2527778	0.99827639	0.0002721533	0.000272154
Dec 21, 2015	0.2527778	0.99607821	0.0002715541	0.0002715548
		Sum	0.0008251796	0.0008251796

Table S.1: CDS Market data.



Cashflows

Date	Act Cashflow	Disc Factor	Survival Prob	Disc Cashflow
06/22/2015	26,111.11	0.99952277	0.9997	26,089.53
09/21/2015	25,277.78	0.99827639	0.9992	25,213.94
12/21/2015	25,277.78	0.99607821	0.9987	25,146.99
Total	76,666.67			76,450.46

Fig. S.14: CDS Price data.

Exercise 11.3

a) We have

$$\begin{aligned}
&\sum_{k=i}^{j-1} \mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1})\exp\left(-\int_t^{T_{k+1}} r(s)ds\right) \,\Big|\, \mathcal{G}_t\right] \\
&= \sum_{k=i}^{j-1} \mathbb{E}\left[\left(\mathbb{1}_{\{T_k<\tau\}} - \mathbb{1}_{\{T_{k+1}<\tau\}}\right)(1-\xi_{k+1})\exp\left(-\int_t^{T_{k+1}} r(s)ds\right) \,\Big|\, \mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}} \sum_{k=i}^{j-1} \mathbb{E}\left[(1-\xi_{k+1})\left(\mathrm{e}^{-\int_t^{T_k}\lambda_s ds} - \mathrm{e}^{-\int_t^{T_{k+1}}\lambda_s ds}\right)\mathrm{e}^{-\int_t^{T_{k+1}} r(s)ds} \,\Big|\, \mathcal{F}_t\right]
\end{aligned}$$

$$= \mathbb{1}_{\{\tau>t\}}(1-\xi)\sum_{k=i}^{j-1} \mathrm{e}^{-\int_t^{T_{k+1}} r(s)ds} \mathbb{E}\left[\mathrm{e}^{-\int_t^{T_k} \lambda_s ds} - \mathrm{e}^{-\int_t^{T_{k+1}} \lambda_s ds} \ \Big| \ \mathcal{F}_t \right]$$

$$= \mathbb{1}_{\{\tau>t\}}(1-\xi)\sum_{k=i}^{j-1} P(t,T_{k+1})\left(Q(t,T_k)-Q(t,T_{k+1})\right).$$

b) We have

$$V^p(t,T) = S_t^{i,j}\sum_{k=i}^{j-1}\delta_k \mathbb{E}\left[\mathbb{1}_{\{\tau>T_{k+1}\}}\exp\left(-\int_t^{T_{k+1}} r(s)ds\right) \ \Big| \ \mathcal{G}_t\right]$$

$$= S_t^{i,j}\sum_{k=i}^{j-1}\delta_k \mathbb{E}\left[\mathbb{1}_{\{T_{k+1}<\tau\}}\exp\left(-\int_t^{T_{k+1}} r(s)ds\right) \ \Big| \ \mathcal{G}_t\right]$$

$$= \mathbb{1}_{\{\tau>t\}}S_t^{i,j}\sum_{k=i}^{j-1}\delta_k \mathbb{E}\left[\exp\left(-\int_t^{T_{k+1}} \lambda_s ds\right)\exp\left(-\int_t^{T_{k+1}} r(s)ds\right) \ \Big| \ \mathcal{F}_t\right]$$

$$= S_t^{i,j}\mathbb{1}_{\{\tau>t\}}\sum_{k=i}^{j-1}\delta_k \exp\left(-\int_t^{T_{k+1}} r(s)ds\right)\mathbb{E}\left[\exp\left(-\int_t^{T_{k+1}} \lambda_s ds\right) \ \Big| \ \mathcal{F}_t\right]$$

$$= \mathbb{1}_{\{\tau>t\}}S_t^{i,j}\sum_{k=i}^{j-1}\delta_k P(t,T_{k+1})Q(t,T_{k+1}).$$

c) By equating the protection and premium legs, we find

$$(1-\xi)\sum_{k=i}^{j-1} P(t,T_{k+1})\left(Q(t,T_k)-Q(t,T_{k+1})\right)$$
$$= S_t^{i,j}\sum_{k=i}^{j-1}\delta_k P(t,T_{k+1})Q(t,T_{k+1}).$$

For $j=i+1$, this yields

$$(1-\xi)P(t,T_{i+1})\left(Q(t,T_i)-Q(t,T_{i+1})\right) = S_t^{i,i+1}\delta_i P(t,T_{i+1})Q(t,T_{i+1}),$$

hence

$$Q(t,T_{i+1}) = \frac{1-\xi}{S_t^{i,i+1}\delta_i+1-\xi},$$

with $Q(t,T_i)=1$, and the recurrence relation

$$(1-\xi)P(t,T_{j+1})\left(Q(t,T_j)-Q(t,T_{j+1})\right)$$

$$+ (1-\xi)\sum_{k=i}^{j-1} P(t,T_{k+1})\left(Q(t,T_k)-Q(t,T_{k+1})\right)$$

$$= S_t^{i,j}\delta_j P(t,T_{j+1})Q(t,T_{j+1}) + S_t^{i,j}\sum_{k=i}^{j-1}\delta_k P(t,T_{k+1})Q(t,T_{k+1}),$$

i.e.

$$Q(t,T_{j+1}) = \frac{(1-\xi)Q(t,T_j)}{1-\xi+S_t^{i,j}\delta_j} + \sum_{k=i}^{j-1}\frac{P(t,T_{k+1})\big((1-\xi)Q(t,T_k)-Q(t,T_{k+1})\big((1-\xi)+\delta_k S_t^{i,j}\big)\big)}{P(t,T_{j+1})(1-\xi+S_t^{i,j}\delta_j)}.$$

Exercise 11.4 (Exercise 11.3 continued). From the terminal data of Figure 11.7, we find the following spread data and survival probabilities:

k	Maturity	T_k	$S_t^{1,k}$ (bp)	$Q(t,T_k)$
1	6M	0.5	10.97	0.999087
2	1Y	1	12.25	0.997961
3	2Y	2	14.32	0.995235
4	3Y	3	19.91	0.990037
5	4Y	4	26.48	0.982293
6	5Y	5	33.29	0.972122
7	7Y	7	52.91	0.937632
8	10Y	10	71.91	0.880602

Table S.2: Spread and survival probabilities.

Background on Probability Theory

Exercise A.1

a) We have

$$\mathbb{E}[X] = \sum_{k\geqslant 0} k\mathbb{P}(X=k) = \mathrm{e}^{-\lambda}\sum_{k\geqslant 0} k\frac{\lambda^k}{k!}$$

$$= \mathrm{e}^{-\lambda}\sum_{k\geqslant 1}\frac{\lambda^k}{(k-1)!} = \lambda\mathrm{e}^{-\lambda}\sum_{k\geqslant 0}\frac{\lambda^k}{k!} = \lambda.$$

b) We have

$$\begin{aligned}\mathbb{E}[X^2] &= \sum_{k\geqslant 0} k^2 \mathbb{P}(X=k) \\ &= \mathrm{e}^{-\lambda} \sum_{k\geqslant 1} k^2 \frac{\lambda^k}{k!} \\ &= \mathrm{e}^{-\lambda} \sum_{k\geqslant 1} k \frac{\lambda^k}{(k-1)!} \\ &= \mathrm{e}^{-\lambda} \sum_{k\geqslant 2} \frac{\lambda^k}{(k-2)!} + \mathrm{e}^{-\lambda} \sum_{k\geqslant 1} \frac{\lambda^k}{(k-1)!} \\ &= \lambda^2 \mathrm{e}^{-\lambda} \sum_{k\geqslant 0} \frac{\lambda^k}{k!} + \lambda \mathrm{e}^{-\lambda} \sum_{k\geqslant 0} \frac{\lambda^k}{k!} \\ &= \lambda^2 + \lambda,\end{aligned}$$

and

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \lambda = \mathbb{E}[X].$$

Exercise A.2 We have

$$\begin{aligned}\mathbb{P}(\mathrm{e}^X > c) = \mathbb{P}(X > \log c) &= \int_{\log c}^\infty \mathrm{e}^{-y^2/(2\eta^2)} \frac{dy}{\sqrt{2\pi\eta^2}} \\ &= \int_{(\log c)/\eta}^\infty \mathrm{e}^{-y^2/2} \frac{dy}{\sqrt{2\pi}} = 1 - \Phi((\log c)/\eta) = \Phi(-(\log c)/\eta).\end{aligned}$$

Exercise A.3

a) Using the change of variable $z = (x-\mu)/\sigma$, we have

$$\begin{aligned}\int_{-\infty}^\infty \varphi(x) dx &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^\infty \mathrm{e}^{-(x-\mu)^2/(2\sigma^2)} dx \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^\infty \mathrm{e}^{-y^2/(2\sigma^2)} dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \mathrm{e}^{-z^2/2} dz.\end{aligned}$$

Next, using the polar change of coordinates $dxdy = rdrd\theta$, we find†

† "In a discussion with Grothendieck, Messing mentioned the formula expressing the integral of e^{-x^2} in terms of π, which is proved in every calculus course. Not only did

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$$\begin{aligned}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-z^2/2}dz\right)^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-y^2/2}dy\int_{-\infty}^{\infty}\mathrm{e}^{-z^2/2}dz\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-(y^2+z^2)/2}dydz\\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} r\mathrm{e}^{-r^2/2}drd\theta\\ &= \int_0^{\infty} r\mathrm{e}^{-r^2/2}dr\\ &= \lim_{R\to+\infty}\int_0^R r\mathrm{e}^{-r^2/2}dy\\ &= -\lim_{R\to+\infty}\left[\mathrm{e}^{-r^2/2}\right]_0^R\\ &= \lim_{R\to+\infty}(1-\mathrm{e}^{-R^2/2})\\ &= 1,\end{aligned}$$

or

$$\int_{-\infty}^{\infty}\mathrm{e}^{-z^2/2}dz = \sqrt{2\pi}.$$

b) We have

$$\begin{aligned}\mathbb{E}[X] &= \int_{-\infty}^{\infty} x\varphi(x)dx\\ &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} x\mathrm{e}^{-(x-\mu)^2/(2\sigma^2)}dx\\ &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}(\mu+y)\mathrm{e}^{-y^2/(2\sigma^2)}dx\\ &= \frac{\mu}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-y^2/2}dy + \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y\mathrm{e}^{-y^2/2}dy\\ &= \frac{\mu}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-y^2/2}dy + \frac{\sigma}{\sqrt{2\pi}}\lim_{A\to+\infty}\int_{-A}^{A} y\mathrm{e}^{-y^2/2}dy\\ &= \frac{\mu}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-y^2/2}dy\\ &= \mu\int_{-\infty}^{\infty}\varphi(y)dy\\ &= \mu\mathbb{P}(X\in\mathbb{R})\\ &= \mu,\end{aligned}$$

by symmetry of the function $y \longmapsto y\mathrm{e}^{-y^2/2}$ on $\mathbb{R}$.

c) Similarly, by integration by parts twice on $\mathbb{R}$, we find

Grothendieck not know the formula, but he thought that he had never seen it in his life". Milne (2005).

$$\begin{aligned}
\mathbb{E}[(X-\mathbb{E}[X])^2] &= \int_{-\infty}^{\infty}(x-\mu)^2\varphi(x)dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} y^2 \mathrm{e}^{-(y-\mu)^2/(2\sigma^2)}dy \\
&= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y \times y\mathrm{e}^{-y^2/2}dy \\
&= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-y^2/2}dy \\
&= \sigma^2.
\end{aligned}$$

d) By a completion of squares argument, we have

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^X] &= \int_{-\infty}^{\infty} \mathrm{e}^{x}\varphi(x)dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} \mathrm{e}^{x-(x-\mu)^2/(2\sigma^2)}dx \\
&= \frac{\mathrm{e}^{\mu}}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} \mathrm{e}^{y-y^2/(2\sigma^2)}dy \\
&= \frac{\mathrm{e}^{\mu}}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} \mathrm{e}^{\sigma^2/2+(y-\sigma^2)^2/(2\sigma^2)}dy \\
&= \frac{\mathrm{e}^{\mu}+\sigma^2/2}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} \mathrm{e}^{x^2/(2\sigma^2)}dy \\
&= \mathrm{e}^{\mu}+\frac{\sigma^2}{2}.
\end{aligned}$$

Exercise A.4

a) We have

$$\begin{aligned}
\mathbb{E}[X^+] &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} x^+\mathrm{e}^{-x^2/(2/\sigma^2)}dx \\
&= \frac{\sigma}{\sqrt{2\pi}}\int_{0}^{\infty} x\mathrm{e}^{-x^2/2}dx \\
&= \frac{\sigma}{\sqrt{2\pi}}\left[-\mathrm{e}^{-x^2/2}\right]_{x=0}^{x=\infty} \\
&= \frac{\sigma}{\sqrt{2\pi}}.
\end{aligned}$$

b) We have

$$\mathbb{E}[(X-K)^+] = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}(x-K)^+\mathrm{e}^{-x^2/(2\sigma^2)}dx$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi\sigma^2}}\int_K^\infty (x-K)\mathrm{e}^{-x^2/(2\sigma^2)}dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\int_K^\infty x\mathrm{e}^{-x^2/(2\sigma^2)}dx - \frac{K}{\sqrt{2\pi\sigma^2}}\int_K^\infty \mathrm{e}^{-x^2/(2\sigma^2)}dx \\
&= \frac{\sigma}{\sqrt{2\pi}}\left[-\mathrm{e}^{-x^2/(2\sigma^2)}\right]_{x=K}^\infty - \frac{K}{\sqrt{2\pi}}\int_{-\infty}^{-K/\sigma} \mathrm{e}^{-x^2/2}dx \\
&= \frac{\sigma}{\sqrt{2\pi}}\mathrm{e}^{-K^2/(2\sigma)^2} - K\Phi\left(-\frac{K}{\sigma}\right).
\end{aligned}
$$

c) Similarly, we have

$$
\begin{aligned}
\mathbb{E}[(K-X)^+] &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^\infty (K-x)^+\mathrm{e}^{-x^2/(2\sigma^2)}dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^K (K-x)\mathrm{e}^{-x^2/(2\sigma^2)}dx \\
&= \frac{K}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^K \mathrm{e}^{-x^2/(2\sigma^2)}dx - \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^K x\mathrm{e}^{-x^2/(2\sigma^2)}dx \\
&= \frac{K}{\sqrt{2\pi}}\int_{-\infty}^{K/\sigma} \mathrm{e}^{-x^2/2}dx - \frac{\sigma}{\sqrt{2\pi}}\left[-\mathrm{e}^{-x^2/(2\sigma^2)}\right]_{-\infty}^{x=K} \\
&= \frac{\sigma}{\sqrt{2\pi}}\mathrm{e}^{-K^2/(2\sigma)^2} + K\Phi\left(\frac{K}{\sigma}\right).
\end{aligned}
$$

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